GLOBAL WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR INTERMITTENT INITIAL DATA IN HALF-SPACE

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ABSTRACT. We prove the existence of global-in-time weak solutions of the incompressible Navier-Stokes equations in the half-space $\mathbb{R}^3_+$ with initial data in a weighted space that allows non-uniformly locally square integrable functions that grow at large scales in an intermittent sense. The space for initial data is built on cubes whose sides R are proportional to the distance to the origin and the square integral of the data is allowed to grow as a power of R. The existence is obtained via a new a priori estimate and a stability result in the weighted space, as well as new pressure estimates. Also, we prove eventual regularity of such weak solutions, up to the boundary, for (x,t) satisfying $t \geq \epsilon_0|x|^2 + M$, where $\epsilon_0 > 0$ is arbitrarily small and $M > 0$. By adding conditions on the data within a weighted L^2 framework, we improve algebraic bounds on the size of this region and we refine the pointwise decay rate of the solution within this region. As an application of the existence theorem, we construct global discretely self-similar solutions, thus extending the theory on the half-space to the same generality as the whole space.

CONTENTS

1. INTRODUCTION

We consider solutions to the three-dimensional incompressible Navier-Stokes equations (NSE)

$$\begin{aligned} &\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

posed on $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_3 > 0\}$ satisfying homogeneous Dirichlet boundary condition on $\partial\mathbb{R}^3_+ \times (0,\infty)$ and the initial condition

$$u(x, 0) = u_0(x),$$

where $u_0 \in L^2_{\text{loc}}(\overline{\mathbb{R}^3_+})$ is divergence-free with $u_3 = 0$ on $\partial\mathbb{R}^3_+$. If $u_0 \in L^2$, then the existence of global-in-time weak solutions satisfying the strong energy inequality has been shown in the fundamental works of Leray [Ler] and Hopf [H] (see also [CF, T, RRS, OP]), and are commonly referred to as *the Leray-Hopf weak solutions*. Such solutions

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Date: March 8, 2022.
2000 *Mathematics Subject Classification.* 35Q30, 35Q35, 76D05.
Key words and phrases. Navier-Stokes equations, weak solutions, half space, global existence, eventual regularity, self-similar solutions, unbounded energy.

enjoy additional structure, such as weak-strong uniqueness. One can also construct Leray-Hopf weak solutions that are suitable (see [CKN, BIC]), that is they satisfy the local energy inequality (see (1.4) below). This provides additional interior regularity, as shown in the celebrated work of Caffarelli, Kohn, and Nirenberg [CKN] (see also [RRS, O]). However, the questions of uniqueness and smoothness of such solutions remain open.

The Leray theory has been extended, in [Le1], to the uniformly locally square integrable data in $\mathbb{R}^3$; see also [KS, KwT] for some extensions. For global existence, these works assume some type of decay of the initial data as $|x| \to \infty$, either a pointwise decay of a locally determined quantity, the decay of the L^2 norm confined to balls of unit radius, or the decay of the oscillation computed over balls of unit radius. In [Le2, BT4, BKT, FL2, KwT], existence results are given in weighted spaces, which allow for a lack of decay in some directions. The papers [BKT, FL2] additionally allow for growth in some directions.

The question of well-posedness of the Navier–Stokes equations on $\mathbb{R}^3_+$ is considerably more difficult than the corresponding question in $\mathbb{R}^3$, due to difficulties caused by the pressure when solving the linear Stokes problem. For a treatment of the Stokes problem by the Fourier transform, see Solonnikov [Sol1, Sol2, Sol3], while a different approach to solvability of the Stokes problem was developed by Ukai in [Uka]. For other works on the solvability of the Stokes problem in the half-space, see also [Ka, KLLT]. Until recently the global-in-time existence of a weak solution for uniformly locally square integrable initial data in $\mathbb{R}^3_+$ was open, one of the main challenges being the treatment of the pressure. In fact, the results in [GS] imply the estimate $t\|\nabla p\|_{L^2} \leq C\|u_0\|_{L^2}$ for solutions of the linear Stokes equations. Using the Poincaré inequality we thus obtain

$$\|p(t) - [p]_{B_1(x_0)\cap\mathbb{R}^3_+}(t)\|_{L^2(B_1(x_0)\cap\mathbb{R}^3_+)} \leq t^{-1}\|u_0\|_{L^2(\mathbb{R}^3_+)}$$

where $[f]_A \coloneqq |A|^{-1}\int_A f$ denotes the average of f over A. The right-hand side of this estimate, however, is not integrable in t near the initial time. Recently, Maekawa, Miura, and Prange used in [MMP1] explicit representation of the kernel for the Stokes equations in $\mathbb{R}^3_+$ due to Desch, Hieber, and Prüss [DHP] to obtain a better estimate,

$$\|p(t)\|_{L^2(B_1(x_0)\cap\mathbb{R}^3_+)} \leq t^{-3/4}\|u_0\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^3_+)}, \tag{1.2}$$

(see (2.20), (2.21) in [MMP1]), which is integrable in time close to $t = 0$. Maekawa, Miura, and Prange [MMP1, MMP2] also established a number of new estimates in the case of $\mathbb{R}^3_+$, and have also proved the global-in-time existence of weak solutions for uniformly locally square integrable initial data u_0 in $\mathbb{R}^3_+$, in the spirit of [Le1].

We note that such solutions further complicate the study of the pressure function, due to the contribution to p coming from large scales. For example, in the case of $\mathbb{R}^3$, one way to deal with the pressure is to decompose it into the *near-field* and the *far-field*. Namely, given an open set $Q \subset \mathbb{R}^3$ one may consider

$$\begin{aligned} p_{\mathrm{near}}(x,t) + p_{\mathrm{far}}(x,t) - p_Q(t) &= -\frac{1}{3}|u(x,t)|^2 + \mathrm{p.v.}\int_{Q^*} K_{ij}(x-y)(u_i(y,t)u_j(y,t))\,dy \\ &\quad + \int_{y\notin Q^*} (K_{ij}(x-y) - K_{ij}(x_Q - y))(u_i(y,t)u_j(y,t))\,dy, \end{aligned} \tag{1.3}$$

as in [BK, BKT], where $p_Q(t)$ is an arbitrary function of time, $K_{ij}(y) = \partial_{ij}(4\pi|y|)^{-1}$, and $x_Q \in Q$ is fixed. In this context, the case of the half-space $\mathbb{R}^3_+$ becomes much more difficult, as no such direct decomposition is available. In fact, in addition to the Helmholtz pressure, that is a solution of the nonhomogeneous Poisson equation

$$\begin{cases} -\Delta p_{\mathrm{H}} &= \partial_{ij}(u_i u_j) \quad \text{in } \mathbb{R}^3_+, \\ \partial_3 p_{\mathrm{H}} &= \mathrm{div}\,(u\, u_3) \quad \text{on } \partial\mathbb{R}^3_+, \end{cases}$$

one also needs to take into account the harmonic pressure, which is a solution of the Laplace equation with Neumann boundary condition,

$$\begin{cases} -\Delta p_{\mathrm{harm}} &= 0 \qquad\qquad \text{in } \mathbb{R}^3_+, \\ \partial_3 p_{\mathrm{harm}} &= \Delta u_3|_{x_3=0} \quad \text{on } \partial\mathbb{R}^3_+, \end{cases}$$

where the boundary condition at $x_3 = 0$ should be understood in the sense of the trace of Δu_3. This part of the pressure function is absent in the case of the whole space $\mathbb{R}^3$, and the Helmholtz pressure p_{H} requires a much more sophisticated analysis than in the case of $\mathbb{R}^3$. note that both parts of the pressure also need to be decomposed into the near and far fields.

In this work we are concerned with the initial data u_0 that involves growth at infinity. Note that for such u_0, the inequality (1.2) does not imply any local well-definiteness of the pressure near $t = 0$. This is an interesting problem, and we address it here for u_0 that can grow at infinity in an "intermittent" sense (see Lemmas 2.1 and 2.2

below). Namely, we consider u_0 such that $\|u_0\|_{L^2(Q)} \lesssim |Q|^{1/3}$ for every dyadic block Q of side-length d such that $\mathrm{dist}(Q,0) \sim d$ (see Definition 1.1 below). In such setting one needs to treat the pressure function with additional attention to the energy coming from the large scales of the velocity field. This can be achieved by developing an appropriate framework consisting of a choice of the function space (see Definition 1.1), an energy functional adapted to large scales (see (1.8)), and an a priori estimate for such energy functional (see Theorem 1.4), as well as estimates of all components of the pressure function using such framework (see (2.15)).

To be more precise, our goal is to establish the global existence of weak solutions in $\mathbb{R}^3_+$ with data built on a dyadic tiling $\mathcal{C}$ of the half-space (see Figure 1 for an illustration projected on a half-plane), allowing for growth as $|x| \to \infty$. These data are on one hand general and on the other also well adapted for the study of self-similar solutions and eventual regularity.

The notion of the large scale intermittent initial data has been considered in [BK, BKT] in the case of $\mathbb{R}^3$. This concept should not be confused with the notion of intermittency in turbulence, although we chose the terminology in analogy with this. In particular, intermittency in turbulence can refer to the fact that the active regions associated with small scales do not occupy the full spatial domain. In our setting, the active region should be interpreted as the region where local L^2 quantitites are large. Intermittent then refers to the fact that, intersected with $B_R(0) \cap \mathbb{R}^3_+$, the volume of this region cannot be growing like R^3.

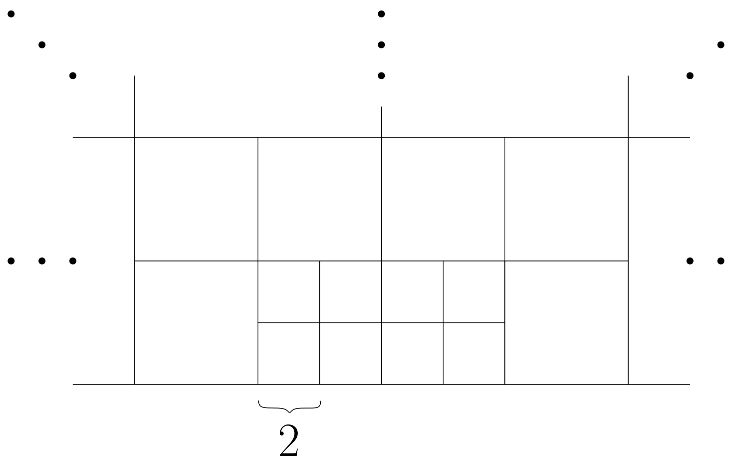

FIGURE 1. The cover of $\mathbb{R}^3_+$ by the collection $\mathcal{C}$. A scaled cover $\mathcal{C}_n$ is obtained by replacing 2 with 2^n.

To state our main results, we first define local energy weak solutions.

Definition 1.1 (Local energy solutions). *A vector field $u \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+} \times [0,T))$, where $0 < T < \infty$, is a local energy solution to (1.1) with divergence-free initial data $u_0 \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+})$ such that $u_{0,3} = 0$ on $\partial\mathbb{R}^3_+$ if the following conditions hold:*

(1) $u \in \bigcap_{R>0} L^\infty(0,T; L^2(B_R(0) \cap \mathbb{R}^3_+))$, $\nabla u \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+} \times [0,T])$ and $u|_{x_3=0} = 0$ a.e. $t \in (0,T)$,
(2) for some $p \in \mathcal{D}'(\mathbb{R}^3_+ \times (0,T))$, the pair (u,p) is a distributional solution to (1.1),
(3) for all compact subsets K of $\mathbb{R}^3_+$ we have $u(t) \to u_0$ in $L^2(K)$ as $t \to 0_+$,
(4) u is suitable in the sense of Caffarelli-Kohn-Nirenberg, i.e., for all non-negative $\phi \in C_c^\infty(\overline{\mathbb{R}^3_+} \times (0,T))$, we have the local energy inequality

$$2 \iint |\nabla u|^2 \phi \, dx \, dt \le \iint |u|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \iint (|u|^2 + 2p)(u \cdot \nabla \phi) \, dx \, dt, \tag{1.4}$$

(5) the function $t \mapsto \int u(x,t) \cdot w(x) \, dx$ is continuous on $[0,T)$ for any compactly supported $w \in L^2(\mathbb{R}^3_+)$,
(6) given a bounded, open set $\Omega \subset \mathbb{R}^3_+$, the pressure satisfies the local pressure expansion*,*

$$p = p_{\mathrm{li,loc}} + p_{\mathrm{li,nonloc}} + p_{\mathrm{loc,H}} + p_{\mathrm{loc,harm}} + p_{\mathrm{nonloc,H}} + p_{\mathrm{harm},\le 1} + p_{\mathrm{harm},\ge 1}, \tag{1.5}$$

which holds a.e. up to a function of time; the terms on the right-hand side are defined in (2.4), (2.7), (2.11), (2.12), and (2.13) below and estimated in (2.15).

We say that u is a local energy solution on $\mathbb{R}^3_+ \times [0,\infty)$ if it is a local energy solution on $\mathbb{R}^3_+ \times [0,T)$ for all $T < \infty$.

This definition is primarily based on the one in [KS]. Some works refer to this class (without the part (6) and with minor modifications) as *local Leray solutions* or *Lemarié-Rieusset type solutions*. This definition contains sufficient properties for work on regularity, see e.g. [CKN, Li, LS, ESS, K, Gr, BS] among others, or in physical applications, see e.g. [DG]. We note that we do not assert any uniform control in L^2_{loc}. The local pressure expansion in (6) above is inspired by the decomposition introduced by [MMP1, MMP2], and is unique up to a function of time; see (2.14).

Our main result is concerned with local energy weak solutions with initial data u_0 belonging to a weighted space that allows growth of the kinetic energy at spatial infinity.

To be precise, given $n \in \mathbb{N}$, we denote by $S_n^{(n)}$ the collection of 32 cubes of side-length 2^n that can be obtained by partitioning $\{x \in \mathbb{R}^3_+ : |x_i| \le 2^{n+1} \text{ for } i = 1,2,3\}$. For $k \ge n+1$, let $R_k = \{x \in \mathbb{R}^3_+ : |x_i| < 2^k; i = 1,2,3\}$ and we denote by $S_k^{(n)}$ the collection of 28 cubes of side-length 2^k that can be obtained by partitioning $\overline{R_{k+1} \setminus R_k}$. Also, set

$$\mathcal{C}_n = \{Q \in S_k, k \ge n\}; \tag{1.6}$$

this is illustrated by Fig. 1 with 2 replaced by 2^n. In other words, $S_k^{(n)}$ is the collection of cubes from $\mathcal{C}_n$ of side-length 2^k. We set $\mathcal{C} = \mathcal{C}_1$.

Definition 1.2. *Given $p \in [1,\infty)$, $q \ge 0$, and $n \ge 1$, we have $f \in M^{p,q}_{\mathcal{C}_n}$ if*

$$\|f\|^p_{M^{p,q}_{\mathcal{C}_n}} = \sup_{Q \in \mathcal{C}_n} \frac{1}{|Q|^{\frac{q}{3}}} \int_Q |f(x)|^p \, dx < \infty.$$

We denote by $\mathring{M}^{p,q}_{\mathcal{C}}$ the closure in $M^{p,q}_{\mathcal{C}}$ of divergence-free, smooth functions, which are compactly supported in $\mathbb{R}^3_+$.

We note that

$$\frac{1}{|Q|^{\frac{q}{3}}} \int_Q |f|^p \, dx \to 0 \text{ as } |Q| \to \infty, Q \in \mathcal{C}, \tag{1.7}$$

for $f \in \mathring{M}^{p,q}_{\mathcal{C}}$, which was shown in [BKT].

In the context of the whole space $\mathbb{R}^3$, the spaces $M^{p,q}_{\mathcal{C}}$ are discussed in detail in [BKT] and from the perspective of interpolation theory in [FL1, Section 7]. The same observations apply here; in particular, the choice of tiling does not matter so long as elements have length scales comparable to their distance from the origin. Additionally, dyadic cubes can be replaced by balls centered at the origin as in [B], i.e., we have

$$\|u_0\|^p_{M^{p,q}_{\mathcal{C}}} \sim \sup_{R \ge 1} \frac{1}{R^q} \int_{B_R(0)} |u_0|^p \, dx,$$

for $u_0 \in L^2_{\mathrm{loc}}$, when $q > 0$. Here and below "$a \sim b$" means "$a \lesssim b$ and $b \lesssim a$". The decay in $\mathring{M}^{p,q}_{\mathcal{C}}$ can also be encoded via Basson's perspective due to the equivalence

$$f \in \mathring{M}^{p,q}_{\mathcal{C}} \iff \lim_{n\to\infty} \|f\|_{M^{p,q}_{\mathcal{C}_n}} = 0 \iff \lim_{R\to\infty} \frac{1}{R^q} \int_{B_R(0)} |u_0|^p \, dx = 0$$

when $q > 0$; see [BKT] for a proof. The space $M^{p,q}_{\mathcal{C}}$ can also be viewed as an inhomogeneous Herz space $K^{-q}_{p,\infty}(\mathbb{R}^3_+)$. The norms in these spaces are defined by

$$\|f\|_{K^q_{p,r}(\mathbb{R}^3_+)} := \begin{cases} \left(\sum_{k\ge 0} 2^{kqr} \|f\|^r_{L^p(A_k)}\right)^{\frac{1}{r}} & \text{for } r < \infty, \\ \sup_{k \ge 0} 2^{kq} \|f\|^r_{L^p(A_k)} & \text{for } r = \infty, \end{cases}$$

where $A_0 := B_0 \cap \mathbb{R}^3_+$, and $A_k := (B_{2^k} \setminus B_{2^{k-1}}) \cap \mathbb{R}^3_+$ for $k \ge 1$. Homogeneous Herz spaces are used to analyze strong solutions to the Navier-Stokes equations in [Tsu].

The first use of the spaces $M^{p,q}_{\mathcal{C}}$ in the analysis of the Navier-Stokes equations was by Basson [B], who constructed local-in-time solutions in two dimensions belonging to the $M^{2,2}_{\mathcal{C}}$ class. In 2D, Basson additionally constructed global in time solutions with L^2_{uloc} initial data by exploiting the maximum principle for the 2D vorticity [B]. Subsequently, the local-in-time existence was addressed in three dimensions in [BK], while the global-in-time existence was obtained in [BKT] for $\mathring{M}^{2,2}_{\mathcal{C}}$. In comparison to [BKT], here we build the divergence-free condition and the decay condition into the space $\mathring{M}^{2,2}_{\mathcal{C}}$ while in [BKT] only the decay condition was built into this space.

Our main theorem is concerned with the global-in-time existence with initial data in $\mathring{M}^{2,q}_{\mathcal{C}}$ where $0 < q \le 2$. In order to state it, for a space-time function u, define

$$\alpha_n(t) = \sup_{s\in[0,t]} \|u(s)\|^2_{M^{2,q}_{\mathcal{C}_n}} \quad \text{and} \quad \beta_n(t) = \sup_{Q\in\mathcal{C}_n} \frac{1}{|Q|^{\frac{q}{3}}} \int_0^t \int_Q |\nabla u|^2 \, dx \, dt, \tag{1.8}$$

for $n \in \mathbb{N}$ and $q \in (0,2]$. Note that both α_n and β_n are non-decreasing functions of t. For simplicity of notation, we omit indicating q in α_n and β_n.

Theorem 1.3 (Global existence of local energy solutions)**.** *Let $q \in (0,2]$ and $u_0 \in \mathring{M}^{2,q}_{\mathcal{C}}$. Then there exists a local energy solution u on $\mathbb{R}^3_+ \times (0,\infty)$ with the initial data u_0 such that*

$$\alpha_1(t) + \beta_1(t) < \infty,$$

for all $t < \infty$.

In comparison with the $L^2_{\mathrm{uloc}}(\mathbb{R}^3_+)$ setting of [MMP1, MMP2], we note that neither of the two spaces $L^2_{\mathrm{uloc}}(\mathbb{R}^3_+)$ and $\mathring{M}^{2,2}_{\mathcal{C}}$ contains the other. For example, if u_0 is a constant (or periodic) function, then $u_0 \in L^2_{\mathrm{uloc}}(\mathbb{R}^3_+) \setminus \mathring{M}^{2,2}_{\mathcal{C}}$, while

$$u_0 := \sum_{k\ge1} 2^{qk/2} \chi_{B_1(2^k e_3)} \in \mathring{M}^{2,q}_{\mathcal{C}} \setminus L^2_{\mathrm{uloc}}(\mathbb{R}^3_+);$$

see [BK]. This example is intermittent at large scales in the sense that the growth is not occurring in all directions simultaneously. The spaces $M^{2,q}$ are well-adapted to "zooming out" dyadically, which makes them suitable for capturing the large scale behavior. In particular, the number q in $M^{2,q}$ measures potential growth at infinity, with larger value of q corresponding to more growth. This implies that $M^{2,q} \subset M^{2,q'}$ for $q < q'$.

In this context, Theorem 1.3 is the first result asserting the global-in-time existence of weak solutions in the half-space that allows intermittent initial data.

Furthermore, the dyadic structure of $M^{2,q}$ makes it possible to quantify the eventual regularity of solutions constructed in Theorem 1.3 (see Theorem 1.7 below). We note that, in the context of scaling of the Navier–Stokes equations, $M^{2,q}$ has the same scaling as $L^{(1-q)/2}$ for large scales. In particular this implies that $M^{2,1}$ includes all self-similar initial data, allowing an extension of Theorem 1.3 to self-similar solutions to the NSE (see Theorem 1.7 below).

In the following theorem, we summarize the bounds which the local energy solutions satisfy.

Theorem 1.4 (Bounds for local energy solutions)**.** *Assume that $q \in (0,2]$ and $u_0 \in \mathring{M}^{2,q}_{\mathcal{C}_1}$. There exists $\gamma_0 > 0$, $\eta = \eta(\|u_0\|_{M^{2,q}_{\mathcal{C}_1}}) > 0$, and $C \ge 1$ with the following property. If (u,p) is a local energy solution on $\mathbb{R}^3_+ \times (0,\infty)$ with the initial data u_0 such that*

$$\alpha_1(t) + \beta_1(t) < \infty,$$

for all $0 < t < \infty$, then for every $n \in \mathbb{R}$, we have

$$\alpha_n(T_n) + \beta_n(T_n) \le C\alpha_n(0), \tag{1.9}$$

for some

$$T_n \ge \eta \min\left\{2^n, \|u_0\|^{-1}_{M^{2,q}_{\mathcal{C}_n}}\right\}^{\gamma_0}. \tag{1.10}$$

Note that $T_n \to \infty$ as $n \to \infty$.

Another important ingredient in the proof of Theorem 1.3 is the following stability result.

Theorem 1.5 (Stability)**.** *Assume $q \in (0,2]$ and $u_0 \in \mathring{M}^{2,q}_{\mathcal{C}}$, and suppose that $\{u_0^{(k)}\}_{k\ge1} \subset M^{2,q}_{\mathcal{C}}$ is such that $\|u_0^{(k)} - u_0\|_{\mathring{M}^{2,q}_{\mathcal{C}}} \to 0$. Moreover, suppose that $\{(u^{(k)}, p^{(k)})\}_{k\ge1}$ is a collection of local energy solutions with initial data $u_0^{(k)}$ that satisfy the assumptions of Theorem 1.4 for every $n \in \mathbb{N}$. Then there exists a subsequence $\{k_l\}_{l\ge1}$ such that $(u^{(k_l)}, p^{(k_l)})$ converges in a weak sense to a global-in-time local energy solution (u,p) with initial data u_0. In addition, for every $n \in \mathbb{N}$, the pair (u,p) satisfies the a priori estimate (1.9), and, given $\Omega \Subset \mathbb{R}^3_+$, each part of the local pressure expansion of $p^{(k_l)}$ converges strongly in $L^{\frac{3}{2}}(\Omega \times (0,T))$ to the corresponding part of the local pressure expansion of p, for every $T > 0$.*

The most difficult part of the proofs of the above two theorems is the treatment of the pressure function. We note that each of the pressure parts $p_{\rm li,loc}$, $p_{\rm li,nonloc}$, $p_{\rm loc,H}$, $p_{\rm loc,harm}$, $p_{\rm nonloc,H}$, $p_{\rm harm,\le 1}$, and $p_{\rm harm,\ge 1}$ is defined in a similar way as in the works [MMP1, MMP2] in the uniformly locally integrable setting. However, there are important differences in our treatment of each of these parts in our estimates (see (2.15)). For example, our pressure estimates are adapted to the energy functional that captures large scales. This results in additional difficulties in balancing the upper bounds against the kinetic energy α and the dissipation energy β. In particular, we cannot afford to control $\nabla p_{\rm loc,harm}$ only in $L^{3/2}L^{9/8}$, which we describe in more detail below Lemma 2.4. A related issue appears in our estimate on the nonlocal harmonic part $p_{\rm harm,\ge 1}$, which is our most difficult estimate, and is balanced against α and β using two auxiliary indices r and δ (see (2.15)). In fact, we use a new method to estimate this part, employing the structure of our tiling $\mathcal{C}$ to handle the part of $p_{\rm harm,\ge 1}$ consisting of double convolution (i.e., the term in (2.13) that is concerned with F_B). We discuss this issue in detail in Steps 2 and 3 of the proof of Lemma 2.6. Moreover, we also use a simpler estimate of the local Helmholtz pressure $p_{\rm loc,H}$ (see (2.22) in [MMP1]) as well as $p_{\rm harm,\le 1}$ (see (2.24) in [MMP1]). Additionally, we do not need any estimates of derivatives of these parts of the pressure.

Our pressure bounds allow us to obtain the a priori estimate (1.9) as well as the strong convergence under the perturbation of the initial data u_0 in $M^{2,2}_{\mathcal{C}}$ mentioned in Theorem 1.5. Consequently, we obtain the explicit representation (1.5) of the pressure for the weak solutions constructed in Theorem 1.3.

Our stability result can also be applied to construct self-similar and discretely self-similar solutions with very rough data. Recall that if u solves (1.1), then so does $u^{(\lambda)}(x,t) := \lambda u(\lambda x, \lambda^2 t)$ for $\lambda > 0$. Self-similar (SS) solutions, i.e., solutions invariant with respect to the scaling of (1.1) for all scaling factors $\lambda > 0$, are noteworthy candidates for the non-uniqueness and could lead to non-uniqueness in the Leray-Hopf class, as demonstrated by [JS1, GuS, ABC]. On the other hand, discretely self-similar (DSS) solutions, i.e., solutions that satisfy the scaling invariance possibly only for some $\lambda > 1$, are candidates for the failure of eventual regularity [BT1, BT4]. For small data, the existence and uniqueness of such solutions follow easily from the classical well-posedness results; see [KT] and the references therein. The more interesting case of large data has been only recently solved by Jia and Šverák [JS2], and some improvements and new approaches have been developed by e.g. [Ts, KTs, BT1, BT3, AB, CW, FL2]. The roughest class of scaling invariant initial data for which existence is known is $L^2_{\rm loc}(\mathbb{R}^3)$ [CW, BT3, Le3, FL2]. Note that if $u_0 \in L^2_{\rm loc}(\mathbb{R}^3)$ is scaling invariant then it belongs to the $\mathbb{R}^3$ version of the space $M^{2,1}_{\mathcal{C}}$, see [BK]. Indeed, this observation led the first and second authors to study these spaces in [BK].

In the case of the half-space, Tsai and Korobkov [KTs] established the original theory of Jia and Šverák [JS2] for smooth, self-similar data via a new method and, later, Tsai and the first author [BT2] addressed rough, discretely self-similar initial data in $L^{3,\infty}$ with arbitrary scaling factor. As a consequence of Theorem 1.5, we prove that any SS/DSS initial data in $\mathring{M}^{2,2}_{\mathcal{C}}$ gives rise to a SS/DSS solution. This class of initial data corresponds to the roughest case for $\mathbb{R}^3$ [CW, BT3, Le3, FL2] with a suitable boundary condition imposed. To see why this is true, assume that $u_0 \in L^2_{\rm loc}(\mathbb{R}^3_+)$ is divergence-free with vanishing normal component at the boundary (this is the boundary condition implicit in the space $\mathring{M}^{2,2}_{\mathcal{C}}$ since $\mathring{M}^{2,2}_{\mathcal{C}}$ is obtained by taking the closure of compactly supported test functions; see [CF, Proposition 1.5]). If, additionally, u_0 is scaling invariant, then, by a re-scaling argument, $u_0 \in M^{2,1}_{\mathcal{C}} \subset M^{2,2}_{\mathcal{C}}$. Furthermore, we have $u_0 \in \mathring{M}^{2,2}_{\mathcal{C}}$ because u_0 decays at spatial infinity (from membership in $M^{2,1}_{\mathcal{C}}$) and satisfies the correct boundary condition.

Theorem 1.6 (Global existence of self-similar solutions)**.** *Assume $u_0 \in \mathring{M}^{2,2}_{\mathcal{C}}$ is divergence-free, satisfies $u_{0,3} = 0$ on $\partial\mathbb{R}^3_+$, and is self-similar (resp. discretely self-similar) for some $\lambda > 1$. Then there exists a global-in-time local energy solution u with data u_0 that is self-similar (resp. discretely self-similar).*

The proof of Theorem 1.6 uses our new a priori bounds to construct discretely self-similar solutions via the stability result of Theorem 1.5 applied to a sequence of solutions given by [BT2]. The approach is similar to the one taken by the first author and Tsai in [BT3] in the case of $\mathbb{R}^3$. However, an important difference is that [BT3] deals with the non-local pressure by exploiting the DSS scaling to localize the far-field part, while in the present paper this technical step is unnecessary because the far-field part of the pressure is controlled using the weighted L^2 framework.

Finally, we show that local energy solutions eventually become regular, up to the boundary, provided u_0 belongs to a subspace of $\mathring{M}^{2,2}_{\mathcal{C}}$. This provides an extension of Theorem D of [CKN] in the setting of half-space $\mathbb{R}^3_+$ by allowing growth of u_0 at spatial infinity. We show that if u_0 is in $\mathring{M}^{2,q}_{\mathcal{C}}$ where $0 < q \le 1$, then an ensuing local energy solution is regular above a parabola.

Theorem 1.7 (Eventual regularity)**.** *Assume* $u_0 \in \mathring{M}^{2,q}_{\mathcal{C}}$ *for some* $q \in (0,1]$*. Then, for any* $\epsilon_0 \in (0,1]$*, there exists* $M > 0$ *so that, if* u *is local energy solution with initial data* u_0 *satisfying*

$$\alpha_1(t) + \beta_1(t) < \infty,$$

for all $0 < t < \infty$*, then* u *is regular (up to the boundary) in the region*

$$\{(x,t) \in \overline{\mathbb{R}^3_+} \times (0,\infty) : t \geq \epsilon_0 |x|^{\frac{2(q+2)}{3}} + M\}. \tag{1.11}$$

Moreover,

$$|u(x,t)| \leq \epsilon_0 t^{-\frac{7}{8}+\frac{q+2}{8\lambda}} \quad \textit{in} \quad \{t \geq \epsilon_0 |x|^{2\lambda} + M\} \tag{1.12}$$

for each $\lambda \in [(q+2)/3, 1]$*.*

Note that (1.12) quantifies the decay of u with respect to q. In fact, given $q \in (0,1]$ the inequality provides a family of quantitative decay estimates: Taking $\lambda = (q+2)/3$ we obtain decay $O(t^{-1/2})$ in the region (1.11), while taking $\lambda > (q+2)/3$ gives a better decay rate in the smaller region (1.12), with the best decay rate $O(t^{\frac{q-5}{8}})$ inside the parabola $\{t \geq \epsilon_0 |x|^2 + M\}$.

Recall from [SSS] that a point $(x,t) \in \overline{\mathbb{R}^3_+} \times (0,\infty)$ is regular if there exists a neighborhood $B \times I \subset \overline{\mathbb{R}^3_+} \times (0,\infty)$ of (x,t) such that u is Hölder continuous in $B \times I$.

We note that an analogous result can be proven in the case of the whole space $\mathbb{R}^3$, complementing the analysis of [BKT]. In fact, in our case the proof is easier, due to a simpler structure (1.3) of the pressure function.

The subject of eventual regularity is classical. For the Leray-Hopf solutions, the global energy inequality makes the matter trivial since $\|u\|_{\dot{H}^1}$ must become small at large times; see also [CKN]. For solutions not satisfying the global energy inequality, the eventual regularity is not generally known. However, if the behavior at the spatial infinity is appropriately controlled, usually via some integrability, then the eventual regularity should hold. For example, it is shown in [BT4] using the ϵ-regularity that any local energy solution on $\mathbb{R}^3$ with data in L^p where $2 < p \leq 3$, or satisfying more general conditions, eventually regularizes. Theorem 1.7, which is a half-space version of a result in [BKT], builds on this idea and identifies the way in which the far-field behavior of the data needs to be controlled within the $M^{2,q}$ framework to ensure the eventual regularity.

The paper is organized as follows. Section 2 contains the study of the local pressure expansion and provides the main estimates (2.15) for all pressure parts. This is then used in Section 3, where we prove the a priori estimate, Theorem 1.4. Section 4 contains the proof of the stability result, Theorem 1.5. We then prove the main existence results, Theorems 1.3 and 1.6 in Section 5. The proof of the eventual regularity result, Theorem 1.7, is provided in Section 6.

2. Pressure formula and estimates

Given $n \geq 1$ and a bounded, open set $\Omega \subset \mathbb{R}^3_+$, let $\overline{m} \geq n$ be the smallest integer for which there exists the largest integer $\underline{m} \in [n, \overline{m}]$ such that

$$\Omega \text{ can be covered using cubes from } S^{(n)}_{\underline{m}} \cup S^{(n)}_{\underline{m}+1} \cup \cdots \cup S^{(n)}_{\overline{m}}.$$

Note that if $\Omega \in \mathcal{C}_n$ for some $n \geq 1$, then $\underline{m} = \overline{m} = m$, where m is such that $\Omega \in S^{(n)}_m$. (Recall (1.6) for the definition of the family $\mathcal{C}_n$, see also Fig. 1.)

Given Ω and n, let Q be the union of (closed) cubes from $\mathcal{C}_n$ that have a nonempty intersection with Ω. Denote by Q^* the union of the neighbors of Q, i.e., the union of Q and all cubes from $\mathcal{C}_n$ that share at least one common boundary point with Q. We similarly define Q^{**} and Q^{***}. We set $\chi \in C^\infty_0(\overline{\mathbb{R}^3_+}, [0,1])$ such that $\chi = 1$ on a neighborhood of Q that includes the union of the $5/4$ homotheties of the cubes included in Q and $\chi = 0$ outside Q^*, and we define χ_* and χ_{**} analogously. In the pressure estimates below, we shall use the following simple geometric fact: If $\xi \in Q$ and $z \in \{\chi < 1\}$ is such that $z \in \tilde{Q}$ for some $\tilde{Q} \in S^{(n)}_k \subset \mathcal{C}_n$, then

$$|\xi' - z'| + \xi_3 + z_3 \gtrsim \begin{cases} 2^{\underline{m}} & k \leq \underline{m}, \\ 2^k & k \geq \underline{m}+1, \end{cases} \tag{2.1}$$

where we used the notation $x = (x', x_3)$ to distinguish the horizontal component x' and the vertical component x_3 of any given point $x \in \mathbb{R}^3_+$. Indeed, if $k \leq \underline{m}$, then either $|\xi' - z'| \gtrsim 2^{\underline{m}}$ (if Q touches the plane $\partial\mathbb{R}^3_+$ and z does not lie in a cube above Q), $z_3 \gtrsim 2^{\underline{m}}$ (if z does lie in a cube above Q) or $\xi_3 \gtrsim 2^{\underline{m}}$ (if Q does not touch the plane). The case

$k\geq \underline{m}+1$ follows similarly as either $|\xi'-z'|\gtrsim 2^k$ (if z lies in a cube touching the plane $\partial\mathbb{R}^3_+$ and ξ does not lie in a cube above it), $\xi_3\gtrsim 2^k$ (if it does) or $z_3\gtrsim 2^k$ (if z lies in a cube not touching the plane). Furthermore, note that

$$|\xi-z|\lesssim 2^{\overline{m}}\qquad \text{for any } \xi,z\in Q^{***}. \tag{2.2}$$

We note that the reason to consider the two indices $\underline{m}$ and $\overline{m}$ is to be able to obtain the local pressure expansion (recall Definition 1.1(6)) for *any bounded and open set* $\Omega\subset\mathbb{R}^3_+$, rather than merely for cubes in the family $\mathcal{C}$, or sets of similar geometry or location. For example, if $\Omega\in C_n$ for some $n\geq 1$, as is the case in most of our applications, then we have $\underline{m}=\overline{m}=m$, where $m\geq n$ is such that 2^m is the side-length of Ω. However, if Ω is a set that stretches through a number of length scales, then $2^{\underline{m}}$ and $2^{\overline{m}}$ should be thought of as the smallest and the largest length scales, respectively, associated to Ω. In other words the indices $\underline{m}$, $\overline{m}$ measure how well the geometry of Ω is adapted to the tiling $\mathcal{C}_n$. One of the features of our pressure estimates (see (2.15) below) is that these two indices are sufficient to describe the dependence of the strength of each of the pressure estimates in terms of geometry of Ω.

We write

$$u=u_{\rm li}+u_{\rm loc}+u_{\rm nonloc},\quad p=p_{\rm li}+p_{\rm loc}+p_{\rm nonloc},$$

where the terms on the right-hand sides are solutions to the linear part

$$\begin{cases}\partial_t u_{\rm li}-\Delta u_{\rm li}+\nabla p_{\rm li}=0,\\ \nabla\cdot u_{\rm li}=0,\\ u_{\rm li}|_{\{z_3=0\}}=0,\\ u_{\rm li}|_{\{t=0\}}=u_0,\end{cases}$$

the local part

$$\begin{cases}\partial_t u_{\rm loc}-\Delta u_{\rm loc}+\nabla p_{\rm loc}=-\nabla\cdot(\chi_{**}u\otimes u),\\ \nabla\cdot u_{\rm loc}=0,\\ u_{\rm loc}|_{\{z_3=0\}}=0,\\ u_{\rm loc}|_{\{t=0\}}=0,\end{cases} \tag{2.3}$$

and the nonlocal part

$$\begin{cases}\partial_t u_{\rm nonloc}-\Delta u_{\rm nonloc}+\nabla p_{\rm nonloc}=-\nabla\cdot((1-\chi_{**})u\otimes u),\\ \nabla\cdot u_{\rm nonloc}=0,\\ u_{\rm nonloc}|_{\{z_3=0\}}=0,\\ u_{\rm nonloc}|_{\{t=0\}}=0.\end{cases}$$

We note that each of the pressure components enters the equation with a gradient, and thus it can be modified by an arbitrary function of t. We have the representation

$$\begin{aligned}p_{\rm li}(x,t)&=\frac{1}{2\pi i}\int_\Gamma {\rm e}^{\lambda t}\int_{\mathbb{R}^3_+}\big(\chi(z)q_\lambda(x'-z',x_3,z_3)\cdot u_0'(z)+(1-\chi(z))q_{\lambda,x,x_\Omega}(z)\cdot u_0'(z)\big)\,dz'\,dz_3\,d\lambda\\ &=p_{\rm li,loc}(x,t)+p_{\rm li,nonloc}(x,t),\end{aligned} \tag{2.4}$$

where $x=(x',x_3)$ and $\Gamma=\{\lambda\in\mathbb{C}:|\arg\lambda|=\eta,\ |\lambda|\geq\kappa\}\cup\{\lambda\in\mathbb{C}\colon|\arg\lambda|\leq\eta,\ |\lambda|=\kappa\}$ with $\eta\in(\pi/2,\pi)$ and $\kappa\in(0,1)$,

$$q_\lambda(x',x_3,z_3)\coloneqq i\int_{\mathbb{R}^2}{\rm e}^{ix'\cdot\xi}{\rm e}^{-|\xi|x_3}{\rm e}^{-\omega_\lambda(\xi)z_3}\left(\frac{\xi}{|\xi|}+\frac{\xi}{\omega_\lambda(\xi)}\right)d\xi,$$

$\omega_\lambda(\xi)\coloneqq\sqrt{\lambda+|\xi|^2}$, and

$$q_{\lambda,x,x_\Omega}(z)\coloneqq q_\lambda(x'-z',x_3,z_3)-q_\lambda(x'_\Omega-z',x_{\Omega,3},z_3);$$

see (2.5)–(2.9) in [MMP1] and (2.8e) in [MMP2]. Above, $x_\Omega=(x'_\Omega,x_{\Omega,3})$ stands for any fixed point of Ω; if Ω is a cube, we denote by x_Ω the center of Ω. Moreover, we have the pointwise estimates

$$|\nabla_x^m q_\lambda(x'-z',x_3,z_3)|\lesssim_m\frac{{\rm e}^{-|\lambda|^{\frac12}z_3}}{(|x'-z'|+x_3+z_3)^{2+m}},\qquad m=0,1,2, \tag{2.5}$$

which are proven in [MMP2, Proposition 3.7].

As for the local (nonlinear) pressure $p_{\rm loc}$, we use the Helmholtz decomposition in the half-space to write

$$\nabla\cdot(\chi_{**}u\otimes u)=\mathbb{P}\nabla\cdot(\chi_{**}u\otimes u)+\nabla p_{\rm loc,H}, \tag{2.6}$$

where $p_{\rm loc,H}$ is the solution to Poisson equation with the Neumann boundary condition

$$\begin{cases} -\Delta p_{\rm loc,H} = \partial_i\partial_j(\chi_{**}u_iu_j) & \text{in } \mathbb{R}^3_+, \\ \partial_3 p_{\rm loc,H} = \partial_i(\chi_{**}u_iu_3) & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

The solution is given by

$$p_{\rm loc,H}(x,t) = c_0\chi_{**}|u|^2(x,t) + \int_{\mathbb{R}^3_+} \partial_{z_i}\partial_{z_j}N(x,z)\chi_{**}(z)u_i(z,t)u_j(z,t)\,dz, \tag{2.7}$$

where c_0 is a constant; also,

$$N(x,z) = \frac{1}{4\pi}\left(\frac{1}{|x-z|} + \frac{1}{|\overline{x}-z|}\right) \tag{2.8}$$

denotes the Neumann kernel for the half-space, where $\overline{x} = (x_1, x_2, -x_3)$ is the reflection of x with respect to the boundary $\partial\mathbb{R}^3_+$. With this definition of the local Helmholtz pressure $p_{\rm loc,H}$, one can use Fourier analytic methods (see (6.2) in [MMP2] and Appendix A.1 in [MMP1]) to deduce that for the Helmholtz projection we have

$$\mathbb{P}\nabla\cdot(\chi_{**}u\otimes u) = F_A + F_B,$$

where F_A is a vector function whose components are finite sums of the terms of the form

$$\partial_j\left(\chi_{**}u_ku_l\right), \tag{2.9}$$

where $j,k,l \in \{1,2,3\}$, and $F_B(z,s) = F_B(z',z_3,s)$ is a finite sum of vectors of the form

$$\mathbf{m}(D')\nabla'\otimes\nabla' \int_0^\infty \int_{\mathbb{R}^2} \left(P(z'-y',|z_3-y_3|) + P(z'-y',z_3+y_3)\right)\left((\chi_{**}v\otimes w)(y',y_3,s)\right)\,dy'\,dy_3, \tag{2.10}$$

where v and w denote various 2D vectors whose components are chosen among u_1, u_2, or u_3; also, $\mathbf{m}(D')$ denotes a multiplier in the horizontal variable z' that is homogeneous of degree 0 and that may be a matrix. Also, $P(x',t) = (2\pi)^{-1}t(t^2+|x'|^2)^{-3/2}$ denotes the 2D Poisson kernel.

Thus, letting $(u_{\rm loc,harm}, p_{\rm loc,harm})$ be a solution to (2.3), but with the right-hand side replaced by $-\mathbb{P}\nabla\cdot(\chi_{**}u\otimes u)$, we see that

$$p_{\rm loc} = p_{\rm loc,H} + p_{\rm loc,harm},$$

by applying the Helmholtz decomposition (2.6) to (2.3), and, from the Duhamel principle

$$p_{\rm loc,harm}(x,t) = \frac{1}{2\pi i}\int_0^t\int_\Gamma e^{(t-s)\lambda}\int_{\mathbb{R}^3_+} q_\lambda(x'-z',x_3,z_3)\cdot(F_A(z,s)+F_B(z,s))\,dz\,d\lambda\,ds. \tag{2.11}$$

As for the nonlocal (nonlinear) pressure $p_{\rm nonloc}$, we use the Helmholtz decomposition to write, similarly as in the case of $p_{\rm loc}$,

$$\nabla\cdot((1-\chi_{**})u\otimes u) = \mathbb{P}\nabla\cdot((1-\chi_{**})u\otimes u) + \nabla p_{\rm nonloc,H},$$

where

$$p_{\rm nonloc,H}(x,t) = \int_{\mathbb{R}^3_+} \partial_{z_i}\partial_{z_j}N_{x,x_\Omega}(z)(1-\chi_{**}(z))u_i(z,t)u_j(z,t)\,dz \tag{2.12}$$

and $N_{x,x_\Omega}(z) = N(x,z) - N(x_\Omega,z)$. Note that by introducing $N(x_\Omega,z)$ we have modified $p_{\rm nonloc,H}$ by a function of t only (see local Helmholtz pressure, $p_{\rm loc,H}$, above), which thus makes no change to $\nabla p_{\rm nonloc,H}$.

Similarly, we can modify the nonlocal harmonic pressure by writing

$$\begin{aligned} p_{\rm harm}(x,t) &= \frac{1}{2\pi i}\int_0^t\int_\Gamma e^{(t-s)\lambda}\int_{\mathbb{R}^3_+} q_\lambda(x'-z',x_3,z_3)\cdot\chi_*F_B(z,s)\,dz'\,dz_3\,d\lambda\,ds \\ &\quad + \frac{1}{2\pi i}\int_0^t\int_\Gamma e^{(t-s)\lambda}\int_{\mathbb{R}^3_+} q_{\lambda,x,x_\Omega}(z)\cdot(1-\chi_*)(F_A(z,s)+F_B(z,s))\,dz\,d\lambda\,ds \\ &= p_{{\rm harm},\le 1}(x,t) + p_{{\rm harm},\ge 1}(x,t) \end{aligned} \tag{2.13}$$

(see also (2.17) in [MMP1]), where F_A and F_B are defined as in (2.9) and (2.10) with χ_{**} replaced by $(1-\chi_{**})$. Note that there is no F_A part in $p_{{\rm harm},\le 1}$ as χ_* vanishes on supp $(1-\chi_{**})$.

We point out that the above representation of the pressure function is *unique* on a given bounded open set $\Omega \subset \mathbb{R}^3_+$, up to a function of time, of any local energy solution u. Indeed, each of the pressure parts $p_{\rm li,loc}$, $p_{\rm li,nonloc}$, $p_{\rm loc,H}$, $p_{\rm loc,harm}$, $p_{\rm nonloc,H}$, $p_{{\rm harm},\le 1}$, and $p_{{\rm harm},\ge 1}$ depends only on u (rather than on our decomposition $u_{\rm li}+u_{\rm loc}+u_{\rm nonloc}$), and so uniqueness (up to a function of time) follows from the distributional form of the Navier-Stokes equations (recall Definition 1.1). In other words, if given $\Omega,\Omega'\subset\mathbb{R}^3_+$ are such that $\Omega\subset\Omega'$ and if we define the pressure functions $p_\Omega, p_{\Omega'}$ as the sum of the above pressure parts (respectively), then

$$p_\Omega - p_{\Omega'} = c_{\Omega,\Omega'}(t) \tag{2.14}$$

on Ω for some $c_{\Omega,\Omega'}$, a function of time only.

In the remaining part of this section, we fix $n\geq 1$ and prove the following estimates on any given bounded open set $\Omega\subset\mathbb{R}^3_+$:

$$\begin{aligned}
\|p_{\mathrm{li,loc}}(t)\|_{L^2(\Omega)} &\lesssim 2^{\frac{q+1}{2}\overline{m}}\|u_0\|_{M^{2,q}}t^{-\frac34},\\
\|p_{\mathrm{li,nonloc}}(t)\|_{L^\infty(\Omega)} &\lesssim 2^{\overline{m}+\frac{q-4}{2}\underline{m}}\|u_0\|_{M^{2,q}}t^{-\frac34},\\
\|p_{\mathrm{loc,H}}(t)\|_{L^{\frac32}(\mathbb{R}^3_+)} &\lesssim \|u(t)\|^2_{L^3(Q^{***})},\\
\|p_{\mathrm{loc,harm}}-\theta\|_{L^{\frac32}((0,t);L^{\frac{17}{10}}(\mathbb{R}^3_+))} &\lesssim 2^{q\overline{m}}\Big(\|\alpha\|^{\frac{13}{34}}_{L^{\frac{39}{5}}(0,t)}\beta(t)^{\frac{21}{34}}+2^{-\frac{21}{17}\underline{m}}\|\alpha\|_{L^{\frac32}(0,t)}\Big),\\
\|p_{\mathrm{nonloc,H}}(t)\|_{L^\infty(\Omega)} &\lesssim 2^{\overline{m}+(q-4)\underline{m}}\|u(t)\|^2_{M^{2,q}},\\
\|p_{\mathrm{harm},\geq 1}\|_{L^r((0,T);L^\infty(\Omega))} &\lesssim 2^{\overline{m}+(q-4)\underline{m}}\bigg(2^{2(\delta-\gamma)\underline{m}}T^\gamma\|\alpha\|^{1-\delta}_{L^{\frac{r(1-\delta)}{1-r\delta}}(0,T)}\beta(T)^\delta\\
&\qquad +\Big(1+T^\gamma 2^{-2\gamma\underline{m}}+T^{\frac12}2^{\overline{m}q-(1+q)\underline{m}}\Big)\|\alpha\|_{L^r(0,T)}\bigg)\\
\|p_{\mathrm{harm},\leq 1}(t)\|_{L^\infty(\Omega)} &\lesssim 2^{\frac{\overline{m}}{4}+(q-4)\underline{m}}\alpha(t)t^{\frac38},
\end{aligned}\tag{2.15}$$

for $t>0$, $r\in[1,\infty)$, $q\in(0,3)$, $\delta\in(0,\min\{1/r,3/4,3q/2\})$, and $\gamma\in(0,\delta/3)$, where θ is a function of t only. The implicit constants depend on q, r, δ, γ, κ, and η; we used the notation (1.8). In particular, the implicit constants do not depend on the choice of x_Ω (recall the above decompositions into the local and nonlocal parts). In fact, as mentioned below (2.2), the dependence of the above estimates on the geometry of Ω is expressed in terms of the indices $\underline{m}$ and $\overline{m}$ only. For simplicity of notation, we have used the abbreviations $M^{2,q}=M^{2,q}_{\mathcal{C}_n}$ and

$$\alpha=\alpha_n,\qquad \beta=\beta_n$$

(recall (1.8)), which we also apply in the remainder of this section.

We note that the implicit constants in (2.15) do not depend on n. If $\Omega\in\mathcal{C}_n$ for some $n\geq 1$, then we have $\underline{m}=\overline{m}=m$, where $m\geq n$ is such that 2^m is the side-length of Q. For such Q the estimates reduce by replacing $2^{\overline{m}}$ and $2^{\underline{m}}$ with $|Q|^{1/3}$. An important property to keep in mind is that if Ω is a cube from $\mathcal{C}_1$ that also belongs to $\mathcal{C}_n$ for some $n>1$, then the estimates get sharper for larger n.

In the estimates below, we write $S_k\equiv S_k^{(n)}$ for brevity.

Lemma 2.1 (Estimate for $p_{\mathrm{li,loc}}$)**.** *For every $t>0$, we have*

$$\|p_{\mathrm{li,loc}}(t)\|_{L^2(\Omega)}\lesssim 2^{\frac{q+1}{2}\overline{m}}\|u_0\|_{M^{2,q}}t^{-\frac34}.$$

Proof of Lemma 2.1. Fix $t>0$, and note that $\Omega\subset\Omega'\times\Omega_3$, where Ω' denotes the projection of Ω onto the (x_1,x_2)-plane, and Ω_3 onto the x_3-axis. For $x_3\in\Omega_3$, we use (2.4) to get

$$\begin{aligned}
\|p_{\mathrm{li,loc}}(\cdot,x_3)\|_{L^2_{x'}(\Omega')} &\lesssim \int_\Gamma \mathrm{e}^{t\,\mathrm{Re}\,\lambda}\left\|\int_{\mathbb{R}^3_+}\chi q_\lambda(x'-z',x_3,z_3)\cdot u_0(z)\,dz'\,dz_3\right\|_{L^2_{x'}(\Omega')}d|\lambda|\\
&\lesssim \int_\Gamma \mathrm{e}^{t\,\mathrm{Re}\,\lambda}\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}\left\|\int_{\mathbb{R}^2}(|x'-z'|+x_3+z_3)^{-2}|\chi u_0(z)|\,dz'\right\|_{L^2_{x'}(\Omega')}dz_3\,d|\lambda|.
\end{aligned}\tag{2.16}$$

Since

$$|x'-z'|\leq|x'|+|z'|\lesssim 2^{\overline{m}}, \tag{2.17}$$

for every $x \in \Omega$ and $z \in Q^*$. Thus it follows that for every z_3

$$\begin{aligned} &\left\| \int_{\mathbb{R}^2} (|x'-z'| + x_3 + z_3)^{-2} |\chi u_0(z)| \, dz' \right\|_{L^2_{x'}(\Omega')} \\ &= \left\| \int_{\mathbb{R}^2} 1_{C2^{\overline{m}}}(x'-z') (|x'-z'| + x_3 + z_3)^{-2} |\chi u_0(z)| \, dz' \right\|_{L^2_{x'}(\Omega')} \\ &\lesssim \|\chi u_0(z', z_3)\|_{L^2_{z'}(\mathbb{R}^2)} \int_{|y'| \le C2^{\overline{m}}} (|y'| + x_3 + z_3)^{-2} \, dy' \\ &\lesssim \|\chi u_0(z', z_3)\|_{L^2_{z'}(\mathbb{R}^2)} 2^{\overline{m}/4} (x_3 + z_3)^{-\frac{1}{4}}, \end{aligned} \tag{2.18}$$

where we used $\int_{|y'| \le a} (|y'| + b)^{-2} \, dy' = 2\pi(-a/(a+b) + \log(1 + a/b)) \le c_\alpha (a/b)^\alpha$ for any $\alpha \in (0,1)$. Therefore,

$$\begin{aligned} &\left\| \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \left\| \int_{\mathbb{R}^2} (|x'-z'| + x_3 + z_3)^{-2} |\chi u_0(z)| \, dz' \right\|_{L^2_{x'}(\Omega')} dz_3 \right\|_{L^2_{x_3}(\Omega_3)} \\ &\lesssim 2^{\frac{\overline{m}}{4}} \|x_3^{-\frac{1}{4}}\|_{L^2_{x_3}(\Omega_3)} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \|\chi u_0(z', z_3)\|_{L^2_{z'}(\mathbb{R}^2)} \, dz_3 \\ &\lesssim 2^{\frac{\overline{m}}{2}} \left(\int_0^\infty \mathrm{e}^{-2|\lambda|^{\frac{1}{2}} z_3} \, dz_3 \right)^{1/2} \|\chi u_0\|_{L^2(\mathbb{R}^3_+)} \\ &\lesssim 2^{\frac{\overline{m}}{2}} \|\chi u_0\|_{L^2(\mathbb{R}^3_+)} |\lambda|^{-\frac{1}{4}}, \end{aligned} \tag{2.19}$$

where we used the Minkowski inequality in the first step. Finally, including the integral in λ, we obtain from (2.16),

$$\|p_{\mathrm{li,loc}}(t)\|_{L^2(\Omega)} \lesssim 2^{\frac{\overline{m}}{2}} \|\chi u_0\|_{L^2(\mathbb{R}^3_+)} \int_\Gamma \mathrm{e}^{t \operatorname{Re} \lambda} |\lambda|^{-\frac{1}{4}} \, d|\lambda| \lesssim 2^{\frac{q+1}{2}\overline{m}} \|u_0\|_{M^{2,q}} t^{-\frac{3}{4}},$$

and the proof is concluded. □

Lemma 2.2 (Estimate for $p_{\mathrm{li,nonloc}}$)**.** *For every $t > 0$, we have*

$$\|p_{\mathrm{li,nonloc}}(t)\|_{L^\infty(\Omega)} \lesssim_q 2^{\overline{m} + \frac{q-4}{2}\underline{m}} \|u_0\|_{M^{2,q}} t^{-\frac{3}{4}},$$

where $q \in (0,3)$.

Proof of Lemma 2.2. For every $x \in \Omega$ we use (2.5) to obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^3_+} q_{\lambda,x,x_\Omega}(z)(1-\chi) u_0(z) \, dz \right| &\lesssim 2^{\overline{m}} \int_0^\infty \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3}}{(|\xi' - z'| + \xi_3 + z_3)^3} |(1-\chi) u_0(z)| \, dz' \, dz_3 \\ &\lesssim 2^{\overline{m}} \Bigg(2^{-3\underline{m}} \sum_{k=n}^{\underline{m}} \sum_{\tilde{Q} \in S_k} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \int_{\mathbb{R}^2} \chi_{\tilde{Q}}(z) |u_0(z)| \, dz' \, dz_3 \\ &\qquad + \sum_{k=\underline{m}+1}^\infty 2^{-3k} \sum_{\tilde{Q} \in S_k} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \int_{\mathbb{R}^2} \chi_{\tilde{Q}}(z) |u_0(z)| \, dz' \, dz_3 \Bigg), \end{aligned}$$

where we write $z = (z', z_3)$ to emphasize the horizontal and vertical components of z and $\xi \in [x, x_\Omega]$, with $[x, x_\Omega]$ denoting the line segment between the points x and x_Ω. We also used (2.2) in the first inequality above and (2.1) in the second. We now apply the Cauchy-Schwarz inequality to the z_3-integral to obtain

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \int_{\mathbb{R}^2} \chi_{\tilde{Q}}(z) |u_0(z)| \, dz' \, dz_3 \le \|\mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3}\|_{L^2_{z_3}(0,\infty)} \left\| \int_{\mathbb{R}^2} \chi_{\tilde{Q}}(z', z_3) |u_0(z', z_3)| \, dz' \right\|_{L^2_{z_3}(0,\infty)} \\ &\lesssim |\lambda|^{-\frac{1}{4}} \|u_0\|_{L^2(\tilde{Q})} 2^k \lesssim |\lambda|^{-\frac{1}{4}} \|u_0\|_{M^{2,q}} 2^{\frac{2+q}{2} k} \end{aligned}$$

for every $\tilde{Q} \in S_k$, where we applied the Cauchy-Schwarz inequality in the z'-integral in the second inequality. Substituting this into the above estimate gives

$$\begin{aligned}\left|\int_{\mathbb{R}^3_+} q_{\lambda,x,x_\Omega}(z)(1-\chi)u_0(z)\,dz\right| &\lesssim 2^{\overline{m}}|\lambda|^{-\frac14}\,\|u_0\|_{M^{2,q}}\left(2^{-3\underline{m}}\sum_{k=n}^{\underline{m}}2^{\frac{2+q}{2}k}+\sum_{k=\underline{m}+1}^{\infty}2^{\frac{q-4}{2}k}\right)\\ &\lesssim 2^{\overline{m}+\frac{q-4}{2}\underline{m}}|\lambda|^{-\frac14}\,\|u_0\|_{M^{2,q}}\,,\end{aligned}$$

from which the lemma follows by integrating in λ and noting that $|\lambda| \geq \kappa$. □

Lemma 2.3 (Estimate for $p_{\mathrm{loc,H}}$)**.** *For every $t > 0$, we have*

$$\|p_{\mathrm{loc,H}}(t)\|_{L^{\frac32}(\mathbb{R}^3_+)} \leq C\|u(t)\|^2_{L^3(Q^{***})}.$$

Proof of Lemma 2.3. This follows directly by the Calderón-Zygmund estimate applied to each of the two components of the Neumann kernel (2.8). □

Lemma 2.4 (Estimate for $p_{\mathrm{loc,harm}}$)**.** *There exists a function θ depending only on t such that*

$$\|p_{\mathrm{loc,harm}}-\theta\|_{L^{\frac32}((0,t);L^{\frac{17}{10}}(\mathbb{R}^3_+))} \lesssim 2^{q\overline{m}}\left(\|\alpha\|^{\frac{13}{34}}_{L^{\frac{39}{5}}(0,t)}\beta(t)^{\frac{21}{34}}+2^{-\underline{m}}\|\alpha\|_{L^{\frac32}(0,t)}\right)$$

for all $q > 0$.

Recall that we use the abbreviations $\alpha = \alpha_n$ and $\beta = \beta_n$; see (1.8).

Let us briefly comment why we estimate $p_{\mathrm{loc,harm}}$ in $L^{3/2}_tL^{17/10}_x$. We are interested in estimating a term of the form $\int_Q u p_{\mathrm{loc,harm}}$ (see Lemma 3.3 below) for a given cube $Q \in \mathcal{C}_n$, for which we can use a bound of the form $\|u\|_{L^{3/2}_tL^{r'}}\|p_{\mathrm{loc,harm}}-\theta\|_{L^{3/2}_tL^r}$, where r' is the conjugate exponent to r. The borderline value of r is $9/5$ as then one can obtain $\|p_{\mathrm{loc,harm}}-\theta\|_{L^{3/2}_tL^{9/5}_x} \lesssim \|\nabla p_{\mathrm{loc,harm}}\|_{L^{3/2}_tL^{9/8}_x} \lesssim |Q|^{q/3}\alpha(t)^{2/3}\beta(t)^{2/3}$, by considering the leading order term only. However, in this case we obtain a power of $\alpha(t)$ on the right-hand side, instead of $\|\alpha\|_{L^p(0,t)}$ for some $p \in [1,\infty)$, which makes it impossible to use an ODE-type argument in the a priori bound; note that Lemma 3.4 below requires that $p < \infty$. Taking $r < 9/5$ replaces the L^∞ norm with a high L^p norm, which makes it possible to use an ODE-type argument, but r also cannot be too low. For example taking $r = 8/5$ one can similarly obtain, up to the leading order, $\|p_{\mathrm{loc,harm}}-\theta\|_{L^{3/2}_tL^{8/5}_x} \lesssim \|\nabla p_{\mathrm{loc,harm}}\|_{L^{3/2}_tL^{24/23}_x} \lesssim |Q|^{q/3}\|\alpha\|^{7/16}_{L^{21/5}(0,t)}\beta(t)^{9/16}$, while a Gagliardo-Nirenberg-Sobolev argument for u gives $\|u\|_{L^3_tL^{8/3}_x} \lesssim |Q|^{q/6}\|\alpha\|^{15/16}_{L^{15/7}(0,t)}\beta(t)^{9/16}$. In this case the total power of β is $9/8 > 1$, which makes it impossible to absorb it by the dissipation term on the left-hand side of the local energy inequality. Therefore we choose $r = 17/10$, as it settles both issues.

Proof of Lemma 2.4. By the Poincaré-Sobolev-Wirtinger inequality (see Theorem II.6.1 in [Ga]) we have, with θ depending only on t,

$$\|p_{\mathrm{loc,harm}}(t)-\theta(t)\|_{L^{\frac{17}{10}}(\mathbb{R}^3_+)} \lesssim \|\nabla p_{\mathrm{loc,harm}}(t)\|_{L^{\frac{51}{47}}(\mathbb{R}^3_+)},$$

and thus, using maximal regularity of the Stokes equation in the half-space ([SvW, GS]),

$$\begin{aligned}\|p_{\mathrm{loc,harm}}-\theta\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac{17}{10}}(\mathbb{R}^3_+))} &\lesssim \|\mathbb{P}\nabla\cdot(\chi_{**}u\otimes u)\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac{51}{47}}(\mathbb{R}^3_+))}\\ &\lesssim \int_0^t \|u(s)\|^{\frac32}_{L^{\frac{102}{43}}(Q^{***})}\|\nabla(\chi_{**}u)(s)\|^{\frac32}_{L^2}\,ds\\ &\lesssim \int_0^t\left(\|u(s)\|^{\frac{39}{34}}_{L^2(Q^{***})}\|\nabla u(s)\|^{\frac{6}{17}}_{L^2(Q^{***})}\|\nabla(\chi_{**}u)(s)\|^{\frac32}_{L^2}+2^{-\frac{6}{17}\underline{m}}\|u(s)\|^{\frac32}_{L^2(Q^{***})}\|\nabla(\chi_{**}u)(s)\|^{\frac32}_{L^2}\right)ds\\ &\lesssim \int_0^t\Bigg(\|u(s)\|^{\frac{39}{34}}_{L^2(Q^{***})}\|\nabla u(s)\|^{\frac{63}{34}}_{L^2(Q^{***})}+2^{-\frac{6}{17}\underline{m}}\|u(s)\|^{\frac32}_{L^2(Q^{***})}\|\nabla u(s)\|^{\frac32}_{L^2(Q^{***})}\\ &\qquad\qquad+2^{-\frac32\underline{m}}\|u(s)\|^{\frac{45}{17}}_{L^2(Q^{***})}\|\nabla u(s)\|^{\frac{6}{17}}_{L^2(Q^{***})}\Bigg)\,ds.\end{aligned}$$

Applying Young's inequality on the last two terms in the integrand, we get

$$
\begin{aligned}
&\|p_{\rm loc,harm}-\theta\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac{17}{10}}(\mathbb{R}^3_+))}\\
&\lesssim \int_0^t \Big(\|u(s)\|^{\frac{39}{34}}_{L^2(Q^{***})}\|\nabla u(s)\|^{\frac{63}{34}}_{L^2(Q^{***})} + 2^{-\frac{63}{34}\underline{m}}\|u(s)\|^3_{L^2(Q^{***})}\Big)\,ds\\
&\lesssim 2^{\frac{3q}{2}\overline{m}}\bigg(\bigg(\bigg(\int_0^t \alpha(s)^{\frac{39}{5}}\,ds\bigg)^{\frac{5}{68}}\beta(t)^{\frac{63}{68}} + 2^{-\frac{63}{34}\underline{m}}\int_0^t \alpha(s)^{\frac32}\,ds\bigg).
\end{aligned}
$$

where we used $2^{3\underline{m}}\lesssim |Q^{***}|\lesssim 2^{3\overline{m}}$, and the proof is concluded. □

Lemma 2.5 (Estimate for $p_{\rm nonloc,H}$)**.** *We have*

$$\|p_{\rm nonloc,H}(t)\|_{L^\infty(\Omega)} \lesssim 2^{\overline{m}+(q-4)\underline{m}}\|u(t)\|^2_{M^{2,q}},$$

for every $t>0$ and $q\in(0,4)$.

Proof of Lemma 2.5. We omit the t variable in the notation. Recall that Q^* is the union of the neighbors of Q, which is a cover of Ω using cubes from $\mathcal{C}_n$. For $x\in Q^*$ and $z\notin Q^{**}$ we have

$$|\partial_{z_i}\partial_{z_j}N_{x,x_\Omega}(z)|\lesssim \frac{|x-x_\Omega|}{|x-z|^4}\lesssim \frac{2^{\overline{m}}}{|x-z|^4}\lesssim \begin{cases} 2^{\overline{m}}2^{-4\underline{m}}, & z\in\tilde Q\in S_k, k\le \underline{m},\\ 2^{\overline{m}}2^{-4k}, & z\in \tilde Q\in S_k, k\ge \underline{m}+1,\end{cases}$$

as in (2.1) and (2.2). Thus, for such x, (2.12) gives

$$
\begin{aligned}
|p_{\rm nonloc,H}(x)| &\lesssim 2^{\overline{m}}\sum_{k\ge n}\sum_{\tilde Q\in S_k,\tilde Q\not\subset Q^{**}}\int_{\tilde Q}\frac{|u(z)|^2}{|x-z|^4}\,dz\\
&\lesssim 2^{\overline{m}}\Bigg(2^{-4\underline{m}}\sum_{k=n}^{\underline{m}}\sum_{\tilde Q\in S_k}\int_{\tilde Q}|u|^2 + \sum_{k\ge \underline{m}+1}\sum_{\tilde Q\in S_k}2^{-4k}\int_{\tilde Q}|u|^2\Bigg)\\
&\lesssim \|u\|^2_{M^{2,q}}2^{\overline{m}}\Bigg(2^{-4\underline{m}}\sum_{k=n}^{\underline{m}}\sum_{\tilde Q\in S_k}2^{qk} + \sum_{k\ge \underline{m}+1}\sum_{\tilde Q\in S_k}2^{(q-4)k}\Bigg)\lesssim \|u\|^2_{M^{2,q}}2^{\overline{m}+(q-4)\underline{m}}
\end{aligned}
$$

for any $q<4$. □

Lemma 2.6 (Estimate for $p_{{\rm harm},\ge 1}$)**.** *If $T>0$, then*

$$
\begin{aligned}
&\|p_{{\rm harm},\ge1}\|_{L^r((0,T);L^\infty(\Omega))}\\
&\lesssim_{r,q,\delta,\gamma} 2^{\overline{m}+(q-4)\underline{m}}\Big(2^{2(\delta-\gamma)\underline{m}}T^\gamma\|\alpha\|^{1-\delta}_{L^{\frac{r(1-\delta)}{1-r\delta}}(0,T)}\beta(T)^\delta + (1+T^\gamma 2^{-2\gamma\underline{m}} + T^{\frac12}2^{\overline{m}q-(1+q)\underline{m}})\|\alpha\|_{L^r(0,T)}\Big)
\end{aligned}
$$

for every $r\in[1,\infty)$, $q\in(0,3)$, $\delta\in(0,\min\{1/r,3/4,3q/2\})$, and $\gamma\in(0,\delta/3)$.

Proof of Lemma 2.6. Recall that by (2.13) we have

$$
\begin{aligned}
p_{{\rm harm},\ge1}(x,t) &= \frac{1}{2\pi i}\int_0^t\int_\Gamma {\rm e}^{(t-s)\lambda}\int_{\mathbb{R}^3_+} q_{\lambda,x,x_\Omega}(z)(1-\chi_*)(F_A(z,s)+F_B(z,s))\,dz\,d\lambda\,ds\\
&= p_{\rm A}(x,t)+p_{\rm B}(x,t).
\end{aligned}
$$

In Step 1 below we provide an estimate for p_A. Next, in Step 2 we show that $\|f_B\|_{L^\infty(\Omega)}\lesssim |Q|^{(q-2)/3}\|u\|^2_{M^{2,q}}$ for every $Q\in\mathcal{C}_n$, where F_B is a sum of terms of the form $\nabla'\otimes\nabla' f_B$ which satisfy

$$\mathbf{m}(D')\nabla'\otimes\nabla'\int_0^\infty \big((P(\cdot,|z_3-y_3|)+P(\cdot,z_3+y_3))*((1-\chi_{**})v\otimes w)(y_3)\big)(z',s)\,dy_3 =: \nabla'\otimes\nabla' f_B,$$

where v and w denote $2D$ vectors whose components are chosen among u_1, u_2, and u_3 (recall (2.10)). We then use this estimate in Step 3 to prove the required bound on $p_{\rm B}$.

Step 1. We show the required estimate for $p_{\rm A}$.
For $x\in\Omega$, we have, using (2.5) with $m=2$, as well as (2.1) and (2.2),

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3_+} q_{\lambda,x,x_\Omega}(z)(1-\chi_*)F_A(z)\,dz \right| \\
&\quad\lesssim 2^{\overline{m}} \int_{\mathbb{R}^3_+} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4} |(1-\chi_*)u\otimes u(z)|\,dz + 2^{-\underline{m}} \int_{\operatorname{supp}\nabla\chi_*} |q_{\lambda,x,x_\Omega}(z)||u(z)|^2\,dz \\
&\quad= 2^{\overline{m}} \int_{2^{\underline{m}}}^\infty \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4} |(1-\chi_*)u\otimes u(z)|\,dz'\,dz_3 \\
&\qquad+ 2^{\overline{m}} \int_0^{2^{\underline{m}}} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4} |(1-\chi_*)u\otimes u(z)|\,dz'\,dz_3 \\
&\qquad+ 2^{-\underline{m}} \int_{\operatorname{supp}\nabla\chi_*} |q_{\lambda,x,x_\Omega}(z)||u(z)|^2\,dz \\
&\quad= 2^{\overline{m}}(I_1+I_2+I_3),
\end{aligned}
\tag{2.20}
$$

where ξ is a point on the line segment joining x and x_Ω. We denote the corresponding (pointwise) bound on $|p_A|$ by $p_{\rm A1}+p_{\rm A2}+p_{\rm A3}$, i.e.,

$$
p_{{\rm A}j}(x,t) = \frac{2^{\overline{m}}}{2\pi} \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} I_j\, d|\lambda|\,ds, \tag{2.21}
$$

for $j=1,2,3$.

For $p_{\rm A1}$ we observe that $\mathrm{e}^{-|\lambda|^{\frac12} z_3} \le \mathrm{e}^{-|\lambda|^{\frac12} 2^{\underline{m}}} \lesssim_\gamma |\lambda|^{-\gamma} 2^{-2\gamma\underline{m}}$ and use (2.1) to obtain

$$
\begin{aligned}
I_1 &\lesssim 2^{-2\gamma\underline{m}} |\lambda|^{-\gamma} \left(2^{-4\underline{m}} \sum_{k=n}^{\underline{m}} \sum_{\tilde Q\in S_k} \int_{\tilde Q} |u|^2 + \sum_{k\ge \underline{m}+1} \sum_{\tilde Q\in S_k} 2^{-4k} \int_{\tilde Q} |u|^2 \right) \\
&\lesssim 2^{-2\gamma\underline{m}} |\lambda|^{-\gamma} \|u\|_{M^{2,q}}^2 \left(2^{-4\underline{m}} \sum_{k=n}^{\underline{m}} 2^{qk} + \sum_{k\ge\underline{m}+1} 2^{(q-4)k} \right) \\
&\lesssim 2^{(q-4-2\gamma)\underline{m}} |\lambda|^{-\gamma} \|u\|_{M^{2,q}}^2 .
\end{aligned}
$$

Thus (2.21) gives

$$
\begin{aligned}
\|p_{\rm A1}(t)\|_{L^\infty(\Omega)} &\lesssim_\gamma 2^{\overline{m}+(q-4-2\gamma)\underline{m}} \alpha(t) \int_0^t \int_\Gamma |\lambda|^{-\gamma} \mathrm{e}^{(t-s)\operatorname{Re}\lambda}\, d|\lambda|\,ds \\
&\lesssim 2^{\overline{m}+(q-4-2\gamma)\underline{m}} \alpha(t) \int_0^t (t-s)^{\gamma-1}\,ds \\
&\lesssim 2^{\overline{m}+(q-4-2\gamma)\underline{m}} \alpha(t) t^\gamma
\end{aligned}
$$

for every $t\in[0,T]$, as required.

Next, we bound $p_{\rm A2}$. We set

$$
a=\frac{1}{1-\gamma}, \qquad b=\frac{3}{2\delta}.
$$

The assumptions on δ and γ guarantee that

$$
a\in(1,2), \qquad a'=\frac{1}{\gamma}>2b, \qquad b>\frac{3r}{2}, \qquad b>2 \qquad \text{and} \qquad q>\frac{1}{b}, \tag{2.22}
$$

where $1/a+1/a'=1$.

Hölder's inequality gives that

$$
\left| \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} \mathrm{e}^{-|\lambda|^{\frac12} z_3}\, d|\lambda| \right| \lesssim_{p,\Gamma} (t-s)^{-\frac1a} z_3^{-\frac{2}{a'}} .
$$

Therefore, using Tonelli's theorem,

$$\begin{aligned} |p_{\mathrm{A}2}(x,t)| &\le 2^{\overline{m}} \int_0^t \int_\Gamma \int_0^{2^{\underline{m}}} \int_{\mathbb{R}^2} \mathrm{e}^{(t-s)\operatorname{Re}\lambda} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4}(1-\chi_*)|u\otimes u(z)|\,dz'\,dz_3\,d|\lambda|\,ds \\ &\lesssim_p 2^{\overline{m}} \int_0^t (t-s)^{-\frac1a} \int_{\mathbb{R}^2}\int_0^{2^{\underline{m}}} \frac{z_3^{-\frac{2}{a'}}}{(|\xi'-z'|+\xi_3+z_3)^4}(1-\chi_*)|u\otimes u(z)|\,dz_3\,dz'\,ds. \end{aligned} \tag{2.23}$$

We write $\mathbb{R}^2 = \bigcup_{k\ge n}\bigcup_{\tilde Q'\in S_k'} \tilde Q'$, where $\tilde Q'$ denotes the projection of $\tilde Q$ onto $\mathbb{R}^2$ and S_k' denotes the collection of projections onto $\partial\mathbb{R}^3_+$ of the cubes from S_k that touch $\partial\mathbb{R}^3_+$. We also set

$$p_{\mathrm{A}2,\tilde Q}(x,t) = \int_0^t (t-s)^{-\frac1a}\int_{\tilde Q'}\int_0^{2^{\underline{m}}} \frac{z_3^{-\frac{2}{a'}}}{(|\xi'-z'|+\xi_3+z_3)^4}(1-\chi_*)|u\otimes u(z)|\,dz_3\,dz'\,ds,$$

so that

$$|p_{\mathrm{A}2}(x,t)| \le 2^{\overline{m}} \sum_{k\ge n}\sum_{\tilde Q'\in S_k'} p_{\mathrm{A}2,\tilde Q}(x,t). \tag{2.24}$$

Letting $b' = b/(b-1) < 2$ we have, for each $\tilde Q\in S_k$,

$$\begin{aligned} \int_0^{2^{\underline{m}}}\int_{\tilde Q'} |u\otimes u(z)| z_3^{-\frac{2}{a'}}\,dz'\,dz_3 &\le 2^{\frac2b k}\left(\int_0^{2^{\underline{m}}} z_3^{-\frac{2b}{a'}}\,dz_3\right)^{\frac1b}\|u\|^2_{L^{2b'}(\tilde Q)} \\ &\lesssim_{a,b} 2^{\left(\frac1b-\frac{2}{a'}\right)\underline{m}} 2^{\frac2b k}\left(\|u\|^{\frac{2b-3}{b}}_{L^2(\tilde Q)}\|\nabla u\|^{\frac3b}_{L^2(\tilde Q)} + 2^{-\frac3b k}\|u\|^2_{L^2(\tilde Q)}\right), \end{aligned} \tag{2.25}$$

where we used the Gagliardo-Nirenberg-Sobolev inequality. Thus, for $\tilde Q\in S_k$ with $k\ge \underline{m}+1$, we use (2.1) to obtain, for any $T>0$,

$$p_{\mathrm{A}2,\tilde Q} \le \int_0^t (t-s)^{-\frac1a}\int_{\tilde Q'}\int_0^{2^{\underline{m}}} \frac{z_3^{-\frac{2}{a'}}}{(|\xi'-z'|+\xi_3+z_3)^4}(1-\chi_*)|u\otimes u(z)|\,dz_3\,dz'\,ds,$$

which implies

$$\begin{aligned} &\|p_{\mathrm{A}2,\tilde Q}\|_{L^r((0,T);L^\infty(\Omega))} \\ &\lesssim_{a,b} 2^{\left(\frac1b-\frac{2}{a'}\right)\underline{m}} 2^{\left(-4+\frac2b\right)k}\left\| \int_0^t (t-s)^{-\frac1a}\left(\|u(s)\|^{\frac{2b-3}{b}}_{L^2(\tilde Q)}\|\nabla u(s)\|^{\frac3b}_{L^2(\tilde Q)} + 2^{-\frac3b k}\|u(s)\|^2_{L^2(\tilde Q)}\right)ds\right\|_{L^r_t(0,T)} \\ &\lesssim_a 2^{\left(\frac1b-\frac{2}{a'}\right)\underline{m}} 2^{\left(-4+\frac2b\right)k} T^{\frac{1}{a'}}\left(\int_0^T \left(\|u(s)\|^{\frac{2b-3}{b}}_{L^2(\tilde Q)}\|\nabla u(s)\|^{\frac3b}_{L^2(\tilde Q)} + 2^{-\frac3b k}\|u(s)\|^2_{L^2(\tilde Q)}\right)^r ds\right)^{\frac1r} \\ &\lesssim_a 2^{\left(\frac1b-\frac{2}{a'}\right)\underline{m}} 2^{\left(-4+\frac2b\right)k} T^{\frac{1}{a'}}\left(\left(\int_0^T \|u(s)\|^{\frac{2r(2b-3)}{2b-3r}}_{L^2(\tilde Q)}\,ds\right)^{\frac{2b-3r}{2br}}\left(\int_0^T \|\nabla u(s)\|^2_{L^2(\tilde Q)}\,ds\right)^{\frac{3}{2b}}\right. \\ &\qquad\qquad \left. + 2^{-\frac3b k}\left(\int_0^T \|u(s)\|^{2r}_{L^2(\tilde Q)}\,ds\right)^{\frac1r}\right) \\ &\lesssim_a 2^{\left(\frac1b-\frac{2}{a'}\right)\underline{m}} 2^{\left(-4+\frac2b+q\right)k} T^{\frac{1}{a'}}\left(\|\alpha\|^{\frac{2b-3}{2b}}_{L^{\frac{r(2b-3)}{2b-3r}}(0,T)}\beta(T)^{\frac{3}{2b}} + 2^{-\frac3b k}\|\alpha\|_{L^r(0,T)}\right), \end{aligned}$$

where we used Young's inequality $\|f*g\|_r \le \|f\|_1\|g\|_r$ in t in the second inequality (which gives the constraint $a>1$), and Hölder's inequality in t in the third (note that $3r<2b$ by (2.22)). For $k\le \underline{m}$ we obtain a similar estimate, except that $2^{\left(-4+\frac{2}{b}+q\right)k}$ is replaced by $2^{-4\underline{m}+\left(\frac{2}{b}+q\right)k}$. Thus (2.24) gives

$$\begin{aligned}\|p_{\mathrm{A}2}\|_{L^r((0,T);L^\infty(\Omega))} &\lesssim_{a,b} 2^{\left(\frac{1}{b}-\frac{2}{a'}\right)\underline{m}}2^{\overline{m}}T^{\frac{1}{a'}}\left(\left(2^{-4\underline{m}}\sum_{k=n}^{\underline{m}}2^{\left(\frac{2}{b}+q\right)k}+\sum_{k\geq\underline{m}+1}2^{\left(-4+\frac{2}{b}+q\right)k}\right)\|\alpha\|^{\frac{2b-3}{2b}}_{L^{\frac{r(2b-3)}{2b-3r}}(0,T)}\beta(T)^{\frac{3}{2b}}\right.\\ &\qquad\left.+\left(2^{-4\underline{m}}\sum_{k=n}^{\underline{m}}2^{\left(q-\frac{1}{b}\right)k}+\sum_{k\geq\underline{m}+1}2^{\left(-4-\frac{1}{b}+q\right)k}\right)\|\alpha\|_{L^r(0,T)}\right)\\ &\lesssim 2^{\overline{m}+\left(\frac{3}{b}-\frac{2}{a'}+q-4\right)\underline{m}}T^{\frac{1}{a'}}\left(\|\alpha\|^{\frac{2b-3}{2b}}_{L^{\frac{r(2b-3)}{2b-3r}}(0,T)}\beta(T)^{\frac{3}{2b}}+2^{-\frac{3}{b}\underline{m}}\|\alpha\|_{L^r(0,T)}\right)\\ &=2^{\overline{m}+(2\delta-2\gamma+q-4)\underline{m}}T^{\gamma}\left(\|\alpha\|^{1-\delta}_{L^{\frac{r(1-\delta)}{1-r\delta}}(0,T)}\beta(T)^{\delta}+2^{-2\delta\underline{m}}\|\alpha\|_{L^r(0,T)}\right),\end{aligned}$$

as required. Note that the infinite sum converges since $-4+2/b+q<0$, by (2.22), and for the finite sum in the second line we use $q>1/b$, recalling (2.22).

Finally, we bound $p_{\mathrm{A}3}$. First, by (2.5), we have

$$I_3=2^{-\overline{m}-\underline{m}}\int_{\operatorname{supp}\nabla\chi_*}|q_{\lambda,x,x_\Omega}(z)||u(z)|^2\,dz\lesssim 2^{-4\underline{m}}\int_{Q^{**}}\mathrm{e}^{-|\lambda|^{\frac{1}{2}}z_3}|u(z)|^2\,dz,$$

where in the second inequality, we used that $||x'-z'|+x_3+z_3|\gtrsim 2^{\underline{m}}$ and $|x-x_\Omega|\lesssim 2^{\overline{m}}$ hold on $\operatorname{supp}\nabla\chi_*$ (see (2.1) and (2.2)). Now, we apply the same analysis as for $p_{\mathrm{A}2}$ yielding the same bound on $p_{\mathrm{A}3}$ as we obtained for $p_{\mathrm{A}2}$. The only difference is that here we do not need to sum in $\tilde{Q}\in\mathcal{C}_n$.

Step 2. We show that, at each time, $\|f_B\|_{L^\infty(Q)}\lesssim|Q|^{\frac{q-2}{3}}\|u\|^2_{M^{2,q}}$ for every $Q\in\mathcal{C}_n$. (Analogously we can obtain $\|F_B\|_{L^\infty(Q)}\lesssim|Q|^{\frac{q-4}{3}}\|u\|^2_{M^{2,q}}$.)
Note that in this step the sets Q, Q^*, Q^{**}, Q^{***} are not related to Ω, but to a fixed cube Q. We shall use the estimate

$$|\mathbf{m}(D')P(y',y_3)|\lesssim\frac{y_3}{(|y'|+y_3)^{3+\alpha}},\tag{2.26}$$

where $\mathbf{m}(D')$ is a multiplier (in the y' variables) that is homogeneous of degree $\alpha>-2$, see [MMP1, p. 576]. Let $z\in Q$, and suppose that $Q\in S_m$. We only consider $P(z_3+y_3)$, as the part with $P(|z_3-y_3|)$ is similar. We have

$$\begin{aligned}|f_B(z)|&\lesssim\sum_{\tilde{Q}\in\mathcal{C}_n,\tilde{Q}\notin Q^{**}}\int_0^\infty\int_{\mathbb{R}^2}\frac{z_3+y_3}{(|z'-y'|+z_3+y_3)^3}(1-\chi_{**})\chi_{\tilde{Q}}|u\otimes u(y',y_3)|\,dy'\,dy_3\\ &\lesssim 2^m\left(2^{-3m}\sum_{k=n}^{m}\sum_{\tilde{Q}\in S_k}\|u\|^2_{L^2(\tilde{Q})}+\sum_{k\geq m+1}\sum_{\tilde{Q}\in S_k}2^{-3k}\|u\|^2_{L^2(\tilde{Q})}\right)\\ &\lesssim 2^m\|u\|^2_{M^{2,q}}\left(2^{-3m}\sum_{k=n}^{m}\sum_{\tilde{Q}\in S_k}2^{qm}+\sum_{k\geq m+1}\sum_{\tilde{Q}\in S_k}2^{(q-3)k}\right)\\ &\lesssim 2^{(q-2)m}\|u\|^2_{M^{2,q}},\end{aligned}$$

(recall (2.10) for the definition of $F_B=\nabla'\otimes\nabla' f_B$) where, in the second inequality, we used

$$\frac{z_3+y_3}{(|z'-y'|+z_3+y_3)^3}\lesssim\begin{cases}2^{-2m}, & k\leq m,\\ 2^{m-3k}, & k\geq m+1\end{cases}$$

whenever $z\in Q$ and $y\in\tilde{Q}$ for some cube $\tilde{Q}\in S_k$ that is disjoint with Q^{**} (which is an analogous claim to (2.1) and (2.2)).

Step 3. We show that $\|p_{\rm B}(t)\|_{L^\infty(\Omega)} \lesssim \alpha(t) 2^{\overline{m}(1+q)-5\underline{m}} t^{\frac12}$ for $q\in(0,3)$.
Note that this, together with Step 1, finishes the proof. We have

$$|p_{\rm B}(x,t)| \lesssim \int_0^t \int_\Gamma {\rm e}^{(t-s)\,{\rm Re}\,\lambda} \int_{\mathbb{R}^3_+} \left(\left|D^2 q_{\lambda,x,x_\Omega}(z)(1-\chi_*)\right| + |\nabla q_{\lambda,x,x_\Omega}(z)\nabla\chi_*| \right. \\ \left. + \left|q_{\lambda,x,x_\Omega}(z) D^2\chi_*\right|\right) |f_B(z,s)| \; dz \; d|\lambda|\, ds. \tag{2.27}$$

Using (2.1), (2.2), and (2.5), we get

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \left|D^2 q_{\lambda,x,x_\Omega}(z)(1-\chi_*) f_B(z,s)\right| \, dz \lesssim 2^{\overline{m}} \sum_{\tilde Q\in\mathcal{C}_n} \int_{\tilde Q} \frac{{\rm e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^5}(1-\chi_*)|f_B(z,s)|\, dz'\, dz_3 \\
&\lesssim 2^{\overline{m}} \left(2^{-5\underline{m}} \sum_{k=n}^{\underline{m}} \sum_{\tilde Q\in S_k} \int_{\tilde Q} {\rm e}^{-|\lambda|^{\frac12} z_3} |f_B(z,s)|\, dz'\, dz_3 \right. \\
&\qquad \left. \sum_{k\ge \underline{m}+1} \sum_{\tilde Q\in S_k} 2^{-5k} \int_{\tilde Q} {\rm e}^{-|\lambda|^{\frac12} z_3} |f_B(z,s)|\, dz'\, dz_3 \right) \\
&\lesssim 2^{\overline{m}} \left(\int_0^\infty {\rm e}^{-|\lambda|^{\frac12} z_3}\, dz_3 \right) \|u\|^2_{M^{2,q}} \left(2^{-5\underline{m}} \sum_{k=n}^{\underline{m}} 2^{qk} + \sum_{k\ge\underline{m}+1} 2^{(q-5)k} \right) \\
&\lesssim 2^{\overline{m}+(q-5)\underline{m}} |\lambda|^{-\frac12} \|u\|^2_{M^{2,q}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} \left(|\nabla q_{\lambda,x,x_\Omega}(z)\nabla\chi_*| + \left|q_{\lambda,x,x_\Omega}(z)D^2\chi_*\right| \right) |f_B(z,s)| \; dz \lesssim 2^{\overline{m}-5\underline{m}} \int_{Q^{**}\setminus Q^*} {\rm e}^{-|\lambda|^{\frac12} z_3} |f_B(z,s)|\, dz'\, dz_3 \\
&\lesssim 2^{\overline{m}-5\underline{m}} \sum_{k=\underline{m}-2}^{\overline{m}+2} \sum_{\tilde Q\in S_k} \int_{\tilde Q} {\rm e}^{-|\lambda|^{\frac12} z_3} |f_B(z,s)|\, dz'\, dz_3 \lesssim 2^{\overline{m}-5\underline{m}} |\lambda|^{-\frac12} \sum_{k=\underline{m}-2}^{\overline{m}+2} 2^{qk} \\
&\lesssim 2^{\overline{m}-5\underline{m}} |\lambda|^{-\frac12} \|u\|^2_{M^{2,q}} \sum_{k=\underline{m}-2}^{\overline{m}+2} 2^{qk} \lesssim 2^{\overline{m}(q+1)-5\underline{m}} |\lambda|^{-\frac12} \|u\|^2_{M^{2,q}}.
\end{aligned}$$

Using these estimates in (2.27), we obtain

$$\|p_{\rm B}(t)\|_{L^\infty(\Omega)} \lesssim \alpha(t) 2^{\overline{m}(1+q)-5\underline{m}} \int_0^t \int_\Gamma {\rm e}^{(t-s)\,{\rm Re}\,\lambda} |\lambda|^{-\frac12}\, d|\lambda| \; ds \lesssim \alpha(t) 2^{\overline{m}(1+q)-5\underline{m}} t^{\frac12},$$

and the proof is complete. □

Lemma 2.7 (Estimate for $p_{{\rm harm},\le 1}$)**.** *For every $t\ge 0$ and $q\in(0,3)$, we have*

$$\|p_{{\rm harm},\le1}(t)\|_{L^\infty(\Omega)} \lesssim 2^{\frac{\overline{m}}{4}+(q-4)\underline{m}} \alpha(t) t^{\frac38}.$$

Recall from (1.8) that $\alpha(t) = \sup_{s\in[0,t]} \|u(s)\|^2_{M^{2,q}_{\mathcal{C}_n}}$.

Proof of Lemma 2.7. By (2.13), we have

$$p_{{\rm harm},\le1}(x,t) = \frac{1}{2\pi i} \int_0^t \int_\Gamma {\rm e}^{(t-s)\lambda} \int_{\mathbb{R}^3_+} q_\lambda(x'-z', x_3, z_3)\chi_* F_B(z,s)\, dz \; d\lambda\, ds,$$

where F_B is as in $p_{\mathrm{harm},\geq 1}$. As in Lemma 2.1, we use (2.5) to obtain

$$
\begin{aligned}
|p_{\mathrm{harm},\leq 1}(x,t)| &\lesssim \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12} z_3} \int_{\mathbb{R}^2} (|x'-z'|+x_3+z_3)^{-2} \, |\chi_* F_B(z,s)| \, dz' \, dz_3 \, d|\lambda| \, ds \\
&\leq \int_0^t \|F_B(s)\|_{L^\infty(Q^*)} \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12} z_3} \int_{\{|z'|\leq C2^{\overline{m}}\}} (|z'|+x_3+z_3)^{-2} \, dz' \, dz_3 \, d|\lambda| \, ds \\
&\lesssim 2^{(q-4)\underline{m}+\frac{\overline{m}}{4}} \alpha(t) \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12} z_3} (x_3+z_3)^{-\frac14} \, dz_3 \, d|\lambda| \, ds \\
&\lesssim 2^{(q-4)\underline{m}+\frac{\overline{m}}{4}} \alpha(t) \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\operatorname{Re}\lambda} |\lambda|^{-\frac38} \, d|\lambda| \, ds \\
&\lesssim 2^{(q-4)\underline{m}+\frac{\overline{m}}{4}} \alpha(t) \int_0^t (t-s)^{-\frac58} \, ds \\
&\lesssim 2^{(q-4)\underline{m}+\frac{\overline{m}}{4}} \alpha(t) t^{\frac38}
\end{aligned}
$$

for every $x \in \Omega$ and $t > 0$, where in the third inequality we used the estimate $\|F_B\|_{L^\infty(\tilde{Q})} \lesssim |\tilde{Q}|^{\frac{q-4}{3}} \|u\|_{M^{2,q}}^2 \lesssim 2^{(q-4)\underline{m}} \|u\|_{M^{2,q}}^2$, for every $\tilde{Q} \subset Q^*$ (recall Step 2 above), as well as the fact $\int_{|y'|\leq a} (|y'|+b)^{-2} \, dy' \lesssim (a/b)^{1/4}$, as in the proof of Lemma 2.1. $\square$

3. A PRIORI BOUND

We now establish our main a priori bound for solutions to (1.1) for initial data in $M^{2,q}_{\mathcal{C}_n}$. We work under the assumption

$$
0 < q \leq 2. \tag{3.1}
$$

Recall from (1.8) the notation

$$
\alpha_n(t) = \sup_{s\in[0,t]} \|u(s)\|^2_{M^{2,q}_{\mathcal{C}_n}} \quad \text{and} \quad \beta_n(t) = \sup_{Q\in\mathcal{C}_n} \frac{1}{|Q|^{\frac{q}{3}}} \int_0^t \int_Q |\nabla u|^2.
$$

Note that, for the sake of brevity, we have omitted "$dx\, ds$" in the last integral. We continue this convention below in the instances that do not cause confusion. Since in Lemma 6.3 below we show that α and β are continuous functions of t, we define $\alpha_n(0) = \|u_0\|^2_{M^{2,q}_{\mathcal{C}_n}}$.

Theorem 1.4 follows from the following statement.

Proposition 3.1 (A priori bound)**.** *Assume that (3.1) holds. There exists $\gamma \geq 1$ with the following property. Let $n \in \mathbb{N}$, suppose that $u_0 \in \mathring{M}^{2,q}_{\mathcal{C}_n}$, and assume that (u,p) is a local energy solution on $\mathbb{R}^3_+ \times (0,\infty)$ with the initial data u_0 such that*

$$
\alpha_n(t) + \beta_n(t) < \infty,
$$

for all $t < \infty$. Let $T = T_n$ be the solution of

$$
a = b(1+T)^\gamma \left((2a)^3 T^{\frac38} + (2a)^{\frac32} T^{\frac{3}{16}} + a 2^{-\frac{n}{2}} T^{\frac18} \right), \tag{3.2}
$$

where $b > 0$ is a constant and $a = \|u_0\|^2_{M^{2,q}_{\mathcal{C}_n}}$. Then

$$
\alpha_n(T_n) + \beta_n(T_n) \leq 2b\alpha_n(0).
$$

The claim of Theorem 1.4 follows form Proposition 3.1 by taking $\gamma_0 = (10\gamma)^{-1}$, for example. Indeed, in order to verify the lower bound (1.10) on the solution T of (3.2), note that if it is false then $T < \eta d^{-\gamma_0}$, where $d = \max\{2^{-n}, a^{1/2}\}$ and $\eta = \eta(\|u_0\|_{M^{2,q}_{\mathcal{C}_1}}) \in (0,1)$, and so

$$
\begin{aligned}
1 &= b(1+T)^\gamma \left(8a^2 T^{\frac38} + 2^{\frac32} a^{\frac12} T^{\frac{3}{16}} + 2^{-\frac{n}{2}} T^{\frac18} \right) \\
&\leq 8b(1+\eta d^{-\frac{1}{10\gamma}})^\gamma \left(\eta^{\frac38} d^{2-\frac{3}{80\gamma}} + \eta^{\frac{3}{16}} d^{\frac12 - \frac{3}{160\gamma}} + \eta^{\frac18} d^{\frac12 - \frac{1}{80\gamma}} \right) \\
&\leq C_\gamma \eta^{\frac18} (1+d)^3,
\end{aligned}
$$

which gives a contradiction if η is chosen sufficiently small so that $C_\gamma \eta^{\frac18}\left(2+\|u_0\|_{M^{2,q}_{\mathcal{C}_1}}\right)^3 \le 1/2$.

Before proving Proposition 3.1, we recall an interpolation-type lemma from [BKT] which enables us to estimate the cubic term appearing on the right-hand side of the local energy inequality (1.4).

Lemma 3.2. *Let $u\colon \mathbb{R}^3_+ \times (0,T) \to \mathbb{R}^3$. Given $\epsilon > 0$, we have*

$$\begin{aligned}\frac{1}{|Q|^{\frac13}}\int_0^t\int_Q |u|^3 \le C_\epsilon |Q|^{q-\frac43}\int_0^t\left(\frac{1}{|Q|^{\frac q3}}\int_Q |u(s)|^2\right)^3 ds \\ + \epsilon\int_0^t\int_Q |\nabla u|^2 + C|Q|^{\frac q2-\frac56}\int_0^t\left(\frac{1}{|Q|^{\frac q3}}\int_Q|u(s)|^2\right)^{\frac32} ds, \qquad t\in(0,T),\end{aligned} \tag{3.3}$$

for any cube $Q \subset \mathbb{R}^3_+$.

The inequality (3.3) implies

$$\frac{1}{|Q|^{\frac13}}\int_0^t\int_Q |u|^3 \le C_\epsilon |Q|^{q-\frac43}\|\alpha\|^3_{L^3(0,t)} + \epsilon|Q|^{\frac q3}\beta(t) + |Q|^{\frac q2-\frac56}\|\alpha\|^{\frac32}_{L^{\frac32}(0,t)} \tag{3.4}$$

for every $Q\in\mathcal{C}_n$ and $t>0$, suppressing the dependence of α and β on n in the rest of the section. Similarly, one can show that

$$\|u\|_{L^3((0,t);L^{\frac{17}{7}}(Q))} \lesssim |Q|^{\frac q6}\|\alpha\|^{\frac{25}{68}}_{L^{\frac{75}{41}}(0,t)}\beta(t)^{\frac{9}{68}} + |Q|^{\frac q6-\frac{3}{34}}\|\alpha\|^{\frac12}_{L^{\frac32}(0,t)}. \tag{3.5}$$

We now use the pressure estimates developed in the previous section to deduce a bound on the pressure term appearing in the local energy inequality (1.4).

Lemma 3.3. *Let $Q\in\mathcal{C}_n$, and let ϕ_Q be such that $\phi_Q = 1$ on Q, $\phi_Q = 0$ outside Q^*, and $|\nabla\phi_Q| \lesssim |Q|^{-\frac13}$. Then*

$$\int_0^t\int p u\cdot\nabla\phi_Q \le C_\epsilon|Q|^{\frac q3}(1+t)^\gamma\left(\|\alpha\|^3_{L^8(0,t)} + \|\alpha\|^{\frac32}_{L^8(0,t)} + |Q|^{-\frac16}\|\alpha\|_{L^8(0,t)}\right) + \epsilon|Q|^{\frac q3}\beta(t)$$

for $0<q\le 2$ and $\epsilon>0$, where $\gamma\ge1$.

Proof of Lemma 3.3. We have

$$\begin{aligned}&\int_0^t\int (p_{\mathrm{li,loc}}+p_{\mathrm{li,nonloc}})u\cdot\nabla\phi_Q \\ &\lesssim |Q|^{-\frac13}\int_0^t \|p_{\mathrm{li,loc}}\|_{L^2(Q^*)}\|u\|_{L^2(Q^*)}\,ds + |Q|^{-\frac13}\int_0^t\|p_{\mathrm{li,nonloc}}\|_{L^\infty(Q^*)}\|u\|_{L^1(Q^*)}\,ds \\ &\lesssim |Q|^{\frac{q-1}{6}}\int_0^t \|u_0\|_{M^{2,q}_{\mathcal{C}_n}}\|u\|_{L^2(Q^*)}s^{-\frac34}\,ds + |Q|^{\frac{q-4}{6}}\int_0^t\|u_0\|_{M^{2,q}_{\mathcal{C}_n}}\|u\|_{L^1(Q^*)}s^{-\frac34}\,ds \\ &\lesssim |Q|^{\frac{q-1}{6}}\int_0^t\|u\|_{L^2(Q^*)}\|u_0\|_{M^{2,q}_{\mathcal{C}_n}}s^{-\frac34}\,ds \\ &\lesssim |Q|^{\frac q3-\frac16}\int_0^t\alpha(s)s^{-\frac34}\,ds \lesssim |Q|^{\frac q3-\frac16}t^{\frac18}\|\alpha\|_{L^8(0,t)},\end{aligned} \tag{3.6}$$

where we used the first two inequalities of (2.15) in the second step.

Next, by the third estimate in (2.15), we may bound

$$\begin{aligned}\int_0^t\int p_{\mathrm{loc,H}}\,u\cdot\nabla\phi_Q &\lesssim |Q|^{-\frac13}\left(\int_0^t\int_{Q^*}|u|^3 + \int_0^t\int_{Q^*}|p_{\mathrm{loc,H}}|^{\frac32}\right) \lesssim |Q|^{-\frac13}\int_0^t\int_{Q^{***}}|u|^3 \\ &\lesssim C_\epsilon|Q|^{\frac q3}|Q|^{\frac{2q}{3}-\frac43}\|\alpha\|^3_{L^3(0,t)} + \epsilon|Q|^{\frac q3}\beta(t) + |Q|^{\frac q3}|Q|^{\frac q6-\frac56}\|\alpha\|^{\frac32}_{L^{\frac32}(0,t)},\end{aligned} \tag{3.7}$$

by (3.4). With $\theta = \theta(t)$ as in (2.15), we write

$$\begin{aligned}
\int_0^t \int p_{\mathrm{loc,harm}}\, u\cdot\nabla\phi_Q &= \int_0^t \int (p_{\mathrm{loc,harm}} - \theta)\, u\cdot\nabla\phi_Q \\
&\lesssim |Q|^{-\frac13}\|u\|_{L^3((0,t);L^{\frac{17}{7}}(Q^*))}\|p_{\mathrm{loc,harm}}-\theta\|_{L^{\frac32}((0,t);L^{\frac{17}{10}}(\mathbb{R}^3_+))} \\
&\lesssim |Q|^{\frac q3}\left(|Q|^{\frac{q-2}{6}}\|\alpha\|^{\frac{25}{68}}_{L^{\frac{75}{41}}(0,t)}\beta(t)^{\frac{9}{68}} + |Q|^{\frac q6-\frac{43}{102}}\|\alpha\|^{\frac12}_{L^{\frac32}(0,t)}\right) \\
&\qquad\times\left(\|\alpha\|^{\frac{13}{34}}_{L^{\frac{39}{5}}(0,t)}\beta(t)^{\frac{21}{34}} + |Q|^{-\frac{4}{51}}\|\alpha\|^{\frac12}_{L^3(0,t)}\beta(t)^{\frac12} + |Q|^{-\frac{21}{51}}\|\alpha\|_{L^{\frac32}(0,t)}\right) \\
&\lesssim |Q|^{\frac q3}|Q|^{\frac{q-2}{6}}\left(t^{\frac{253}{1632}}\|\alpha\|^{\frac{25}{68}}_{L^8(0,t)}\beta(t)^{\frac{9}{68}} + |Q|^{-\frac{3}{34}}t^{\frac{13}{48}}\|\alpha\|^{\frac12}_{L^8(0,t)}\right) \\
&\qquad\times\left(t^{\frac{1}{816}}\|\alpha\|^{\frac{13}{34}}_{L^8(0,t)}\beta(t)^{\frac{21}{34}} + |Q|^{-\frac{4}{51}}t^{\frac{5}{48}}\|\alpha\|^{\frac12}_{L^8(0,t)}\beta(t)^{\frac12} + |Q|^{-\frac{21}{51}}t^{\frac{13}{24}}\|\alpha\|_{L^8(0,t)}\right),
\end{aligned} \tag{3.8}$$

where we applied (3.5) in the second inequality. Therefore,

$$\begin{aligned}
\int_0^t \int p_{\mathrm{loc,harm}}\, u\cdot\nabla\phi_Q
&\lesssim |Q|^{\frac q3}(1+t)\Big(\|\alpha\|^{\frac34}_{L^8(0,t)}\beta(t)^{\frac34} + \|\alpha\|^{\frac{59}{68}}_{L^8(0,t)}\beta(t)^{\frac{43}{68}} + \|\alpha\|^{\frac{93}{68}}_{L^8(0,t)}\beta(t)^{\frac{9}{68}} \\
&\qquad + \|\alpha\|^{\frac{15}{17}}_{L^8(0,t)}\beta(t)^{\frac{21}{34}} + \|\alpha\|_{L^8(0,t)}\beta(t)^{\frac12} + \|\alpha\|^{\frac32}_{L^8(0,t)}\Big) \\
&\lesssim C_\epsilon|Q|^{\frac q3}(1+t)^C\left(\|\alpha\|^3_{L^8(0,t)} + \|\alpha\|^{\frac32}_{L^8(0,t)}\right) + \epsilon|Q|^{\frac q3}\beta(t),
\end{aligned}$$

where we used $|Q|\gtrsim 1$ to remove $|Q|^{\frac{q-2}{6}}$ and other non-positive powers of $|Q|$ in the parentheses in the first inequality.

Next,

$$\begin{aligned}
\int_0^t\int p_{\mathrm{nonloc,H}}\, u\cdot\nabla\phi_Q &\lesssim |Q|^{-\frac13}\int_0^t \|u(s)\|_{L^1(Q^*)}\|p_{\mathrm{nonloc,H}}(s)\|_{L^\infty(Q^*)}\,ds \\
&\lesssim |Q|^{\frac q3-\frac56}\int_0^t \|u(s)\|_{L^2(Q^*)}\|u(s)\|^2_{M^{2,q}_{\mathcal{C}_n}}\,ds \lesssim |Q|^{\frac q2-\frac56}\int_0^t \alpha^{\frac32}(s)\,ds \\
&= |Q|^{\frac q3}|Q|^{\frac q6-\frac56}\|\alpha\|^{\frac32}_{L^{\frac32}(0,t)}.
\end{aligned} \tag{3.9}$$

Using the estimate for $p_{\mathrm{harm},\geq1}$ in (2.15) with $r=3/2$, $\delta=q/6$, and $\gamma=q/12$ we have

$$\begin{aligned}
\int_0^t\int p_{\mathrm{harm},\geq1}\, u\cdot\nabla\phi_Q &\lesssim |Q|^{-\frac13}\|u\|_{L^3((0,t);L^1(Q^*))}\|p_{\mathrm{harm},\geq1}\|_{L^{\frac32}((0,t);L^\infty(Q^*))} \\
&\lesssim |Q|^{\frac16}\|u\|_{L^3((0,t);L^2(Q^*))}\|p_{\mathrm{harm},\geq1}\|_{L^{\frac32}((0,t);L^\infty(Q^*))} \\
&\lesssim |Q|^{\frac16+\frac q6}\|\alpha\|^{\frac12}_{L^{\frac32}(0,t)}\|p_{\mathrm{harm},\geq1}\|_{L^{\frac32}((0,t);L^\infty(Q^*))} \\
&\lesssim |Q|^{\frac q2-\frac56}\|\alpha\|^{\frac12}_{L^{\frac32}(0,t)}\left(|Q|^{\frac{q}{18}}t^{\frac{q}{12}}\|\alpha\|^{\frac{6-q}{6}}_{L^{\frac{6-q}{4-q}}(0,t)}\beta(t)^{\frac q6} + \left(1+t^{\frac{q}{12}}|Q|^{-\frac{q}{18}} + t^{\frac12}|Q|^{-\frac13}\right)\|\alpha\|_{L^{3/2}(0,t)}\right) \\
&\lesssim |Q|^{\frac q2-\frac56}\|\alpha\|^{\frac12}_{L^3(0,t)}t^{\frac12}\left(|Q|^{\frac{q}{18}}t^{-\frac{1}{36}}\|\alpha\|^{\frac{6-q}{6}}_{L^3(0,t)}\beta(t)^{\frac q6} + \left(1+t^{\frac12}|Q|^{-\frac13}\right)\|\alpha\|_{L^3(0,t)}\right) \\
&\lesssim C_\epsilon|Q|^{\frac{-q^2+10q-15}{18-3q}}t^{\frac{18-q}{36-6q}}\|\alpha\|^{\frac{9-q}{6-q}}_{L^3(0,t)} + \epsilon|Q|^{\frac q3}\beta(t) + |Q|^{\frac{3q-5}{6}}t^{\frac12}\left(1+t^{\frac12}|Q|^{-\frac13}\right)\|\alpha\|^{\frac32}_{L^3(0,t)} \\
&\lesssim C_{\epsilon,q}|Q|^{\frac q3}(1+t)\left(\|\alpha\|^8_{L^3(0,t)} + \|\alpha\|^{\frac32}_{L^3(0,t)}\right) + \epsilon|Q|^{\frac q3}\beta(t),
\end{aligned}$$

and, from the last bound in (2.15),

$$\begin{aligned}\int_0^t \int p_{\mathrm{harm},\le 1}\, u\cdot\nabla\phi_Q &\lesssim |Q|^{-\frac{1}{3}} \int_0^t \|u(s)\|_{L^1(Q^*)} \|p_{\mathrm{harm},\le 1}(s)\|_{L^\infty(Q^*)}\, ds \\ &\lesssim |Q|^{\frac{q}{3}-\frac{13}{12}} \int_0^t \|u(s)\|_{L^2(Q^*)}\alpha(s)s^{\frac{3}{8}}\, ds \lesssim |Q|^{\frac{q}{2}-\frac{13}{12}} \int_0^t \alpha(s)^{\frac{3}{2}} s^{\frac{3}{8}}\, ds \\ &\lesssim |Q|^{\frac{q}{3}} |Q|^{\frac{q}{6}-\frac{13}{12}} t^{\frac{19}{16}} \|\alpha\|_{L^8(0,t)}^{\frac{3}{2}},\end{aligned} \tag{3.10}$$

and the proof is complete. □

The following lemma contains the necessary barrier statement needed for the a priori bound in Proposition 3.1.

Lemma 3.4. *Suppose that $f \in L^\infty_{\mathrm{loc}}([0,T_0);[0,\infty))$ satisfies*

$$f(t) \le a + b(1+t)^\gamma \left(\|f\|_{L^p(0,t)}^{\overline{p}} + \|f\|_{L^p(0,t)}^{\overline{q}} + c\|f\|_{L^p(0,t)} \right),$$

where $\overline{p},\overline{q} > 1$, $p \in [1,\infty)$, $\gamma \ge 0$ and $a,b,c > 0$. Then

$$f(t) \le 2a,$$

for $t \le \min\{T, T_0\}$, where $T > 0$ is the solution of

$$a = b(1+T)^\gamma \big((2a)^{\overline{p}} T^{\frac{\overline{p}}{p}} + (2a)^{\overline{q}} T^{\frac{\overline{q}}{p}} + 2acT^{\frac{1}{p}} \big). \tag{3.11}$$

Observe that $T \to \infty$ if $\max\{a,c\} \to 0$.

Proof of Lemma 3.4. By (3.11), the function $g(t) = 2a$ satisfies

$$g(t) \ge a + b(1+t)^\gamma \left(\|g\|_{L^p(0,t)}^{\overline{p}} + \|g\|_{L^p(0,t)}^{\overline{q}} + c\|g\|_{L^p(0,t)} \right),$$

for $t \in [0,T_1]$, where $T_1 = \min\{T,T_0\}$. The inequality $f(t) \le g(t)$ for $t \in [0,T_1]$ then follows by a standard barrier argument. □

Proof of Proposition 3.1. Let $Q \in \mathcal{C}_n$. Using the local energy inequality (1.4) with $\phi(x,t) = \phi_Q(x)\psi_m(t)$ where ψ_m is a suitable sequence of functions, and weak continuity in time, we obtain

$$\int |u(t)|^2\phi_Q + 2\int_0^t \int |\nabla u|^2 \phi_Q \le \int |u(0)|^2\phi_Q + \int_0^t \int |u|^2\Delta\phi_Q + \int_0^t \int (|u|^2 + p) u\cdot\nabla\phi_Q, \tag{3.12}$$

from where, using (3.4) and Lemma 3.3.

$$\begin{aligned}&\int |u(t)|^2\phi_Q + 2\int_0^t \int |\nabla u|^2\phi_Q \\ &\qquad \le C|Q|^{\frac{q}{3}}\alpha(0) + C_\epsilon |Q|^{\frac{q}{3}}(1+t)^\gamma \left(\|\alpha\|_{L^8(0,t)}^3 + \|\alpha\|_{L^8(0,t)}^{\frac{3}{2}} + |Q|^{-\frac{1}{6}}\|\alpha\|_{L^8(0,t)} \right) + \epsilon|Q|^{\frac{q}{3}}\beta(t),\end{aligned}$$

for all $t > 0$, where we used the restriction $q \le 2$ to write $|Q|^{\frac{2q}{3}-\frac{4}{3}} \lesssim 1$. Dividing by $|Q|^{\frac{q}{3}}$, taking $\sup_{Q\in\mathcal{C}_n}$, and absorbing the last term on the right-hand side, we obtain

$$\alpha(t) + \beta(t) \le C\alpha(0) + C(1+t)^\gamma \Big(\|\alpha\|_{L^8(0,t)}^3 + \|\alpha\|_{L^8(0,t)}^{\frac{3}{2}} + 2^{-\frac{n}{2}} \|\alpha\|_{L^8(0,t)} \Big), \tag{3.13}$$

where $\gamma \ge 0$ is a constant and we used the fact that $|Q| \gtrsim 2^{3n}$. The claim now follows from Lemma 3.4, applied with $f(s) = \alpha(s) + \beta(s)$, $a = C\alpha(0)$, $b = C$, $c = 2^{-\frac{n}{2}}$, $p = 8$, $\overline{p} = 3$, and $\overline{q} = 3/2$, where C is the constant in (3.13). Note that the definition (3.11) of T given by the lemma then becomes (3.2), as required. □

Note that, using the Gagliardo-Nirenberg inequality, we have

$$\frac{1}{|Q|}\int_0^{T_n} \int_Q |u|^3 \lesssim T_n^{\frac{1}{4}} |Q|^{\frac{q-2}{2}} \alpha(T_n)^{\frac{3}{4}} \beta(T_n)^{\frac{3}{4}} + |Q|^{\frac{q-3}{2}} \|\alpha\|_{L^{3/2}(0,T_n)}^{\frac{3}{2}}. \tag{3.14}$$

This inequality is used in Section 6 below.

4. Stability

In this section, we prove the stability theorem. In the following, we use the notation

$$\|(f,g)\|_X = 2\max\{\|f\|_X, \|g\|_X\}.$$

Proof of Theorem 1.5. Consider $n \in \mathbb{N}$. Since $u_0^{(k)} \to u_0$ in $M^{2,q}_{\mathcal{C}}$ and, using (1.7), we have that for any $\epsilon > 0$ there exists $K(n)$ so that

$$\|u_0^{(k)} - u_0\|_{M^{2,q}_{\mathcal{C}_n}} < \epsilon$$

for all $k \geq K(n)$. Let T_n be given by (3.2) for $\|u_0\|_{M^{2,q}_{\mathcal{C}_n}}$, and let $T_n^{(k)}$ be the same for $\|u_0^{(k)}\|_{M^{2,q}_{\mathcal{C}_n}}$. The above observation implies that, for sufficiently large k, we have $T_n^{(k)} \geq T_n/2$. Since we will ultimately pass to a subsequence, we ignore the finitely many terms not satisfying $T_n^{(k)} \geq T_n/2$. Hence, we can apply Theorem 1.4 to conclude that $u^{(k)}$ are uniformly bounded in $L^\infty(0,T_n/2;L^2(B_n \cap \mathbb{R}^3_+)) \cap L^2(0,T_n/2;H^1(B_n \cap \mathbb{R}^3_+))$, where $\{B_n\}_{n\geq 1}$ is a sequence of expanding balls (e.g. $B_n = B(0,n)$). Note that T_n is an unbounded, non-decreasing sequence.

We need to show that $\partial_t u^{(k)}$ are uniformly bounded in the dual space of $L^5(0,T_n/2;W_0^{1,3}(B_n \cap \mathbb{R}^3_+))$. Let Q be a cube containing $B_n \cap \mathbb{R}^3_+$. Based on local estimates for $u^{(k)}$ and pressure estimates in Section 2, this follows nearly identically to [MMP1, p. 561]. The only added work here involves our treatment of $p^{(k)}_{\mathrm{loc,harm}}$, for which we have

$$\begin{aligned}\left|\int_0^{T_n/2}\int_{B_n} p^{(k)}_{\mathrm{loc,harm}}\nabla\cdot\phi\right| &\leq \left(\int_0^{T_n/2}\left(\int_Q |p^{(k)}_{\mathrm{loc,harm}}|^{17/10}\,dx\right)^{\frac{10}{17}\frac{3}{2}} dt\right)^{2/3} \\ &\qquad \cdot \left(\int_0^{T_n/2}\left(\int_Q |\nabla\phi|^{\frac{17}{7}}\,dx\right)^{\frac{7}{17}\cdot 3} dt\right)^{\frac{1}{3}} \\ &\lesssim_{Q,T_n} \left(\int_0^{T_n/2}\left(\int_Q |p^{(k)}_{\mathrm{loc,harm}}|^{17/10}\,dx\right)^{\frac{10}{17}\frac{3}{2}} dt\right)^{2/3} \|\nabla\phi\|_{L^5(0,T_n/2;L^3(B_n\cap\mathbb{R}^3_+))},\end{aligned}$$

where we have used Hölder's inequality in both space and time variables; here $\phi \in C_0^\infty(B_n)^3$. The terms involving $p^{(k)}_{\mathrm{loc,harm}}$ are uniformly bounded by Lemma 2.4. Hence, following [MMP1, p. 561], we have obtained the claimed uniform bound on $\partial_t u^{(k)}$.

By repeatedly applying the Lions-Aubin lemma and using a diagonalization argument, we obtain that there exists $u\colon \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}^3$ such that, for every $n \in \mathbb{N}$,

$$\begin{aligned} u^{(k)} &\to u \quad \text{weakly-}* \text{ in } L^\infty(0,T_n/2;L^2(B_n\cap\mathbb{R}^3_+)),\\ u^{(k)} &\to u \quad \text{weakly in } L^2(0,T_n/2;H^1(B_n\cap\mathbb{R}^3_+)),\\ u^{(k)} &\to u \quad \text{strongly in } L^2(0,T_n/2;L^2(B_n\cap\mathbb{R}^3_+)),\end{aligned}$$

after passing to a subsequence of $\{u^{(k)}\}$. By interpolation, the strong convergence can be extended to

$$u^{(k)} \to u \text{ strongly in } L^r(0,T_n/2;L^p(B_n\cap\mathbb{R}^3_+)), \tag{4.1}$$

for any $p,r > 2$ such that $2/r + 3/p < 3/2$ and $p < 6$. By (3.14), the convergence in $L^3(0,T_n/2;L^3(B_n \cap \mathbb{R}^3_+))$ implies

$$\sup_{Q\in\mathcal{C}_n}\frac{1}{|Q|^{\frac{q}{2}}}\int_0^{T_n/2}\int_Q |u|^3 \leq CT_n^{\frac{1}{4}}\|u_0\|^3_{M^{2,q}_{\mathcal{C}_n}} + \frac{CT_n}{2^{3n/2}}\|u_0\|^2_{M^{2,q}_{\mathcal{C}_n}},$$

since this estimate is satisfied by all $u^{(k)}$ for sufficiently large k.

Note that, since $T_n \to \infty$ as $n \to \infty$, the convergence properties listed above on $B_n \cap \mathbb{R}^3_+ \times (0,T_n/2)$ extend to any cube $Q \subset \mathbb{R}^3_+$ and $T > 0$. We next show that for any $n \in \mathbb{N}$ and $T > 0$, letting α_n and β_n be as in (1.8) for u, we have $\alpha_n(T) + \beta_n(T) < \infty$. By our remark about convergence in $Q \times (0,T)$, for any $Q \in \mathcal{C}_n$,

$$\sup_{0<s<T}\frac{1}{|Q|^{q/3}}\|u(s)\|^2_{L^2(Q)} \leq \limsup_{k\to\infty}\sup_{0<s<T}\frac{1}{|Q|^{q/3}}\|u^{(k)}(s)\|^2_{L^2(Q)}, \tag{4.2}$$

which is uniformly bounded in k by the bounds for $u^{(k)}$ in $M^{2,q}_{\mathcal{C}}$. For the gradient terms, the weak convergence implies

$$\frac{1}{|Q|^{q/3}}\int_0^T\int_Q |\nabla u|^2 \leq \liminf_{k\to\infty}\frac{1}{|Q|^{q/3}}\int_0^T\int_Q |\nabla u^{(k)}|^2, \tag{4.3}$$

which is again uniformly bounded. We now take a supremum over $Q \in \mathcal{C}_n$ to obtain the boundedness of $\alpha_n(T)$ and $\beta_n(T)$.

Given a bounded open set $\Omega \subset \mathbb{R}^3_+$ we define p via the local pressure expansion,

$$p := p_{\text{li,loc}} + p_{\text{li,nonloc}} + p_{\text{loc,H}} + p_{\text{loc,harm}} + p_{\text{nonloc,H}} + p_{\text{harm},\leq 1} + p_{\text{harm},\geq 1},$$

where we use the formulas (2.4), (2.7), (2.11), (2.12), and (2.13) for each of the pressure parts. We similarly define $p^{(k)}$ as the local pressure expansion of each $u^{(k)}$, where $k \geq 1$, and we set

$$\bar{p}^{(k)} = p^{(k)} - p.$$

We also use analogous notation for the differences between each part of the local pressure expansion. We now claim that, for every compact set $K \subset \overline{\mathbb{R}^3_+}$,

- the following parts converge to zero in $L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(K))$: $\bar{p}^{(k)}_{\text{loc,H}}$, $\bar{p}^{(k)}_{\text{nonloc,H}}$, $\bar{p}^{(k)}_{\text{harm},\geq 1}$, $\bar{p}^{(k)}_{\text{harm},\leq 1}$,
- the following parts converge to zero in $L^p(0,T;L^2(K))$ for $p < \frac{4}{3}$: $\bar{p}^{(k)}_{\text{li,loc}}$, $\bar{p}^{(k)}_{\text{li,nonloc}}$.
- the sequence $\bar{p}^{(k)}_{\text{loc,harm}}$ is bounded in $L^p(0,T;L^p(K))$ for every $p < 3/2$ and that $\|\bar{p}^{(k)}_{\text{loc,harm}}\|_{L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(\tilde{K}))} \to 0$ as $k \to \infty$ for every $\tilde{K} \Subset \mathbb{R}^3_+$ (i.e., locally away from the boundary).

These convergence properties guarantee that (u,p) satisfies the Navier-Stokes equations (1.1) in the sense of distributions on $\mathbb{R}^3_+$ as well as the local energy inequality (1.4) for non-negative test functions $\phi \in C_c^\infty(\overline{\mathbb{R}^3_+} \times [0,\infty))$. Indeed, using the above convergence modes (of $u^{(k)}$ and of each part of the pressure function) the only nontrivial convergence is

$$\int_0^t \int p^{(k)}_{\text{loc,harm}} u^{(k)} \cdot \nabla\phi \to \int_0^t \int p_{\text{loc,harm}} u \cdot \nabla\phi,$$

for each $t > 0$ and each nonnegative $\phi \in C_c^\infty(\overline{\mathbb{R}^3_+} \times [0,\infty))$. For this, let $K \Subset \overline{\mathbb{R}^3_+}$ be such that $\operatorname{supp}\phi(s) \subset K$ for all s, fix $\epsilon > 0$, and let $\tilde{K} \Subset \mathbb{R}^3_+$ be such that

$$\|u\|_{L^{\frac{19}{6}}((K\setminus\tilde{K})\times(0,t))} \leq \epsilon \left(2\|(p^{(k)}_{\text{loc,harm}}, p_{\text{loc,harm}})\|_{L^{\frac{19}{13}}(K\times(0,t))} \right)^{-1}.$$

Then

$$\begin{aligned}
\int_0^t \int_K \left| p^{(k)}u^{(k)} - pu \right| &\leq \int_0^t \int_K \left| u(p^{(k)} - p) \right| + \int_0^t \int_K \left| (u^{(k)} - u)p^{(k)} \right| \\
&\leq \|u\|_{L^{\frac{19}{6}}((K\setminus\tilde{K})\times(0,t))} \|(p^{(k)},p)\|_{L^{\frac{19}{13}}} + \|u\|_{L^{19/6}} \|p^{(k)} - p\|_{L^{\frac{19}{13}}(\tilde{K}\times(0,t))} + \|u^{(k)} - u\|_{L^{\frac{19}{6}}} \|p^{(k)}\|_{L^{\frac{19}{13}}} \\
&\leq \frac{\epsilon}{2} + C\left(\|p^{(k)} - p\|_{L^{\frac{19}{13}}(\tilde{K}\times(0,t))} + \|u^{(k)} - u\|_{L^{\frac{19}{6}}} \right) \\
&\leq \epsilon
\end{aligned}$$

for sufficiently large k, as required, where for brevity we omitted the notation "loc,harm" and we set $L^q = L^q(K \times (0,t))$.

Moreover, the local pressure expansion for u (recall Definition 1.1(6)) then follows for every open and bounded $\Omega \subset \mathbb{R}^3_+$ by the uniqueness argument as in (2.14). The remaining properties of local energy solutions (i.e., that $u(t) \to u_0$ in L^2_{loc} and that $u(t)$ is weakly continuous in L^2_{loc}) can be proven using well-known techniques, see e.g. [KS, KwT]. Finally, since we have shown (u,p) satisfies the assumptions of Theorem 1.4, the asserted a priori bounds now follows by applying Theorem 1.4 to (u,p).

We now fix $K \Subset \mathbb{R}^3_+$ and prove the convergence properties for the pressure listed above.

For $p_{\text{li,loc}}$ and $p_{\text{li,nonloc}}$, we have

$$\|\bar{p}^{(k)}_{\text{li,loc}}(t)\|_{L^2(K)} \lesssim_{K,Q} \|u_0^{(k)} - u_0\|_{L^2(Q^*)} t^{-\frac{3}{4}}$$

and

$$\|\bar{p}^{(k)}_{\text{li,nonloc}}(t)\|_{L^\infty(K)} \lesssim_{K,Q} \|u_0^{(k)} - u_0\|_{M^{2,2}_{\mathcal{C}}} t^{-\frac{3}{4}},$$

as in Lemmas 2.1 and 2.2. Since $u_0^{(k)} \to u_0 \in M^{2,q}_{\mathcal{C}}$, it follows that $p^{(k)}_{\text{li}} \to p_{\text{li}} \in L^p(0,T;L^2(Q))$ for every $T < \infty$ and $p < \frac{4}{3}$.

The local part of the pressure is expanded into the harmonic and the Helmholtz parts. For the Helmholtz part, we have

$$\int_0^T \|\bar{p}^{(k)}_{\mathrm{loc,H}}(t)\|^{\frac32}_{L^{\frac32}(\mathbb{R}^3_+)}\,dt \lesssim \int_0^T \|(u^{(k)}\otimes u^{(k)} - u\otimes u)(t)\|^{\frac32}_{L^{\frac32}(Q^{***})}\,dt$$
$$\lesssim \|(u,u^{(k)})\|^{\frac32}_{L^3(0,T;L^3(Q^{***}))}\|u^{(k)}-u\|^{\frac32}_{L^3(0,T;L^3(Q^{***}))},$$

where the first inequality follows as in Lemma 2.3. The right-hand side vanishes as $k\to\infty$ for every $T<\infty$ by (4.1). Thus $p^{(k)}_{\mathrm{loc,H}} \to p_{\mathrm{loc,H}} \in L^{3/2}(0,T;L^{3/2}(Q))$.

We next treat $\overline{p}^{(k)}_{\mathrm{loc,harm}}$. For other pressure components, we are able to refer heavily to the work in Section 2. This term requires a different approach. We have

$$p_{\mathrm{loc,harm}}(x,t) = p_A + p_B := \frac{1}{2\pi i}\int_0^t\int_\Gamma \mathrm{e}^{(t-s)\lambda}\int_{\mathbb{R}^3_+} q_\lambda(x'-z',x_3,z_3)\cdot(F_A(z,s)+F_B(z,s))\,dz\,d\lambda\,ds,$$

where, recalling (2.11), F_A is a 2D vector function whose components are sums of terms of the form $\partial_j\,(\chi_{**}u_ku_l) =: \partial_j f_A$, where $j,k,l\in\{1,2,3\}$, and $F_B(z,s)=F_B(z',z_3,s)$ is a sum of 2D vectors of the form

$$\mathbf{m}(D')\nabla'\otimes\nabla'\int_0^\infty \left((P(\cdot,|z_3-y_3|)+P(\cdot,z_3+y_3))*(\chi_{**}v\otimes w)(y_3)\right)(z',s)\,dy_3$$
$$=: \nabla'\otimes f_B(z,s),$$

where v and w denotes various 2D vectors whose components are chosen among u_1, u_2, or u_3; also, $\mathbf{m}(D')$ denotes a multiplier in the horizontal variable z' that is homogeneous of degree 0, and $\widehat{P}(\xi',t) := \mathrm{e}^{-t|\xi'|}$, i.e., P is the 2D Poisson kernel. Thus, using $\|\mathbf{m}(D')P(\cdot,s)\|_{L^1(\mathbb{R}^2)}\lesssim 1$, which is a consequence of (2.26) and $|y_3|\lesssim_Q 1$, we obtain

$$\|f_B(z',z_3,s)\|_{L^p_{z'}(\mathbb{R}^2)} \lesssim_Q \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^p(\mathbb{R}^3_+)}$$

for every $z_3>0$, $s>0$, $p\ge 1$. Thus, by (2.5), we get

$$\begin{aligned}
\|p_{\mathrm{B}}(t)\|_{L^p(\mathbb{R}^3_+)} &\lesssim \int_0^t\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\left\|\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}\left\|\int_{\mathbb{R}^2}(|x'-z'|+z_3+x_3)^{-3}f_B(z,s)\,dz'\right\|_{L^p_{x'}(\mathbb{R}^2)}dz_3\right\|_{L^p_{x_3}} d|\lambda|\,ds\\
&\lesssim \int_0^t \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^p(\mathbb{R}^3_+)}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}\left\|(x_3+z_3)^{-1}\right\|_{L^p_{x_3}(0,\infty)}dz_3\,d|\lambda|\,ds\\
&\lesssim \int_0^t \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^p(\mathbb{R}^3_+)}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}z_3^{-1+\frac1p}\,dz_3\,d|\lambda|\,ds\\
&\lesssim \int_0^t \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^p(\mathbb{R}^3_+)}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}|\lambda|^{-\frac{1}{2p}}\,d|\lambda|\,ds\\
&\lesssim \int_0^t \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^p(\mathbb{R}^3_+)}(t-s)^{\frac{1-2p}{2p}}\,ds,
\end{aligned} \tag{4.4}$$

where we used $\int_{\mathbb{R}^2}(|y'|+a)^{-3}\,dy'\lesssim a^{-1}$ and $\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}v}v^b\,dv\lesssim|\lambda|^{-\frac{1+b}{2}}$. Thus

$$\|p_{\mathrm{B}}\|_{L^p((\mathbb{R}^3_+)\times(0,T))}\lesssim \|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^1((0,T);L^p(\mathbb{R}^3_+))}T^{-1+\frac{3}{2p}}.$$

Note that we have

$$\|\nabla'(\chi_{**}v\otimes w)(s)\|_{L^1((0,T);L^p(\mathbb{R}^3_+))} \lesssim_Q \int_0^T \|u(s)\|_{L^{\frac{2p}{2-p}}}\|\nabla u(s)\|_{L^2(Q^{***})}\,ds + \|u\|^2_{L^2((0,T);L^{2p}(Q^{***}))}$$
$$\lesssim \|u\|_{L^2((0,T);L^{\frac{2p}{2-p}}(Q^{***}))}\|\nabla u\|_{L^2((0,T);L^2(Q^{***}))} + \|u\|^2_{L^2((0,T);L^{2p}(Q^{***}))},$$

which is bounded due to (4.1) for $p<3/2$. Thus both $p_{\rm B}$ and $p_{\rm B}^{(k)}$, $k\geq 1$, are bounded in $L^p(0,T;L^p(K))$, for every $p<3/2$. On the other hand, on a set $\tilde K = K'\times K_3$ compactly embedded in $\mathbb{R}^3_+$,

$$\begin{aligned}\|\overline{p}_B^{(k)}(\cdot ,x_3,t)\|_{L^{\frac32}_{x'}(K')} &\lesssim \int_0^t \||u^{(k)}(s)|^2-|u(s)|^2\|_{L^{\frac32}(Q^{***})}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}(x_3+z_3)^{-2}\,dz_3\;d|\lambda|\;ds\\ &\lesssim_K \int_0^t \||u^{(k)}(s)|^2-|u(s)|^2\|_{L^{\frac32}(Q^{***})}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}|\lambda|^{-\frac12}\;d|\lambda|\;ds\\ &\lesssim \int_0^t \||u^{(k)}(s)|^2-|u(s)|^2\|_{L^{\frac32}(Q^{***})}(t-s)^{-\frac12}\;ds\end{aligned}$$

for every $x_3\in K_3$ and $t>0$, where the first inequality follows in the same way as the first two inequalities in (4.4), except that we now put both derivatives from $\nabla'\otimes\nabla'$ onto q_λ (rather than one onto q_λ and one onto $(\chi_{**}v\otimes w)$), and the second inequality follows simply by bounding $x_3+z_3\gtrsim_K 1$. Thus

$$\|\overline{p}_B^{(k)}\|_{L^{\frac32}((0,T);L^{\frac32}(\tilde K))}\lesssim_K T^{\frac12}\||u^{(k)}(s)|^2-|u(s)|^2\|_{L^{\frac32}((0,T);L^{\frac32}(Q^{***}))}$$

for every $T>0$, which vanishes in the limit $k\to\infty$ due to (4.1).

For $\overline{p}_A^{(k)}$, we have that $f_A^{(k)}$ converges to f_A in $L^{\frac32}((0,T);L^{\frac32}(\mathbb{R}^3_+)$ by (4.1); recall that f_A is a sum of the term of the form $\chi_{**}u_lu_m$, $l,m=1,2,3$, and $f_A^{(k)}$ is defined analogously. For every $t\in(0,T)$, $T>0$, and every $q\in(3/2,5/3)$,

$$\begin{aligned}\|p_A(t)\|_{L^{\frac32}(K)} &\lesssim \int_0^t\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\left\|\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}\left\|\int_{\mathbb{R}^2}(|x'-z'|+z_3+x_3)^{-3}f_A(z,s)\,dz'\right\|_{L^{\frac32}_{x'}(\mathbb{R}^2)}dz_3\right\|_{L^{\frac32}_{x_3}}d|\lambda|\;ds\\ &\lesssim_{K,Q}\int_0^t\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\left\|\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}(x_3+z_3)^{-1}\,\|f_A(\cdot,z_3,s)\|_{L^{\frac32}_{z'}(\mathbb{R}^2)}\;dz_3\right\|_{L^{\frac32}_{x_3}(K_3)}d|\lambda|\;ds\\ &\lesssim_K\int_0^t\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}\int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12}z_3}z_3^{-\frac13}\,\|f_A(\cdot,z_3,s)\|_{L^{\frac32}_{z'}(\mathbb{R}^2)}\;dz_3\,d|\lambda|\;ds\\ &\lesssim_{Q,q}\int_0^t\|f_A(s)\|_{L^q(\mathbb{R}^3_+)}\int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda}|\lambda|^{\frac16-\frac1{2q'}}\;d|\lambda|\;ds\\ &\lesssim\int_0^t\|f_A(s)\|_{L^q(\mathbb{R}^3_+)}\,(t-s)^{\frac1{2q'}-\frac76}\;ds,\end{aligned}$$

where $q'\in(5/2,3)$ is the conjugate exponent to q. Therefore, we have the required estimate

$$\|p_A\|_{L^{\frac32}((0,T);L^{\frac32}(K))}\lesssim_{K,Q,q} T^{\frac1{2q'}-\frac16}\|f_A\|_{L^q((0,T);L^q(Q^{***}))}$$

for any $q\in(3/2,5/3)$ (we can choose any such q), and similarly for $p_A^{(k)}$. By replacing f_A by $f_A^{(k)}-f_A$ we also obtain the convergence $\|\overline{p}_A^{(k)}\|_{L^{\frac32}((0,T);L^{\frac32}(K))}\to 0$, as required.

The nonlocal components are $\overline{p}^{(k)}_{\rm nonloc,H}$, $\overline{p}^{(k)}_{{\rm harm},\geq 1}$, and $\overline{p}^{(k)}_{{\rm harm},\leq 1}$. The first of these is similar to the case of $\mathbb{R}^3$ in [BK, BKT], but we include the details to illustrate the main approach. We set

$$A_T=\sup_{t\in(0,T)}\sup_k\|(u(t),\,u^{(k)}(t))\|^2_{M^{2,2}_{\mathcal C}},\qquad B_T=\sup_k\sup_{Q\in\mathcal C}|Q|^{-\frac23}\int_0^T\int_Q\left(|\nabla u^{(k)}|^2+|\nabla u|^2\right).$$

Note that although the statement of Theorem 1.5 is for $M^{2,q}_{\mathcal C}$, due to $M^{2,q}_{\mathcal C}\subset M^{2,2}_{\mathcal C}$, we also have uniform bounds for A_T and B_T.

Recalling the details of the proof of Lemma 2.5, we have

$$\begin{aligned}|\overline{p}^{(k)}_{\rm nonloc,H}(x,t)| &\lesssim_K \sum_{l>M}\sum_{\tilde Q\in S_l}\int_{\tilde Q}\frac{|u^{(k)}(z)\otimes u^{(k)}(z)-u(z)\otimes u(z)|}{|x-z|^4}\;dz+\int_{Q_M}|u^{(k)}\otimes u^{(k)}-u\otimes u|\,dz\\ &\lesssim A_T\sum_{l>M}2^{-2l}+\int_{Q_M}|u^{(k)}\otimes u^{(k)}-u\otimes u|\;dz\end{aligned}$$

for every $x \in K$, where $M \in \mathbb{N}$ is such that $2^M \gg \mathrm{dist}(K,0)$, and $Q_M = \bigcup_{k\le M} \bigcup_{\tilde{Q}\in S_k} \tilde{Q}$. Since the series above can be estimated by 2^{-2M}, we can choose M sufficiently large so that

$$\int_0^t \|\bar{p}^{(k)}_{\mathrm{nonloc,H}}(s)\|_{L^\infty(K)}\, ds \le \frac{\epsilon}{2} + C_K \int_0^t \int_{Q_M(0)} |u^{(k)} \otimes u^{(k)} - u\otimes u|\; dz\; ds \le \epsilon$$

for any preassigned ϵ, where we have taken large k in the last inequality. Due to the uniform bounds of both $p^{(k)}_{\mathrm{nonloc,H}}$ and $p_{\mathrm{nonloc,H}}$ in $L^\infty(0,T;L^\infty(K))$, due to Lemma 2.5, we obtain the required convergence $\bar{p}^{(k)}_{\mathrm{nonloc,H}} \to 0$ in $L^{3/2}(0,T;L^{3/2}(K))$.

For $\overline{p}^{(k)}_{\mathrm{harm},\ge 1}$, we write

$$|\overline{p}^{(k)}_{\mathrm{harm},\ge 1}| \le \overline{p}^{(k)}_{\mathrm{A1}} + \overline{p}^{(k)}_{\mathrm{A2}} + \overline{p}^{(k)}_{\mathrm{A3}} + |\overline{p}^{(k)}_{\mathrm{B}}|,$$

as in the proof of Lemma 2.6, except that now we apply a cutoff at $z_3 = 1$, instead of $z_3 = 2^{\underline{m}}$ in the z_3 integral (recall (2.20)). For example, for $\overline{p}^{(k)}_{\mathrm{A1}}$ we have for $x \in K$ that

$$\begin{aligned} &|\bar{p}^{(k)}_{\mathrm{A1}}(x,t)| \\ &\quad \lesssim_K \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} \int_1^\infty \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4} (1-\chi_*)|u^{(k)}\otimes u^{(k)} - u\otimes u|(z)\, dz'\, dz_3\, d|\lambda|\, ds, \end{aligned} \tag{4.5}$$

where ξ belongs to the line segment between x and x_K, which is a fixed point inside K. Denote by I the double integral in (4.5) with respect to the variables z' and z_3. As in Lemma 2.6 (recall the calculation below (2.21)), we have $\mathrm{e}^{-|\lambda|^{\frac12} z_3} \lesssim |\lambda|^{-\frac12}$, which gives

$$\begin{aligned} I &\lesssim_K |\lambda|^{-\frac12} \sum_{l>M} \sum_{\tilde{Q}\in S_l} 2^{-4l} \int_{\tilde{Q}} (|u|^2 + |u^{(k)}|^2) + |\lambda|^{-\frac12} \int_{Q_M} (1-\chi_*(z)) \frac{|u^{(k)}\otimes u^{(k)} - u\otimes u|(z)}{(|\xi'-z'|+\xi_3+z_3)^4}\, dz \\ &\lesssim_K |\lambda|^{-\frac12} A_T \sum_{l>M} 2^{-2l} + |\lambda|^{-\frac12} \|u^{(k)}\otimes u^{(k)} - u\otimes u\|_{L^1(Q_M)} \\ &\lesssim |\lambda|^{-\frac12} A_T 2^{-2M} + |\lambda|^{-\frac12} \||u^{(k)}\otimes u^{(k)} - u\otimes u|\|_{L^1(Q_M)} \end{aligned}$$

for large M, where we used (2.1) in the first inequality. Inserting this into the above integral in λ and s leads to

$$\|\bar{p}^{(k)}_{\mathrm{A1}}(t)\|_{L^\infty(K)} \lesssim_K t^{\frac12} A_T 2^{-2M} + \int_0^t (t-s)^{-\frac12} \|u^{(k)} - u\|_{L^2(Q_M)} \|(u,u^{(k)})\|_{L^2(Q_M)}\, ds.$$

Thus, applying Young's inequality for the convolution in time we obtain

$$\|\bar{p}^{(k)}_{\mathrm{A1}}\|_{L^2((0,T);L^\infty(K))} \lesssim_{K,T} A_T 2^{-2M} + \sup_{0<s<T} \|(u(s),u^{(k)}(s))\|_{L^2(Q_M)} \|u^{(k)} - u\|_{L^2(Q_M\times(0,T))},$$

which converges to 0 (by first choosing large M and then large k).

For $\bar{p}^{(k)}_{\mathrm{A2}}$, we have, similarly to (2.23),

$$|\bar{p}^{(k)}_{\mathrm{A2}}(x,t)| \lesssim_K \int_0^t (t-s)^{-\frac{9}{10}} \int_{\mathbb{R}^2} \int_0^1 \frac{z_3^{-\frac15}}{(|\xi'-z'|+\xi_3+z_3)^4} (1-\chi_*)|u^{(k)}\otimes u^{(k)}(z) - u\otimes u(z)|\, dz_3\; dz'\; ds,$$

and, similarly to (2.25) considered in the case $a = \frac{10}{9}$, $a' = 10$, $\delta = \frac12$, $b = 3$, $r = \frac32$, we have, for every $\tilde{Q} \in S_m$,

$$\begin{aligned} &\int_0^1 \int_{\tilde{Q}} |u^{(k)}\otimes u^{(k)} - u\otimes u| z_3^{-\frac15}\, dz'\, dz_3 \lesssim \|u^{(k)}\otimes u^{(k)} - u\otimes u\|_{L^{\frac32}(\tilde{Q})} \\ &\quad \lesssim 2^{\frac23 m} \big(\|(u,\, u^{(k)})\|_{L^2(\tilde{Q})} \|(\nabla u,\, \nabla u^{(k)})\|_{L^2(\tilde{Q})} + 2^{-m} \|(u,\, u^{(k)})\|^2_{L^2(\tilde{Q})} \big) \lesssim 2^{\frac83 m} \left((A_T B_T)^{\frac12} + A_T \right). \end{aligned}$$

Using Young's inequality for the convolution in time, we thus obtain

$$\|\bar{p}^{(k)}_{\mathrm{A2}}\|_{L^{\frac32}((0,T);L^\infty(K))} \lesssim_{K,T} \left((A_T B_T)^{\frac12} + A_T \right) \sum_{m>M} 2^{-\frac43 m} + C_M \|u^{(k)}\otimes u^{(k)} - u\otimes u\|_{L^{\frac32}((0,T);L^{\frac32}(Q_M))},$$

where, for the first term, we used $(|\xi'-z'|+\xi_3+z_3) \gtrsim 2^m$ for $z \in \tilde{Q} \in S_m$ for $m \ge M$ (recall (2.1)); for the second term, we used the first step in the inequality above. We now choose M sufficiently large, and then a large k to obtain the required convergence.

For $\bar{p}^{(k)}_{\rm A3}$ we have, using (2.21),

$$\begin{aligned}|\bar{p}^{(k)}_{\rm A3}(x,t)| &\lesssim_K \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} \int_{\mathrm{supp}\ \nabla\chi_*} \frac{\mathrm{e}^{-|\lambda|^{\frac12} z_3}}{(|\xi'-z'|+\xi_3+z_3)^4} |u^{(k)}\otimes u^{(k)}(z,s) - u\otimes u(z,s)|\, dz\, d|\lambda|\, ds\\ &\lesssim_K \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} |\lambda|^{-\frac16}\, d|\lambda| \|u^{(k)}\otimes u^{(k)}(s) - u\otimes u(s)\|_{L^{\frac32}(Q^{**})}\, ds\\ &\lesssim \int_0^t (t-s)^{-\frac56} \|u^{(k)}\otimes u^{(k)}(s) - u\otimes u(s)\|_{L^{\frac32}(Q^{**})}\, ds,\end{aligned}$$

which gives $\|\bar{p}^{(k)}_{\rm A3}\|_{L^{\frac32}((0,T);L^{\frac32}(K))} \to 0$ by applying Young's inequality for the convolution in time.

As for $\bar{p}^{(k)}_B$, recalling (2.27), we write

$$|\bar{p}^{(k)}_B(x,t)| \lesssim \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} \int_{\mathbb{R}^3_+} \left| q_{\lambda,x,x_K}(z)(1-\chi_*)\bar{F}^{(k)}_B(z,s)\right|\, dz\ d|\lambda|\, ds,$$

where $\bar{F}^{(k)}_B$ (defined in the same way as F_B in Step 2 of the proof of Lemma 2.6, but with $u\otimes u$ replaced by $u^{(k)}\otimes u^{(k)} - u\otimes u$), can be estimated by

$$|\bar{F}^{(k)}_B(z)| \lesssim_Q 2^{-M} A_T + C_M \|u^{(k)}\otimes u^{(k)} - u\otimes u\|_{L^1(Q_M)} \tag{4.6}$$

for any fixed $Q\subset \mathbb{R}^3_+$, and $z\in Q$, where $M>0$ large enough so that $Q\subset Q_{M/2}$. This can be obtained by an easy modification of Step 2 of the proof of Lemma 2.6 by separating the integration region into Q_M and the rest, as above. We then obtain

$$\begin{aligned}|\bar{p}^{(k)}_B(x,t)| &\lesssim_Q \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} |\lambda|^{-\frac12} \left(\sum_{l>L} \sum_{\tilde{Q}\in S_l} 2^{-2l} \|\bar{F}^{(k)}_B(s)\|_{L^\infty(\tilde{Q})} + C_L \|\bar{F}^{(k)}_B(s)\|_{L^\infty(Q_L)} \right) d|\lambda|\, ds,\\ &\lesssim \int_0^t (t-s)^{-\frac12} \left(A_T \sum_{l>L} 2^{-4l} + C_L A_T 2^{-M} + C_L C_M \|u^{(k)}\otimes u^{(k)}(s) - u\otimes u(s)\|_{L^1(Q_M)} \right) ds,\end{aligned}$$

where, in the second line, we have used the estimate $\|\bar{F}^{(k)}_B(s)\|_{L^\infty(\tilde{Q})} \lesssim |\tilde{Q}|^{-\frac23} A_T$ (from Step 2 of the proof of Lemma 2.6) in the summation, and have assumed that $M>2L$ in order to use (4.6). This gives

$$\|\bar{p}^{(k)}_{\rm B}\|_{L^{\frac32}((0,T);L^\infty(K))} \lesssim_{K,T} A_T 2^{-4L} + C_L A_T 2^{-M} + C_L C_M \|u^{(k)}\otimes u^{(k)} - u\otimes u\|_{L^{\frac32}((0,T);L^{\frac32}(Q_M))},$$

which provides the required convergence by first choosing large L, then M and k.

Finally, for the remaining component of the nonlocal pressure, $\bar{p}^{(k)}_{\mathrm{harm},\le 1}$, we have

$$\begin{aligned}\left|\bar{p}^{(k)}_{\mathrm{harm},\le 1}(x,t)\right| &= \frac{1}{2\pi} \left| \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\,\mathrm{Re}\,\lambda} \int_{\mathbb{R}^3_+} q_\lambda(x'-z',x_3,z_3)\chi_* \bar{F}^{(k)}_B(z,s)'\, dz\ d\lambda\, ds \right|\\ &\lesssim_K \int_0^t \int_\Gamma \mathrm{e}^{(t-s)\mathbb{R}\mathrm{e}\,\lambda} \|\bar{F}^{(k)}_B(s)\|_{L^\infty(Q^*)} \int_{Q^*} |q_\lambda(x'-z',x_3,z_3)|\ dz\ d|\lambda|\, ds\end{aligned}$$

for every $x\in K$. Thus noting that $\int_{Q^*} |q_\lambda(x'-z',x_3,z_3)|\ dz \lesssim_{Q^*} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac12} z_3} z_3^{-\frac14}\, dz_3 \lesssim |\lambda|^{-\frac38}$ (recall (2.5) and (2.18)) and using (4.6) gives

$$\left|\bar{p}^{(k)}_{\mathrm{harm},\le 1}(x,t)\right| \lesssim_K \int_0^t (t-s)^{-\frac58} \left(A_T 2^{-M} + C_M \|u^{(k)}\otimes u^{(k)}(s) - u\otimes u(s)\|_{L^1(Q_M)} \right) ds,$$

where M is chosen in analogy to prior cases. This implies the convergence $\|\bar{p}^{(k)}_{\mathrm{harm},\le 1}\|_{L^{\frac32}((0,T);L^\infty(K))} \to 0$, as required. □

5. EXISTENCE

In this section we apply the stability result of Section 4 to obtain the existence of a global weak solution when $u_0 \in \mathring{M}^{2,2}_{\mathcal{C}}$ and of scaling invariant solutions when u_0 is additionally scaling invariant. Note that $u_0 \in \mathring{M}^{2,2}_{\mathcal{C}}$ implies $u_{0,3}|_{x_3=0} = 0$.

We start with the general case.

Proof of Theorem 1.3. Let $u_0^{(k)} \in C_0^\infty(\overline{\mathbb{R}^3_+})$ be divergence-free, satisfy $u_{0,3}^{(k)}|_{x_3=0} = 0$ and be such that $u_0^{(k)} \to u_0 \in M^{2,2}_{\mathcal{C}}$. By the Leray-Hopf theory, we obtain global-in-time finite energy solutions $u^{(k)}$ and pressure $p^{(k)}$ in the Leray-Hopf class satisfying the local energy inequality (see [Tsai2, Chapter 3] for an exposition on the Leray-Hopf weak solutions on $\mathbb{R}^3_+$; the local energy inequality is not included for $\mathbb{R}^3_+$ in [Tsai2] but follows as a consequence of the construction by adapting ideas from [MMP1].) Since these solutions satisfy the global energy inequality, $\alpha_k(t) + \beta_k(t) < \infty$ for every $t > 0$, where α_k and β_k are the quantities corresponding to (1.8) for $u^{(k)}$. These solutions are also local Leray solutions in the sense of [MMP1], and, following [MMP1, Proposition 3.1], satisfy the local pressure expansion tailored to $\mathcal{C}$ and $\mathcal{C}_n$. Hence, these solutions satisfy the a priori bounds in Theorem 1.4 for $q = 2$. The asserted global solution exists due to Theorem 1.5. □

We now address Theorem 1.6. For our foundation, we use the scaling invariant solutions of [BT2]. These belong to the energy perturbed class which we now recall.

Definition 5.1 (EP-solutions to (1.1))**.** *The vector field u defined on $\mathbb{R}^3_+ \times (0,\infty)$ is an* energy perturbed solution to *(1.1), abbreviated 'EP-solution,' with divergence-free initial data $u_0 \in L^{3,\infty}(\mathbb{R}^3_+)$ if*

$$\int_0^\infty \big((u,\partial_s f) - (\nabla u, \nabla f) - (u\cdot\nabla u, f)\big)\,ds = 0,$$

for all $f \in \{f \in C_0^\infty(\mathbb{R}^3_+ \times (0,\infty)) : \nabla \cdot f = 0\}$, we have

$$u - Su_0 \in L^\infty(0,T;L^2(\mathbb{R}^3_+)) \cap L^2(0,T;H^1(\mathbb{R}^3_+)),$$

for any $T > 0$, and

$$\lim_{t\to 0^+} \|u(t) - Su_0(t)\|_{L^2(\mathbb{R}^3_+)} = 0,$$

where $Su_0(t) \in L^\infty(0,\infty;L^{3,\infty}(\mathbb{R}^3_+))$ is the solution to the time-dependent Stokes system with initial data u_0 and zero boundary value.

Energy perturbed solutions have played a role in several recent papers on the existence and regularity theory for the Navier-Stokes equation. Most relevantly, the construction of self-similar solutions in [BT1] led naturally to this structure. Additionally, Barker and Seregin [BS] used such solutions to show that, on the half-space, the L^3 norm must become infinite at a potential singularity, extending a result of [S1] from the whole-space. Energy perturbed solutions and a modified argument compared to [S1] were needed in [BS] because the local Leray theory had not been extended to the half-space. This has since been achieved in [MMP1]. Later the a priori estimates from [BS] were extended to construct global weak solutions on the whole-space and analyze regularity and uniqueness issues for initial data in scaling critical spaces [SS, BSS, AB]. Some of these ideas have been extended to the half-space in [TP].

The main theorem of [BT2] is the following.

Theorem 5.2 ([BT2])**.** *If $u_0 \in L^{3,\infty}(\mathbb{R}^3_+)$ is SS (resp. λ-DSS), is divergence-free and such that $u_{0,3}|_{x_3=0} = 0$, then there exists an EP-solution u on $\mathbb{R}^3_+ \times [0,\infty)$ with initial data u_0, which is SS (resp. λ-DSS). Moreover, $u(x_1,x_2,0,t) = 0$ for almost every $t > 0$.*

Our proof of Theorem 1.6 is by stability using the solutions of Theorem 5.2 as approximations. To connect these with a scaling invariant datum in $\mathring{M}^{2,2}_{\mathcal{C}}$ we need the following lemma.

Lemma 5.3. *Assume $u_0 \in \mathring{M}^{2,2}_{\mathcal{C}}$ is λ-DSS, is divergence-free and such that $u_{0,3}|_{x_3=0} = 0$. Then there exists a sequence $\{u_0^{(k)}\} \subset L^{3,\infty}(\mathbb{R}^3_+)$ so that $u_{0,3}^{(k)}|_{x_3=0} = 0$, all $u_0^{(k)}$ are λ-DSS, divergence-free and $u_0^{(k)} \to u_0$ in $M^{2,2}_{\mathcal{C}}$. If u_0 is self-similar, then $u_0^{(k)}$ can also be taken to be self-similar.*

The proof of this is similar to the proof of [BT3, Lemma 4.1] and the details are omitted.

Proof of Theorem 1.6. Concerning the solutions of Theorem 5.2, it is an easy exercise to check that they are local energy solutions. Indeed, Su_0 is smooth and decays in the sense that it belongs to $\mathcal{L}^2_{\mathrm{uloc}}$, where we adopt the notation of [MMP1]. The L^2 part also enjoys this decay. This is a sufficient condition for u to have the local pressure expansion which follows by adapting [MMP1, Proof of Proposition 3.1] to $\mathcal{C}$ and $\mathcal{C}_n$. Additionally, the quantities α_n and β_n defined in (1.8) are finite for solutions of Theorem 5.2, a claim we now justify. Let u be an energy perturbed solution. Then $u - Su_0 \in L^\infty(0,T;L^2(\mathbb{R}^3_+)) \cap L^2(0,T;H^1(\mathbb{R}^3_+))$. On the other hand, it can be shown that, for any $t > 0$,

$$\sup_{s\in[0,t]} \|Su_0(s)\|^2_{M^{2,2}_{\mathcal{C}_n}} + \sup_{Q\in\mathcal{C}_n} \frac{1}{|Q|^{\frac{2}{3}}} \int_0^t \int_Q |\nabla Su_0|^2\,dx\,ds < \infty,$$

by approximating u_0 in $M^{2,2}_{\mathcal{C}_n}$ by elements of C^∞_c and then extending analogous estimates for the solutions of the Stokes equations with the approximated initial data to Su_0 as in our proof of Theorem 1.5—see (4.2) and (4.3). Taken together, this shows that $u = (u - Su_0) + Su_0$ satisfies (1.8).

Based on this, the solutions from Theorem 5.2 satisfy the conditions of Theorem 1.5.

Given u_0, from Lemma 5.3 we obtain a sequence $u_0^{(k)}$ which converges to u_0 in $M^{2,2}_{\mathcal{C}}$. By Theorem 5.2 we obtain for each k a global EP-solution $u^{(k)}$. These solutions and data satisfy the assumptions of Theorem 1.5. Hence, there exists a local energy solution u for initial data u_0 which is a limit of $u^{(k)}$ in the sense given in the proof of Theorem 1.5. This convergence is sufficient to guarantee u is DSS. The argument is identical when u is self-similar. □

6. Eventual Regularity

The goal of this section is to prove Theorem 1.7, which asserts regularity in a parabolic region with an arbitrarily small leading coefficient.

Let $0 < q \le 1$, and suppose that

$$u_0 \in \mathring{M}^{2,q}_{\mathcal{C}},$$

and assume that u is a local energy solution with initial data u_0. As in (1.8) we set

$$\alpha_n(t) = \sup_{s\in[0,t]} \|u(s)\|^2_{M^{2,q}_{\mathcal{C}_n}} \quad \text{and} \quad \beta_n(t) = \sup_{Q\in\mathcal{C}_n} \frac{1}{|Q|^{\frac{q}{3}}} \int_0^t \int_Q |\nabla u|^2.$$

Assume that $\alpha_1(t) + \beta_1(t) < \infty$ for all $0 < t < \infty$. This implies that, for every $n \in \mathbb{N}$ and $0 < t < \infty$,

$$\alpha_n(t) + \beta_n(t) < \infty.$$

Observe that $\alpha_n(0) \to 0$ as $n \to \infty$ by (1.7).

In the proof of Theorem 1.7, we shall use the following version of the Gronwall lemma.

Lemma 6.1. *Suppose that $f\colon [0,T_0] \to [0,\infty)$ is a nonnegative increasing continuous function, which satisfies*

$$f(t) \le af(0) + \left(1 + \frac{t}{T_0}\right)\left(\frac{1}{8}f(t) + af(t)^p\right),$$

where $p > 1$. There exists $\varepsilon > 0$, depending on a and p such that if $f(0) \le \varepsilon$, then $f(t) \le 4af(0)$ for $t \in [0,T_0]$.

Proof of Lemma 6.1. The proof is obtained by the barrier argument, comparing the solution $f(t)$ with $4af(0)$. □

Another important ingredient in the proof of Theorem 1.7 is the following estimate on the pressure term in the energy inequality.

Lemma 6.2. *Let $q \in (0,1]$ and $n \in \mathbb{N}$. If u_0 and u are as above, then*

$$\frac{1}{|Q|^{\frac{q}{3}}} \int_0^t \int p\,u \cdot \nabla\phi_Q \le C\alpha_n(0) + \left(1 + t|Q|^{-\frac{2}{3}}\right)^2 \left(\frac{1}{8}\alpha_n(t) + C(\alpha_n(t) + \beta_n(t))^3\right) \tag{6.1}$$

for all $Q \in \mathcal{C}_n$.

Proof of Lemma 6.2. In this proof we write $\alpha = \alpha_n$ and $\beta = \beta_n$ for brevity. We also set

$$\tilde{t} = \frac{t}{|Q|^{\frac{2}{3}}}.$$

From (3.6), we have

$$\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int (p_{\rm li,loc}+p_{\rm li,nonloc})u\cdot\nabla\phi_Q \le C|Q|^{-\frac16}t^{\frac14}\alpha(0)^{\frac12}\alpha(t)^{\frac12}=C\tilde{t}^{\frac14}\alpha(0)^{\frac12}\alpha(t)^{\frac12}\le \frac{1}{16}(1+\tilde{t})\alpha(t)+C\alpha(0).$$

By (3.7), we have

$$\begin{aligned}\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int p_{\rm loc,H}\,u\cdot\nabla\phi_Q &\lesssim |Q|^{-\frac{q+1}{3}}\int_0^t\int_{Q^{***}}|u|^3 \lesssim |Q|^{\frac{q}{6}-\frac13}t^{\frac14}\alpha(t)^{\frac34}\beta(t)^{\frac34}+|Q|^{\frac{q}{6}-\frac56}t\alpha(t)^{\frac32}\\ &=\tilde{t}^{\frac14}|Q|^{\frac{q-1}{6}}\alpha(t)^{\frac34}\beta(t)^{\frac34}+\tilde{t}|Q|^{\frac{q-1}{6}}\alpha(t)^{\frac32},\end{aligned}$$

where we also used (3.14). Next, by (3.8),

$$\begin{aligned}&\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int p_{\rm loc,harm}\,u\cdot\nabla\phi_Q\\ &\quad\lesssim |Q|^{\frac{q-2}{6}}\left(t^{\frac{253}{1632}}\|\alpha\|_{L^8(0,t)}^{\frac{25}{68}}\beta(t)^{\frac{9}{68}}+|Q|^{-\frac{3}{34}}t^{\frac{13}{48}}\|\alpha\|_{L^8(0,t)}^{\frac12}\right)\\ &\qquad\qquad\times\left(t^{\frac{1}{816}}\|\alpha\|_{L^8(0,t)}^{\frac{13}{34}}\beta(t)^{\frac{21}{34}}+|Q|^{-\frac{4}{51}}t^{\frac{5}{48}}\|\alpha\|_{L^8(0,t)}^{\frac12}\beta(t)^{\frac12}+|Q|^{-\frac{21}{51}}t^{\frac{13}{24}}\|\alpha\|_{L^8(0,t)}\right)\\ &\quad\lesssim |Q|^{\frac{q-1}{6}}(1+\tilde{t})(\alpha(t)+\beta(t))^{\frac32},\end{aligned}$$

where we used that $\alpha(t)$ is nondecreasing in the second inequality.

From (3.9), we have

$$\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int p_{\rm nonloc,H}\,u\cdot\nabla\phi_Q\lesssim |Q|^{\frac{q}{6}-\frac56}\|\alpha\|_{L^{\frac32}(0,t)}^{\frac32}\lesssim |Q|^{\frac{q}{6}-\frac16}\tilde{t}\alpha(t)^{\frac32}.$$

For $p_{{\rm harm},\ge 1}$ we take $r=3/2$ in Lemma 2.6 to obtain

$$\begin{aligned}\|p_{{\rm harm},\ge1}\|_{L^{\frac32}((0,t);L^\infty(Q))}&\lesssim_{\gamma,\delta}|Q|^{\frac{q-3}{3}}\left(|Q|^{\frac{2\delta-2\gamma}{3}}t^\gamma\|\alpha\|^{1-\delta}_{L^{\frac{3(1-\delta)}{2-3\delta}}(0,t)}\beta(t)^\delta+(1+t^\gamma|Q|^{-\frac{2\gamma}{3}}+t^{\frac12}|Q|^{-\frac13})\|\alpha\|_{L^{\frac32}(0,t)}\right)\\ &\lesssim |Q|^{\frac{q-3}{3}}\left(|Q|^{\frac{2\delta-2\gamma}{3}}t^{\gamma+\frac{2-3\delta}{3}}\alpha(t)^{1-\delta}\beta(t)^\delta+(1+t^{\frac12}|Q|^{-\frac13})t^{\frac23}\alpha(t)\right)\\ &\lesssim |Q|^{\frac{q-3}{3}}t^{\frac23}(\alpha(t)+\beta(t))\left(1+(t^{\frac12}|Q|^{-\frac13})^{2\gamma-2\delta}+t^{\frac12}|Q|^{-\frac13}\right)\end{aligned}\tag{6.2}$$

for any $q\in(0,3)$, $\delta\in(0,\min\{2/3,3q/2\})$ and $\gamma\in(0,\delta/3)$. This gives

$$\begin{aligned}\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int_Q p_{{\rm harm},\ge1}\,u\cdot\nabla\phi_Q&\lesssim |Q|^{-\frac{q}{3}-\frac13}\|p_{{\rm harm},\ge1}\|_{L^{\frac32}((0,t);L^\infty(\Omega))}\|u\|_{L^3((0,t);L^1(\Omega))}\\ &\lesssim |Q|^{-\frac{q}{3}-\frac13}|Q|^{\frac{q-3}{3}}(\alpha(t)+\beta(t))t^{\frac23}\left(1+(t^{\frac12}|Q|^{-\frac13})^{2\gamma-2\delta}+t^{\frac12}|Q|^{-\frac13}\right)|Q|^{\frac12+\frac{q}{6}}\|\alpha\|^{\frac12}_{L^{\frac32}(0,t)}\\ &\lesssim |Q|^{\frac{q}{6}-\frac56}(\alpha(t)+\beta(t))^{\frac32}t\left(1+(t^{\frac12}|Q|^{-\frac13})^{2\gamma-2\delta}+t^{\frac12}|Q|^{-\frac13}\right)\\ &=|Q|^{\frac{q-1}{6}}(\alpha(t)+\beta(t))^{\frac32}\tilde{t}\left(1+\tilde{t}^{\gamma-\delta}+\tilde{t}^{\frac12}\right)\\ &\lesssim_q(\alpha(t)+\beta(t))^{\frac32}(1+\tilde{t})^{\frac32},\end{aligned}$$

where, in the last step, we have chosen $\gamma=\delta/6$ with $\delta\in(0,\min\{2/3,3q/2\})$ sufficiently small so that $1+\gamma-\delta>0$. Finally, we use (3.10) to obtain

$$\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int_Q p_{{\rm harm},\le1}\,u\cdot\nabla\phi_Q\lesssim |Q|^{\frac{q}{6}-\frac{13}{12}}t^{\frac{19}{16}}\|\alpha\|^{\frac32}_{L^8(0,t)}\lesssim |Q|^{\frac{q-1}{6}}\tilde{t}^{\frac{11}{8}}\alpha(t)^{\frac32}.$$

Summing the above inequalities, we obtain (6.1), as required. ☐

Before the proof of Theorem 1.7, we also need the following fact.

Lemma 6.3. *Let u be as above. Then $\alpha_n+\beta_n$ is a continuous function of t.*

Proof of Lemma 6.3. The proof is similar to [BK, Proof of Lemma 3.2]. We only sketch the continuity of α_n on $[0,T]$, where $T>0$ is fixed, as the argument for β_n is simpler. First, for every $Q\in\mathcal{C}_n$, the function $\sup_{s\in[0,t]}\int_Q |u|^2\phi$ is continuous in t. The rest follows by

$$\lim_{n\to\infty}\sup_{Q\in\mathcal{C}_n;|Q|\geq 2^m}\frac{1}{|Q|^{\frac{q}{3}}}\sup_{t\in[0,T]}\int_Q |u(\cdot,t)|^2\phi=0$$

by finding $m\in\mathbb{N}$ such that T_m in Theorem 1.4 satisfies $T_m\geq T$ and by applying (1.7) and Theorem 1.4 with $m\to\infty$. □

We are now ready to prove the main theorem on eventual regularity.

Proof of Theorem 1.7. By (3.14), we have

$$\frac{1}{|Q|^{\frac{2}{3}}}\int_0^t\int_Q |u|^3\lesssim t^{\frac14}|Q|^{\frac{q}{2}-\frac{2}{3}}\alpha_n(t)^{\frac34}\beta_n(t)^{\frac34}+t|Q|^{\frac{q}{2}-\frac{7}{6}}\alpha_n(t)^{\frac32}. \tag{6.3}$$

for all $Q\in\mathcal{C}_n$, from where

$$\begin{aligned}\frac{1}{|Q|^{\frac{q}{3}}}\int_0^t\int_Q |u|^2u\cdot\nabla\phi_Q &\lesssim t^{\frac14}|Q|^{\frac{q-2}{6}}\alpha_n(t)^{\frac34}\beta_n(t)^{\frac34}+t|Q|^{\frac{q-5}{6}}\alpha_n(t)^{\frac32}\\ &=(t|Q|^{-\frac23})^{\frac14}|Q|^{\frac{q-1}{6}}\alpha_n(t)^{\frac34}\beta_n(t)^{\frac34}+(t|Q|^{-\frac23})|Q|^{\frac{q-1}{6}}\alpha_n(t)^{\frac32}.\end{aligned} \tag{6.4}$$

Note that the both terms on the right-hand side are dominated by the right-hand side of (6.1) by $q\leq 1$. Thus, applying Lemma 6.2 and (6.4) in the energy inequality (3.12), and estimating the linear term $|Q|^{-q/3}\int_0^t\int|u|^2\Delta\phi_Q$ by a constant multiple of $t|Q|^{-2/3}\alpha_n(t)$, we get

$$\alpha_n(t)+\beta_n(t)\leq C\alpha_n(0)+\left(1+\sigma t|Q|^{-\frac23}\right)^2\left(\frac18(\alpha_n(t)+\beta_n(t))+C(\alpha_n(t)+\beta_n(t))^3\right) \tag{6.5}$$

for some $\sigma\geq 1$, since the term $t|Q|^{-\frac23}\alpha_n(t)$ is also dominated by the right-hand side of (6.5).

We now fix

$$\lambda\in\left[\frac{q+2}{3},1\right], \tag{6.6}$$

(in particular $\lambda\in(2/3,1]$), and let $\nu\in(0,1/2]$ be sufficiently small so that

$$\bigcup_{n\geq n_0}((-2^n,2^n)^3\cap\mathbb{R}^3_+)\times\left(\frac{2^{2n\lambda}}{4\nu\sigma},\frac{2^{2n\lambda}}{\nu\sigma},\right)\supset\left\{(x,t)\in\mathbb{R}^3_+\times(0,\infty):t\geq\epsilon_0|x|^{2\lambda}+M\right\} \tag{6.7}$$

for $n_0=1$, where $M>0$ is a constant independent of the choice of λ. By letting M depend on n_0 we see that then (6.7) holds for all $n_0\geq 1$. With $\epsilon\in(0,1]$ to be determined, we find $n_0\in\mathbb{N}$ such that

$$\nu\sigma 2^{2n}\geq 2^{2n\lambda}\qquad\text{and}\qquad \alpha_n(0)\leq\epsilon\qquad\text{for } n\geq n_0.$$

(Recall (1.7) that $\alpha_n(0)\to 0$ as $n\to\infty$.) For every $n\geq n_0$ we consider $Q\in\mathcal{S}_n^{(n)}$, i.e., $Q\in\mathcal{C}_n$ with the side-length 2^n, and

$$t=\frac{2^{2n\lambda}}{\nu\sigma}.$$

Due to our choice of n_0 we see that $t|Q|^{-2/3}\leq 1$. Moreover, applying Lemma 6.1 with $f(t)=\alpha_n(t)+\beta_n(t)$, which is continuous by Lemma 6.3, we obtain that

$$\alpha_n(s)+\beta_n(s)\lesssim\epsilon,\qquad 0\leq s\leq t,\quad n\geq n_0,$$

if $\epsilon>0$ is sufficiently small. Note also that, having fixed n_0 we have also fixed $M>0$ in (6.7). We show below that there exists $\theta(t)$ such that

$$\frac1t\|u\|^3_{L^3((0,t);L^3(Q))}+\frac1t\|p-\theta(t)\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^{\frac32}(Q))}\lesssim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32}, \tag{6.8}$$

for all $n\geq n_0$. This, (6.7), and the boundary partial regularity criterion due to Seregin et al [SSS, Theorem 1.1] shows that fixing ϵ sufficiently small gives regularity of (u,p), together with an upper bound

$$|u(x,t)|\lesssim\epsilon^{\frac13}t^{\frac{q+2-3\lambda}{8\lambda}-\frac12}=\epsilon^{\frac13}t^{-\frac78+\frac{q+2}{8\lambda}}$$

in $\{(x,t)\in\overline{\mathbb{R}^3_+}\times(0,\infty): t\geq \epsilon_0|x|^{2\lambda}+M\}$, which proves Theorem 1.7. We note that the regularity criterion in [SSS] requires the integrability condition

$$\nabla p, D^2u \in L^{\frac32}_{\mathrm{loc}}((0,\infty); L^{\frac98}_{\mathrm{loc}}(\overline{\mathbb{R}^3_+})),$$

which is the content of Lemma 6.4 stated next.

It remains to verify (6.8). For the velocity field, we use (6.3) to estimate

$$\frac{1}{t}\|u\|^3_{L^3((0,t);L^3(Q))} \lesssim \left(\frac{|Q|^{\frac{q}{2}}}{t^{\frac34}} + |Q|^{\frac{q-1}{2}}\right)\epsilon^{\frac32} \sim \left(t^{\frac{3q-3\lambda}{4\lambda}} + t^{\frac{3q-3}{4\lambda}}\right)\epsilon^{\frac32} \lesssim t^{\frac{3q-3\lambda}{4\lambda}}\epsilon^{\frac32}, \tag{6.9}$$

where we used

$$|Q|\sim t^{\frac{3}{2\lambda}}. \tag{6.10}$$

For the local linear part of the pressure, $p_{\mathrm{li,loc}}$, we use Hölder's inequality and the first inequality in (2.15) to get

$$\frac{1}{t}\|p_{\mathrm{li,loc}}\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^{\frac32}(Q))} \lesssim \frac{|Q|^{\frac14}}{t}\|p_{\mathrm{li,loc}}\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^2(Q))} \lesssim \frac{|Q|^{\frac{q+2}{4}}}{t^{\frac98}}\epsilon^{\frac32} \sim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32},$$

where we used (6.10). Similarly, we have

$$\frac{1}{t}\|p_{\mathrm{li,nonloc}}\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^{\frac32}(Q))} \lesssim \frac{|Q|}{t}\|p_{\mathrm{li,nonloc}}\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^\infty(Q))} \lesssim \frac{|Q|^{\frac{q+2}{4}}}{t^{\frac98}}\epsilon^{\frac32} \sim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32}.$$

By the third inequality in (2.15) and (6.9), we have

$$\frac{1}{t}\|p_{\mathrm{loc,H}}\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac32}(Q))} \lesssim t^{\frac{3q-3\lambda}{4\lambda}}\epsilon^{\frac32}.$$

Next, by the fourth inequality in (2.15), we have

$$\begin{aligned}\frac{1}{t}\|p_{\mathrm{loc,harm}}-\theta\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac32}(Q))} &\lesssim \frac{|Q|^{\frac{2}{17}}}{t}\|p_{\mathrm{loc,harm}}-\theta\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac{17}{10}}(Q))}\\ &\lesssim \frac{|Q|^{\frac{q}{2}+\frac{2}{17}}}{t}\left(t^{\frac{5}{68}}+\frac{t}{|Q|^{\frac{21}{34}}}\right)\epsilon^{\frac32} = \left(\frac{|Q|^{\frac{q}{2}+\frac{2}{17}}}{t^{\frac{63}{68}}}+|Q|^{\frac{q-1}{2}}\right)\epsilon^{\frac32}\\ &\sim \left(t^{\frac{3q}{4\lambda}+\frac{3}{17\lambda}-\frac{63}{68}}+t^{\frac{3q-3}{4\lambda}}\right)\epsilon^{\frac32} \lesssim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32},\end{aligned}$$

where we used (6.6), along with $q\leq 1$ in the last step. By the fifth inequality in (2.15), we have

$$\begin{aligned}\frac{1}{t}\|p_{\mathrm{nonloc,H}}\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac32}(Q))} &\lesssim \frac{|Q|}{t}\|p_{\mathrm{nonloc,H}}\|^{\frac32}_{L^{\frac32}((0,t);L^\infty(Q))} \lesssim \frac{|Q|}{t}|Q|^{\frac{q-3}{2}}\epsilon^{\frac32}\\ &= \frac{|Q|^{\frac{q-1}{2}}}{t}\epsilon^{\frac32} \sim t^{\frac{3q-3}{4\lambda}-1}\epsilon^{\frac32}.\end{aligned}$$

Using inequality (6.2) with $\gamma=\delta/6$, $\delta=q/8$, and noting that $t|Q|^{-2/3}=2^{2n(q-1)/3}/\nu\sigma$ gives

$$\begin{aligned}\frac{1}{t}\|p_{\mathrm{harm},\geq 1}\|^{\frac32}_{L^{\frac32}((0,t);L^{\frac32}(Q))} &\lesssim |Q|^{\frac{q-1}{2}}(\alpha_n(t)+\beta_n(t))^{\frac32}\left(1+(t|Q|^{-\frac23})^{\gamma-\delta}+t^{\frac12}|Q|^{-\frac13}\right)^{\frac32}\\ &\lesssim |Q|^{\frac{q-1}{2}}\left(t|Q|^{-\frac23}\right)^{-\frac{5q}{32}}\epsilon^{\frac32} \sim t^{\frac{29q}{32\lambda}-\frac{3}{4\lambda}-\frac{5q}{32}}\epsilon^{\frac32} \lesssim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32}.\end{aligned}$$

Finally, by the last inequality in (2.15), we get

$$\frac{1}{t}\|p_{\mathrm{harm},\leq 1}\|^{\frac32}_{L^{\frac32}(0,t);L^{\frac32}(Q))} \lesssim \frac{|Q|}{t}\|p_{\mathrm{harm},\leq 1}\|^{\frac32}_{L^{\frac32}((\frac18 t,t);L^\infty(Q))} \lesssim |Q|^{\frac{q}{2}-\frac78}t^{\frac{9}{16}}\epsilon^{\frac32} \sim t^{\frac{3q}{4\lambda}-\frac{21}{16\lambda}+\frac{9}{16}}\epsilon^{\frac32} \lesssim t^{\frac{3q+6-9\lambda}{8\lambda}}\epsilon^{\frac32},$$

which completes the proof of (6.8). □

Lemma 6.4 (Integrability of ∇p and D^2u)**.** *For any local energy solution (u,p), we have*

$$\nabla p, D^2u \in L^{\frac32}((t_0,T); L^{\frac98}_{\mathrm{loc}}(\overline{\mathbb{R}^3_+})),$$

for every $T>0$ and $t_0\in(0,T)$.

The lemma can be proven using local boundary regularity of the Stokes system, due to [S2]. However, a more direct proof can be obtained using the representation formulae (2.4)–(2.13) of the pressure function as well as maximal parabolic regularity. We present this argument for the sake of completeness.

Proof of Lemma 6.4. Let $\Omega \subset \mathbb{R}^3_+$ be a bounded open set. We review the pressure estimates (2.15) and observe that the gradients of each nonlocal part of the pressure, i.e., $\nabla p_{\mathrm{li,loc}}$, $\nabla p_{\mathrm{nonloc,H}}$, $\nabla p_{\mathrm{harm},\geq 1}$, belong to $L^{\frac{3}{2}}((t_0,T);L^{\frac{9}{8}}(\Omega))$, as the derivative falling onto the kernel q_λ only improves the estimate (since we obtain faster decay on the pointwise bound on q_λ (2.5)). For the local parts, we obtain the required regularity by reexamining their estimates from (2.15), as follows.

For $p_{\mathrm{li,loc}}$, we argue as in (2.16)–(2.19), with the L^2 norms replaced by $L^{\frac{9}{8}}$, to obtain

$$\begin{aligned}
\|\nabla p_{\mathrm{li,loc}}(t)\|_{L^{\frac{9}{8}}(\Omega)} &\lesssim \int_\Gamma \mathrm{e}^{t\operatorname{Re}\lambda} \int_0^\infty \mathrm{e}^{-|\lambda|^{\frac{1}{2}} z_3} \|\chi u_0(\cdot, z_3)\|_{L^{\frac{9}{8}}_{z'}} \left(\int_0^{c(\Omega)} (x_3+z_3)^{-\frac{9}{8}}\, dx_3 \right)^{\frac{8}{9}} dz_3\, d|\lambda| \\
&\lesssim_\Omega \|\chi u_0\|_{L^2} \int_\Gamma \mathrm{e}^{t\operatorname{Re}\lambda} \left(\int_0^\infty \mathrm{e}^{-2|\lambda|^{\frac{1}{2}} z_3} z_3^{-\frac{2}{9}}\, dz_3 \right)^{\frac{1}{2}} d|\lambda| \\
&= \|\chi u_0\|_{L^2} \int_\Gamma \mathrm{e}^{t\operatorname{Re}\lambda} |\lambda|^{-\frac{7}{36}}\, d|\lambda| \\
&\lesssim \|\chi u_0\|_{L^2} t^{-\frac{29}{36}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second inequality.

For $p_{\mathrm{loc,harm}}$, we argue as in (2.17) to obtain

$$\|\nabla p_{\mathrm{loc,harm}}\|_{L^{\frac{3}{2}}((0,T);L^{\frac{9}{8}}(\mathbb{R}^3_+))} \lesssim \|\mathbb{P}\nabla\cdot(\chi_{**} u\otimes u)\|_{L^{\frac{3}{2}}((0,T);L^{\frac{9}{8}}(\mathbb{R}^3_+))} \lesssim_\Omega \alpha_n(T) + \beta_n(T),$$

where the dependence on Ω is via the cutoff function χ_{**}.

The estimate on $\nabla p_{\mathrm{loc,H}}$ follows by direct calculation and Calderón-Zygmund estimates, and the estimate on $\nabla p_{\mathrm{harm},\leq 1}$ follows in the same way as $\nabla p_{\mathrm{li,loc}}$ above, by observing that $\|\chi_* F_B(t)\|_{L^2} \lesssim_\Omega \alpha_n(T)$ for every $t\in(0,T)$ (which can be obtained in the same way as Step 2 of the proof of Lemma 2.6) and by integration in time. This gives $\|\nabla p_{\mathrm{harm},\leq 1}(t)\|_{L^{\frac{9}{8}}(\Omega)} \lesssim_\Omega \alpha_n(T) t^{\frac{7}{36}}$, and so $\nabla p_{\mathrm{harm},\leq 1} \in L^{\frac{3}{2}}((t_0,T);L^{\frac{9}{8}}(\Omega))$, as required.

In order to get the integrability assertion for $D^2 u$, let $\phi \in C_0^\infty(\overline{\mathbb{R}^3_+}\times(0,\infty))$ be arbitrary. Then we have

$$\partial_t(u\phi) - \Delta(u\phi) = f, \tag{6.11}$$

where $f = -\phi u\cdot\nabla u - \phi\nabla p + u(\partial_t\phi - \Delta\phi) - 2\nabla\phi\cdot\nabla u$. By the first part of the proof, we have $\phi\nabla p \in L^{\frac{3}{2}}(([0,\infty);L^{\frac{9}{8}}(\overline{\mathbb{R}^3_+}))$. Also, $\phi u\cdot\nabla u \in L^{\frac{3}{2}}((0,\infty);L^{\frac{9}{8}}(\overline{\mathbb{R}^3_+}))$ since $u\in L^6_{\mathrm{loc}}((0,\infty);L^{\frac{18}{7}}(\overline{\mathbb{R}^3_+}))$ and $\nabla u \in L^2_{\mathrm{loc}}((0,\infty);L^2(\overline{\mathbb{R}^3_+}))$. Using also the local square integrability of u and ∇u, we get $f\in L^{\frac{3}{2}}([0,\infty);L^{\frac{9}{8}}(\overline{\mathbb{R}^3_+}))$. Applying the maximal parabolic regularity (see Section D.5 in [RRS], for example) to the equation (6.11) with zero initial data, we obtain $D^2(u\phi) \in L^{\frac{3}{2}}([0,\infty);L^{\frac{9}{8}}(\overline{\mathbb{R}^3_+}))$. By local square integrability of $u, \nabla u$, this implies $\phi D^2 u \in L^{\frac{3}{2}}([0,\infty);L^{\frac{9}{8}}(\overline{\mathbb{R}^3_+}))$, and since ϕ was an arbitrary test function supported in $\overline{\mathbb{R}^3_+}\times(0,\infty)$, the proof is complete. □

ACKNOWLEDGMENTS

We thank the referees for careful reading of the manuscript and many insightful comments and suggestions. ZB and WO were supported in part by the Simons Foundation, while IK was supported in part by the NSF grant DMS-1907992.

DATA AVAILABILITY STATEMENT

This research does not have any associated data.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

HUMAN AND ANIMAL RIGHTS

This research does not involve Human Participants and/or Animals. The manuscript complies to the Ethical Rules applicable for the Archive for Rational Mechanics and Analysis.

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