Mach Limits in Analytic Spaces

Juhi Jang, Igor Kukavica, and Linfeng Li

ABSTRACT. We address the Mach limit problem for the Euler equations in the analytic spaces. We prove that, given analytic data, the solutions to the compressible Euler equations are uniformly bounded in a suitable analytic norm and then show that the convergence toward the incompressible Euler solution holds in the analytic norm. We also show that the same results hold more generally for Gevrey data with the convergence in the Gevrey norms.

1. Introduction

The low Mach number limit problem, which concerns the passage from slightly compressible flows to incompressible flows, is a classical singular limit problem in mathematical fluid dynamics. The problem has both physical and mathematical importance. There have been many significant works on the subject and a great deal of progress made in recent decades [A1, A3, As, E, FKM, I, Is1, Is2, Is3, KM1, KM2, MS, S1, S2, U]. The main difficulty of the problem is the presence of different wave speeds, which play a significant role in the limit process. In particular, one has to address the vanishing of the acoustic waves in the limit. A study of the low Mach number limit involves two parts: the uniform bounds and existence of slightly compressible flows for a time-independent of Mach numbers and convergence to solutions of the limiting equations. Interestingly, the analysis of such a singular limit problem significantly changes depending whether compressible fluids are isentropic or non-isentropic, if compressible fluids are inviscid or viscous, if initial data are well-prepared or not, if the problem is set in the whole space or domains with boundaries, or which regularity space of data is considered. In this paper, we address the low Mach number limit of the non-isentropic compressible Euler flows in $\mathbb{R}^3$ in analytic and, more generally, in Gevrey spaces.

Before describing the results, we briefly review prior relevant works (cf. [A1, A3, MS] for more extensive reviews). For isentropic flows or well-prepared initial data, it is well-known that solutions of the compressible Euler equations with low Mach numbers exist in Sobolev spaces for a time interval independent of the Mach numbers [KM1, KM2, S1]. When initial data are well-prepared, solutions converge to the solutions of the corresponding incompressible Euler equations with the limiting initial data [KM1, KM2, S1]. For the isentropic flows with general initial data, the convergence is not uniform for times close to zero and initial layers are present [As, U, I, Is1, Is2, Is3]. On the other hand, the non-isentropic problem with general initial data is much more involved. In this case, the pressure depends not only on the density but also on the entropy that enters into the coefficients of the linearized equations, and the convergence is more subtle because the acoustic waves are governed by a wave equation with variable coefficients. The first existence and convergence of the non-isentropic problem were given in [MS] and the existence result for general domains with boundary and the convergence result for exterior domains were obtained in [A1]. The results above were obtained in Sobolev spaces. Recently the low Mach number limit was studied in [FKM] starting from dissipative measure-valued solutions of the isentropic Euler equations. Also, the Mach limit in the domains with evolving boundary was addressed in [DE, DL], while for the dissipative case, see [A2, D1, D2, DG, DM, F, FN, H, LM, M]. For other works on analyticity for the equations involving fluids, see [B, BB, BGK, CKV, LO], while for different approaches to analyticity, cf. [Bi, BF, BoGK, FT, G, GK, KP, OT].

This paper concerns the non-isentropic equations with general analytic or Gevrey initial data in $\mathbb{R}^3$ and convergence holding in these strong norms. The first result provides a uniform in ϵ bound of the analytic solution, where

$\epsilon > 0$ represents the Mach number, while the second result asserts the convergence of the solution to the limiting equation as ϵ tends to zero. The main difficulty is in obtaining the uniform analytic bound. The Mach limit in an analytic norm is then proven by interpolating the uniform boundedness result and the convergence in the Sobolev space due to Métivier and Schochet in [**MS**].

For the isentropic case, the standard energy estimate method can be applied to the velocity equation to obtain analytic estimates. However, for the non-isentropic case, the problem is more difficult since the matrix, $E(\epsilon u^\epsilon, S^\epsilon)$ (cf. the formulation (2.14)–(2.15)), also depends on S^ϵ, and thus spatial derivative bounds cannot be obtained solely by the fundamental energy estimates. Moreover, the non-isentropic Euler flows feature intriguing wave-transport structure: The divergence component of the modified velocity is governed by nonlinear acoustic equations, while the curl component and entropy are transported, and their interactions are coupled. Thus a careful analysis that captures the coupled structure of the modified velocity and the entropy is required. To accomplish these, we use the elliptic regularity for the velocity to reduce the spatial derivative to divergence and curl components. The key to the former is that the divergence equation for the velocity is properly balanced with the analytic energy solution, which motivates us to include time derivatives using $\epsilon\partial_t$ to our analytic norm; for the latter, we appeal to the transport equation of the curl component, which can be treated in a similar way as the entropy. Thus, the pure time analytic norm needs to be treated differently than the one which also involves the spatial derivatives (cf. Sections 6.2 and 6.3 respectively). It is important to include the analytic weight κ in (3.3), which ultimately balances the time and the spatial derivatives. The main difficulty in our approach is the handling of the vorticity ω, which can not be treated directly. Instead, as in [**A1**], we need to consider the equation for the modified vorticity $\operatorname{curl}(r_0 v)$, where r_0 is a certain function of the entropy (cf. Section 6.1 below). The product and chain rules then lead to complicated analytic coupling among the entropy, divergence, vorticity, and $\operatorname{curl}(r_0 v)$.

The paper is organized as follows. In Section 2, we introduce the Mach number limit problem and then formulate the symmetrized version of the compressible Euler equations. In Section 3, we define the analytic norm and state the main results. The first theorem relies on Lemma 3.3, the proof of which is given at the end of Section 6. We present the energy estimate for the transport equation in Section 4. Product rule and chain rules in analytic spaces are provided in Section 5. In Section 6, we estimate the curl, divergence, and time-derivative components of the velocity. In Section 7, we prove the convergence theorem. In Section 8, we establish the finiteness of the space-time analytic norm at the initial time under the assumption that the initial data is real-analytic in the spatial variable. In Section 9, we provide the Mach limit theorem in any Gevrey space.

2. Set-up

We consider the compressible Euler equations describing the motion of an inviscid, non-isentropic gaseous fluid in $\mathbb{R}^3$

$$\partial_t \rho + v \cdot \nabla \rho + \rho \nabla \cdot v = 0 \tag{2.1}$$

$$\rho\,(\partial_t v + v \cdot \nabla v) + \nabla P = 0 \tag{2.2}$$

$$\partial_t S + v \cdot \nabla S = 0, \tag{2.3}$$

where $\rho = \rho(x,t) \in \mathbb{R}_+$ is the density, $v = v(x,t) \in \mathbb{R}^3$ is the velocity, $P = P(x,t) \in \mathbb{R}_+$ is the pressure, and $S = S(x,t) \in \mathbb{R}$ is the entropy of the fluid. The system (2.1)–(2.3) is closed with the equation of state

$$P = P(\rho, S). \tag{2.4}$$

For instance, the equation of state for an ideal gas takes the form

$$P(\rho, S) = \rho^\gamma e^S, \tag{2.5}$$

where $\gamma > 1$ is the adiabatic exponent.

To address the low Mach number limit, we introduce the rescalings

$$\tilde{t} = \epsilon t, \qquad \tilde{x} = x, \qquad \tilde{\rho} = \rho, \qquad \tilde{v} = \frac{v}{\epsilon}, \qquad \tilde{P} = P, \qquad \tilde{S} = S,$$

where $\epsilon > 0$ represents the Mach number, the ratio of the typical fluid speed to the typical sound speed. We assume that the typical sound speed is $O(1)$. For simplicity of notation, we omit tilde, and obtain the rescaled system

$$\partial_t \rho + v \cdot \nabla \rho + \rho \nabla \cdot v = 0 \tag{2.6}$$

$$\rho \left(\partial_t v + v \cdot \nabla v \right) + \frac{1}{\epsilon^2} \nabla P = 0 \tag{2.7}$$

$$\partial_t S + v \cdot \nabla S = 0. \tag{2.8}$$

The goal of this paper is to obtain the low Mach number limit of (2.6)–(2.8) in analytic spaces.

2.1. Reformulation. Now, consider P, instead of ρ, as an independent variable, we may write (2.4) as

$$\rho = \rho(P, S),$$

and (2.6) is then replaced by

$$A_0 \left(\partial_t P + v \cdot \nabla P \right) + \nabla \cdot v = 0, \tag{2.9}$$

where

$$A_0 = A_0(S, P) = \frac{1}{\rho(S, P)} \frac{\partial \rho(S, P)}{\partial P}.$$

The equation of state for an ideal gas in (2.5) then reads as

$$\rho(P, S) = P^{\frac{1}{\gamma}} e^{-\frac{S}{\gamma}}.$$

To symmetrize the Euler equations, we set

$$P = \bar{P} e^{\epsilon p},$$

for a positive constant $\bar{P}$ which represents the reference state at the spatial infinity so that $P = \bar{P} + O(\epsilon)$. Using $\partial_t P = \epsilon P \partial_t p$ and $\nabla P = \epsilon P \nabla p$, we rewrite (2.9) and (2.7) as

$$a \left(\partial_t p + v \cdot \nabla p \right) + \frac{1}{\epsilon} \nabla \cdot v = 0 \tag{2.10}$$

$$r \left(\partial_t v + v \cdot \nabla v \right) + \frac{1}{\epsilon} \nabla p = 0, \tag{2.11}$$

respectively, where

$$a = a(S, \epsilon p) = A_0(S, \bar{P} e^{\epsilon p}) \bar{P} e^{\epsilon p} \tag{2.12}$$

and

$$r = r(S, \epsilon p) = \frac{\rho(S, \bar{P} e^{\epsilon p})}{\bar{P} e^{\epsilon p}}. \tag{2.13}$$

In the case of an ideal gas, from $\rho(P, S) = P^{\frac{1}{\gamma}} e^{-\frac{S}{\gamma}}$, we have the expression

$$a = \frac{1}{\gamma}$$

for a, and

$$r = (\bar{P} e^{\epsilon p})^{\frac{1}{\gamma} - 1} e^{-\frac{S}{\gamma}}$$

for r. Thus we have obtained the symmetrized version of the compressible Euler equation for non-isentropic fluids in $\mathbb{R}^3$, which reads

$$E(S, \epsilon u)(\partial_t u + v \cdot \nabla u) + \frac{1}{\epsilon} L(\partial_x) u = 0, \tag{2.14}$$

$$\partial_t S + v \cdot \nabla S = 0, \tag{2.15}$$

where $u = (p, v)$ and

$$E(S, \epsilon u) = \begin{pmatrix} a(S, \epsilon u) & 0 \\ 0 & r(S, \epsilon u) I_3 \end{pmatrix}, \qquad L(\partial_x) = \begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{pmatrix}. \tag{2.16}$$

After transforming (2.1)–(2.3) to the symmetrized form (2.14)–(2.15), we now focus on the formulation (2.14)–(2.15). In view of (2.12) and (2.13), we assume

$$a(S,\epsilon u) = f_1(S)g_1(\epsilon u) \tag{2.17}$$

and

$$r(S,\epsilon u) = f_2(S)g_2(\epsilon u), \tag{2.18}$$

where f_1, f_2, g_1, and g_2 are positive entire real-analytic functions.

3. The main results

We assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies

$$\|(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{H^5} \le M_0 \tag{3.1}$$

and

$$\sum_{m=0}^{\infty}\sum_{|\alpha|=m} \|\partial^\alpha(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{L^2}\frac{\tau_0^{(m-3)_+}}{(m-3)!} \le M_0, \tag{3.2}$$

where $\tau_0, M_0 > 0$ are fixed constants. For $\tau > 0$, define the mixed weighted analytic space

$$A(\tau) = \{u \in C^\infty(\mathbb{R}^3) : \|u\|_{A(\tau)} < \infty\},$$

where

$$\|u\|_{A(\tau)} = \sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}\tau(t)^{(m-3)_+}}{(m-3)!}; \tag{3.3}$$

here, $\tau \in (0,1]$ represents the mixed space-time analyticity radius and where $\kappa > 0$. It is convenient that the term with $\|u\|_{L^2}$ is not included in the norm. In (3.2) and below we use the convention $n! = 1$ when $n \in -\mathbb{N}$. As shown in Section 8 below, (3.2) implies that with $\kappa = 1$

$$\|(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{A(\tilde\tau_0)} \le Q(M_0) \tag{3.4}$$

for some function Q, where $\tilde\tau_0 = \tau_0/Q(M_0)$ is a sufficiently small constant. Note that the time derivatives of the initial data are defined iteratively by differentiating the equations (2.14)–(2.15) and evaluating at $t = 0$ (cf. Section 8 below for details). Also observe that the norm in (3.3) is an increasing function of κ, and thus (3.4) holds for any $\kappa \in (0,1]$. We define the analyticity radius function

$$\tau(t) = \tau(0) - Kt, \tag{3.5}$$

where $\tau(0) \le \min\{1, \tilde\tau_0\}$ is a sufficiently small parameter (different from $\tilde\tau_0$), and $K \ge 1$ is a sufficiently large parameter, both to be determined below.

The first theorem provides a uniform in ϵ boundedness of the analytic norm on a time interval, which is independent of ϵ.

THEOREM 3.1. *Assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies (3.1)–(3.2), where $M_0, \tau_0 > 0$. There exist sufficiently small constants $\kappa, \tau(0), \epsilon_0, T_0 > 0$, depending on M_0, such that*

$$\|(p^\epsilon, v^\epsilon, S^\epsilon)(t)\|_{A(\tau)} \le M, \qquad 0 < \epsilon \le \epsilon_0, \qquad t \in [0, T_0], \tag{3.6}$$

where τ is as in (3.5) and K and M are sufficiently large constants depending on M_0.

We now turn to the Mach limit for solutions of (2.14)–(2.15) in $\mathbb{R}^3$ as $\epsilon \to 0$. Denote

$$\delta = \frac{\kappa\tau(0)}{C_0}, \tag{3.7}$$

where $\tau(0), \kappa \in (0,1]$ are fixed constants chosen in the proof of Theorem 3.1, and $C_0 > 1$ is a sufficiently large constant to be chosen in Section 7. We introduce the spatial analytic norm

$$\|u\|_{X_\delta} = \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^2} \frac{\delta^{(m-3)_+}}{(m-3)!}, \tag{3.8}$$

where $\delta > 0$ is as in (3.7). Note that this is a part of our main analytic A-norm, (3.2).

By Theorem 3.1, for a given M_0 and $\tau_0 > 0$, the solutions $(p^\epsilon, v^\epsilon, S^\epsilon)$ are uniformly bounded by M in the norm of $C^0([0,T_0], X_\delta)$ for fixed parameters κ, T_0, and $\epsilon \in (0, \epsilon_0]$. The second main theorem shows that solutions of (2.14)–(2.15) converge to the solution of the stratified incompressible Euler equations

$$r(S,0)(\partial_t v + v \cdot \nabla v) + \nabla \pi = 0, \tag{3.9}$$

$$\operatorname{div} v = 0, \tag{3.10}$$

$$\partial_t S + v \cdot \nabla S = 0, \tag{3.11}$$

as $\epsilon \to 0$.

THEOREM 3.2. *Let $\delta > 0$, and assume that the initial data $(v_0^\epsilon, S_0^\epsilon)$ converges to (v_0, S_0) in X_δ and in L^2 as $\epsilon \to 0$, and S_0^ϵ decays sufficiently rapidly at infinity in the sense*

$$|S_0^\epsilon(x)| \le C|x|^{-1-\zeta}, \qquad |\nabla S_0^\epsilon(x)| \le C|x|^{-2-\zeta},$$

for $0 < \epsilon \le \epsilon_0$ and some constants C and $\zeta > 0$. Then $(v^\epsilon, p^\epsilon, S^\epsilon)$ converges to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $C^0([0,T_0], X_\delta)$, where $(v^{(\mathrm{inc})}, S^{(\mathrm{inc})})$ is the solution to (3.9)–(3.11) with the initial data (w_0, S_0), and w_0 is the unique solution of

$$\operatorname{div} w_0 = 0,$$
$$\operatorname{curl}(r_0 w_0) = \operatorname{curl}(r_0 v_0),$$

with $r_0 = r(S_0, 0)$.

In the rest of the paper, the constant C depends only on M_0 and τ_0, and it may vary from relation to relation; we omit the superscript ϵ, and we write S, u for S^ϵ, u^ϵ.

Theorem 3.2 is proven in Section 7 below as a consequence of Theorem 3.1. The proof of Theorem 3.1 consists of a priori estimates performed on the solutions. The a priori estimates are easily justified by simply restricting the sum (3.2) to $m \le m_0$ where $m_0 \in \{6, 7, \dots\}$ is arbitrary. The estimates on the finite sums are justified since boundedness of solutions in any Sobolev norm is known by [**A1**].

The proof of Theorem 3.1 relies on analytic a priori estimates on the entropy S and the (modified) velocity u. The a priori estimate needed to prove Theorem 3.1 is the following.

LEMMA 3.3. Let $M_0 > 0$. For any $\kappa \le 1$, there exist constants C, τ_1, ϵ_0, T_0 and a nonnegative continuous function Q such that for all $\epsilon \in (0, \epsilon_0]$, the norm

$$M_{\epsilon,\kappa}(T) = \sup_{t \in [0,T]} \left(\|S(t)\|_{A(\tau(t))} + \|u(t)\|_{A(\tau(t))}\right) \tag{3.12}$$

satisfies the estimate

$$M_{\epsilon,\kappa}(t) \le C + (t + \epsilon + \kappa + \tau(0))\, Q(M_{\epsilon,\kappa}(t)), \tag{3.13}$$

for $t \in [0, T_0]$ and $\tau(0) \in (0, \tau_1]$, provided

$$K \ge Q(M_{\epsilon,\kappa}(T_0)) \tag{3.14}$$

holds.

With $\tau = \tau(t)$ as in (3.5), we use the notation (3.12). The constant K depends on M (and thus ultimately on M_0), i.e., $K = Q(M)$. We shall work on an interval of time such that

$$T_0 \le \frac{\tau(0)}{2K}. \tag{3.15}$$

Thus we have $\tau(0)/2 \leq \tau(t) \leq \tau(0)$ for $t \in [0, T_0]$.

From here on, we denote by Q a positive increasing continuous function, which may change from inequality to inequality; importantly, the function Q does not depend on ϵ, κ, and t. The estimates are performed on an interval of time $[0, T]$ where (3.5) holds and is such that

$$T \leq \frac{\tau(0)}{2K}.$$

In the rest of the paper, we allow all the constants to depend on τ_0.

PROOF OF THEOREM 3.1 GIVEN LEMMA 3.3. Let $M_0 > 0$ be as in (3.1)–(3.2). Also, fix C_0 and Q_0 as the constant C and the function Q appearing in the statement of Lemma 3.3, respectively. Now, choose and fix

$$M_1 > \max\{C_0, Q_0(M_0)\}.$$

Then select $\kappa \leq 1$ sufficiently small, $\tau(0) \leq \min\{1, \tilde{\tau}_0, \tau_1\}$, $T_1 \in (0, T_0]$, and $\epsilon \in (0, \epsilon_0]$ sufficiently small, so that

$$C_0 + (T_1 + \epsilon + \kappa + \tau(0))\, Q_0(M_1) < M_1.$$

Next, set

$$T_2 = \min\left\{T_1, \frac{\tau(0)}{2Q_0(M_1)}\right\}. \tag{3.16}$$

In view of (3.14), this last condition ensures

$$\frac{\tau(0)}{2} \leq \tau(t) \leq \tau(0), \qquad t \in [0, T_2].$$

Note that $M_{\epsilon,\kappa}(0) \leq M_0$. By (3.13)–(3.16) and the continuation principle, we get

$$M_{\epsilon,\kappa}(t) \leq M_1, \qquad t \in [0, T_2],$$

and Theorem 3.1 is proven. □

Sections 4–6 are devoted to the proof of Lemma 3.3, thus completing the proof of Theorem 3.1.

REMARK 3.4 (Boundedness of Sobolev norms). By [**A1**, Theorem 1.1] the H^5 norm of $(p^\epsilon, v^\epsilon, S^\epsilon)$ can be estimated by a constant on a time interval $[0, T_0]$, where T_0 only depends on the H^5 norm of the initial data. More precisely, for given initial data satisfying (3.1), there exists $T_0 > 0$ and a constant C such that

$$\sup_{0 \leq m \leq 5, 0 \leq j \leq m, |\alpha| = j} \|\partial^\alpha (\epsilon \partial_t)^{m-j} (p^\epsilon, v^\epsilon, S^\epsilon)(t)\|_{L^2} \leq C, \qquad t \in [0, T_0], \qquad \epsilon \in (0, 1].$$

In the rest of the paper, we always work on an interval of time $[0, T]$ such that $0 < T \leq T_0$.

REMARK 3.5. (Boundedness of functions of solutions). If F is a smooth function of u and S, then from Remark 3.4 there exists some constant C depending on the function F such that

$$\|F(\epsilon u(t), S(t))\|_{L^\infty} \leq C, \qquad t \in [0, T_0], \qquad \epsilon \in (0, 1].$$

4. Analytic estimate of the entropy

The following statement provides an analytic estimate for the entropy S.

LEMMA 4.1. *Let $M_0 > 0$. For any $\kappa \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ such that if $0 < \tau(0) \leq \tau_1$, then*

$$\|S(t)\|_{A(\tau(t))} \leq C + tQ(M_{\epsilon,\kappa}(t)), \qquad t \in (0, T_0], \tag{4.1}$$

for all $\epsilon \in (0, 1]$, provided K in (3.5) satisfies

$$K \geq Q(M_{\epsilon,\kappa}(T_0)),$$

where $T_0 > 0$ is a sufficiently small constant depending on M_0.

PROOF OF LEMMA 4.1. Fix $m \in \mathbb{N}$ and $|\alpha| = j$ where $0 \le j \le m$. We apply $\partial^\alpha(\epsilon\partial_t)^{m-j}$ to the equation (2.15) and take the L^2-inner product with $\partial^\alpha(\epsilon\partial_t)^{m-j}S$ obtaining

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha(\epsilon\partial_t)^{m-j}S\|_{L^2}^2 + \big\langle v\cdot\nabla\partial^\alpha(\epsilon\partial_t)^{m-j}S, \partial^\alpha(\epsilon\partial_t)^{m-j}S\big\rangle = \big\langle [v\cdot\nabla, \partial^\alpha(\epsilon\partial_t)^{m-j}]S, \partial^\alpha(\epsilon\partial_t)^{m-j}S\big\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in L^2. Using the Cauchy-Schwarz inequality and summing over $|\alpha| = j$, we obtain

$$\frac{d}{dt}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}S\|_{L^2} \le C\|\nabla v\|_{L^\infty_x}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}S\|_{L^2} + C\sum_{|\alpha|=j}\|[v\cdot\nabla, \partial^\alpha(\epsilon\partial_t)^{m-j}]S\|_{L^2}.$$

With the notation (3.2), the above estimate implies

$$\begin{aligned}\frac{d}{dt}\|S\|_{A(\tau)} &= \dot\tau(t)\|S\|_{\tilde A(\tau)} + \sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\frac{\kappa^{(j-3)_+}\tau^{(m-3)_+}}{(m-3)!}\frac{d}{dt}\|\partial^\alpha(\epsilon\partial_t)^{m-j}S\|_{L^2}\\ &\le \dot\tau(t)\|S\|_{\tilde A(\tau)} + C\|\nabla v\|_{L^\infty_x}\|S\|_{A(\tau)} + C\sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{\substack{k=0\\1\le l+k}}^{m-j}\mathcal{C}_{m,j,l,\alpha,\beta,k},\end{aligned}\tag{4.2}$$

where

$$\mathcal{C}_{m,j,l,\alpha,\beta,k} = \frac{\kappa^{(j-3)_+}\tau^{(m-3)_+}}{(m-3)!}\binom{\alpha}{\beta}\binom{m-j}{k}\|\partial^\beta(\epsilon\partial_t)^k v\cdot\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2}$$

with

$$\|u\|_{\tilde A(\tau)} = \sum_{m=4}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}(m-3)\tau(t)^{m-4}}{(m-3)!}$$

denoting the dissipative analytic norm corresponding to (3.2). In the above sums as well as below, the multiindexes $\alpha, \beta, \dots$ are assumed to belong to $\mathbb{N}_0^3$. The third term on the far right side of (4.2) equals

$$\begin{aligned}\mathcal{C} &= C\sum_{m=1}^{4}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{\substack{k=0\\1\le l+k}}^{m-j}\mathcal{C}_{m,j,l,\alpha,\beta,k}\\ &\quad + C\sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{\substack{k=0\\1\le l+k\le[m/2]}}^{m-j}\mathcal{C}_{m,j,l,\alpha,\beta,k}\\ &\quad + C\sum_{m=7}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{\substack{k=0\\ [m/2]+1\le l+k\le m-3}}^{m-j}\mathcal{C}_{m,j,l,\alpha,\beta,k}\\ &\quad + C\sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{k=0}^{m-j}\mathcal{C}_{m,j,l,\alpha,\beta,k}\mathbb{1}_{\{m-2\le l+k\le m\}}\\ &= \mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4,\end{aligned}$$

where we split the sum according to the low and high values of $l+k$ and m. We claim that there exists $T_0 > 0$, such that for any $\kappa \in (0,1]$, there is $\tau_1 \in (0,1]$ such that if $0 < \tau(0) \le \tau_1$, then

$$\mathcal{C}_1 \le C, \tag{4.3}$$

$$\mathcal{C}_2 \le C\|v\|_{A(\tau)}\|S\|_{\tilde A(\tau)}, \tag{4.4}$$

$$\mathcal{C}_3 \le C\|v\|_{A(\tau)}\|S\|_{\tilde A(\tau)}, \tag{4.5}$$

$$\mathcal{C}_4 \le C\|v\|_{A(\tau)}. \tag{4.6}$$

Proof of (4.3): Using Hölder's and the Sobolev inequalities, $\mathcal{C}_1$ may be estimated by low-order mixed space-time derivatives, and (4.3) follows by appealing to Remark 3.4.

Proof of (4.4): Using Hölder's and the Sobolev inequalities we arrive at

$$\mathcal{C}_2 \leq C \sum_{m=5}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{\substack{\beta\leq\alpha \\ |\beta|=l}} \sum_{\substack{k=0 \\ 1\leq l+k\leq[m/2]}}^{m-j} \frac{\kappa^{(j-3)_+}\tau^{m-3}}{(m-3)!} \binom{\alpha}{\beta} \binom{m-j}{k} \|\partial^\beta (\epsilon\partial_t)^k v\|_{L^2}^{1/4}$$
$$\times \|D^2 \partial^\beta (\epsilon\partial_t)^k v\|_{L^2}^{3/4} \|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k} \nabla S\|_{L^2},$$

and thus

$$\begin{aligned} \mathcal{C}_2 \leq C \sum_{m=5}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{\substack{\beta\leq\alpha \\ |\beta|=l}} \sum_{\substack{k=0 \\ 1\leq l+k\leq[m/2]}}^{m-j} \kappa^a \tau^b \left(\|\partial^\beta (\epsilon\partial_t)^k v\|_{L^2} \frac{\kappa^{(l-3)_+}\tau^{(l+k-3)_+}}{(l+k-3)!} \right)^{1/4} \\ \times \left(\|D^2\partial^\beta (\epsilon\partial_t)^k v\|_{L^2} \frac{\kappa^{(l-1)_+}\tau^{(l+k-1)_+}}{(l+k-1)!} \right)^{3/4} \\ \times \left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2} \frac{\kappa^{(j-l-2)_+}(m-k-l-2)\tau^{m-k-l-3}}{(m-k-l-2)!} \right) \mathcal{A}_{m,j,l,\alpha,\beta,k}, \end{aligned} \tag{4.7}$$

where

$$\mathcal{A}_{m,j,l,\alpha,\beta,k} = \binom{\alpha}{\beta}\binom{m-j}{k} \frac{(l+k-3)!^{1/4}(l+k-1)!^{3/4}(m-k-l-2)!}{(m-k-l-2)(m-3)!} \tag{4.8}$$

and

$$\begin{aligned} a &= (j-3)_+ - \left(\frac{l-3}{4}\right)_+ - \left(\frac{3l-3}{4}\right)_+ - (j-l-2)_+, \\ b &= m-3 - \left(\frac{l+k-3}{4}\right)_+ - \left(\frac{3l+3k-3}{4}\right)_+ - (m-k-l-3). \end{aligned} \tag{4.9}$$

For simplicity, we omitted indicating the dependence of a and b on j, k, and l. Since $l+k\geq 1$ and $0\leq l\leq j$, one can readily check that $-3/2\leq a\leq 3/2$ and $1\leq b\leq 3/2$, which implies

$$\kappa^a\tau^b \leq C \tag{4.10}$$

if

$$\tau(0)\leq\kappa^3. \tag{4.11}$$

Recall the combinatorial inequality

$$\binom{\alpha}{\beta} \leq \binom{|\alpha|}{|\beta|}, \tag{4.12}$$

which may also be written as

$$\binom{j}{l}\binom{m-j}{k} \leq \binom{m}{l+k}, \tag{4.13}$$

from where we obtain

$$\begin{aligned} \mathcal{A}_{m,j,l,\alpha,\beta,k} &\leq \frac{Cm!}{(l+k)!(m-l-k)!} \frac{(l+k-3)!(l+k)^{3/2}(m-k-l-3)!}{(m-3)!} \\ &\leq \frac{Cm^3}{(m-l-k)^3} \leq C, \end{aligned} \tag{4.14}$$

since $l+k \le [m/2]$. Using

$$\sum_{|\alpha|=j} \sum_{\substack{\beta\le\alpha\\ |\beta|=l}} x_\beta y_{\alpha-\beta} = \left(\sum_{|\beta|=l} x_\beta\right)\left(\sum_{|\gamma|=j-l} y_\gamma\right) \tag{4.15}$$

from [**KV**, Lemma 4.2], together with (4.7), (4.8)–(4.10), (4.14) and the discrete Hölder inequality, we obtain

$$\begin{aligned} \mathcal{C}_2 \le C \sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j} \sum_{\substack{k=0\\ 1\le l+k\le[m/2]}}^{m-j} &\left(\sum_{|\beta|=l} \|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2} \frac{\kappa^{(l-3)_+}\tau^{(l+k-3)_+}}{(l+k-3)!}\right)^{1/4} \\ &\times \left(\sum_{|\beta|=l} \|D^2\partial^\beta(\epsilon\partial_t)^k v\|_{L^2} \frac{\kappa^{(l-1)_+}\tau^{(l+k-1)_+}}{(l+k-1)!}\right)^{3/4} \\ &\times \left(\sum_{|\gamma|=j-l} \|\partial^\gamma(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2} \frac{\kappa^{(j-l-2)_+}(m-k-l-2)\tau^{m-k-l-3}}{(m-k-l-2)!}\right) \\ \le C\|v\|_{A(\tau)}\|S\|_{\tilde{A}(\tau)}, \end{aligned} \tag{4.16}$$

where the last inequality follows from the discrete Young inequality.

Proof of (4.5): We reverse the roles of $l+k$ and $m-l-k$ and proceed as above, arriving at

$$\begin{aligned} \mathcal{C}_3 \le C \sum_{m=7}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}} \sum_{\substack{k=0\\ [m/2]+1\le l+k\le m-3}}^{m-j} &\kappa^a\tau^b \left(\|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2} \frac{\kappa^{(l-3)_+}\tau^{l+k-3}}{(l+k-3)!}\right) \\ \times \left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2} \frac{\kappa^{(j-l-2)_+}(m-l-k-2)\tau^{m-l-k-3}}{(m-l-k-2)!}\right)^{1/4} \\ \times \left(\|D^2\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2} \frac{\kappa^{(j-l)_+}(m-l-k)\tau^{m-l-k-1}}{(m-l-k)!}\right)^{3/4} \mathcal{B}_{m,j,l,\alpha,\beta,k}, \end{aligned} \tag{4.17}$$

where we denote

$$\mathcal{B}_{m,j,l,\alpha,\beta,k} = \binom{\alpha}{\beta}\binom{m-j}{k} \frac{(l+k-3)!(m-l-k-2)!^{1/4}(m-l-k)!^{3/4}}{(m-l-k-2)^{1/4}(m-l-k)^{3/4}(m-3)!}$$

and

$$\begin{aligned} a &= (j-3)_+ - (l-3)_+ - \left(\frac{j-l-2}{4}\right)_+ - \left(\frac{3j-3l}{4}\right)_+, \\ b &= m-3-(l+k-3) - \frac{(m-l-k-3)}{4} - \frac{3(m-l-k-1)}{4}. \end{aligned}$$

Since $0\le l\le j$, it is readily seen that $-5/2 \le a \le 1/2$ and $b = 3/2$, which implies

$$\kappa^a\tau^b \le C \tag{4.18}$$

if (4.11) holds. Using (4.12)–(4.13), we obtain

$$\begin{aligned} \mathcal{B}_{m,j,l,\alpha,\beta,k} &\le \frac{Cm!}{(l+k)!(m-l-k)!} \frac{(l+k-3)!(m-l-k-2)!(m-l-k-1)^{1/2}}{(m-3)!} \\ &\le \frac{Cm^3}{(l+k)^3} \le C, \end{aligned} \tag{4.19}$$

since $[m/2]+1 \le l+k$. Combining (4.17)–(4.19) and proceeding as in (4.16), we obtain

$$\mathcal{C}_3 \le C\|v\|_{A(\tau)}\|S\|_{\tilde{A}(\tau)}.$$

Proof of (4.6): We split $\mathcal{C}_4$ into three sums according to the value of $l+k$ being equal to $m-2$, $m-1$, or m, and denote them by $\mathcal{C}_{41}$, $\mathcal{C}_{42}$, and $\mathcal{C}_{43}$, respectively.

For $\mathcal{C}_{41}$, we use Hölder's and the Sobolev inequalities and obtain

$$\begin{aligned}
\mathcal{C}_{41} \le C \sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{k=0}^{m-j} &\frac{\kappa^{(j-3)_+}\tau^{m-3}}{(m-3)!}\binom{\alpha}{\beta}\binom{m-j}{k}\mathbb{1}_{\{l+k=m-2\}}\\
&\times \|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2}\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^\infty}\\
\le C\sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{k=0}^{m-j} &\left(\sum_{|\beta|=l}\|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2}\frac{\kappa^{(l-3)_+}\tau^{m-5}}{(m-5)!}\right)\left(\sum_{|\gamma|=j-l}\|D^2\partial^\gamma(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2}\right)^{3/4}\\
&\times\left(\sum_{|\gamma|=j-l}\|\partial^\gamma(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2}\right)^{1/4}\frac{m!}{(m-2)!}\frac{(m-5)!}{(m-3)!}\mathbb{1}_{\{l+k=m-2\}}\\
\le C\|v\|_{A(\tau)},&
\end{aligned}$$

where in the second inequality we applied (4.12)–(4.13), (4.15) and we used $\tau,\kappa\le C$; in the last inequality, we estimated the low-order mixed space-time Sobolev norm of S by C using Remark 3.4.

For $\mathcal{C}_{42}$ and $\mathcal{C}_{43}$, we proceed as in above, by writing

$$\begin{aligned}
\mathcal{C}_{42} \le C \sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{k=0}^{m-j} &\frac{\kappa^{(j-3)_+}\tau^{m-3}}{(m-3)!}\binom{\alpha}{\beta}\binom{m-j}{k}\mathbb{1}_{\{l+k=m-1\}}\\
&\times \|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2}\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^\infty}\\
\le C\sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{k=0}^{m-j} &\left(\sum_{|\beta|=l}\|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2}\frac{\kappa^{(l-3)_+}\tau^{m-4}}{(m-4)!}\right)\left(\sum_{|\gamma|=j-l}\|D^2\partial^\gamma(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2}\right)^{3/4}\\
&\times\left(\sum_{|\gamma|=j-l}\|\partial^\gamma(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^2}\right)^{1/4}\frac{m!}{(m-1)!}\frac{(m-4)!}{(m-3)!}\mathbb{1}_{\{l+k=m-1\}}\\
\le C\|v\|_{A(\tau)}&
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_{43} \le C \sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{\substack{\beta\le\alpha\\|\beta|=l}}\sum_{k=0}^{m-j} &\frac{\kappa^{(j-3)_+}\tau^{m-3}}{(m-3)!}\binom{\alpha}{\beta}\binom{m-j}{k}\mathbb{1}_{\{l+k=m\}}\\
&\times \|\partial^\beta(\epsilon\partial_t)^k v\|_{L^2}\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla S\|_{L^\infty}\\
\le C\sum_{m=5}^{\infty}\sum_{j=0}^{m}\sum_{|\beta|=j}&\left(\|\partial^\beta(\epsilon\partial_t)^{m-j}v\|_{L^2}\frac{\kappa^{(j-3)_+}\tau^{m-3}}{(m-3)!}\right)\|D^2\nabla S\|_{L^2}^{3/4}\|\nabla S\|_{L^2}^{1/4}\\
\le C\|v\|_{A(\tau)}.&
\end{aligned}$$

Combining (4.2)–(4.6) and Remark 3.4 to bound $\|\nabla v\|_{L^\infty_x}$, we get

$$\frac{d}{dt}\|S\|_{A(\tau)} \le \|S\|_{\tilde{A}(\tau)}(\dot\tau + C\|v\|_{A(\tau)}) + C\|S\|_{A(\tau)} + C\|v\|_{A(\tau)} + C. \tag{4.20}$$

Now, determine K in (3.5) to be sufficiently large so that

$$\dot\tau(t) + C\|v\|_{A(\tau)} \le 0, \qquad 0\le t\le T_0, \tag{4.21}$$

where $T_0 > 0$ satisfies (3.15). The lemma is then proven by integrating (4.20) on $[0, T_0]$, using (4.21), and applying the Gronwall lemma. $\square$

After Section 5, we work with derivatives of the solution and thus instead of the norms (3.2) we use

$$\|u\|_{B(\tau)} = \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon\partial_t)^{m-j} u\|_{L^2} \frac{\kappa^{(j-2)_+} \tau(t)^{(m-2)_+}}{(m-2)!} \tag{4.22}$$

and the corresponding dissipative analytic norm

$$\|u\|_{\tilde{B}(\tau)} = \sum_{m=3}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon\partial_t)^{m-j} u\|_{L^2} \frac{\kappa^{(j-2)_+} (m-2) \tau(t)^{m-3}}{(m-2)!}. \tag{4.23}$$

It turns out that the curl component of the velocity satisfies an equation similar to the one for the entropy, but with the nonzero right-hand side. Thus we now consider the inhomogeneous transport equation

$$\partial_t \tilde{S} + v \cdot \nabla \tilde{S} = G,$$

where $\tilde{S} = \tilde{S}(x,t)$, $v = v(x,t)$, and $G = G(x,t)$.

LEMMA 4.2. *For any $\kappa \in (0,1]$, there exists $\tau_1 \in (0,1]$ such that if $0 < \tau(0) \le \tau_1$, then*

$$\|\tilde{S}\|_{A(\tau)} \le \|\tilde{S}(0)\|_{A(\tau)} + C \int_0^t \left(\|G(s)\|_{A(\tau)} + \|v(s)\|_{A(\tau)} \right) ds + Ct, \qquad t \in [0, T_0], \tag{4.24}$$

for some constant C and sufficiently small $T_0 > 0$, provided K in (3.5) satisfies

$$K \ge C \|v(t)\|_{A(\tau)}, \qquad t \in [0, T_0], \tag{4.25}$$

where T_0 is chosen sufficiently small so that (3.15) holds. Similarly, for any $\kappa \le 1$, there exists $\tau(0) > 0$ such that

$$\|\tilde{S}\|_{B(\tau)} \le \|\tilde{S}(0)\|_{B(\tau)} + C \int_0^t \left(\|G(s)\|_{B(\tau)} + \|v(s)\|_{B(\tau)} \right) ds + Ct, \qquad t \in [0, T_0], \tag{4.26}$$

for some constant C and sufficiently small $T_0 > 0$, provided K satisfies

$$K \ge C \|v(t)\|_{B(\tau)}, \qquad t \in [0, T_0], \tag{4.27}$$

where T_0 is chosen sufficiently small so that (3.15) holds.

Note that from definitions (3.2) and (4.22), we have

$$\|v\|_{B(\tau)} \le \|v\|_{A(\tau)}$$

for all v, and thus (4.25) implies (4.27).

PROOF. We proceed exactly as in the proof of Lemma 4.1. Using the Cauchy-Schwarz inequality with the inhomogeneous part G, we obtain

$$\frac{d}{dt} \|\tilde{S}\|_{A(\tau)} \le \|\tilde{S}\|_{\tilde{A}(\tau)} (\dot{\tau} + C\|v\|_{A(\tau)}) + C(\|\tilde{S}\|_{A(\tau)} + \|v\|_{A(\tau)} + \|G\|_{A(\tau)}) + C.$$

The estimate (4.24) then follows by using (4.21) and the Gronwall inequality. Analogously, we use the analytic shift $(m-2)!$ instead of $(m-3)!$ and proceed as in the proof of Lemma 4.1, we conclude

$$\frac{d}{dt} \|\tilde{S}\|_{B(\tau)} \le \|\tilde{S}\|_{\tilde{B}(\tau)} (\dot{\tau} + C\|v\|_{B(\tau)}) + C(\|\tilde{S}\|_{B(\tau)} + \|v\|_{B(\tau)} + \|G\|_{B(\tau)}) + C.$$

The assertion (4.26) may then be obtained by setting

$$\dot{\tau}(t) + C\|v\|_{B(\tau)} \le 0$$

with C sufficiently large and using the Gronwall inequality. $\square$

5. Analytic estimate of $\partial_t E$

In order to bound the velocity, we first need to obtain an analytic estimate for $\partial_t E$, which in turn requires the bound on the entropy. We first provide a product rule for the type B norm.

LEMMA 5.1. *Let $k \in \{2, 3, \ldots\}$ and $\tau > 0$. For $f_1, \ldots, f_k \in B(\tau)$, and any $\kappa \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ and $T_0 > 0$ such that if $0 < \tau(0) \le \tau_1$, then*

$$\left\|\prod_{i=1}^{k} f_i\right\|_{B(\tau)} \le C^k \sum_{i=1}^{k} \left(\|f_i\|_{B(\tau)} \prod_{1\le j\le k; j\neq i} (\|f_j\|_{B(\tau)} + \|f_j\|_{L^2}) \right),$$

for $k \ge 2$, where the constant is independent of k.

PROOF OF LEMMA 5.1. By induction, it is sufficient to prove the inequality

$$\|fg\|_{B(\tau)} \le C\|f\|_{B(\tau)}(\|g\|_{B(\tau)} + \|g\|_{L^2}) + C(\|f\|_{B(\tau)} + \|f\|_{L^2})\|g\|_{B(\tau)}, \tag{5.1}$$

for f and g such that the respective right hand sides are finite. To prove the estimate (5.1), we use the Leibniz rule and write

$$\begin{aligned} \|fg\|_{B(\tau)} &= \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=j} \frac{\kappa^{(j-2)_+} \tau^{(m-2)_+}}{(m-2)!} \|\partial^\alpha (\epsilon\partial_t)^{m-j} (fg)\|_{L^2} \\ &\le \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{k=0}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k}, \end{aligned} \tag{5.2}$$

where

$$\mathcal{H}_{m,j,l,\alpha,\beta,k} = \binom{\alpha}{\beta} \binom{m-j}{k} \frac{\kappa^{(j-2)_+} \tau^{(m-2)_+}}{(m-2)!} \|\partial^\beta (\epsilon\partial_t)^k f \partial^{\alpha-\beta} (\epsilon\partial_t)^{m-j-k} g\|_{L^2}.$$

We split the sum on the right side of (5.2) according to the low and high values of $l + k$ and m, and we claim

$$\sum_{m=1}^{2} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{k=0}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k} \le C(\|f\|_{B(\tau)} + \|f\|_{L^2})\|g\|_{B(\tau)} + C(\|g\|_{B(\tau)} + \|g\|_{L^2})\|f\|_{B(\tau)}, \tag{5.3}$$

$$\sum_{m=3}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{k=0}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k} \mathbb{1}_{\{l+k=0\}} \le C(\|f\|_{B(\tau)} + \|f\|_{L^2})\|g\|_{B(\tau)}, \tag{5.4}$$

$$\sum_{m=3}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{\substack{k=0 \\ 1\le l+k\le [m/2]}}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k} \le C(\|f\|_{B(\tau)} + \|f\|_{L^2})\|g\|_{B(\tau)}, \tag{5.5}$$

$$\sum_{m=3}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{\substack{k=0 \\ [m/2]+1\le l+k\le m-1}}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k} \le C\|f\|_{B(\tau)}(\|g\|_{B(\tau)} + \|g\|_{L^2}). \tag{5.6}$$

$$\sum_{m=3}^{\infty} \sum_{j=0}^{m} \sum_{l=0}^{j} \sum_{|\alpha|=j} \sum_{|\beta|=l, \beta\le\alpha} \sum_{k=0}^{m-j} \mathcal{H}_{m,j,l,\alpha,\beta,k} \mathbb{1}_{\{l+k=m\}} \le C\|f\|_{B(\tau)}(\|g\|_{B(\tau)} + \|g\|_{L^2}), \tag{5.7}$$

Proof of (5.3): For $m=1$, we use Hölder's and the Sobolev inequalities and arrive at

$$
\begin{aligned}
&\sum_{j=0}^{1}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{k=0}^{1-j}\mathcal{H}_{1,j,l,\alpha,\beta,k} \le C\|f\epsilon\partial_t g\|_{L^2}+C\|fDg\|_{L^2}+C\|g\epsilon\partial_t f\|_{L^2}+C\|gDf\|_{L^2}\\
&\quad\le C\|D^2 f\|_{L^2}^{3/4}\|f\|_{L^2}^{1/4}\|\epsilon\partial_t g\|_{L^2}+C\|D^2 f\|_{L^2}^{3/4}\|f\|_{L^2}^{1/4}\|Dg\|_{L^2}\\
&\qquad+C\|D^2 g\|_{L^2}^{3/4}\|g\|_{L^2}^{1/4}\|\epsilon\partial_t f\|_{L^2}+C\|D^2 g\|_{L^2}^{3/4}\|g\|_{L^2}^{1/4}\|Df\|_{L^2}\\
&\quad\le C(\|f\|_{B(\tau)}+\|f\|_{L^2})\|g\|_{B(\tau)}+C(\|g\|_{B(\tau)}+\|g\|_{L^2})\|f\|_{B(\tau)}.
\end{aligned}
\tag{5.8}
$$

For $m=2$, by Leibniz rule we write

$$
\begin{aligned}
&\sum_{j=0}^{2}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{k=0}^{2-j}\mathcal{H}_{2,j,l,\alpha,\beta,k}\\
&\quad\le C\|f(\epsilon\partial_t)^2 g\|_{L^2}+C\|fD\epsilon\partial_t g\|_{L^2}+C\|fD^2 g\|_{L^2}\\
&\qquad+C\|Df\epsilon\partial_t g\|_{L^2}+C\|DfDg\|_{L^2}+C\|\epsilon\partial_t f(\epsilon\partial_t)g\|_{L^2}+C\|\epsilon\partial_t fDg\|_{L^2}\\
&\qquad+C\|g(\epsilon\partial_t)^2 f\|_{L^2}+C\|gD\epsilon\partial_t f\|_{L^2}+C\|gD^2 f\|_{L^2}.
\end{aligned}
\tag{5.9}
$$

All the terms in (5.9) are estimated using Hölder and Sobolev inequalities. For illustration, we treat the fifth term, for which we write

$$
\|DfDg\|_{L^2}\le\|Df\|_{L^4}\|Dg\|_{L^4}\le C\|D^2 f\|_{L^2}^{3/4}\|Df\|_{L^2}^{1/4}\|D^2 g\|_{L^2}^{3/4}\|Dg\|_{L^2}^{1/4}\le C\|f\|_{B(\tau)}\|g\|_{B(\tau)}. \tag{5.10}
$$

Collecting the estimates (5.8)–(5.10), we obtain (5.3).

Proof of (5.4): Using Hölder and Sobolev inequalities, we obtain

$$
\begin{aligned}
&\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{k=0}^{m-j}\mathcal{H}_{m,j,l,\alpha,\beta,k}\mathbb{1}_{\{l+k=0\}}\\
&\quad\le C\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|f\|_{L^\infty}\|\partial^\alpha(\epsilon\partial_t)^{m-j}g\|_{L^2}\frac{\kappa^{(j-2)_+}\tau^{(m-2)_+}}{(m-2)!}\\
&\quad\le C(\|f\|_{B(\tau)}+\|f\|_{L^2})\|g\|_{B(\tau)}.
\end{aligned}
\tag{5.11}
$$

Proof of (5.5): Using Hölder and Sobolev inequalities, we obtain

$$
\begin{aligned}
&\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{\substack{k=0\\0\le l+k\le[m/2]}}^{m-j}\mathcal{H}_{m,j,l,\alpha,\beta,k}\\
&\quad\le C\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{\substack{k=0\\0\le l+k\le[m/2]}}^{m-j}\left(\|\partial^\beta(\epsilon\partial_t)^k f\|_{L^2}\frac{\kappa^{(l-2)_+}\tau^{(l+k-2)_+}}{(l+k-2)!}\right)^{1/4}\\
&\qquad\times\left(\|D^2\partial^\beta(\epsilon\partial_t)^k f\|_{L^2}\frac{\kappa^{l_+}\tau^{(l+k)_+}}{(l+k)!}\right)^{3/4}\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}g\|_{L^2}\frac{\kappa^{(j-l-2)_+}\tau^{(m-l-k-2)_+}}{(m-l-k-2)!}\right),
\end{aligned}
$$

where we bound the constant coefficient by C analogously as in (4.14), and bound the τ and κ term by C analogously as in (4.9)–(4.10). Therefore, using the discrete Hölder and Young inequalities, we obtain (5.5).

Proof of (5.6): We reverse the roles of l and $m-l-k$ and proceed as in the above argument, obtaining

$$\begin{aligned}
&\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{\substack{k=0\\ [m/2]+1\le l+k\le m}}^{m-j}\mathcal{H}_{m,j,l,\alpha,\beta,k}\\
&\le C\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{\substack{k=0\\ [m/2]+1\le l+k\le m}}^{m-j}\left(\|\partial^{\beta}(\epsilon\partial_t)^k f\|_{L^2}\frac{\kappa^{(l-2)_+}\tau^{(l+k-2)_+}}{(l+k-2)!}\right)\\
&\quad\times\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}g\|_{L^2}\frac{\kappa^{(j-l-2)_+}\tau^{(m-l-k-2)_+}}{(m-l-k-2)!}\right)^{1/4}\\
&\quad\times\left(\|D^2\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}g\|_{L^2}\frac{\kappa^{(j-l)_+}\tau^{(m-l-k)_+}}{(m-l-k)!}\right)^{3/4}.
\end{aligned}$$

Therefore, using the discrete Hölder and Young inequalities, we obtain (5.6).

Proof of (5.7): We proceed as in (5.11), obtaining

$$\begin{aligned}
&\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{l=0}^{j}\sum_{|\alpha|=j}\sum_{|\beta|=l,\beta\le\alpha}\sum_{k=0}^{m-j}\mathcal{H}_{m,j,l,\alpha,\beta,k}\mathbb{1}_{\{l+k=m\}}\\
&\le C\sum_{m=3}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|g\|_{L^\infty}\left(\|\partial^{\alpha}(\epsilon\partial_t)^{m-j}f\|_{L^2}\frac{\kappa^{(j-2)_+}\tau^{(m-2)_+}}{(m-2)!}\right)\\
&\le C\|f\|_{B(\tau)}(\|g\|_{B(\tau)}+\|g\|_{L^2}).
\end{aligned}$$

Combining (5.3)–(5.7), we obtain (5.2). □

Similarly to (5.1) and Lemma 5.1, with analytic shift $(m-3)!$ rather than $(m-2)!$, we also have

$$\|fg\|_{A(\tau)}\le C\|f\|_{A(\tau)}(\|g\|_{A(\tau)}+\|g\|_{L^2})+C(\|f\|_{A(\tau)}+\|f\|_{L^2})\|g\|_{A(\tau)}. \tag{5.12}$$

In the case when f belongs to L^∞ but is not square integrable, we have variant formulas

$$\|fg\|_{A(\tau)}\le C\|f\|_{A(\tau)}(\|g\|_{A(\tau)}+\|g\|_{L^2})+C\|f\|_{L^\infty}\|g\|_{A(\tau)}, \tag{5.13}$$

and

$$\|fg\|_{B(\tau)}\le C\|f\|_{B(\tau)}(\|g\|_{B(\tau)}+\|g\|_{L^2})+C\|f\|_{L^\infty}\|g\|_{B(\tau)}. \tag{5.14}$$

The proofs are similar to (5.1), where the modification of the proof for the variant formula (5.13) is to treat the term $\|f\partial^\alpha(\epsilon\partial_t)^{m-j}g\|_{L^2}$ by Hölder's inequality with exponents ∞ and 2.

The next lemma provides an analytic estimate for composition of functions.

LEMMA 5.2. *Assume that f is an entire real-analytic function. Then there exists a function Q such that*

$$\|f(S(t))\|_{B(\tau)}\le Q(\|S(t)\|_{A(\tau)}+\|S(t)\|_{L^2}), \tag{5.15}$$

and

$$\|f(S(t))\|_{A(\tau)}\le Q(\|S(t)\|_{A(\tau)}+\|S(t)\|_{L^2}), \tag{5.16}$$

where Q also depends on f.

PROOF. First we prove (5.15). Since f is entire, for every $R>0$ there exists $N(R)>0$ such that

$$|f^{(k)}(x)|\le\frac{Nk!}{R^k},\qquad x\in\mathbb{R},\qquad k\in\mathbb{N}_0$$

and

$$f(S(t)) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)S(t)^k}{k!}. \tag{5.17}$$

By Lemma 5.1, we obtain

$$\left\| \frac{f^{(k)}(0)}{k!} S^k \right\|_{B(\tau)} \le \frac{NC^k k}{R^k} \|S\|_{B(\tau)} (\|S\|_{B(\tau)} + \|S\|_{L^2})^{k-1}. \tag{5.18}$$

Summing (5.18) in $k \in \mathbb{N}$ and using the Taylor expansion (5.17), we arrive at

$$\begin{aligned}\|f(S(t))\|_{B(\tau)} &\le \sum_{k=1}^{\infty} \left\| \frac{f^{(k)}(0)}{k!} S(t)^k \right\|_{B(\tau)} \le CN \frac{\|S(t)\|_{B(\tau)}}{\|S\|_{B(\tau)} + \|S\|_{L^2}} \sum_{k=1}^{\infty} \left(\frac{C(\|S\|_{B(\tau)} + \|S\|_{L^2})}{R} \right)^k \\ &\le CN \sum_{k=1}^{\infty} \left(\frac{C(\|S\|_{B(\tau)} + \|S\|_{L^2})}{R} \right)^k .\end{aligned}$$

Choosing $R = 2C\|S(t)\|_{B(\tau)} + 2C\|S(t)\|_{L^2}$, we obtain $\|f(S(t))\|_{B(\tau)} \le CN$, where N depends on $\|S(t)\|_{B(\tau)} + \|S(t)\|_{L^2}$. Finally, observe that $\|S(t)\|_{B(\tau)} \le \|S(t)\|_{A(\tau)}$, by the definition of the norms, concluding the proof of (5.15). The estimate (5.16) is proven analogously by using (5.12), and we omit the details here. □

For the next two lemmas, assume that $\tilde{e}$ is one of components of the matrix E in (2.16), i.e., either r or one of the components of a. By the assumptions (2.17) and (2.18), we have

$$\tilde{e}(S, \epsilon u) = f(S) g(\epsilon u), \tag{5.19}$$

where f and g are positive entire real-analytic functions.

The first lemma gives the estimate of the derivative of the component of the matrix E.

LEMMA 5.3. *Given $M_0 > 0$, and (5.19), where f and g are as above. Then*

$$\|\partial_t \tilde{e}\|_{B(\tau)} \le Q(\|u\|_{A(\tau)} + \|u\|_{L^2}, \|S\|_{A(\tau)} + \|S\|_{L^2}) \tag{5.20}$$

for some function Q.

PROOF. By (2.15), the chain rule, and product rule, we obtain

$$\partial_t \tilde{e} = f'(S)\partial_t S g(\epsilon u) + f(S)\nabla g(\epsilon u) \cdot \epsilon \partial_t u = -f'(S) v \cdot \nabla S g(\epsilon u) + f(S) \nabla g(\epsilon u) \cdot \epsilon \partial_t u.$$

Therefore,

$$\|\partial_t \tilde{e}\|_{B(\tau)} \le \|f'(S) v \cdot \nabla S g(\epsilon u)\|_{B(\tau)} + \|f(S)\nabla g(\epsilon u) \cdot \epsilon \partial_t u\|_{B(\tau)} = \mathcal{G}_1 + \mathcal{G}_2.. \tag{5.21}$$

By repeated use of (5.1) and (5.14) and Remark 3.5, we arrive at

$$\begin{aligned}\mathcal{G}_1 &\le \|f'(S)\|_{B(\tau)} (\|v \cdot \nabla S g(\epsilon u)\|_{B(\tau)} + \|v \cdot \nabla S g(\epsilon u)\|_{L^2}) + \|f'(S)\|_{L^\infty} \|v \cdot \nabla S g(\epsilon u)\|_{B(\tau)} \\ &\le \|f'(S)\|_{B(\tau)} \left(\|g(\epsilon u)\|_{B(\tau)} (\|v \cdot \nabla S\|_{B(\tau)} + \|v \cdot \nabla S\|_{L^2}) + C\|v \cdot \nabla S\|_{B(\tau)} + \|v \cdot \nabla S\|_{L^2} \right) \\ &\quad + C\|g(\epsilon u)\|_{B(\tau)} \left(\|v \cdot \nabla S\|_{B(\tau)} + \|v \cdot \nabla S\|_{L^2} \right) + C\|v \cdot \nabla S\|_{B(\tau)}.\end{aligned} \tag{5.22}$$

For the term $\|v \cdot \nabla S\|_{B(\tau)}$, we again appeal to (5.1), obtaining

$$\|v \cdot \nabla S\|_{B(\tau)} \le C\|v\|_{B(\tau)} \left(\|\nabla S\|_{B(\tau)} + \|\nabla S\|_{L^2} \right) + C\|\nabla S\|_{B(\tau)} \left(\|v\|_{B(\tau)} + \|v\|_{L^2} \right).$$

By the definition of the analytic norms in (3.2) and (4.22), we have

$$\begin{aligned}\|\nabla S\|_{B(\tau)} &= \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon \partial_t)^{m-j} \nabla S\|_{L^2} \frac{\kappa^{(j-2)_+} \tau^{(m-2)_+}}{(m-2)!} \\ &\le C \sum_{m=1}^{\infty} \sum_{j=0}^{m} \sum_{|\alpha|=j+1} \|\partial^\alpha (\epsilon \partial_t)^{m-j} S\|_{L^2} \frac{\kappa^{(j-2)_+} \tau^{(m-2)_+}}{(m-2)!} \le C\|S\|_{A(\tau)}.\end{aligned} \tag{5.23}$$

Collecting estimates (5.22)–(5.23), we obtain

$$\mathcal{G}_1 \le Q(\|u\|_{A(\tau)} + \|u\|_{L^2}, \|S\|_{A(\tau)} + \|S\|_{L^2}). \tag{5.24}$$

Using analogous arguments, we also get

$$\mathcal{G}_2 \le Q(\|u\|_{A(\tau)} + \|u\|_{L^2}, \|S\|_{A(\tau)} + \|S\|_{L^2}) \tag{5.25}$$

since by definition

$$\begin{aligned}\|\epsilon\partial_t u\|_{B(\tau)} &= \sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\frac{\kappa^{(j-2)_+}\tau^{(m-2)_+}}{(m-2)!}\\ &\le C\sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}\|_{L^2}\frac{\kappa^{(j-3)_+}\tau^{(m-3)_+}}{(m-3)!} \le C\|u\|_{A(\tau)}.\end{aligned}$$

Therefore, (5.20) is proven by combining (5.21), (5.24), and (5.25). ☐

The second lemma gives the analytic estimates for the component of the matrix E.

LEMMA 5.4. *Assume* (5.19)*, where f and g are as above. Then*

$$\|\tilde{e}(t)\|_{A(\tau)} \le Q(\|u\|_{A(\tau)} + \|u\|_{L^2}, \|S\|_{A(\tau)} + \|S\|_{L^2}), \tag{5.26}$$

for some function Q.

PROOF. Since $\tilde{e}(t) = f(S)g(\epsilon u)$, the proof of the estimate (5.26) may be carried out by appealing to Lemmas 5.1 and 5.2 in the A-norm. ☐

6. Estimates on the velocity

Recall that

$$E(S,\epsilon u) = \begin{pmatrix} a(S,\epsilon u) & 0\\ 0 & r(S,\epsilon u)\mathbb{I}_3\end{pmatrix},$$

where $a(S,\epsilon u)$ and $r(S,\epsilon u)$ are as in Section 2.1.

6.1. Estimate on the curl. We first need to rewrite the equation (2.11) so to be able to estimate the curl of the velocity v. Introduce $r_0(x) = r(x,0)$ (i.e., $r_0(S) = r(S,0)$), and note that, by our assumptions,

$$r_0(S) = f_2(S)g_2(0).$$

Define

$$\tilde{f}(x,y) = 1 - \frac{r_0(x)}{r(x,y)} = 1 - \frac{g_2(0)}{g_2(y)}.$$

Since $\tilde{f}$ is a function of y only and vanishes at $y=0$, there exists a bounded entire function h such that

$$\tilde{f}(x,y) = yh(y).$$

Denoting

$$\tilde{h}(x) = xh(\epsilon x),$$

we then have

$$\tilde{f}(S,\epsilon u) = \epsilon\tilde{h}(u).$$

Since $\partial_t S + v\cdot\nabla S = 0$, the equation (2.11) for v is equivalent to the nonlinear transport equation

$$(\partial_t + v\cdot\nabla)(r_0 v) + \frac{1}{\epsilon}\nabla p = \tilde{h}\nabla p.$$

Applying curl to the above equation and using $\operatorname{curl}\nabla p = 0$, we arrive at

$$(\partial_t + v\cdot\nabla)\operatorname{curl}(r_0 v) = [v\cdot\nabla, \operatorname{curl}](r_0 v) + [\operatorname{curl}, \tilde{h}]\nabla p. \tag{6.1}$$

To treat (6.1), we would like to use (4.26) from Lemma 4.2 and thus we need to estimate the forcing term

$$G = [v\cdot\nabla, \operatorname{curl}](r_0 v) + [\operatorname{curl}, \tilde{h}]\nabla p = G_1 + G_2$$

in the analytic norm (4.22). Since $([v_j\partial_j, \operatorname{curl}]w)_i = \epsilon_{ikm}\partial_m v_j \partial_j w_k$, where ϵ_{ikm} is the permutation symbol, we may apply Lemma 5.1 and obtain

$$\|G_1\|_{B(\tau)} \le C(\|\nabla(r_0 v)\|_{B(\tau)} + \|\nabla(r_0 v)\|_{L^2})(\|\nabla v\|_{B(\tau)} + \|\nabla v\|_{L^2}). \tag{6.2}$$

From (5.13), (5.23), (5.26) and (6.2), we get $\|G_1\|_{B(\tau)} \le Q(M_{\epsilon,\kappa}(t))$. The term G_2 may be estimated in an analogous way since $([\operatorname{curl}, \tilde{h}]\nabla p)_i = \epsilon_{ijk}\partial_j\tilde{h}\partial_k p$, leading to

$$\|G_2\|_{B(\tau)} \le C(\|\nabla p\|_{B(\tau)} + \|\nabla p\|_{L^2})(\|\nabla\tilde{h}\|_{B(\tau)} + \|\nabla\tilde{h}\|_{L^2}) \le Q(M_{\epsilon,\kappa}(t)).$$

Proceeding as in Lemma 4.2, we obtain

$$\frac{d}{dt}\|\operatorname{curl}(r_0 v)\|_{B(\tau)} = \dot{\tau}\|\operatorname{curl}(r_0 v)\|_{\tilde{B}(\tau)} + \sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\frac{\kappa^{(j-2)_+}\tau^{(m-2)_+}}{(m-2)!}\frac{d}{dt}\|\partial^\alpha(\epsilon\partial_t)^{m-j}\operatorname{curl}(r_0 v)\|_{L^2}, \tag{6.3}$$

where we used the notation (4.22)–(4.23). By (4.26) from Lemma 4.2, we get

$$\begin{aligned}\|\operatorname{curl}(r_0 v)(t)\|_{B(\tau)} &\le \|\operatorname{curl}(r_0 v)(0)\|_{B(\tau)} + Ct\sup_{s\in(0,t)}\|G(s)\|_{B(\tau)} + Ct\sup_{s\in(0,t)}\|v(s)\|_{B(\tau)} + Ct\\ &\le C + tQ(M_{\epsilon,\kappa}(t)).\end{aligned} \tag{6.4}$$

Next, we estimate $\operatorname{curl} v$ in the analytic norm $B(\tau)$. Denoting

$$R_0 = \frac{1}{r_0},$$

we rewrite

$$\|\operatorname{curl} v\|_{B(\tau)} \le \|R_0\operatorname{curl}(r_0 v)\|_{B(\tau)} + \|[\operatorname{curl}, R_0]r_0 v\|_{B(\tau)} = \xi_1 + \xi_2. \tag{6.5}$$

For the term ξ_1, we use (5.14) and the curl estimate (6.4), obtaining

$$\xi_1 \le C\|R_0\|_{B(\tau)}(\|\operatorname{curl}(r_0 v)\|_{B(\tau)} + \|\operatorname{curl}(r_0 v)\|_{L^2}) + C\|R_0\|_{L^\infty}\|\operatorname{curl}(r_0 v)\|_{B(\tau)}. \tag{6.6}$$

Since R_0 satisfies the homogeneous transport equation $\partial_t R_0 + v\cdot\nabla R_0 = 0$, the inequality (4.26) from Lemma 4.2 implies

$$\|R_0(S(t))\|_{B(\tau)} \le \|R_0(S(0))\|_{B(\tau)} + Ct\sup_{s\in(0,t)}\|v(s)\|_{B(\tau)} + Ct \le C + tQ(M_{\epsilon,\kappa}(t)). \tag{6.7}$$

Combining (6.4), (6.6), and (6.7), we obtain

$$\xi_1 \le C + tQ(M_{\epsilon,\kappa}(t)). \tag{6.8}$$

For ξ_2, we first rewrite it as

$$[\operatorname{curl}, R_0]r_0 v = R_1\nabla S\times v,$$

where $R_1 = -r_0'/r_0$. Applying Lemma 5.1 and (5.14), we get

$$\begin{aligned}\xi_2 = \|[\operatorname{curl}, R_0]r_0 v\|_{B(\tau)} &\le C\|R_1\|_{B(\tau)}(\|\nabla S\|_{B(\tau)} + \|\nabla S\|_{L^2})(\|v\|_{B(\tau)} + \|v\|_{L^2})\\ &\quad + C\|\nabla S\|_{B(\tau)}(\|R_1\|_{B(\tau)} + \|R_1\|_{L^\infty})(\|v\|_{B(\tau)} + \|v\|_{L^2})\\ &\quad + C\|v\|_{B(\tau)}(\|R_1\|_{B(\tau)} + \|R_1\|_{L^\infty})(\|\nabla S\|_{B(\tau)} + \|\nabla S\|_{L^2}).\end{aligned}$$

To bound the right hand side, it suffices to estimate $\|R_1\|_{B(\tau)}$, $\|\nabla S\|_{B(\tau)}$, and $\|v\|_{B(\tau)}$, as the rest are bounded by C (cf. Remark 3.4). For $\|R_1\|_{B(\tau)}$, since $R_1 = -r_0'/r_0$ depends only on the entropy S, it satisfies the homogeneous transport equation $\partial_t R_1 + v\cdot\nabla R_1 = 0$ and thus by Lemma 4.2,

$$\|R_1(S(t))\|_{B(\tau)} \le \|R_1(S(0))\|_{B(\tau)} + Ct + Ct\sup_{s\in(0,t)}\|v(s)\|_{B(\tau)} \le C + tQ(M_{\epsilon,\kappa}(t)).$$

For $\|\nabla S\|_{B(\tau)}$, by (5.23) and Lemma 4.1, we obtain

$$\|\nabla S\|_{B(\tau)} \le C\|S\|_{A(\tau)} + tQ(M_{\epsilon,\kappa}(t)).$$

For $\|v\|_{B(\tau)}$, by the norm relation we have

$$\begin{aligned}\|v\|_{B(\tau)} &\leq \sum_{m=1}^{3}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|\partial^j(\epsilon\partial_t)^{m-j}v\|_{L^2} + \sum_{m=4}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|\partial^j(\epsilon\partial_t)^{m-j}v\|_{L^2}\frac{\kappa^{(j-2)_+}\tau(t)^{m-2}}{(m-2)!}\\ &\leq C + C\tau\|v\|_{A(\tau)} \leq C + \tau Q(M_{\epsilon,\kappa}(t)).\end{aligned}$$

By combining the above estimates, we deduce that

$$\xi_2 \leq C + (t+\tau)Q(M_{\epsilon,\kappa}(t)).$$

Therefore, together with (6.5) and (6.8) we arrive at

$$\|\operatorname{curl} v\|_{B(\tau)} \leq C + (t+\tau)\,Q(M_{\epsilon,\kappa}(t)). \tag{6.9}$$

6.2. Energy equation for the pure time derivatives (A_1 norm). In this section, we estimate the pure time-analytic norm

$$\|u\|_{A_1(\tau)} = \sum_{m=1}^{\infty}\|(\epsilon\partial_t)^m u\|_{L^2}\frac{\tau(t)^{(m-3)_+}}{(m-3)!}$$

with the corresponding dissipative analytic norm

$$\|u\|_{\tilde{A}_1(\tau)} = \sum_{m=4}^{\infty}\|(\epsilon\partial_t)^m u\|_{L^2}\frac{(m-3)\tau(t)^{m-4}}{(m-3)!}.$$

Consider the partially linearized equation

$$E(\partial_t\dot{u} + v\cdot\nabla\dot{u}) + \frac{1}{\epsilon}L(\partial_x)\dot{u} = F, \tag{6.10}$$

where $\dot{u} = (\dot{p},\dot{v})$ and $E = E(S,\epsilon u)$.

The next lemma provides a differential inequality that is used for pure time derivatives of u.

LEMMA 6.1. *For all $(\dot{u},F)$ satisfying (6.10), we have*

$$\frac{d}{dt}\|E^{1/2}\dot{u}\|_{L^2} \leq C(\|\dot{u}\|_{L^2} + \|F\|_{L^2}),$$

for a constant $C \geq 1$.

PROOF. We multiply the equation (6.10) by $\dot{u}$ and integrate in $\mathbb{R}^3$. Since $L(\partial_x)$ is skew-symmetric, we have

$$\frac{1}{\epsilon}\langle L(\partial_x)\dot{u},\dot{u}\rangle = 0,$$

i.e., the term with $1/\epsilon$ cancels out. Using also the Cauchy-Schwarz inequality, we get

$$\langle E\partial_t\dot{u},\dot{u}\rangle \leq C\|\nabla(Ev)\|_{L^\infty_x}\|\dot{u}\|_{L^2}^2 + C\|F\|_{L^2}\|\dot{u}\|_{L^2}, \tag{6.11}$$

and thus by Hölder's inequality and since E is a positive definite symmetric matrix, we obtain from (6.11)

$$\begin{aligned}\frac{d}{dt}\|E^{1/2}\dot{u}\|_{L^2}^2 &= \frac{d}{dt}\langle E\dot{u},\dot{u}\rangle = \langle\partial_t E\dot{u},\dot{u}\rangle + 2\langle E\partial_t\dot{u},\dot{u}\rangle\\ &\leq \|\partial_t E\|_{L^\infty_x}\|\dot{u}\|_{L^2}^2 + C\|\nabla(Ev)\|_{L^\infty_x}\|\dot{u}\|_{L^2}^2 + C\|F\|_{L^2}\|\dot{u}\|_{L^2}.\end{aligned} \tag{6.12}$$

On the other hand,

$$\frac{d}{dt}\|E^{1/2}\dot{u}\|_{L^2}^2 = 2\|E^{1/2}\dot{u}\|_{L^2}\frac{d}{dt}\|E^{1/2}\dot{u}\|_{L^2}. \tag{6.13}$$

Now, we combine (6.12)–(6.13), and using that the low-order Sobolev norms of $\partial_t E$, $\partial_x E$, $\partial_x v$, and $E^{-1/2}$ may be estimated by C (cf. Remark 3.4 and 3.5). We arrive at

$$\frac{d}{dt}\|E^{1/2}\dot{u}\|_{L^2} \leq \frac{C\|\dot{u}\|_{L^2}^2}{\|E^{1/2}\dot{u}\|_{L^2}} + \frac{C\|F\|_{L^2}\|\dot{u}\|_{L^2}}{\|E^{1/2}\dot{u}\|_{L^2}} \leq C(\|\dot{u}\|_{L^2} + \|F\|_{L^2}),$$

where we appealed to

$$\|\dot u\|_{L^2} = \|E^{-1/2}E^{1/2}\dot u\|_{L^2} \le C\|E^{-1/2}\|_{L^\infty}\|E^{1/2}\dot u\|_{L^2} \le C\|E^{1/2}\dot u\|_{L^2},$$

and the lemma is proven. □

Using the previous lemma, the next statement provides a pure time derivative analytic estimate for the solution u in the $A_1(\tau)$ norm.

LEMMA 6.2. *There exist $t_0 > 0$ sufficiently small depending only on M_0 and $\epsilon_1 > 0$ sufficiently small depending on $M_{\epsilon,\kappa}(T)$ such that for $t \in (0, t_0)$ and $\epsilon \in (0, \epsilon_1)$, we have*

$$\|u(t)\|_{A_1} \le C + tQ(M_{\epsilon,\kappa}(t)), \tag{6.14}$$

for a function Q.

PROOF. For $m \in \mathbb{N}$, we apply $(\epsilon\partial_t)^m$ to the equation (2.14). Then $\dot u = (\epsilon\partial_t)^m u$ satisfies (6.10) with

$$F = [E, (\epsilon\partial_t)^m]\partial_t u + [Ev, (\epsilon\partial_t)^m]\nabla u. \tag{6.15}$$

Denote

$$\|u\|_{A_E} = \sum_{m=1}^{\infty} \|E^{1/2}(\epsilon\partial_t)^m u\|_{L^2} \frac{\tau^{(m-3)_+}}{(m-3)!} \tag{6.16}$$

with the corresponding dissipative norm

$$\|u\|_{\tilde A_E} = \sum_{m=4}^{\infty} \|E^{1/2}(\epsilon\partial_t)^m u\|_{L^2} \frac{(m-3)\tau^{m-4}}{(m-3)!}. \tag{6.17}$$

By Lemma 6.1 and using the notation (6.16)–(6.17), we obtain

$$\begin{aligned} \frac{d}{dt}\|u\|_{A_E} &= \dot\tau\|u\|_{\tilde A_E} + \sum_{m=1}^{\infty} \frac{\tau^{(m-3)_+}}{(m-3)!}\frac{d}{dt}\|E^{1/2}(\epsilon\partial_t)^m u\|_{L^2} \\ &\le \dot\tau\|u\|_{\tilde A_E} + C\|u\|_{A_1} + C\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}\|F\|_{L^2}, \end{aligned} \tag{6.18}$$

where F is given in (6.15). Note that

$$\begin{aligned} \|F\|_{L^2} &\le \sum_{j=1}^{m}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} + \sum_{j=1}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2} \\ &= F_{1,m} + F_{2,m}. \end{aligned} \tag{6.19}$$

For the first sum in (6.19), we have

$$\begin{aligned} \sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{1,m} &= \sum_{m=1}^{4}\sum_{j=1}^{m}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} \\ &\quad + \sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} \\ &\quad + \sum_{m=5}^{\infty}\sum_{j=[m/2]+1}^{m}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} \\ &= \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3, \end{aligned} \tag{6.20}$$

where we split the sum according to the low and high values of j. We claim

$$\mathcal{D}_1 \le C, \tag{6.21}$$

$$\mathcal{D}_2 \le C\|\partial_t E\|_{B(\tau)}\|u\|_{\tilde A_1(\tau)} + C\|u\|_{\tilde A_1(\tau)}, \tag{6.22}$$

$$\mathcal{D}_3 \le C\|\partial_t E\|_{B(\tau)}\|u\|_{A(\tau)}. \tag{6.23}$$

Proof of (6.21): Using Hölder's and the Sobolev inequalities, we may estimate $\mathcal{D}_1$ using low order mixed space time derivative of u and S, and from Remark 3.4, we obtain (6.21).

Proof of (6.22): Using the approach as in the estimate for S, we have

$$\begin{aligned}
\mathcal{D}_1 &= \sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]} \frac{\tau^{m-3}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} \\
&\le C\tau^a \sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]} \left(\frac{\tau^{(j-3)_+}}{(j-3)!}\|(\epsilon\partial_t)^{j-1}\partial_t E\|_{L^2}\right)^{1/4}\left(\frac{\tau^{(j-1)_+}}{(j-1)!}\|D^2(\epsilon\partial_t)^{j-1}\partial_t E\|_{L^2}\right)^{3/4} \\
&\quad\times\left(\frac{(m-j-2)\tau^{m-j-3}}{(m-j-2)!}\|(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\right)\mathcal{A}_{j,m},
\end{aligned} \tag{6.24}$$

where

$$\mathcal{A}_{j,m} = \frac{m!}{j!(m-j)!(m-3)!}\frac{(j-3)!^{1/4}(j-1)!^{3/4}(m-j-2)!}{m-j-2} \le \frac{Cm^3}{(m-j)^3} \le C,$$

and

$$a = m-3-\left(\frac{j-3}{4}\right)_+ - \left(\frac{3j-3}{4}\right)_+ - (m-j-3) \ge 1, \tag{6.25}$$

since $1 \le j \le [m/2]$. By (6.24)–(6.25) and the discrete Young inequality, we obtain

$$\begin{aligned}
\mathcal{D}_2 &\le C\sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\left(\frac{\tau^{(j-3)_+}}{(j-3)!}\|(\epsilon\partial_t)^{j-1}\partial_t E\|_{L^2}\right)\left(\frac{(m-j-2)\tau^{m-j-3}}{(m-j-2)!}\|(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\right) \\
&\quad + C\sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\left(\frac{\tau^{(j-1)_+}}{(j-1)!}\|D^2(\epsilon\partial_t)^{j-1}\partial_t E\|_{L^2}\right)\left(\frac{(m-j-2)\tau^{m-j-3}}{(m-j-2)!}\|(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\right) \\
&\le C\left(\|\partial_t E\|_{B(\tau)} + \|\partial_t E\|_{L^2}\right)\|u\|_{\tilde{A}_1(\tau)} \\
&\le C\|\partial_t E\|_{B(\tau)}\|u\|_{\tilde{A}_1(\tau)} + C\|u\|_{\tilde{A}_1(\tau)}.
\end{aligned}$$

Proof of (6.23): Reversing the roles of j and $m-j$ and proceeding as in the above argument, we may write

$$\begin{aligned}
\mathcal{D}_3 &\le C\tau^b\sum_{m=5}^{\infty}\sum_{j=[m/2]+1}^{m}\left(\frac{\tau^{(j-3)_+}}{(j-3)!}\|(\epsilon\partial_t)^{j-1}\partial_t E\|_{L^2}\right)\left(\frac{\tau^{(m-j-2)_+}}{(m-j-2)!}\|(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\right)^{1/4} \\
&\quad\times\left(\frac{\tau^{(m-j)_+}}{(m-j)!}\|D^2(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\right)^{3/4}\mathcal{B}_{j,m},
\end{aligned} \tag{6.26}$$

where

$$\mathcal{B}_{j,m} = \frac{m!}{j!(m-j)!}\frac{(j-3)!(m-j-2)!^{1/4}(m-j)!^{3/4}}{(m-3)!} \le \frac{Cm^3}{j^3} \le C,$$

and

$$b = m-3-(j-3)_+ - \left(\frac{m-j-2}{4}\right)_+ - \left(\frac{3m-3j}{4}\right)_+ \ge 0, \tag{6.27}$$

since $m \ge j \ge [m/2]+1$. From (6.26)–(6.27), we obtain

$$\mathcal{D}_3 \le C\|\partial_t E\|_{B(\tau)}\|u\|_{A(\tau)}. \tag{6.28}$$

Analogously, the second sum in (6.19) may be separated according to low and high values of j, obtaining

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{2,m} &= \sum_{m=1}^{4}\sum_{j=1}^{[m/2]}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\\ &\quad+\sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\\ &\quad+\sum_{m=5}^{\infty}\sum_{j=[m/2]+1}^{m}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\\ &=\mathcal{D}_4+\mathcal{D}_5+\mathcal{D}_6.\end{aligned} \tag{6.29}$$

We claim

$$\mathcal{D}_4 \le C, \tag{6.30}$$

$$\mathcal{D}_5 \le C\|Ev\|_{A(\tau)}\|\nabla u\|_{A_1(\tau)}, \tag{6.31}$$

$$\mathcal{D}_6 \le C\|Ev\|_{A(\tau)}\|u\|_{A(\tau)}. \tag{6.32}$$

Proof of (6.30): Proceeding as in the proof of (6.30), we obtain that the low-order mixed space-time derivatives may be estimated by C.

Proof of (6.31): As in (6.24), we have

$$\begin{aligned}\mathcal{D}_5 &\le C\sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)\|_{L^2}^{1/4}\|D^2(\epsilon\partial_t)^j(Ev)\|_{L^2}^{3/4}\|(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\\ &\le C\sum_{m=5}^{\infty}\sum_{j=1}^{[m/2]}\left(\|(\epsilon\partial_t)^j(Ev)\|_{L^2}\frac{\tau^{(j-3)_+}}{(j-3)!}\right)^{1/4}\left(\|D^2(\epsilon\partial_t)^j(Ev)\|_{L^2}\frac{\tau^{(j-1)_+}}{(j-1)!}\right)^{3/4}\\ &\qquad\times\left(\|(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\frac{\tau^{(m-j-3)_+}}{(m-j-3)!}\right)\\ &\le C\|Ev\|_{A(\tau)}\|\nabla u\|_{A_1(\tau)}.\end{aligned}$$

Proof of (6.32): As in (6.26), we arrive at

$$\begin{aligned}\mathcal{D}_6 &\le C\sum_{m=5}^{\infty}\sum_{j=[m/2]+1}^{m}\frac{\tau^{(m-3)_+}}{(m-3)!}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)\|_{L^2}\|(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}^{1/4}\|_{L^2}\|D^2(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}^{3/4}\\ &\le C\sum_{m=5}^{\infty}\sum_{j=[m/2]+1}^{m}\left(\|(\epsilon\partial_t)^j(Ev)\|_{L^2}\frac{\tau^{(j-3)_+}}{(j-3)!}\right)\left(\|(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\frac{\tau^{(m-j-2)_+}}{(m-j-2)!}\right)^{1/4}\\ &\qquad\times\left(\|D^2(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\frac{\tau^{(m-j)_+}}{(m-j)!}\right)^{3/4}\\ &\le C\|Ev\|_{A(\tau)}\|u\|_{A(\tau)}.\end{aligned}$$

Collecting the above estimates (6.20)–(6.23) and (6.29)–(6.32), we obtain from (6.19),

$$\begin{aligned}&\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{1,m}+\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{2,m}\le\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3+\mathcal{D}_4+\mathcal{D}_5+\mathcal{D}_6\\ &\quad\le C+C\|\partial_t E\|_{B(\tau)}\|u\|_{\tilde{A}_1(\tau)}+C\|u\|_{\tilde{A}_1}+C\|\partial_t E\|_{B(\tau)}\|u\|_{A(\tau)}\\ &\qquad+C\|Ev\|_{A(\tau)}\|\nabla u\|_{A_1(\tau)}+C\|Ev\|_{A(\tau)}\|u\|_{A(\tau)},\end{aligned} \tag{6.33}$$

where we estimate $\|\partial_t E\|_{B(\tau)}$ using Lemma 5.3, and $\|Ev\|_{A(\tau)}$ with (5.13) and Lemma 5.4.

In order to estimate the dissipative term $\|\nabla u\|_{A_1(\tau)}$, we recall the elliptic regularity for the div-curl system

$$\|\nabla v\|_{L^2} \le C\|\operatorname{div} v\|_{L^2} + C\|\operatorname{curl} v\|_{L^2}, \tag{6.34}$$

which, together with the definition of the $A_1(\tau)$ norm, leads to

$$\|\nabla u\|_{A_1(\tau)} \le C\|L(\partial_x)u\|_{A_1(\tau)} + C\|\operatorname{curl} v\|_{A_1(\tau)}. \tag{6.35}$$

To treat the divergence part of the dissipative term, we rewrite the equation (2.14) as

$$L(\partial_x)u = -E(S,\epsilon u)(\epsilon\partial_t u + \epsilon v\cdot\nabla u). \tag{6.36}$$

For $m\in\mathbb{N}$, we apply $(\epsilon\partial_t)^m$ to the equation (6.36), obtaining

$$\begin{aligned}\|(\epsilon\partial_t)^m L(\partial_x)u\|_{L^2} \le{}& C\sum_{j=0}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j E(\epsilon\partial_t)^{m-j+1}u\|_{L^2} + C\epsilon\|Ev\|_{L^\infty}\|(\epsilon\partial_t)^m\nabla u\|_{L^2}\\ &+ C\epsilon\sum_{j=1}^{m}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}.\end{aligned}$$

From here we arrive at

$$\begin{aligned}\|L(\partial_x)u\|_{A_1(\tau)} &= \sum_{m=1}^{\infty}\|(\epsilon\partial_t)^m L(\partial_x)u\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\le \sum_{m=1}^{2}\|E(\epsilon\partial_t)^{m+1}u\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!} + \sum_{m=3}^{\infty}\|E\|_{L^\infty}\|(\epsilon\partial_t)^{m+1}u\|_{L^2}\frac{\tau^{m-3}}{(m-3)!}\\ &\quad + C\epsilon\sum_{m=1}^{\infty}\sum_{j=1}^{m}\binom{m}{j}\|(\epsilon\partial_t)^{j-1}\partial_t E(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\quad + C\epsilon\sum_{m=1}^{\infty}\|(\epsilon\partial_t)^m\nabla v\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\quad + C\epsilon\sum_{m=1}^{\infty}\sum_{j=1}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j(Ev)(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\le C + C\|E\|_{L^\infty}\|u\|_{\tilde{A}_1} + C\epsilon\|\nabla v\|_{A_1(\tau)} + C\epsilon\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{1,m} + C\epsilon\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}F_{2,m},\end{aligned} \tag{6.37}$$

where we used the notation from (6.19) in the last inequality. The third term on the far right side of (6.37) may be absorbed in the left side of (6.35) when ϵ is sufficiently small, and the fourth and fifth terms may be absorbed into the left side of (6.33) when ϵ is sufficiently small depending on $M_{\epsilon,\kappa}(T)$.

To treat the curl part of the dissipative term, using the similar technique for the curl estimate above, we have

$$\begin{aligned}&\|(\epsilon\partial_t)^m\operatorname{curl} v\|_{L^2} = \|(\epsilon\partial_t)^m\operatorname{curl}(R_0 r_0 v)\|_{L^2}\\ &\le \sum_{j=0}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j R_0(\epsilon\partial_t)^{m-j}\operatorname{curl}(r_0 v)\|_{L^2} + \sum_{j=0}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j\nabla R_0(\epsilon\partial_t)^{m-j} r_0 v\|_{L^2}.\end{aligned}$$

We use a similar technique as in the proofs of (4.4)–(4.5), obtaining

$$\begin{aligned}\|\operatorname{curl} v\|_{A_1(\tau)} &\le \sum_{m=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j R_0(\epsilon\partial_t)^{m-j}\operatorname{curl}(r_0 v)\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\quad+\sum_{m=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j}\|(\epsilon\partial_t)^j\nabla R_0(\epsilon\partial_t)^{m-j}r_0 v\|_{L^2}\frac{\tau^{(m-3)_+}}{(m-3)!}\\ &\le C\|R_0\|_{A(\tau)}\|\operatorname{curl}(r_0 v)\|_{\tilde B(\tau)}+C\|R_0\|_{A(\tau)}\|\operatorname{curl}(r_0 v)\|_{B(\tau)}\\ &\quad+C\|R_0\|_{A(\tau)}\|r_0 v\|_{A(\tau)}+C\|R_0\|_{\tilde A(\tau)}\|r_0 v\|_{A(\tau)}.\end{aligned} \tag{6.38}$$

Since R_0 is a function of S, it satisfies the inhomogeneous transport equation

$$\partial_t R_0 + v\cdot\nabla R_0 = 0.$$

Then Lemma 4.1 and (4.20) imply that

$$\frac{d}{dt}\|R_0\|_{A(\tau)} \le \|R_0\|_{\tilde A(\tau)}(\dot\tau + C\|v\|_{A(\tau)}) + C\|R_0\|_{A(\tau)} + C\|v\|_{A(\tau)} + C. \tag{6.39}$$

Coupling (6.3), (6.18), and (6.39), we arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{A_E}+\|R_0\|_{A(\tau)}+\|\operatorname{curl}(r_0 v)\|_{B(\tau)}\right)\\ &\le \dot\tau\|u\|_{\tilde A_E}+C\|u\|_{A_1(\tau)}+C\sum_{m=1}^{\infty}\frac{\tau^{(m-3)_+}}{(m-3)!}\|F\|_{L^2}+\|R_0\|_{\tilde A(\tau)}(\dot\tau+C\|v\|_{A(\tau)})+C\|R_0\|_{A(\tau)}\\ &\quad+C\|v\|_{A(\tau)}+\dot\tau\|\operatorname{curl}(r_0 v)\|_{\tilde B(\tau)}+C\|G\|_{B(\tau)}+C\|v\|_{B(\tau)}+C.\end{aligned} \tag{6.40}$$

Collecting the estimates (6.18), (6.33), (6.35), (6.37), (6.38), and (6.40), we arrive at

$$\begin{aligned}&\frac{d}{dt}\left(\|u\|_{A_E}+\|R_0\|_{A(\tau)}+\|\operatorname{curl}(r_0 v)\|_{B(\tau)}\right)\\ &\le \|u\|_{\tilde A_E}\left(\dot\tau+C\|\partial_t E\|_{B(\tau)}+C+C\|Ev\|_{A(\tau)}\right)+C\|\partial_t E\|_{B(\tau)}\|u\|_{A(\tau)}+C\|Ev\|_{A(\tau)}\|u\|_{A(\tau)}\\ &\quad+C\|Ev\|_{A(\tau)}+C\|Ev\|_{A(\tau)}\|R_0\|_{A(\tau)}\|\operatorname{curl}(r_0 v)\|_{B(\tau)}+C\|Ev\|_{A(\tau)}\|R_0\|_{A(\tau)}\|r_0 v\|_{A(\tau)}\\ &\quad+\|R_0\|_{\tilde A(\tau)}\left(\dot\tau+C\|v\|_{A(\tau)}+C\|Ev\|_{A(\tau)}\|r_0 v\|_{A(\tau)}\right)+C\|R_0\|_{A(\tau)}+C\|v\|_{A(\tau)}\\ &\quad+\|\operatorname{curl}(r_0 v)\|_{\tilde B(\tau)}\left(\dot\tau+C\|Ev\|_{A(\tau)}\|R_0\|_{A(\tau)}\right)+C\|G\|_{B(\tau)}+C\|v\|_{B(\tau)}+C,\end{aligned}$$

where we appealed to

$$\|u\|_{\tilde A_1(\tau)} \le C\|u\|_{\tilde A_E}$$

by the boundedness of $\|E^{-1/2}\|_{L^\infty}$.

Now, assume that the radius $\tau(t)$ decreases sufficiently fast so that the factors next to $\|u\|_{\tilde A_E}$, $\|R_0\|_{\tilde A(\tau)}$, and $\|\operatorname{curl}(r_0 v)\|_{\tilde B(\tau)}$ are less than or equal to 0. Integrating the resulting inequality on $[0,t]$, we get

$$\|u\|_{A_E} \le \|u(0)\|_{A_E}+\|R_0(0)\|_{A(\tau)}+\|\operatorname{curl}(r_0 v)(0)\|_{B(\tau)}+tQ(M_{\epsilon,\kappa}(t)) \le C+tQ(M_{\epsilon,\kappa}(t)),$$

and since $\|u\|_{A_1} \le C\|u\|_{A_E}$, the proof is concluded. □

6.3. Energy equation for the mixed derivatives (A_2 norm). Here we estimate the mixed space-time analytic norm. For this purpose, denote

$$\|u\|_{A_2(\tau)} = \sum_{m=1}^{\infty}\sum_{j=1}^{m}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}\tau(t)^{(m-3)_+}}{(m-3)!},$$

and let

$$\|u\|_{\tilde{A}_2(\tau)} = \sum_{m=4}^{\infty}\sum_{j=1}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}(m-3)\tau(t)^{m-4}}{(m-3)!}$$

be the corresponding dissipative analytic norm. Note that

$$\|u(t)\|_A = \|u(t)\|_{A_1} + \|u(t)\|_{A_2}.$$

LEMMA 6.3. *Given $M_0 > 0$, there exist a function Q and ϵ_0 such that for $\epsilon \in (0,\epsilon_0)$, $\kappa < 1$, the solution of* (2.14) *satisfies*

$$\|u(t)\|_{A_2} \le C + (t+\epsilon+\tau+\kappa)\,Q(M_{\epsilon,\kappa}(t)). \tag{6.41}$$

PROOF OF LEMMA 6.3. By the definition of the $A_2(\tau)$ norm, we have

$$\begin{aligned}
\|u\|_{A_2(\tau)} &= \sum_{m=1}^{\infty}\sum_{j=1}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}\tau^{(m-3)_+}}{(m-3)!}\\
&= \sum_{m=1}^{3}\sum_{j=1}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{1}{(m-3)!} + \sum_{m=4}^{\infty}\sum_{j=4}^{m}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{j-3}\tau^{m-3}}{(m-3)!}\\
&\quad+ \sum_{m=4}^{\infty}\sum_{j=1}^{3}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\tau^{m-3}}{(m-3)!} = \mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3,
\end{aligned}$$

where we split the sum according to the high and low values of j and m. We claim

$$\mathcal{P}_1 \le C, \tag{6.42}$$
$$\mathcal{P}_2 \le C + (t+\epsilon+\kappa+\tau)\,Q(M_{\epsilon,\kappa}(t)), \tag{6.43}$$
$$\mathcal{P}_3 \le C + (t+\epsilon+\tau)\,Q(M_{\epsilon,\kappa}(t)). \tag{6.44}$$

Firstly, (6.42) follows by using Sobolev inequalities and Remark 3.4.

Proof of (6.43): We rewrite equation (2.14) as

$$L(\partial_x)u = -E(S,\epsilon u)(\epsilon\partial_t u + \epsilon v\cdot\nabla u). \tag{6.45}$$

For $m \ge 3$, and $|\alpha| = j$ where $3 \le j \le m$, we commute $\partial^\alpha(\epsilon\partial_t)^{m-j}$ with (6.45), and using div-curl regularity (6.34), we obtain

$$\begin{aligned}
\|\nabla\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2} &\le C\|L(\partial_x)\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2} + C\|\operatorname{curl}(\partial^\alpha(\epsilon\partial_t)^{m-j}v)\|_{L^2}\\
&\le C\sum_{k=0}^{m-j}\sum_{l=0}^{j}\sum_{|\beta|=l,\beta\le\alpha}\binom{\alpha}{\beta}\binom{m-j}{k}\|\partial^\beta(\epsilon\partial_t)^k E\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k+1}u\|_{L^2}\\
&\quad+ C\epsilon\|Ev\|_{L^\infty}\|\partial^\alpha(\epsilon\partial_t)^{m-j}\nabla u\|_{L^2}\\
&\quad+ C\epsilon\sum_{l=0}^{j}\sum_{\substack{k=0\\ l+k\ge1}}^{m-j}\sum_{|\beta|=l,\beta\le\alpha}\binom{\alpha}{\beta}\binom{m-j}{k}\|\partial^\beta(\epsilon\partial_t)^k(Ev)\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla u\|_{L^2}\\
&\quad+ C\|\partial^\alpha(\epsilon\partial_t)^{m-j}\operatorname{curl} v\|_{L^2}.
\end{aligned} \tag{6.46}$$

The second term on the far right side of (6.46) can be absorbed into the left side when ϵ is sufficiently small. Multiply the above estimate with appropriate weights and then sum, with change of variables we obtain

$$\begin{aligned}
\mathcal{P}_2 &= \sum_{m=4}^{\infty}\sum_{j=4}^{m}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{j-3}\tau^{m-3}}{(m-3)!}\\
&\le C\sum_{m=3}^{\infty}\sum_{j=3}^{m}\sum_{|\alpha|=j}\|\nabla\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{j-2}\tau^{m-2}}{(m-2)!}\\
&\le C\kappa\sum_{m=4}^{\infty}\sum_{j=3}^{m-1}\sum_{|\alpha|=j}\sum_{k=0}^{m-j-1}\sum_{l=0}^{j}\sum_{|\beta|=l,\beta\le\alpha}\frac{\kappa^{j-3}\tau^{m-3}}{(m-3)!}\binom{\alpha}{\beta}\binom{m-j-1}{k}\|\partial^\beta(\epsilon\partial_t)^kE\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}u\|_{L^2}\\
&\quad+C\epsilon\sum_{m=3}^{\infty}\sum_{j=3}^{m}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\substack{k=0\\ l+k\ge1}}^{m-j}\sum_{|\beta|=l,\beta\le\alpha}\frac{\kappa^{j-2}\tau^{m-2}}{(m-2)!}\binom{\alpha}{\beta}\binom{m-j}{k}\\
&\qquad\times\|\partial^\beta(\epsilon\partial_t)^k(Ev)\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla u\|_{L^2}\\
&\quad+C\sum_{m=3}^{\infty}\sum_{j=3}^{m}\sum_{|\alpha|=j}\frac{\kappa^{j-2}\tau^{m-2}}{(m-2)!}\|\partial^\alpha(\epsilon\partial_t)^{m-j}\operatorname{curl}v\|_{L^2}\\
&\le C\kappa\|Eu\|_{A(\tau)}+C\epsilon(\|Ev\|_{A(\tau)}+\|Ev\|_{L^2})\|u\|_{A(\tau)}+C\|\operatorname{curl}v\|_{B(\tau)}\\
&\le C+(t+\kappa+\tau+\epsilon)\,Q(M_{\epsilon,\kappa}(t)),
\end{aligned}\tag{6.47}$$

where the last inequality follows from the estimates (5.13) and (6.9).

Proof of (6.44): For $m\ge4$ and $|\alpha|=j$ where $1\le j\le3$, we proceed as in (6.46)–(6.47) and obtain

$$\begin{aligned}
\mathcal{P}_3 &= \sum_{m=4}^{\infty}\sum_{j=1}^{3}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\tau^{m-3}}{(m-3)!}\\
&\le C\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\|\nabla\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\tau^{m-2}}{(m-2)!}\\
&\le C\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\|L(\partial_x)\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\tau^{m-2}}{(m-2)!}+C\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\|\operatorname{curl}(\partial^\alpha(\epsilon\partial_t)^{m-j}u)\|_{L^2}\frac{\tau^{m-2}}{(m-2)!}.
\end{aligned}\tag{6.48}$$

Therefore,

$$\begin{aligned}
\mathcal{P}_3 &\le C\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{|\beta|=l,\beta\le\alpha}\frac{\tau^{m-2}}{(m-2)!}\|\partial^\beta E\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j+1}u\|_{L^2}\\
&\quad+C\epsilon\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\sum_{k=1}^{m-j}\sum_{l=0}^{j}\sum_{|\beta|=l,\beta\le\alpha}\frac{\tau^{m-2}}{(m-2)!}\binom{m-j}{k}\|\partial^\beta(\epsilon\partial_t)^{k-1}\partial_tE\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k+1}u\|_{L^2}\\
&\quad+C\epsilon\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\substack{k=0\\ l+k\ge1}}^{m-j}\sum_{|\beta|=l,\beta\le\alpha}\frac{\tau^{m-2}}{(m-2)!}\binom{m-j}{k}\|\partial^\beta(\epsilon\partial_t)^k(Ev)\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-j-k}\nabla u\|_{L^2}\\
&\quad+C\sum_{m=3}^{\infty}\sum_{j=0}^{2}\sum_{|\alpha|=j}\frac{\tau^{m-2}}{(m-2)!}\|\partial^\alpha(\epsilon\partial_t)^{m-j}\operatorname{curl}v\|_{L^2}.
\end{aligned}\tag{6.49}$$

The second term on the far right side is estimated by using Lemmas 5.1 and 5.3, while the third and fourth term can be estimated analogously as in (6.47). For the first term, denoted by $\mathcal{P}_{31}$, we use the div-curl regularity to reduce the spatial derivative. We split it according to the values of j, obtaining

$$\begin{aligned}\mathcal{P}_{31} &= \sum_{m=3}^{\infty}\sum_{|\alpha|=0}\frac{\tau^{m-2}}{(m-2)!}\|E(\epsilon\partial_t)^{m+1}u\|_{L^2} + \sum_{m=3}^{\infty}\sum_{|\alpha|=1}\sum_{l=0}^{1}\sum_{|\beta|=l,\beta\le\alpha}\frac{\tau^{m-2}}{(m-2)!}\|\partial^\beta E\partial^{\alpha-\beta}(\epsilon\partial_t)^m u\|_{L^2}\\ &\quad+\sum_{m=3}^{\infty}\sum_{|\alpha|=2}\sum_{l=0}^{2}\sum_{|\beta|=l,\beta\le\alpha}\frac{\tau^{m-2}}{(m-2)!}\|\partial^\beta E\partial^{\alpha-\beta}(\epsilon\partial_t)^{m-1}u\|_{L^2}\\ &\le C\|u\|_{A_1(\tau)} + C\sum_{m=2}^{\infty}\|\nabla(\epsilon\partial_t)^m u\|_{L^2}\frac{\tau^{m-2}}{(m-2)!} + C\sum_{m=3}^{\infty}\|\nabla^2(\epsilon\partial_t)^{m-1}u\|_{L^2}\frac{\tau^{m-2}}{(m-2)!}.\end{aligned}\tag{6.50}$$

The first term on the far side is bounded by $C + tQ(M_{\epsilon,\kappa}(t))$ by Lemma 6.2, while the second and third terms can be estimated analogously to (6.48)–(6.50). Combining the resulting inequalities, we obtain

$$\mathcal{P}_3 \le C + (t+\epsilon+\tau)\,Q(M_{\epsilon,\kappa}(t)),$$

and the lemma then follows by (6.42)–(6.44). □

PROOF OF LEMMA 3.3. The inequality (3.13) follows by using (4.1), (6.14), and (6.41). □

7. The Mach limit

In this section, we prove the second main theorem on the Mach limit in the space X.

PROOF OF THEOREM 3.2. Let $\delta > 0$ be a small constant, which is to be determined below. For the sake of contradiction, we assume that $(v^\epsilon, p^\epsilon, S^\epsilon)$ does not converge to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $C([0,T], X_\delta)$. Then there exists a sequence $(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})$ which does not converge to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $C([0,T], X_\delta)$ as $\epsilon_n \to 0$. Recall from [MS, Theorem 1.4] that $(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})$ converges to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $L^\infty([0,T], L^2(\mathbb{R}^3))$ as $\epsilon_n \to 0$. For $k, n \in \mathbb{N}$, we define $v_{kn}(t) = v^{\epsilon_k}(t) - v^{\epsilon_n}(t)$. For $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^3$, using integration by parts and the Cauchy-Schwarz inequality leads to

$$\|\partial^\alpha v_{kn}\|_{L^2}^2 = \langle \partial^\alpha v_{kn}, \partial^\alpha v_{kn}\rangle = (-1)^{|\alpha|}\langle v_{kn}, \partial^{2\alpha} v_{kn}\rangle \le \|v_{kn}\|_{L^2}\|\partial^{2\alpha}v_{kn}\|_{L^2}.$$

Summing over $|\alpha| = m$ with $m \in \mathbb{N}$ such that $m \ge 4$, we obtain

$$\begin{aligned}\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\|\partial^\alpha v_{kn}\|_{L^2}\frac{\delta^{(m-3)_+}}{(m-3)!} &\le \|v_{kn}\|_{L^2}^{1/2}\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\|\partial^{2\alpha}v_{kn}\|_{L^2}^{1/2}\frac{\delta^{(m-3)_+}}{(m-3)!}\\ &= \|v_{kn}\|_{L^2}^{1/2}\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\left(\|\partial^{2\alpha}v_{kn}\|_{L^2}\frac{\kappa^{(2m-3)_+}\tau^{(2m-3)_+}}{(2m-3)!}\right)^{1/2}\frac{(2m-3)!^{1/2}\delta^{(m-3)_+}}{(m-3)!\kappa^{(2m-3)_+/2}\tau^{(2m-3)_+/2}}\\ &\le CM^{1/2}\|v_{kn}\|_{L^2}^{1/2}\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\frac{(2m-3)!^{1/2}\delta^{(m-3)_+}}{(m-3)!\kappa^{(2m-3)_+/2}\tau^{(2m-3)_+/2}}\\ &\le M^{1/2}\|v_{kn}\|_{L^2}^{1/2}\sum_{m=4}^{\infty}\frac{C^m(2m-3)!^{1/2}\delta^{(m-3)_+}}{(m-3)!\kappa^{(2m-3)_+/2}\tau^{(2m-3)_+/2}},\end{aligned}\tag{7.1}$$

where $C > 0$ is a fixed universal constant and M is as in (3.6). Now choose $\delta > 0$ sufficiently small so that $\delta/\kappa\tau \le 1/CC_0$ on the whole time interval $[0, T_0]$, where C_0 is sufficiently large, and obtain

$$\sum_{m=4}^{\infty}\sum_{|\alpha|=m}\|\partial^\alpha v_{kn}\|_{L^2}\frac{\delta^{(m-3)_+}}{(m-3)!} \le M^{1/2}\delta^{-3/2}\|v_{kn}\|_{L^2}^{1/2}\sum_{m=4}^{\infty}\frac{(2m-3)!^{1/2}}{C_0^m(m-3)!} \le M^{1/2}\delta^{-3/2}\|v_{kn}\|_{L^2}^{1/2},\tag{7.2}$$

where we used Stirling's formula and assumed C_0 to be sufficiently large so the sum converges. Analogously, we set $p_{kn}(t) = p^{\epsilon_k}(t) - p^{\epsilon_n}(t)$ and $S_{kn}(t) = S^{\epsilon_k}(t) - S^{\epsilon_n}(t)$ and proceed as in above obtaining

$$\sum_{m=4}^{\infty} \sum_{|\alpha|=m} \|\partial^\alpha p_{kn}\|_{L^2} \frac{\delta^{m-3}}{(m-3)!} \le M^{1/2}\delta^{-3/2}\|v_{kn}\|_{L^2}^{1/2}, \tag{7.3}$$

and

$$\sum_{m=4}^{\infty} \sum_{|\alpha|=m} \|\partial^\alpha S_{kn}\|_{L^2} \frac{\delta^{m-3}}{(m-3)!} \le M^{1/2}\delta^{-3/2}\|S_{kn}\|_{L^2}^{1/2}. \tag{7.4}$$

Note also that $\|v_{kn}\|_{H^3} \le C\|v_{kn}\|_{H^4}^{3/4}\|v_{kn}\|_{L^2}^{1/4} \le C\|v_{kn}\|_{L^2}^{1/4}$, by Remark 3.4, with analogous inequalities for p_{kn} and S_{kn}. Since M and δ are fixed constants, we infer from (7.2)–(7.4) that the sequence $\{(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})\}$ is Cauchy in $C([0,T_0], X_\delta)$ which implies that it converges in $C([0,T_0], X_\delta)$, which is a contradiction. Therefore, $\{(v^\epsilon, p^\epsilon, S^\epsilon)\}$ is convergent and converges to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $C([0,T_0], X_\delta)$ as $\epsilon \to 0$. ☐

8. Analyticity assumptions on the initial data

In this section, we assume that the initial data satisfies (3.2), and intend to prove that for smaller n we have

$$\sum_{n=0}^{3}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n u(0)\|_{L^2} \frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!} \le \Gamma, \tag{8.1}$$

and

$$\sum_{n=0}^{3}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n S(0)\|_{L^2} \frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!} \le \Gamma, \tag{8.2}$$

where $\Gamma > 0$ is a sufficiently large constant depending on M_0; for larger values of n, we claim that there exists a sufficiently small parameter $\lambda > 0$ depending on M_0, such that for all $k \ge 4$ we have

$$\sum_{n=4}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n u(0)\|_{L^2} \frac{\lambda^{n-3}\tau_0^{(j+n-3)_+}}{(j+n-3)!} \le 1 \tag{8.3}$$

and

$$\sum_{n=4}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n S(0)\|_{L^2} \frac{\lambda^{n-3}\tau_0^{(j+n-3)_+}}{(j+n-3)!} \le 1. \tag{8.4}$$

In (8.3) and (8.4) we then choose $\tilde{\tau}_0 = \lambda\tau_0/2$ and using (8.1)–(8.4), we get

$$\begin{aligned}
\|(p_0, v_0, S_0)\|_{A(\tilde{\tau}_0)} &= \sum_{n=0}^{\infty}\sum_{j=0, n+j\ge 1}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n (u,S)(0)\|_{L^2} \frac{\tilde{\tau}_0^{(j+n-3)_+}}{(j+n-3)!} \\
&\le \sum_{n=0}^{3}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n (u,S)(0)\|_{L^2} \frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!} \\
&\quad + \sum_{n=4}^{\infty}\frac{1}{2^{n-3}}\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(\epsilon\partial_t)^n (u,S)(0)\|_{L^2} \frac{\lambda^{n-3}\tau_0^{(j+n-3)_+}}{(j+n-3)!} \le \Gamma + \sum_{n=4}^{\infty}\frac{1}{2^{n-3}} \le C,
\end{aligned}$$

obtaining (3.4). In the remainder of this section, we prove (8.1)–(8.4).

For $n = 0$, we use the assumption (3.2) on the initial data to obtain

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha(u,S)(0)\|_{L^2} \frac{\tau_0^{(j-3)_+}}{(j-3)!} \le \Gamma_0, \tag{8.5}$$

for some constant $\Gamma_0 > 0$. Next, for $n = 1$, we apply ∂^α to (2.15) where $|\alpha| = j \in \mathbb{N}_0$, which leads to

$$\partial^\alpha \partial_t S = -\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta v \cdot \partial^{\alpha-\beta} \nabla S.$$

Therefore,

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha \epsilon\partial_t S\|_{L^2} \frac{\tau_0^{(j-2)_+}}{(j-2)!}\\ &\qquad\le C\epsilon \sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l} \binom{\alpha}{\beta} \|\partial^\beta v\cdot\partial^{\alpha-\beta}\nabla S\|_{L^2} \frac{\tau_0^{(j-2)_+}}{(j-2)!}.\end{aligned} \tag{8.6}$$

We split the right side of (8.6) according to low and high values l. By Hölder and Sobolev inequalities, we have

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha \epsilon\partial_t S\|_{L^2} \frac{\tau_0^{(j-2)_+}}{(j-2)!}\\ &\quad\le C\epsilon\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{0\le l\le[j/2]}\sum_{\beta\le\alpha,|\beta|=l}\left(\|D^2\partial^\beta v\|_{L^2}\frac{\tau_0^{(l-1)_+}}{(l-1)!}\right)^{3/4}\left(\|\partial^\beta v\|_{L^2}\frac{\tau_0^{(l-3)_+}}{(l-3)!}\right)^{1/4}\\ &\qquad\times\left(\|\partial^{\alpha-\beta}\nabla S\|_{L^2}\frac{\tau_0^{(j-l-2)_+}}{(j-l-2)!}\right)\frac{(l-1)!^{3/4}(l-3)!^{1/4}(j-l-2)!j!}{(j-2)!(j-l)!l!}\\ &\quad+C\epsilon\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{[j/2]+1\le l\le j}\sum_{\beta\le\alpha,|\beta|=l}\left(\|\partial^\beta v\|_{L^2}\frac{\tau_0^{(l-3)_+}}{(l-3)!}\right)\left(\|D^2\partial^{\alpha-\beta}\nabla S\|_{L^2}\frac{\tau_0^{(j-l)_+}}{(j-l)!}\right)^{3/4}\\ &\qquad\times\left(\|\partial^{\alpha-\beta}\nabla S\|_{L^2}\frac{\tau_0^{(j-l-2)_+}}{(j-l-2)!}\right)^{1/4}\frac{(j-l)!^{3/4}(j-l-2)!^{1/4}(l-3)!j!}{(j-2)!(j-l)!l!}.\end{aligned} \tag{8.7}$$

One may check that

$$\frac{(l-1)!^{3/4}(l-3)!^{1/4}(j-l-2)!j!}{(j-2)!(j-l)!l!} \le C,$$

for $l \le [j/2]$, while

$$\frac{(j-l)!^{3/4}(j-l-2)!^{1/4}(l-3)!j!}{(j-2)!(j-l)!l!} \le C, \tag{8.8}$$

for $l \ge [j/2]+1$. Collecting the estimates (8.5) and (8.7)–(8.8), we obtain

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha \epsilon\partial_t S\|_{L^2} \frac{\tau_0^{(j-2)_+}}{(j-2)!} \le C(\|v\|_{A_0(\tau_0)} + \|v\|_{L^2})\|S\|_{A_0(\tau_0)} \le \Gamma_1,$$

where $\Gamma_1 = Q(\Gamma_0)$ and

$$\|u\|_{A_0(\tau_0)} = \sum_{j=1}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha u\|_{L^2} \frac{\tau_0^{(j-3)_+}}{(j-3)!}.$$

As for (8.1), we rewrite the equation (2.14) as

$$\epsilon\partial_t u = -\epsilon v\cdot\nabla u - \tilde{E}L(\partial_x)u, \tag{8.9}$$

where we denoted $\tilde{E}(S,\epsilon u) = E^{-1}(S,\epsilon u)$. Applying ∂^α to (8.9), where $|\alpha| = j \geq 0$, we get

$$\begin{aligned}\|\partial^\alpha \epsilon\partial_t u\|_{L^2} \leq C\epsilon \sum_{l=0}^{j} \sum_{\beta\leq\alpha,|\beta|=l} \binom{\alpha}{\beta} \|\partial^\beta v\cdot\partial^{\alpha-\beta}\nabla u\|_{L^2} \\ + C\sum_{l=0}^{j} \sum_{\beta\leq\alpha,|\beta|=l} \binom{\alpha}{\beta} \|\partial^\beta \tilde{E}\partial^{\alpha-\beta}\nabla u\|_{L^2},\end{aligned}$$

from where

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\partial^\alpha\epsilon\partial_t u\|_{L^2}\frac{\tau_0^{(j-2)_+}}{(j-2)!} \\ &\quad\leq C\epsilon\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\leq\alpha,|\beta|=l}\binom{\alpha}{\beta}\|\partial^\beta v\cdot\partial^{\alpha-\beta}\nabla u\|_{L^2}\frac{\tau_0^{(j-2)_+}}{(j-2)!} \\ &\quad\quad + C\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\leq\alpha,|\beta|=l}\binom{\alpha}{\beta}\|\partial^\beta\tilde{E}\partial^{\alpha-\beta}\nabla u\|_{L^2}\frac{\tau_0^{(j-2)_+}}{(j-2)!} = I_1 + I_2.\end{aligned} \tag{8.10}$$

The term I_1 can be estimated analogously as in (8.6)–(8.8), obtaining $\mu_1 \leq Q(\Gamma_0)$. For the term I_2, we proceed as in (8.6)–(8.8), obtaining $\mu_2 \leq C\|\tilde{E}\|_{A_0(\tau_0)}\|u\|_{A_0(\tau_0)} + C\|\tilde{E}\|_{L^\infty}\|u\|_{A_0(\tau_0)}$. One may easily check that the product rules in Lemmas 5.1 and 5.2 hold for the norm $A_0(\tau_0)$. Thus we have

$$\|\tilde{E}\|_{A_0(\tau_0)} \leq Q(\|u\|_{A_0(\tau_0)} + \|u\|_{L^2}, \|S\|_{A_0(\tau_0)} + \|S\|_{L^2}) \leq Q(\Gamma_0). \tag{8.11}$$

Combining (8.10)–(8.11), we may write

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha\epsilon\partial_t u\|_{L^2}\frac{\tau_0^{(j-2)_+}}{(j-2)!} \leq \Gamma_1,$$

where $\Gamma_1 = Q(\Gamma_0)$.

For $n = 2$ and $n = 3$, the proof is completely analogous and we obtain

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n S\|_{L^2}\frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!} \leq \Gamma_n$$

and

$$\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n u\|_{L^2}\frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!} \leq \Gamma_n,$$

for sufficiently large Γ_n depending on Γ_0. Summing over n from 0 to 3, we obtain (8.1) and (8.2) for sufficiently large $\Gamma = Q(\Gamma_0)$. We fix Γ for the rest of the proof.

Next, we prove (8.3) and (8.4) for all $k \geq 4$ using induction and starting with the case $k = 4$. First, we apply $\partial^\alpha(\epsilon\partial_t)^3$ to (2.15), where $|\alpha| = j \geq 0$, obtaining

$$\partial^\alpha(\epsilon\partial_t)^3\partial_t S = -\sum_{\beta\leq\alpha}\sum_{n=0}^{3}\binom{\alpha}{\beta}\binom{3}{n}\partial^\beta(\epsilon\partial_t)^n v\cdot\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla S.$$

Using the splitting argument as in (8.6)–(8.7),

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^4 S\|_{L^2}\frac{\lambda\tau_0^{j+1}}{(j+1)!}\\
&\le C\epsilon\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{0\le l\le[j/2]}\sum_{\beta\le\alpha,|\beta|=l}\sum_{n=0}^{3}\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla S\|_{L^2}\frac{\tau_0^{(j-n-l+1)_+}}{(j-n-l+1)!}\right)\\
&\qquad\times\left(\|\partial^\beta(\epsilon\partial_t)^n v\|_{L^2}\frac{\tau_0^{(l+n-3)_+}}{(l+n-3)!}\right)^{1/4}\left(\|D^2\partial^\beta(\epsilon\partial_t)^n v\|_{L^2}\frac{\tau_0^{(l+n-1)_+}}{(l+n-1)!}\right)^{3/4}\\
&\quad+C\epsilon\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{[j/2]+1\le l\le j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{n=0}^{3}\left(\|\partial^\beta(\epsilon\partial_t)^n v\|_{L^2}\frac{\tau_0^{(l+n-3)_+}}{(l+n-3)!}\right)\\
&\qquad\times\left(\|D^2\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla S\|_{L^2}\frac{\tau_0^{(j-n-l+3)_+}}{(j-n-l+3)!}\right)^{3/4}\\
&\qquad\times\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla S\|_{L^2}\frac{\tau_0^{(j-n-l+1)_+}}{(j-n-l+1)!}\right)^{1/4}.
\end{aligned}
\tag{8.12}
$$

Appealing to (8.1) and (8.2), we arrive at

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^4 S\|_{L^2}\frac{\lambda\tau_0^{j+1}}{(j+1)!}\\
&\le C\lambda\sum_{n=0}^{3}\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n u\|_{L^2}\frac{\tau_0^{(j+n-3)_+}}{(j+n-3)!}\right)\\
&\qquad\times\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{3-n} S\|_{L^2}\frac{\tau_0^{(j-n)_+}}{(j-n)!}\right)\le\frac{1}{2},
\end{aligned}
\tag{8.13}
$$

where we set $\lambda=1/Q(\Gamma)$, concluding the proof of (8.4) for $k=4$. As for (8.3), we apply $\partial^\alpha(\epsilon\partial_t)^3$ to (8.9), where $|\alpha|=j\ge 0$, obtaining

$$
\begin{aligned}
\|\partial^\alpha(\epsilon\partial_t)^4 u\|_{L^2}&\le C\epsilon\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\binom{\alpha}{\beta}\binom{3}{n}\|\partial^\beta(\epsilon\partial_t)^n v\cdot\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla u\|_{L^2}\\
&\quad+C\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\binom{\alpha}{\beta}\binom{3}{n}\|\partial^\beta(\epsilon\partial_t)^n\tilde{E}\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla u\|_{L^2}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^4 u\|_{L^2}\frac{\lambda\tau_0^{j+1}}{(j+1)!}\\
&\le C\epsilon\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{n=0}^{3}\binom{\alpha}{\beta}\binom{3}{n}\|\partial^\beta(\epsilon\partial_t)^n v\cdot\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla u\|_{L^2}\frac{\tau_0^{j+1}}{(j+1)!}\\
&\quad+C\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{n=0}^{3}\binom{\alpha}{\beta}\binom{3}{n}\|\partial^\beta(\epsilon\partial_t)^n\tilde{E}\partial^{\alpha-\beta}(\epsilon\partial_t)^{3-n}\nabla u\|_{L^2}\frac{\tau_0^{j+1}}{(j+1)!}\\
&=I_{41}+I_{42}.
\end{aligned}
\tag{8.14}
$$

The term I_{41} can be estimated as in (8.12)–(8.13), obtaining $I_{41} \le 1/2$, while I_{42} can be treated in a similar fashion as in (8.11), arriving at $I_{42} \le C\lambda \|\tilde{E}\|_{A_3(\tau_0)} \|u\|_{A_3(\tau_0)} + C\lambda \|\tilde{E}\|_{L^\infty} \|u\|_{A_3(\tau_0)}$, where for each $k \ge 3$, we denote

$$\|u\|_{A_k(\tau_0)} = \sum_{n=0}^{k} \sum_{j=0, j+n\ge 1}^{\infty} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon\partial_t)^n u\|_{L^2} \frac{\lambda^{(n-3)_+} \tau_0^{(j+n-3)_+}}{(j+n-3)!}. \tag{8.15}$$

One can easily check that Lemma 5.1 and 5.2 hold for the $A_k(\tau_0)$-norm for each $k \ge 3$. Therefore, $I_{42} \le 1/2$ by choosing $\lambda = 1/Q(\Gamma)$. There, we obtain (8.3) for $k = 4$.

Now we assume that we have (8.3) and (8.4) for some $k \ge 4$, and prove them for $k+1$. For $n \ge 3$, we apply $\partial^\alpha (\epsilon\partial_t)^n$ to (2.15), where $|\alpha| = j \ge 0$, obtaining

$$\partial^\alpha (\epsilon\partial_t)^n \partial_t S = -\sum_{\beta\le\alpha} \sum_{m=0}^{n} \partial^\beta (\epsilon\partial_t)^m v \cdot \partial^{\alpha-\beta} (\epsilon\partial_t)^{n-m} \nabla S,$$

from where

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon\partial_t)^{n+1} S\|_{L^2} \frac{\lambda^{n-2} \tau_0^{(j+n-2)_+}}{(j+n-2)!} \\
&\quad \le C\epsilon \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \sum_{l=0}^{j} \sum_{\beta\le\alpha, |\beta|=l} \sum_{m=0}^{n} \binom{\alpha}{\beta} \binom{n}{m} \\
&\qquad\qquad \times \|\partial^\beta (\epsilon\partial_t)^m v \cdot \partial^{\alpha-\beta} (\epsilon\partial_t)^{n-m} \nabla S\|_{L^2} \frac{\lambda^{n-2} \tau_0^{(j+n-2)_+}}{(j+n-2)!}.
\end{aligned}$$

We split the above sum according to the low and high values of $l+n$. Using a similar argument as in (8.12), we get

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{|\alpha|=j} \|\partial^\alpha (\epsilon\partial_t)^{n+1} S\|_{L^2} \frac{\lambda^{n-2} \tau_0^{(j+n-2)_+}}{(j+n-2)!} \\
&\le C\epsilon\lambda \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \sum_{l=0}^{j} \sum_{\beta\le\alpha, |\beta|=l} \sum_{m=0}^{n} \left(\|D^2 \partial^\beta (\epsilon\partial_t)^m v\|_{L^2} \frac{\lambda^{(m-3)_+} \tau_0^{(l+m-1)_+}}{(l+m-1)!} \right)^{3/4} \\
&\qquad \times \left(\|\partial^\beta (\epsilon\partial_t)^m v\|_{L^2} \frac{\lambda^{(m-3)_+} \tau_0^{(l+m-3)_+}}{(l+m-3)!} \right)^{1/4} \\
&\qquad \times \left(\|\partial^{\alpha-\beta} (\epsilon\partial_t)^{n-m} \nabla S\|_{L^2} \frac{\lambda^{(n-m-3)_+} \tau_0^{(j+n-l-m-2)_+}}{(j+n-l-m-2)!} \right) \mathbb{1}_{\{0\le l+m\le [(j+n)/2]\}} \\
&\quad + C\epsilon\lambda \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \sum_{l=0}^{j} \sum_{\beta\le\alpha, |\beta|=l} \sum_{m=0}^{n} \left(\|D^2 \partial^{\alpha-\beta} (\epsilon\partial_t)^{n-m} \nabla S\|_{L^2} \frac{\lambda^{(n-m-3)_+} \tau_0^{(j+n-l-m)_+}}{(j+n-l-m)!} \right)^{3/4} \\
&\qquad \times \left(\|\partial^{\alpha-\beta} (\epsilon\partial_t)^{n-m} \nabla S\|_{L^2} \frac{\lambda^{(n-m-3)_+} \tau_0^{(j+n-l-m-2)_+}}{(j+n-l-m-2)!} \right)^{1/4} \\
&\qquad \times \left(\|\partial^\beta (\epsilon\partial_t)^m v\|_{L^2} \frac{\lambda^{(m-3)_+} \tau_0^{(l+m-3)_+}}{(l+m-3)!} \right) \mathbb{1}_{\{[(j+n)/2]+1\le l+m\le j+n\}},
\end{aligned}$$

from where

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n+1}S\|_{L^2}\frac{\lambda^{n-2}\tau_0^{(j+n-2)_+}}{(j+n-2)!}\\
&\le C\epsilon\lambda\sum_{m=0}^{n}\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m}v\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right)\\
&\qquad\times\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n-m}S\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-m-3)_+}}{(j+n-m-3)!}\right).
\end{aligned} \tag{8.16}$$

Summing the above estimate in n from 3 to k, we get

$$\sum_{n=4}^{k+1}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n}S\|_{L^2}\frac{\lambda^{n-3}\tau_0^{(j+n-3)_+}}{(j+n-3)!}=\sum_{n=3}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n+1}S\|_{L^2}\frac{\lambda^{n-2}\tau_0^{(j+n-2)_+}}{(j+n-2)!},$$

which is bounded from above by

$$\begin{aligned}
&C\lambda\sum_{n=3}^{k}\sum_{m=0}^{n}\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m}v\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right)\\
&\qquad\times\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n-m}S\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-m-3)_+}}{(j+n-m-3)!}\right)\\
&\le C\lambda\left(\sum_{m=0}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m}v\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right)\\
&\qquad\times\left(\sum_{m=0}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m}S\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right).
\end{aligned} \tag{8.17}$$

By (8.1) and (8.2), and the inductive hypothesis (8.3)–(8.4) for k, we arrive at

$$\sum_{n=4}^{k+1}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n}S\|_{L^2}\frac{\lambda^{n-3}\tau_0^{(j+n-3)_+}}{(j+n-3)!}\le\frac{1}{2},$$

where we choose $\lambda=1/Q(\Gamma)$, which leads to (8.4) for $k+1$.

As for (8.3), we apply $\partial^\alpha(\epsilon\partial_t)^n$ to (8.9) where $|\alpha|=j\ge 0$ and $n\ge 3$. Similarly to (8.10), we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n+1}u\|_{L^2}\frac{\lambda^{n-2}\tau_0^{j+n-2}}{(j+n-2)!}\\
&\le C\epsilon\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{m=0}^{n}\binom{\alpha}{\beta}\binom{n}{m}\|\partial^\beta(\epsilon\partial_t)^m v\cdot\partial^{\alpha-\beta}(\epsilon\partial_t)^{n-m}\nabla u\|_{L^2}\frac{\lambda^{n-3}\tau_0^{j+n-2}}{(j+n-2)!}\\
&\quad+C\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{m=0}^{n}\binom{\alpha}{\beta}\binom{n}{m}\|\partial^\beta(\epsilon\partial_t)^m\tilde{E}\partial^{\alpha-\beta}(\epsilon\partial_t)^{n-m}\nabla u\|_{L^2}\frac{\lambda^{n-3}\tau_0^{j+n-2}}{(j+n-2)!}\\
&=J_{1n}+J_{2n}.
\end{aligned} \tag{8.18}$$

For the term J_{1n}, we proceed as in (8.16)–(8.17), obtaining

$$\sum_{n=3}^{k}J_{1n}\le\frac{1}{2}. \tag{8.19}$$

For the term J_{2n}, we split the sum according to the low and high values of $l+n$. Proceeding as in (8.16), we arrive at

$$\begin{aligned} J_{2n} &\le C\lambda \sum_{j=0}^{\infty}\sum_{|\alpha|=j} \|\tilde{E}\|_{L^\infty}\|\partial^\alpha(\epsilon\partial_t)^n\nabla u\|_{L^2}\frac{\lambda^{n-3}\tau_0^{j+n-2}}{(j+n-2)!} \\ &\quad + C\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{m=0}^{n}\left(\|D^2\partial^\beta(\epsilon\partial_t)^m\tilde{E}\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(l+m-1)_+}}{(l+m-1)!}\right)^{3/4} \\ &\qquad \times\left(\|\partial^\beta(\epsilon\partial_t)^m\tilde{E}\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(l+m-3)_+}}{(l+m-3)!}\right)^{1/4} \\ &\qquad \times\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{n-m}\nabla u\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-l-m-2)_+}}{(j+n-l-m-2)!}\right)\mathbb{1}_{\{1\le l+m\le[(j+n)/2]\}} \\ &\quad + C\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\sum_{l=0}^{j}\sum_{\beta\le\alpha,|\beta|=l}\sum_{m=0}^{n}\left(\|D^2\partial^{\alpha-\beta}(\epsilon\partial_t)^{n-m}\nabla u\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-l-m)_+}}{(j+n-l-m)!}\right)^{3/4} \\ &\qquad \times\left(\|\partial^{\alpha-\beta}(\epsilon\partial_t)^{n-m}\nabla u\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-l-m-2)_+}}{(j+n-l-m-2)!}\right)^{1/4} \\ &\qquad \times\left(\|\partial^\beta(\epsilon\partial_t)^m\tilde{E}\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(l+m-3)_+}}{(l+m-3)!}\right)\mathbb{1}_{\{[(j+n)/2]+1\le l+m\le j+n\}}, \end{aligned}$$

and thus

$$\begin{aligned} J_{2n} &\le C\lambda\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n u\|_{L^2}\frac{\lambda^{(n-3)_+}\tau_0^{(j+n-3)_+}}{(j+n-3)!} \\ &\quad + C\lambda\sum_{m=0}^{n}\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j,m+j\ge1}\|\partial^\alpha(\epsilon\partial_t)^m\tilde{E}\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right) \\ &\qquad \times\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n-m}u\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-m-3)_+}}{(j+n-m-3)!}\right). \end{aligned}$$

Summing the above estimate in n from 3 to k , we obtain

$$\begin{aligned} \sum_{n=3}^{k} J_{2n} &\le C\lambda\sum_{n=3}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n u\|_{L^2}\frac{\lambda^{(n-3)_+}\tau_0^{(j+n-3)_+}}{(j+n-3)!} \\ &\quad + C\lambda\sum_{n=3}^{k}\sum_{m=0}^{n}\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j,m+j\ge1}\|\partial^\alpha(\epsilon\partial_t)^m\tilde{E}\|_{L^2}\frac{\lambda^{(m-3)_+}\tau_0^{(j+m-3)_+}}{(j+m-3)!}\right) \\ &\qquad \times\left(\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n-m}u\|_{L^2}\frac{\lambda^{(n-m-3)_+}\tau_0^{(j+n-m-3)_+}}{(j+n-m-3)!}\right) \\ &\le C\lambda\|u\|_{A_k(\tau_0)} + C\lambda\|\tilde{E}\|_{A_k(\tau_0)}(\|u\|_{A_k(\tau_0)}+\|u\|_{L^2}), \end{aligned} \tag{8.20}$$

where we used the $A_k(\tau_0)$ norm in (8.15). The first term on the right side of above can be estimated by $1/4$, for sufficiently small $\lambda = 1/Q(\Gamma)$. For the second term of the right-hand side of (8.20), it is easy to check that the product rules in Lemmas 5.1 and 5.2 hold for the norm $A_k(\tau_0)$, and the function Q in Lemma 5.2 is independent of k. Therefore, from (8.20) and the inductive hypothesis (8.3)–(8.4) for k, we obtain

$$\begin{aligned}\sum_{n=3}^{k} J_{2n} &\le \frac{1}{4} + C\lambda\|\tilde{E}\|_{A_k(\tau_0)}(\|u\|_{A_k(\tau_0)} + \|u\|_{L^2}) \\ &\le \frac{1}{4} + \lambda Q(\|u\|_{A_k(\tau_0)} + \|u\|_{L^2}, \|S\|_{A_k(\tau_0)} + \|S\|_{L^2}) \le \frac{1}{2}.\end{aligned} \tag{8.21}$$

Finally, combining (8.18), (8.19), and (8.21),

$$\begin{aligned}&\sum_{n=4}^{k+1}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^n u\|_{L^2}\frac{\lambda^{n-3}\tau_0^{j+n-3}}{(j+n-3)!} \\ &= \sum_{n=3}^{k}\sum_{j=0}^{\infty}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{n+1} u\|_{L^2}\frac{\lambda^{n-2}\tau_0^{j+n-2}}{(j+n-2)!} \le \sum_{n=3}^{k} J_{1n} + \sum_{n=3}^{k} J_{2n} \le 1,\end{aligned}$$

concluding the proof of (8.3) for $k+1$.

9. The Mach limit in a Gevrey norm

Theorem 3.1 shows that if the initial data is analytic, then the Mach limit holds in an analytic norm. In this section, we show that if, more generally, the initial data is Gevrey, then the Mach limit holds in the Gevrey norm.

Thus, assume the initial data is Gevrey regular that satisfies

$$\sum_{m=0}^{\infty}\sum_{|\alpha|=m}\|\partial^\alpha(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{L^2}\frac{\tau_0^{(m-3)_+}}{(m-3)!^s} \le M_0, \tag{9.1}$$

where $s \ge 1$ is the Gevrey index and , $\tau_0, M_0 > 0$ are fixed constants. Note that when $s = 1$ we recover the class of real-analytic functions. Also, for the Sobolev regularity, we assume that we have (3.1).

Similarly to (3.3), we define the mixed weighted Gevrey norm

$$\|u\|_{G(\tau)} = \sum_{m=1}^{\infty}\sum_{j=0}^{m}\sum_{|\alpha|=j}\|\partial^\alpha(\epsilon\partial_t)^{m-j}u\|_{L^2}\frac{\kappa^{(j-3)_+}\tau(t)^{(m-3)_+}}{(m-3)!^s},$$

where $\tau \in (0,1]$ represents the mixed space-time Gevrey radius and $\kappa \in (0,1]$ is a fixed parameter depending on M_0. Proceeding as in Section 8, we can prove that with $\kappa = 1$ we have

$$\|(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{G(\tilde{\tau}_0)} \le Q(M_0) \tag{9.2}$$

for some $\tilde{\tau}_0 > 0$ depending on τ_0 and M_0. Thus (9.2) holds for any $\kappa \in (0,1]$ as it is an increasing function of κ. We also define the analyticity radius function as

$$\tau(t) = \tau(0) - Kt, \tag{9.3}$$

where $\tau(0) \le \min\{\tilde{\tau}_0, 1\}$ is a sufficiently small parameter, and $K \ge 1$ is a sufficiently large parameter depending on M_0. We shall work on the time interval $[0, T_0]$ where $T_0 > 0$ respects (3.15) and Remark 3.4.

The first theorem generalizes Theorem 3.1 by showing uniform boundedness in the Gevrey norms.

THEOREM 9.1. *Assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies (3.1) and (9.1), where $s \ge 1$ and $\tau_0, M_0 > 0$. There exist sufficiently small constants $\kappa, \tau(0), \epsilon_0, T_0 > 0$, depending on τ_0, s, and M_0, such that*

$$\|(p^\epsilon, v^\epsilon, S^\epsilon)(t)\|_{G(\tau)} \le M, \qquad 0 < \epsilon \le \epsilon_0, \qquad t \in [0, T_0], \tag{9.4}$$

where τ is as in (9.3) and K and M are sufficiently large constants depending on s and M_0.

PROOF OF THEOREM 9.1. We proceed exactly as in Sections 4–6, obtaining the a priori estimates analogous to (3.13). Then we use a similar argument as in Section 3 to prove (9.4) We omit further details. □

Similarly to (3.8), we introduce the spatial Gevrey norm

$$\|u\|_{Y_\delta} = \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^2} \frac{\delta^{(m-3)_+}}{(m-3)!^s},$$

where $\delta > 0$ is as in (3.7).

The next theorem provides convergence of the solution in (9.1) to the corresponding incompressible Euler equation in the Gevrey space.

THEOREM 9.2. *Let $\delta > 0$ be as in (3.7), and assume that the initial data $(v_0^\epsilon, S_0^\epsilon)$ converges to (v_0, S_0) in Y_δ and in L^2 as $\epsilon \to 0$, and S_0^ϵ decays sufficiently rapidly at infinity in the sense*

$$|S_0^\epsilon(x)| \le C|x|^{-1-\zeta}, \qquad |\nabla S_0^\epsilon(x)| \le C|x|^{-2-\zeta},$$

for $0 < \epsilon \le \epsilon_0$ and some constants C and $\zeta > 0$. Then $(v^\epsilon, p^\epsilon, S^\epsilon)$ converges to $(v^{(\mathrm{inc})}, 0, S^{(\mathrm{inc})})$ in $C([0,T_0], Y_\delta)$, where $(v^{(\mathrm{inc})}, S^{(\mathrm{inc})})$ is the solution to (3.9)–(3.11) with the initial data (w_0, S_0), and w_0 is the unique solution of

$$\begin{aligned} &\operatorname{div} w_0 = 0, \\ &\operatorname{curl}(r_0 w_0) = \operatorname{curl}(r_0 v_0), \end{aligned}$$

with $r_0 = r(S_0, 0)$.

PROOF OF THEOREM 9.2. Theorem 9.2 follows by using arguments analogous to those in Section 7. □

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References

[A1] T. Alazard, *Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions*, Adv. Differential Equations **10** (2005), no. 1, 19–44.

[A2] T. Alazard, *Low Mach number limit of the full Navier-Stokes equations*, Arch. Ration. Mech. Anal. **180** (2006), no. 1, 1–73.

[A3] T. Alazard, *A minicourse on the Low Mach number limit*, Discret. Contin. Dyn. Syst. Ser. S **1** (2008), no. 3, 365–404.

[As] K. Asano, *On the incompressible limit of the compressible Euler equation*, Japan J. Appl. Math. **4** (1987), 455–488.

[B] C. Bardos, *Analyticité de la solution de l'équation d'Euler dans un ouvert de R^n*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 5, Aii, A255–A258.

[BB] C. Bardos and S. Benachour, *Domaine d'analycité des solutions de l'équation d'Euler dans un ouvert de R^n*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 4, 647–687.

[Bi] A. Biswas, *Local existence and Gevrey regularity of 3-D Navier-Stokes equations with l_p initial data*, J. Differential Equations **215** (2005), no. 2, 429–447.

[BF] A. Biswas and C. Foias, *On the maximal space analyticity radius for the 3D Navier-Stokes equations and energy cascades*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 3, 739–777.

[BGK] Z. Bradshaw, Z. Grujić, and I. Kukavica, *Local analyticity radii of solutions to the 3D Navier-Stokes equations with locally analytic forcing*, J. Differential Equations **259** (2015), no. 8, 3955–3975.

[BoGK] J.L. Bona, Z. Grujić, and H. Kalisch, *A KdV-type Boussinesq system: from the energy level to analytic spaces*, Discrete Contin. Dyn. Syst. **26** (2010), no. 4, 1121–1139.

[CKV] G. Camliyurt, I. Kukavica, and V. Vicol, *Analyticity up to the boundary for the Stokes and the Navier-Stokes systems*, Trans. Amer. Math. Soc. **373** (2020), no. 5, 3375–3422.

[D1] R. Danchin, *Zero Mach number limit for compressible flows with periodic boundary conditions*, Amer. J. Math. **124** (2002), no. 6, 1153–1219.

[D2] R. Danchin, *Zero Mach number limit in critical spaces for compressible Navier-Stokes equations*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 1, 27–75.

[DM] R. Danchin and P.B. Mucha, *From compressible to incompressible inhomogeneous flows in the case of large data*, Tunis. J. Math. **1** (2019), no. 1, 127–149.

[DG] B. Desjardins and E. Grenier, *Low Mach number limit of viscous compressible flows in the whole space*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455** (1999), no. 1986, 2271–2279.

[DE] M.M. Disconzi and D.G. Ebin, *Motion of slightly compressible fluids in a bounded domain, II*, Commun. Contemp. Math. **19** (2017), no. 4, 1650054, 57.

[DL] M.M. Disconzi and C. Luo, *On the incompressible limit for the compressible free-boundary Euler equations with surface tension in the case of a liquid*, Arch. Ration. Mech. Anal. **237** (2020), no. 2, 829–897.

[E] D.B. Ebin, *The motion of slightly compressible fluids viewed as a motion with strong constraining force*, Ann. Math. **105** (1977), 141–200.

[F] E. Feireisl, *Flows of viscous compressible fluids under strong stratification: incompressible limits for long-range potential forces*, Math. Models Methods Appl. Sci. **21** (2011), no. 1, 7–27.

[FKM] E. Feireisl, C. Klingenberg, and S. Markfelder, *On the low Mach number limit for the compressible Euler system*, SIAM J. Math. Anal. **51** (2019), no. 2, 1496–1513

[FN] Eduard Feireisl and Antonín Novotný, *Inviscid incompressible limits of the full Navier-Stokes-Fourier system*, Comm. Math. Phys. **321** (2013), no. 3, 605–628.

[FT] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. **87** (1989), no. 2, 359–369.

[G] Y. Giga, *Time and spatial analyticity of solutions of the Navier-Stokes equations*, Comm. Partial Differential Equations **8** (1983), no. 8, 929–948.

[GK] Z. Grujić and I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in L^p*, J. Funct. Anal. **152** (1998), no. 2, 447–466.

[H] D. Hoff, *The zero-Mach limit of compressible flows*, Comm. Math. Phys. **192** (1998), no. 3, 543–554.

[I] T. Iguchi, *The incompressible limit and the initial layer of the compressible Euler equation in $\mathbf{R}^n_+$*, Math. Methods Appl. Sci. **20** (1997), 945–958.

[Is1] H. Isozaki, *Wave operators and the incompressible limit of the compressible Euler equation*, Commun. Math. Phys. **110** (1987), 519–524.

[Is2] H. Isozaki, *Singular limits for the compressible Euler equation in an exterior domain*, J. Reine Angew. Math. **381** (1987), 1–36.

[Is3] H. Isozaki, *Singular limits for the compressible Euler equation in an exterior domain. II. Bodies in a uniform flow*, Osaka J. Math. **26** (1989), 399–410.

[KV] I. Kukavica and V.C. Vicol, *The domain of analyticity of solutions to the three-dimensional Euler equations in a half space*, Discrete Contin. Dyn. Syst. **29** (2011), no. 1, 285–303.

[KM1] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math. **34** (1981), no. 4, 481–524.

[KM2] S. Klainerman and A. Majda, *Compressible and incompressible fluids*, Comm. Pure Appl. Math. **35** (1982), no. 5, 629–651.

[KP] S.G. Krantz and H.R. Parks, *A primer of real analytic functions*, Basler Lehrbücher [Basel Textbooks], vol. 4, Birkhäuser Verlag, Basel, 1992.

[LM] P.-L. Lions and N. Masmoudi, *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl. (9) **77** (1998), no. 6, 585–627.

[LO] C.D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Differential Equations **133** (1997), no. 2, 321–339.

[M] N. Masmoudi, *Incompressible, inviscid limit of the compressible Navier-Stokes system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), no. 2, 199–224.

[MS] G. Métivier and S. Schochet, *The incompressible limit of the non-isentropic Euler equations*, Arch. Ration. Mech. Anal. **158** (2001), no. 1, 61–90.

[OT] M. Oliver and E.S. Titi, *On the domain of analyticity of solutions of second order analytic nonlinear differential equations*, J. Differential Equations **174** (2001), no. 1, 55–74.

[S1] S. Schochet, *The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit*, Comm. Math. Phys. **104** (1986), no. 1, 49–75.

[S2] S. Schochet, *The mathematical theory of low Mach number flows*, M2AN Math. Model. Numer. Anal. **39** (2005), no. 3, 441–458.

[U] S. Ukai, *The incompressible limit and the initial layer of the compressible Euler equation*, J. Math. Kyoto Univ. **26** (1986), no. 2, 323–331.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: `juhijang@usc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: `kukavica@usc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089
E-mail address: `lli265@usc.edu`