A CHARACTERIZATION OF SPACES OF HOMOGENEOUS TYPE INDUCED BY CONTINUOUS ELLIPSOID COVERS OF $\mathbb{R}^n$

MARCIN BOWNIK, BAODE LI, AND TAL WEISSBLAT

Abstract. We study the relationship between the concept of a continuous ellipsoid Θ cover of $\mathbb{R}^n$, which was introduced by Dahmen, Dekel, and Petrushev [7, 8, 11], and the space of homogeneous type induced by Θ. We characterize the class of quasi-distances on $\mathbb{R}^n$ (up to equivalence) which correspond to continuous ellipsoid covers. This places firmly continuous ellipsoid covers as a subclass of spaces of homogeneous type on $\mathbb{R}^n$ satisfying quasi-convexity and 1-Ahlfors-regularity.

1. Introduction

Discrete and continuous ellipsoid covers of $\mathbb{R}^n$ were introduced by Dahmen, Dekel, and Petrushev in the construction and analysis of multilevel preconditioners for partition of unity methods applied to elliptic boundary value problems [7] and in the study of Besov spaces with pointwise variable anisotropy [8, 9], see also the survey [11]. A continuous ellipsoid cover consists of ellipsoids $\theta_{x,t}$ with centers $x \in \mathbb{R}^n$ and scales $t \in \mathbb{R}$ satisfying a natural shape condition. Dekel, Han, and Petrushev [10] have shown that an ellipsoid cover Θ defines a space of homogeneous type in the sense of Coifman and Weiss [5, 6] with a quasi-distance ρ_Θ given by

$$\rho_\Theta(x,y) := \inf_{\theta\in\Theta,\ x,y\in\theta} |\theta|. \tag{1.1}$$

More precisely, $\mathbb{R}^n$ equipped with the Lebesgue measure and quasi-distance ρ_Θ is 1-Ahlfors regular, i.e., Lebesgue measure of balls satisfy $|B_{\rho_\Theta}(x,r)| \sim r$ for all $x \in \mathbb{R}^n$ and $r > 0$. Subsequently, Dekel, Petrushev, and Weissblat [12] have developed the Hardy spaces $H^p(\Theta)$ associated with a continuous ellipsoid cover Θ for the entire range of $0 < p \le 1$. Among the results shown in this setting are grand maximal function characterization, atomic decomposition, and classification of Hardy spaces [12], the duality of Hardy spaces [13], molecular decomposition [1], and boundedness of Calderón-Zygmund singular integral operators [4]. In contrast with the general theory of Hardy spaces on spaces of homogenous type [2, 6, 14], these results work in the full range $0 < p \le 1$. This is actually the largest class of spaces of homogeneous type on $\mathbb{R}^n$ equipped with Lebesgue measure, where such complete H^p theory has been developed so far.

A natural question arises about the relationship between ellipsoid covers and spaces of homogeneous type on $\mathbb{R}^n$. What quasi-distances on $\mathbb{R}^n$ are induced by continuous ellipsoid

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covers? In this paper we answer this question by characterizing all quasi-distances (up to equivalence) which correspond to continuous ellipsoid covers via the formula (1.1). In addition that ρ is 1-Ahlfors regular, we impose that ρ is quasi-convex. That is, there exists a constant $Q \geq 1$ such that for every $x \in \mathbb{R}^n$ and $r > 0$ there exists an ellipsoid $\xi = \xi_x^r$ with center x such that

$$\xi_x^r \subseteq B_\rho(x, r) \subseteq Q \cdot \xi_x^r, \tag{1.2}$$

where $Q \cdot \xi = Q(\xi - x) + x$ is a dilate of an ellipsoid ξ by a factor Q. The famous maximal volume ellipsoid theorem of John [3, 16, 19] attests that every convex body in $\mathbb{R}^n$ is Q-quasi-convex with $Q = n$. Hence, the above definition is a natural generalization of convexity reminiscent of the concept of a quasi-conformal mapping [15].

The main result of the paper shows that there is one-to-one correspondence between quasi-convex, 1-Ahlfors-regular quasi-distances and continuous ellipsoid covers in $\mathbb{R}^n$. In this correspondence we identify equivalent quasi-distances and likewise equivalent ellipsoid covers. In other words, a quasi-convex, 1-Ahlfors-regular quasi-distance ρ gives rise to a continuous ellipsoid cover $\Xi = \{\xi_x^r : x \in \mathbb{R}^n, r > 0\}$, where ξ_x^r satisfies (1.2). In turn, a quasi-distance ρ_Ξ, which is induced by Ξ and given by (1.1), is quasi-convex and 1-Ahlfors-regular, and ρ_Ξ is equivalent to ρ.

While the methods of the proof are quite elementary and require mostly basic properties of ellipsoids, some of them could not be found in the existing literature such as Theorem 2.1. The most demanding arguments revolve around the inner property which guarantees appropriate growth of balls $B_\rho(x, r)$ as $r \to \infty$. It turns out that this property is automatically implied by the quasi-convexity and 1-Ahlfors-regularity of ρ. In turn, the inner property plays a key role in showing that $\Xi = \{\xi_x^r : x \in \mathbb{R}^n, r > 0\}$ satisfies the shape condition, which is the key requirement for Ξ to be a continuous ellipsoid cover.

This article is organized as follows. Section 2 is devoted to proving basic properties of ellipsoids such as Theorem 2.1. In Section 3 we introduce the notion of a continuous ellipsoid cover, recall some of its known properties and prove new ones. In Section 4 we study quasi-convexity and the inner property and show the main characterization result of the paper, Theorem 4.9. Finally, in Section 5 we give applications and examples of quasi-distances illustrating our main result.

2. Ellipsoids in $\mathbb{R}^n$

In this section we recall some basic properties of ellipsoids in $\mathbb{R}^n$. An *ellipsoid* ξ in $\mathbb{R}^n$ is an image of the closed Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ under an affine map, i.e.,

$$\xi = M_\xi(\mathbb{B}^n) + c_\xi,$$

where M_ξ is an $n \times n$ nonsingular matrix and $c_\xi \in \mathbb{R}^n$ is the center of ellipsoid ξ. For any ellipsoid ξ and $\lambda > 0$, define a dilated ellipsoid by

$$\lambda \cdot \xi := \lambda M_\xi(\mathbb{B}^n) + c_\xi.$$

The following elementary theorem shows that if one ellipsoid is contained in the other, then we have a reverse inclusion relation for a dilated ellipsoid.

Theorem 2.1. *If two ellipsoids η and ξ in $\mathbb{R}^n$ satisfy $\eta \subseteq \xi$, then*

$$\xi \subseteq 2\frac{|\xi|}{|\eta|} \cdot \eta.$$

Moreover, if η and ξ have the same center, then the above holds without the factor 2.

Since we could not find Theorem 2.1 in the literature, we will give its proof using three more elementary lemmas.

Lemma 2.2. *Let $D := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a diagonal matrix. If $\mathbb{B}^n \subseteq D(\mathbb{B}^n) + c$ with $c \in \mathbb{R}^n$, then $\mathbb{B}^n \subseteq D(\mathbb{B}^n)$.*

Proof. Assume that $\mathbb{B}^n \subseteq D(\mathbb{B}^n) + c$, $c \in \mathbb{R}^n$. We only need to verify $|\lambda_i| \geq 1$ for any $i = 1, \ldots, n$. Let $e_1, \ldots, e_n$ be a standard basis of $\mathbb{R}^n$. Note that

$$D(\mathbb{B}^n) \subseteq \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq |\lambda_i| \quad \text{for all } i = 1, \ldots, n\}.$$

Since $\mathbb{B}^n - c \subseteq D(\mathbb{B}^n)$, then the absolute value of i'th coordinate of $e_i - c$ or $-e_i - c$ is ≥ 1. Hence, by the above inclusion we have $|\lambda_i| \geq 1$ for every $i = 1, \ldots, n$. □

Lemma 2.3. *Let A be a nonsigular matrix. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of AA^T and $D := \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$. If $\mathbb{B}^n \subseteq A(\mathbb{B}^n) + c$ with $c \in \mathbb{R}^n$, then there exists an orthogonal matrix U such that $\mathbb{B}^n \subseteq UA(\mathbb{B}^n) = D(\mathbb{B}^n)$. In particular, $\lambda_i \geq 1$ for all $i = 1, \ldots, n$.*

Proof. Since AA^T is a positive symmetric matrix, then there exists an orthogonal matrix U such that $UAA^TU^T = UA(UA)^T = D^2$. Therefore,

$$\begin{aligned} D(\mathbb{B}^n) &= \{Dx \in \mathbb{R}^n : x^T x \leq 1\} = \{x \in \mathbb{R}^n : x^T (D^2)^{-1} x \leq 1\} \\ &= \{x \in \mathbb{R}^n : x^T (UA(UA)^T)^{-1} x \leq 1\} = UA(\mathbb{B}^n). \end{aligned}$$

Suppose that $\mathbb{B}^n \subseteq A(\mathbb{B}^n) + c$ with $c \in \mathbb{R}^n$. We have

$$\mathbb{B}^n = U(\mathbb{B}^n) \subseteq UA(\mathbb{B}^n) + Uc = D(\mathbb{B}^n) + Uc.$$

Hence, by Lemma 2.2, we have $\mathbb{B}^n \subseteq UA(\mathbb{B}^n) = D(\mathbb{B}^n)$. □

Lemma 2.4. *If two ellipsoids satisfy $\eta \subseteq \xi$, then $\eta - c_\eta \subseteq \xi - c_\xi$, where c_η and c_ξ are centers of η and ξ, respectively.*

Proof. Without loss of generality, we can assume that $c_\eta = 0$ by using translations. Let

$$\eta := M_\eta(\mathbb{B}^n), \qquad \xi := M_\xi(\mathbb{B}^n) + c_\xi,$$

for some nonsingular matrices M_η and M_ξ. Since

$$\mathbb{B}^n = (M_\eta)^{-1}\eta \subseteq (M_\eta)^{-1}\xi = (M_\eta)^{-1}M_\xi\mathbb{B}^n + (M_\eta)^{-1}c_\xi,$$

by Lemma 2.3 we have $\mathbb{B}^n \subseteq (M_\eta)^{-1}M_\xi\mathbb{B}^n$, which yields the required conclusion. □

Proof of Theorem 2.1. Take any two ellipsoids $\eta := M_\eta(\mathbb{B}^n) + c_\eta \subseteq \xi := M_\xi(\mathbb{B}^n) + c_\xi$. Without loss of generality, we may assume that $c_\eta = 0$ by using translations. Let $A = (M_\eta)^{-1}M_\xi$. By Lemma 2.4, we have

$$\mathbb{B}^n = (M_\eta)^{-1}(\eta) \subseteq (M_\eta)^{-1}(\xi - c_\xi) = A\mathbb{B}^n. \tag{2.3}$$

Let $D := \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})$, where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of AA^T. By Lemma 2.3, there exits an orthogonal matrix U such that $UA(\mathbb{B}^n) = D(\mathbb{B}^n)$ and hence

$$U^{-1}D(\mathbb{B}^n) = (M_\eta)^{-1}(\xi - c_\xi). \tag{2.4}$$

Since $|\det U| = 1$ and $\lambda_i \geq 1$ for all $i = 1, \dots, n$, we have

$$\frac{|\xi|}{|\eta|} = \frac{|(M_\eta)^{-1}(\xi - c_\xi)|}{|(M_\eta)^{-1}(\eta)|} = \frac{|U^{-1}(D(\mathbb{B}^n))|}{|\mathbb{B}^n|} = \prod_{i=1}^n \sqrt{\lambda_i} \geq \max_{1\leq i\leq n} \sqrt{\lambda_i}. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we obtain

$$(M_\eta)^{-1}(\xi - c_\xi) = U^{-1}D(\mathbb{B}^n) \subseteq U^{-1} \max_{1\leq i\leq n} \sqrt{\lambda_i}\mathbb{B}^n = \max_{1\leq i\leq n} \sqrt{\lambda_i}\mathbb{B}^n \subseteq \frac{|\xi|}{|\eta|}(M_\eta)^{-1}(\eta). \tag{2.6}$$

Moreover, using the assumption $\eta \subseteq \xi$, we get

$$\mathbb{B}^n = (M_\eta)^{-1}(\eta) \subseteq (M_\eta)^{-1}(\xi) = (M_\eta)^{-1}(M_\xi(\mathbb{B}^n) + c_\xi).$$

By this and $UA(\mathbb{B}^n) = D(\mathbb{B}^n)$, we have

$$\mathbb{B}^n = U(\mathbb{B}^n) \subseteq U(M_\eta)^{-1}(M_\xi(\mathbb{B}^n) + c_\xi) = D(\mathbb{B}^n) + U(M_\eta)^{-1}c_\xi.$$

This implies that $\mathbb{B}^n - U(M_\eta)^{-1}c_\xi \subseteq D(\mathbb{B}^n)$ and hence

$$U(M_\eta)^{-1}c_\xi \in -D(\mathbb{B}^n) \subseteq -\max_{1\leq i\leq n} \sqrt{\lambda_i}\mathbb{B}^n = \max_{1\leq i\leq n} \sqrt{\lambda_i}\mathbb{B}^n.$$

Combining this with (2.5) yields

$$(M_\eta)^{-1}c_\xi \in \max_{1\leq i\leq n} \sqrt{\lambda_i}\mathbb{B}^n \subseteq \frac{|\xi|}{|\eta|}(M_\eta)^{-1}(\eta).$$

Hence, by (2.6) we have

$$(M_\eta)^{-1}(\xi) \subseteq \frac{|\xi|}{|\eta|}(M_\eta)^{-1}(\eta) + (M_\eta)^{-1}c_\xi \subseteq 2\frac{|\xi|}{|\eta|}(M_\eta)^{-1}(\eta).$$

Applying M_η to both sides we finally obtain $\xi \subseteq 2\frac{|\xi|}{|\eta|} \cdot \eta$.

Finally, if $\eta \subseteq \xi$ have the same center, then we may assume that $c_\eta = c_\xi = 0$. Hence, (2.6) alone implies that $\xi \subseteq \frac{|\xi|}{|\eta|}\eta$. □

3. Ellipsoid covers and quasi-distances on $\mathbb{R}^n$

In this section we recall the properties of a continuous ellipsoid cover Θ, which was originally introduced by Dahmen, Dekel, and Petrushev [8]. This includes properties of quasi-distance ρ_Θ which is induced by the cover Θ. Moreover, we translate the shape condition of Θ into a geometric form involving only containment of dilates of ellipsoids in Θ.

Definition 3.1. We say that

$$\Theta := \{\theta_{x,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

is a *continuous ellipsoid cover* of $\mathbb{R}^n$, or shortly a cover, if there exist positive constants $\mathbf{p}(\Theta) := \{a_1, \dots, a_6\}$ such that:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exists an ellipsoid $\theta_{x,t} := M_{x,t}(\mathbb{B}^n) + x$, where $M_{x,t}$ is a real $n \times n$ nonsingular matrix, satisfying

$$a_1 2^{-t} \le |\theta_{x,t}| \le a_2 2^{-t}. \tag{3.7}$$

(ii) Intersecting ellipsoids in Θ satisfy the *shape condition* requiring that for any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$, then

$$a_3 2^{-a_4 s} \le 1/\|(M_{y,t+s})^{-1} M_{x,t}\| \le \|(M_{x,t})^{-1} M_{y,t+s}\| \le a_5 2^{-a_6 s}. \tag{3.8}$$

Here, $\|\cdot\|$ is the matrix norm given by $\|A\| := \max_{|x|=1} |Ax|$ for any nonsingular matrix A.

It is worth emphasizing that we do not assume any measurability or continuity condition on a continuous ellipsoid cover Θ. Indeed, by [4, Theorem 2.2] there exists an equivalent ellipsoid cover such that its corresponding matrix valued function $x \mapsto M_{x,t}$ is continuous for any $t \in \mathbb{R}$.

Remark 3.2. The shape condition (ii) in Definition 3.1 has the following equivalent formulation by reversing scales. For any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta_{x,t} \cap \theta_{y,t-s} \neq \emptyset$, then

$$\frac{1}{a_5} 2^{a_6 s} \le 1/\|(M_{y,t-s})^{-1} M_{x,t}\| \le \|(M_{x,t})^{-1} M_{y,t-s}\| \le \frac{1}{a_3} 2^{a_4 s}. \tag{3.9}$$

Indeed, (3.9) follows from (3.8) applied to $\theta_{y,t-s}$ and $\theta_{x,t}$ in place of $\theta_{x,t}$ and $\theta_{y,t+s}$, respectively. Reversing this argument, shows the converse implication.

The shape condition (3.8) can be also equivalently restated in terms of dilates of the ellipsoids in Θ without referring to scale parameter t.

Lemma 3.3. *Let $\Theta = \{\theta_{x,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$ be a collection of ellipsoids satisfying* (3.7)*. Then, the shape condition* (3.8) *holds if and only if there exists constants a_3' and a_5' such that for any two ellipsoids ξ, $\eta \in \Theta$, if $|\eta| \le |\xi|$ and $\xi \cap \eta \neq \emptyset$, then*

$$a_3' \left(\frac{|\eta|}{|\xi|}\right)^{a_4} (\xi - c_\xi) \subseteq \eta - c_\eta \subseteq a_5' \left(\frac{|\eta|}{|\xi|}\right)^{a_6} (\xi - c_\xi), \tag{3.10}$$

where c_ξ and c_η are the centers of ξ and η, respectively.

Proof. By (3.7) for any $t, s \in \mathbb{R}$ we have

$$\frac{a_1}{a_2} 2^{-s} \le \frac{|\theta_{y,t+s}|}{|\theta_{x,t}|} \le \frac{a_2}{a_1} 2^{-s} \tag{3.11}$$

Hence, if $|\theta_{y,t+s}| \le |\theta_{x,t}|$, then $s \ge -s_0$, where $s_0 := \log_2(a_2/a_1) \ge 0$. As a partial converse, if $s \ge s_0$, then $|\theta_{y,t+s}| \le |\theta_{x,t}|$.

Suppose that the shape condition (3.8) holds for $t \in \mathbb{R}$, $s \ge 0$, and $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$. First, we shall show that the same condition also holds for $s \ge -s_0$, albeit for some new constants a_3' and a_5'. Indeed, if $s \ge 0$, then there is nothing new to show. Suppose next $-s_0 \le s \le 0$. Then, by the reverse form of (3.8), see Remark 3.2, we have

$$\frac{1}{a_5} 2^{-a_6 s} \le 1/\|(M_{y,t+s})^{-1} M_{x,t}\| \le \|(M_{x,t})^{-1} M_{y,t+s}\| \le \frac{1}{a_3} 2^{-a_4 s}.$$

Take $\tilde{a}_3 = \min(a_3, 1/a_5)$ and $\tilde{a}_5 = \max(a_5, (1/a_3)2^{(a_6-a_4)s_0})$. Since $a_6 \le a_4$, for $-s_0 \le s \le 0$ we have

$$\tilde{a}_3 2^{-a_4 s} \le \frac{1}{a_5} 2^{-a_6 s} \quad \text{and} \quad \frac{1}{a_3} 2^{-a_4 s} \le \tilde{a}_5 2^{-a_6 s}.$$

Therefore, if $t \in \mathbb{R}$, $s \ge -s_0$, and $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$, then

$$\tilde{a}_3 2^{-a_4 s} \le 1/\|(M_{y,t+s})^{-1} M_{x,t}\| \le \|(M_{x,t})^{-1} M_{y,t+s}\| \le \tilde{a}_5 2^{-a_6 s}. \tag{3.12}$$

Now suppose we have two ellipsoids $\xi, \eta \in \Theta$ such that $|\eta| \le |\xi|$ and $\xi \cap \eta \neq \emptyset$. We write $\eta = \theta_{y,t+s}$ and $\xi = \theta_{x,t}$ for some $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$. Since $|\eta| \le |\xi|$, we necessarily have $s \ge -s_0$. By the right-hand side inequality of (3.12) we have $(M_{x,t})^{-1} M_{y,t+s}(\mathbb{B}^n) \subseteq \tilde{a}_5 2^{-a_6 s} \mathbb{B}^n$. Hence, (3.11) implies that

$$M_{y,t+s}(\mathbb{B}^n) \subseteq a_5' \left(\frac{|\theta_{y,t+s}|}{|\theta_{x,t}|} \right)^{a_6} M_{x,t}(\mathbb{B}^n), \tag{3.13}$$

where $a_5' = \tilde{a}_5 (a_2/a_1)^{a_6}$. Applying the same argument for the left-hand side inequality of (3.12) yields

$$a_3' \left(\frac{|\theta_{y,t+s}|}{|\theta_{x,t}|} \right)^{a_4} M_{x,t}(\mathbb{B}^n) \subseteq M_{y,t+s}(\mathbb{B}^n), \tag{3.14}$$

where $a_3' = \tilde{a}_3 (a_1/a_2)^{a_4}$. This shows (3.10).

Conversely, suppose that (3.10) holds for $\xi, \eta \in \Theta$, $|\eta| \le |\xi|$, and $\xi \cap \eta \neq \emptyset$. We claim that the same condition holds when $|\eta| \le (a_2/a_1)|\xi|$ and $\xi \cap \eta \neq \emptyset$, albeit for some new constants $\check{a}_3$ and $\check{a}_5$. Indeed, if $|\eta| \le |\xi|$, then there is nothing new to show. Suppose that $|\xi| \le |\eta| \le (a_2/a_1)|\xi|$. Then, by (3.10) and by reversing order of inclusions we have

$$\frac{1}{a_5'} \left(\frac{|\eta|}{|\xi|} \right)^{a_6} (\xi - c_\xi) \subseteq \eta - c_\eta \subseteq \frac{1}{a_3'} \left(\frac{|\eta|}{|\xi|} \right)^{a_4} (\xi - c_\xi).$$

Hence, (3.10) holds with constants $\check{a}_3 = \min(a_3', 1/a_5')$ and $\check{a}_5 = \max(a_5', (1/a_3)(a_2/a_1)^{a_4 - a_6})$ in place of a_3' and a_5', respectively. Now, take any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $s \ge 0$ such that $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$. Letting $\eta = \theta_{y,t+s}$ and $\xi = \theta_{x,t}$, (3.10) yields (3.13) and (3.14). Converting these inclusions into norm inequalities using (3.11) yields (3.8) for appropriate constants a_3 and a_5 . □

Remark 3.4. As a consequence of Lemma 3.3 we propose the alternative geometric definition of an ellipsoid cover Θ, which will be used in our consideration in Section 4. A collection $\Theta = \{\xi_x^r : x \in \mathbb{R}^n, r > 0\}$ is a continuous ellipsoid cover if there exist positive constants $\mathbf{p}(\Theta) := \{a_1, \ldots, a_6\}$ such that:

(i) for every $x \in \mathbb{R}^n$ and $r > 0$, ξ_x^r is an ellipsoid with center x and volume satisfying

$$a_1 r \le |\xi_x^r| \le a_2 r,$$

(ii) for any ellipsoids ξ, $\eta \in \Theta$, such that $|\eta| \le |\xi|$ and $\xi \cap \eta \neq \emptyset$, we have (3.10).

To translate between two formulations involving scale $t \in \mathbb{R}$ and "radius" $r > 0$, it suffices to take $\theta_{x,\,t} = \xi_x^r$, where $r = 2^{-t}$, and then apply Lemma 3.3.

The following lemma from [12, Lemma 2.2] is a direct consequence of the shape condition (3.8).

Lemma 3.5. *Let Θ be a continuous ellipsoid cover. Then there exists $c > 0$ depending only on $\mathbf{p}(\Theta)$ such that for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\lambda \geq 1$, we have $\lambda \cdot \theta_{x,t} \subseteq \theta_{x,t-c\lambda}$.*

The following lemma is a continuous analogue of [8, Lemma 2.8], which was originally shown for discrete ellipsoid covers. Hence, for the sake of completeness we include its proof.

Lemma 3.6. *Let Θ be a continuous ellipsoid cover. Then there exists a constant $s^* \geq 0$ depending only on $\mathbf{p}(\Theta)$ such that for any ellipsoids $\theta_{x,t}$ and $\theta_{y,t+s}$ with $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$, where x, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \geq 0$, we have $\theta_{x,t} \cup \theta_{y,t+s} \subseteq \theta_{x,t-\ell}$ for any $\ell \geq s^*$,.*

Proof. We write $\theta_{x,t} := M_{x,t}(\mathbb{B}^n)+x$, $\theta_{y,t+s} := M_{y,t+s}(\mathbb{B}^n)+y$, and let $\omega := (M_{x,t})^{-1}(\theta_{y,t+s} - x)$. Then by the shape condition (3.8) and $s \geq 0$, we have

$$\begin{aligned}\operatorname{diam}(\omega) &:= \sup_{z,z' \in \omega} |z - z'| = \sup_{z,z' \in \mathbb{B}^n} |(M_{x,t})^{-1} M_{y,t+s}(z - z')| \\ &\leq 2\|(M_{x,t})^{-1} M_{y,t+s}\| \leq 2a_5 2^{-a_6 s} \leq 2a_5.\end{aligned}$$

This, together with $\theta_{x,t} \cap \theta_{y,t+s} \neq \emptyset$, implies that

$$(M_{x,t})^{-1}[(\theta_{x,t} \cup \theta_{y,t+s}) - x] = \mathbb{B}^n \cup \omega \subseteq (1 + 2a_5)\mathbb{B}^n. \tag{3.15}$$

Therefore, we have

$$\theta_{x,t} \cup \theta_{y,t+s} \subseteq (1 + 2a_5) \cdot \theta_{x,t}$$

On the other hand, by Lemma 3.5 we have for any $\lambda \geq 1 + 2a_5$,

$$(1 + 2a_5) \cdot \theta_{x,t} \subseteq \lambda \cdot \theta_{x,t} \subseteq \theta_{x,t-\lambda c}.$$

Hence, Lemma 3.6 holds for $s^* = (1 + 2a_5)c$. □

Next we move to exploring the relationship between continuous ellipsoid covers and quasi-distances on $\mathbb{R}^n$.

Definition 3.7. A mapping $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is called a *quasi-distance* if there exists a positive constant $\kappa \geq 1$ such that for all $x, y, z \in \mathbb{R}^n$,

(i) $\rho(x, y) = 0 \Leftrightarrow x = y$;
(ii) $\rho(x, y) = \rho(y, x)$;
(iii) $\rho(x, z) \leq \kappa(\rho(x, y) + \rho(y, z))$.

Dahmen, Dekel, and Petrushev have shown that an ellipsoid cover Θ induces a quasi-distance ρ_Θ on $\mathbb{R}^n$, see [8, Proposition 2.7]. Moreover, $\mathbb{R}^n$ equipped with the quasi-distance ρ_Θ and the Lebesgue measure is a space of homogeneous type which is Ahlfors 1-regular [8, Proposition 2.10]. These results can be summarized as follows.

Proposition 3.8. *Let Θ be a continuous ellipsoid cover. The function $\rho_\Theta : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by*

$$\rho_\Theta(x, y) := \inf_{\theta \in \Theta} \{|\theta| : x, y \in \theta\} \tag{3.16}$$

is a quasi-distance on $\mathbb{R}^n$. Moreover, the Lebesgue measure of balls

$$B_{\rho_\Theta}(x, r) = \{y \in \mathbb{R}^n : \rho_\Theta(x, y) < r\} \tag{3.17}$$

with respect to the quasi-distance ρ_Θ satisfies

$$|B_{\rho_\Theta}(x, r)| \sim r \qquad \text{for all } x \in \mathbb{R}^n,\ r > 0, \tag{3.18}$$

with equivalence constants depending only on $\mathbf{p}(\Theta)$.

The condition (3.18) states the Lebesgue measure is 1-Ahlfors regular with respect the quasi-distance ρ_Θ. This immediately implies the doubling property $|B_{\rho_\Theta}(x, 2r)| \lesssim |B_{\rho_\Theta}(x, r)|$, which is a defining feature of spaces of homogeneous type introduced by Coifman and Weiss [5, 6].

The following result is stated without the proof in [12, Theorem 2.7]. Its proof can be found in [4, Proposition 2.10].

Proposition 3.9. *Let Θ be a continuous ellipsoid cover and let ρ_Θ be a quasi-distance as in (3.16). For any ball $B_{\rho_\Theta}(x, r)$ with $x \in \mathbb{R}^n$ and $r > 0$, there exist $t_1, t_2 \in \mathbb{R}$ such that*

$$\theta_{x,\,t_1} \subset B_{\rho_\Theta}(x, r) \subset \theta_{x,\,t_2} \qquad \text{and} \qquad |\theta_{x,\,t_1}| \sim |\theta_{x,\,t_2}| \sim r,$$

where equivalence constants depend only on $\mathbf{p}(\Theta)$.

Using Proposition 3.9 we can introduce a more convenient variant of a quasi-distance induced by a continuous ellipsoid cover.

Proposition 3.10. *Let Θ be an ellipsoid cover. For any $x, y \in \mathbb{R}^n$, define*

$$\rho_1(x, y) := \inf_{y \in \theta_{x,\,t} \in \Theta} |\theta_{x,\,t}| \qquad \text{and} \qquad \rho_2(x, y) := \inf_{x \in \theta_{y,\,t} \in \Theta} |\theta_{y,\,t}|.$$

Then the map $\widetilde{\rho}_\Theta(x, y) := \min\{\rho_1(x, y), \rho_2(x, y)\}$ is a quasi-distance which is equivalent to $\rho_\Theta(x, y)$ as in (3.16).

Proof. It suffices to show that

$$\rho_\Theta(x, y) \sim \rho_1(x, y) \quad \text{for any} \quad x, y \in \mathbb{R}^n. \tag{3.19}$$

Indeed, if (3.19) holds, then by symmetry we have $\rho_\Theta(x, y) \sim \rho_2(x, y)$, and therefore

$$\rho_\Theta(x, y) \sim \min\{\rho_1(x, y), \rho_2(x, y)\} = \widetilde{\rho}_\Theta(x, y).$$

Since $\widetilde{\rho}_\Theta(x, y) = \widetilde{\rho}_\Theta(y, x)$, the fact that ρ_Θ is a quasi-distance (see Proposition 3.8), implies that $\widetilde{\rho}_\Theta$ is also a quasi-distance which is equivalent to ρ_Θ.

Let $x, y \in \mathbb{R}^n$. Obviously, $\rho_\Theta(x, y) \le \rho_1(x, y)$, so it remains to prove that there exists a constant $C > 0$ such that $\rho_1(x, y) \le C\rho_\Theta(x, y)$. Let $r := \rho_\Theta(x, y)$. By Proposition 3.9, there exist two ellipsoids $\theta_{x,\,t_1}$, $\theta_{x,\,t_2}$ with $|\theta_{x,\,t_1}| \sim |\theta_{x,\,t_2}| \sim r$ such that

$$\theta_{x,\,t_1} \subseteq B_{\rho_\Theta}(x, 2r) \subseteq \theta_{x,\,t_2}.$$

Since $y \in B_{\rho_\Theta}(x, 2r)$, by the definition of $\rho_1(x, y)$, it follows that

$$\rho_1(x, y) \le |\theta_{x,\,t_2}| \sim r = \rho_\Theta(x, y),$$

which completes the proof of Proposition 3.10. □

4. Quasi-convex quasi-distances on $\mathbb{R}^n$

In this section we show that the quasi-distance ρ_Θ, induced by a continuous ellipsoid cover Θ, is not only 1-Ahlfors-regular, but it also satisfies two other crucial properties: quasi-convexity and the inner property. We also show the converse statement that any quasi-convex, 1-Ahlfors-regular quasi-distance ρ automatically satisfies the inner property and generates a continuous ellipsoid cover Ξ. In addition, the quasi-distance ρ_Ξ, induced by Ξ, is equivalent to ρ. This constitutes the main result of the paper.

We start by recalling properties of convex bodies in $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. Fritz John [16, p. 202, Theorem III] proved that

every convex body in $\mathbb{R}^n$ contains a unique ellipsoid of maximal volume. The dilate by the dimension n of such ellipsoid contains the original convex body, see [3] and [19, Theorem 3.13].

Theorem 4.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body. Then there exists a unique ellipsoid $\xi \subseteq \mathbb{R}^n$ of maximal volume such that $\xi \subseteq K$. Moreover, $K \subseteq n \cdot \xi$.*

Motivated by Theorem 4.1 and the concept of quasiconformal mapping [15] we can generalize the notion of convexity.

Definition 4.2. Let $Q \geq 1$. We say that a subset $K' \subseteq \mathbb{R}^n$ is *Q-quasi-convex* with respect to $x \in K'$, if there exists an ellipsoid $\xi \subseteq \mathbb{R}^n$ with center $c_\xi = x$ such that

$$\xi \subseteq K' \subseteq Q \cdot \xi. \tag{4.20}$$

By Theorem 4.1, any convex body in $\mathbb{R}^n$ is Q-quasi-convex with respect the center of the unique maximal volume ellipsoid contained in the convex body, where $Q = n$. Notice that we do not impose uniqueness in the above definition. Namely, for a given set K' there could be two different ellipsoids (even of maximal volume) satisfying (4.20).

Definition 4.3. Given a quasi-distance $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, we say that ρ is *quasi-convex* if there exists $Q \geq 1$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, the ball

$$B_\rho(x, r) := \{y \in \mathbb{R}^n : \rho(x, y) < r\}$$

is Q-quasi-convex with respect to x. That is, there exists an ellipsoid ξ_x^r with center x such that

$$\xi_x^r \subseteq B_\rho(x, r) \subseteq Q \cdot \xi_x^r. \tag{4.21}$$

In this case we define the corresponding family of ellipsoids

$$\Xi_\rho := \{\xi_x^r : x \in \mathbb{R}^n, r > 0\}. \tag{4.22}$$

Lemma 4.4. *For any continuous ellipsoid cover Θ, the induced quasi-distance ρ_Θ given by* (3.16) *is quasi-convex.*

Proof. For any ball $B_{\rho_\Theta}(x, r)$, by Proposition 3.9, there exist two ellipsoids $\theta_{x,t_1}, \theta_{x,t_2} \in \Theta$ and two constants $d_2 \geq d_1 > 0$, which depend only on $\mathbf{p}(\Theta)$, such that $\theta_{x,t_1} \subseteq B_{\rho_\Theta}(x, r) \subseteq \theta_{x,t_2}$ and

$$d_1 r \leq |\theta_{x,t_1}| \leq |B_{\rho_\Theta}(x, r)| \leq |\theta_{x,t_2}| \leq d_2 r.$$

Since $\theta_{x,t_1} \subseteq \theta_{x,t_2}$, by Theorem 2.1 we conclude that $\theta_{x,t_2} \subseteq \frac{|\theta_{x,t_2}|}{|\theta_{x,t_1}|}\theta_{x,t_1} \subseteq \frac{d_2}{d_1}\theta_{x,t_1}$. Therefore, we have

$$\theta_{x,t_1} \subseteq B_{\rho_\Theta}(x, r) \subseteq \frac{d_2}{d_1}\theta_{x,t_1}.$$

This proves that the induced quasi-distance ρ_Θ is quasi-convex with $Q = d_2/d_1$. □

We introduce yet another property of a quasi-distance which will play an important role in our considerations.

Definition 4.5. We say that a quasi-distance ρ on $\mathbb{R}^n$ satisfies the *inner property* if there exist constants $a = a(\rho), b = b(\rho) > 0$ such that for any $x \in \mathbb{R}^n$, $r > 0$ and $\lambda \geq 1$,

$$a\lambda^b(B_\rho(x,\, r) - x) \subseteq B_\rho(x,\, \lambda r) - x. \tag{4.23}$$

The inner property is stronger than the reverse doubling property [20] since it immediately implies that

$$a^n \lambda^{bn} |B_\rho(x,\, r)| \le |B_\rho(x,\, \lambda r)| \qquad \text{for all } \lambda \ge 1.$$

While the inner property (4.23) of ρ is formulated in terms of balls, it can also be equivalently phrased in terms of ellipsoids in Ξ_ρ.

Lemma 4.6. *Let ρ be a quasi-distance on $\mathbb{R}^n$, which is quasi-convex. Let ξ_x^r be the corresponding ellipsoids as in Definition 4.3. Then ρ satisfies the inner property if and only if there exist positive constants $a_1, b_1 > 0$ such that for any $x \in \mathbb{R}^n$, $r > 0$ and $\lambda \ge 1$,*

$$a_1 \lambda^{b_1} \cdot \xi_x^r \subseteq \xi_x^{\lambda r}. \tag{4.24}$$

Proof. Since ρ is quasi convex, for every $x \in \mathbb{R}^n$ and $r > 0$, there exists an ellipsoid ξ_x^r such that (4.21) holds. By (4.23), it follows that, for any $x \in \mathbb{R}^n$, $r > 0$ and $\lambda \ge 1$,

$$a\lambda^b(\xi_x^r - x) \subseteq a\lambda^b(B_\rho(x,\, r) - x) \subseteq B_\rho(x,\, \lambda r) - x \subseteq Q(\xi_x^{\lambda r} - x).$$

Hence, (4.24) holds true with $a_1 = a/Q$ and $b_1 = b$. Similarly we can show that (4.24) implies (4.23) with $a = a_1/Q$ and $b = b_1$. □

The following lemma implies that intersecting ellipsoids in Ξ_ρ of comparable volume have similar shapes.

Lemma 4.7. *Let ρ be a quasi-distance which is quasi-convex and 1-Ahlfors-regular. That is, there exists a constant $c_1 \ge 1$ such that*

$$\frac{1}{c_1} r \le |B_\rho(x, r)| \le c_1 r \qquad \text{for all } x \in \mathbb{R}^n, r > 0. \tag{4.25}$$

Let Ξ_ρ be a family of ellipsoids corresponding to ρ as in Definition 4.3 and let $c_2 \ge 1$. Suppose that $\xi = \xi_x^r$, $\eta = \xi_y^s \in \Xi_\rho$, $x, y \in \mathbb{R}^n$, $r, s > 0$, are such that

$$B_\rho(x, r) \cap \eta \ne \emptyset \qquad \text{and} \qquad |\eta| \le c_2 |\xi|. \tag{4.26}$$

Then there exists a constant $c \ge 1$, which depends only on c_1, c_2, the triangle inequality constant κ, and the quasi-convexity parameter Q, such that $\eta \subseteq c \cdot \xi$.

Proof. By the quasi-convexity of ρ, there exists $Q \ge 1$ such that

$$\xi \subseteq B_\rho(x, r) \subseteq Q \cdot \xi, \qquad \eta \subseteq B_\rho(y, s) \subseteq Q \cdot \eta. \tag{4.27}$$

Hence, by (4.25) we have

$$\frac{1}{Q^n c_1} r \le |\xi| \le c_1 r \qquad \text{and} \qquad \frac{1}{Q^n c_1} s \le |\eta| \le c_1 s. \tag{4.28}$$

Combining (4.26), (4.27), and (4.28) we have

$$s \le c_3 r, \qquad \text{where } c_3 := Q^n (c_1)^2 c_2.$$

Since $B_\rho(x, r) \cap B_\rho(y, s) \ne \emptyset$, the triangle inequality of ρ, and the quasi-convex property of ρ, implies that

$$\eta \subseteq B_\rho(y, s) \subseteq B_\rho(x, \kappa(r + 2\kappa c_3 r)) \subseteq B_\rho(x, 3\kappa^2 c_3 r) \subseteq Q \cdot \xi_x^{3\kappa^2 c_3 r}.$$

By (4.28) we have

$$\frac{|Q \cdot \xi_x^{3\kappa^2 c_3 r}|}{|\xi|} = \frac{|Q|^n |\xi_x^{3\kappa^2 c_3 r}|}{|\xi|} \le c := Q^{2n} (c_1)^2 3\kappa^2 c_3.$$

Since ξ and $Q \cdot \xi_x^{3\kappa^2 c_3 r}$ have the same center, Theorem 2.1 yields

$$\eta \subseteq Q \cdot \xi_x^{3\kappa^2 c_3 r} \subseteq \frac{|Q \cdot \xi_x^{3\kappa^2 c_3 r}|}{|\xi|} \cdot \xi \subseteq c \cdot \xi,$$

which completes the proof of Lemma 4.7. □

Next we show that the inner property holds automatically for quasi-convex and 1-Ahlfors-regular quasi-distances.

Theorem 4.8. *Let ρ be a quasi-distance on $\mathbb{R}^n$ which is quasi-convex and 1-Ahlfors-regular. Then ρ satisfies the inner property.*

Proof. First, we will show that there exists $d = d(\rho) > 1$ such that for every $x \in \mathbb{R}^n$ and $r > 0$,

$$d(B_\rho(x,\, r) - x) \subseteq B_\rho(x,\, 2\kappa r) - x. \tag{4.29}$$

Indeed, let $x \in \mathbb{R}^n$ and $r > 0$. Since ρ is Q-quasi-convex, there exists an ellipsoid $\xi_x^r \in \Xi_\rho$ such that

$$\xi_x^r - x \subseteq B_\rho(x, r) - x \subseteq Q(\xi_x^r - x),$$

and for any $y \in B_\rho(x, r)$ there exists an ellipsoid $\xi_y^r \in \Xi_\rho$ such that

$$\xi_y^r - y \subseteq B_\rho(y, r) - y \subseteq Q(\xi_y^r - y).$$

By (4.28) we have

$$\frac{1}{(c_1)^2 Q} \leq \frac{|\xi_x^r|}{|\xi_y^r|} \leq (c_1)^2 Q.$$

Since $B_\rho(x, r) \cap \xi_y^r \neq \emptyset$, by Lemma 4.7, there exists a positive constant c such that $\xi_y^r \subseteq c \cdot \xi_x^r$. By Lemma 2.4 we have $\xi_y^r - y \subseteq c(\xi_x^r - x)$. Hence, by Theorem 2.1 we have

$$c(\xi_x^r - x) \subseteq \frac{|c\xi_x^r|}{|\xi_y^r|}(\xi_y^r - y) = c^n (c_1)^2 Q(\xi_y^r - y). \tag{4.30}$$

Let $d > 1$ be such that $(d-1)c^{n-1}(c_1)^2 Q^2 = 1$. Take any $z \in d(B_\rho(x, r) - x) + x$. Let $y \in B_\rho(x, r)$ be such that

$$z = d(y - x) + x = y + (d-1)(y - x).$$

By (4.30) and our choice of d we have

$$(d-1)Q(\xi_x^r - x) \subseteq (d-1)Qc^{n-1}(c_1)^2 Q(\xi_y^r - y) \subseteq B_\rho(y, r) - y.$$

Since $y - x \in Q(\xi_x^r - x)$, we further deduce that

$$z = y + (d-1)(y - x) \in y + (B_\rho(y, r) - y) = B_\rho(y, r).$$

By the triangle inequality

$$\rho(z,\, x) \leq \kappa(\rho(x,\, y) + \rho(y,\, z)) \leq 2\kappa r.$$

This implies that $z \in B_\rho(x,\, 2\kappa r)$ and hence (4.29) holds.

Now we can verify the inner property of ρ. Take $\epsilon > 0$ such that $d = (2\kappa)^\epsilon$. Let $\lambda \geq 1$. There exists $\ell \in \mathbb{N}_0$ and

$$(2\kappa)^\ell \leq \lambda < (2\kappa)^{\ell+1}.$$

Hence, by (4.29) it follows that

$$B_\rho(x,\, \lambda r) - x \supseteq B_\rho(x,\, (2\kappa)^\ell r) - x \supseteq d^\ell (B_\rho(x,\, r) - x) \supseteq d^{-1}\lambda^\epsilon (B_\rho(x,\, r) - x).$$

Therefore, the inner property (4.23) holds with $a = d^{-1}$ and $b = \epsilon$. □

The main result of the paper shows a 1-to-1 correspondence between equivalence classes of continuous ellipsoid covers and quasi-convex, 1-Ahlfors-regular quasi-distances on $\mathbb{R}^n$.

Theorem 4.9. *(i) For any continuous ellipsoid cover Θ, the induced quasi-distance ρ_Θ given by* (3.16) *is quasi-convex and* 1*-Ahlfors-regular.*

(ii) Conversely, for any quasi-convex and 1*-Ahlfors-regular quasi-distance ρ on $\mathbb{R}^n$, the corresponding family $\Xi = \Xi_\rho$, given by Definition 4.3, is a continuous ellipsoid cover. Moreover, its induced quasi-distance*

$$\rho_\Xi(x,\, y) := \inf_{\xi \in \Xi} \{|\xi| : x,\, y \in \xi\}, \qquad x, y \in \mathbb{R}^n, \tag{4.31}$$

is equivalent to the original quasi-distance ρ.

Proof. Part (i) follows by Proposition 3.8 and Lemma 4.4. Moreover, by Theorem 4.8 we can deduce that ρ_Θ has the inner property.

To prove (ii), we first verify that a family Ξ_ρ induced by quasi-distance ρ is a continuous ellipsoid cover. By (4.28), there exists a constant $c_1 > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\frac{1}{Q^n c_1} r \le |\xi_x^r| \le c_1 r. \tag{4.32}$$

Therefore, by letting $\theta_{x,\,t} := \xi_x^r$ with $t = -\log_2 r$, we obtain (3.7) for $a_1 := \frac{1}{Q^n c_1}$ and $a_2 := c_1$.

To show that Ξ_ρ satisfies the shape condition (3.8), by Lemma 3.3 it suffices to verify (3.10). Consider two ellipsoids $\xi = \xi_x^r$, $\eta = \xi_y^s$ in Ξ_ρ, where $x,\, y \in \mathbb{R}^n$, $r, s > 0$, such that $\xi \cap \eta \neq \emptyset$ and $|\eta| \le |\xi|$. By Lemma 4.7, there exists a constant $c > 1$ such that $\eta \subseteq c \cdot \xi$. Hence, by Lemma 2.4, we have

$$\eta - y \subseteq c \cdot \xi - x = c(\xi - x). \tag{4.33}$$

Applying Theorem 2.1 yields

$$c(\xi - x) \subseteq \frac{|c\xi|}{|\eta|}(\eta - y) = c^n \frac{|\xi|}{|\eta|}(\eta - y).$$

Thus,

$$c^{1-n}\frac{|\eta|}{|\xi|}(\xi - x) \subseteq \eta - y.$$

This shows the left-hand side inclusion of (3.10) with $a_3' := c^{1-n}$ and $a_4' := 1$.

Next we show the right-hand side inclusion of (3.10). By Theorem 4.8 quasi-distance ρ satisfies the inner property. Hence, by Lemma 4.6 there exists positive constants a_1 and b_1 such that (4.24) holds. Note that we necessarily have $a_1 \le 1$ by letting $\lambda = 1$. Assume first that

$$\frac{r}{s} > {a_1}^{-1/b_1} \ge 1. \tag{4.34}$$

Then the inner property (4.24) for $\lambda = r/s$, implies

$$\xi_y^s \subseteq a_1 \lambda^{b_1} \cdot \xi_y^s \subseteq \xi_y^{\lambda s} = \xi_y^r. \tag{4.35}$$

Hence, $\xi^r_x \cap \xi^r_y \neq \emptyset$. Moreover, by (4.32)

$$|\xi^r_y| \leq c_1 r \leq (c_1)^2 Q^n |\xi^r_x|.$$

Hence, by Lemma 4.7 applied for $c_2 = (c_1)^2 Q^n$, there exists a constant c' such that that $\xi^r_y \subseteq c' \cdot \xi^r_x$. Combining this with (4.35) and Lemma 2.4 we have

$$a_1 \left(\frac{r}{s}\right)^{b_1} (\xi^s_y - y) \subseteq c'(\xi^r_x - x). \tag{4.36}$$

On other hand, by (4.32) we have

$$\frac{|\xi^r_x|}{|\xi^s_y|} \leq (c_1)^2 Q^n \frac{r}{s}. \tag{4.37}$$

Therefore, remembering that $\xi = \xi^r_x$ and $\eta = \xi^s_y$, (4.36) and (4.37) imply that

$$\eta - y \subseteq ((c_1)^2 Q^n)^{b_1} \frac{c'}{a_1} \left(\frac{|\eta|}{|\xi|}\right)^{b_1} (\xi - x).$$

This shows the left-hand side inclusion of (3.10) with $a'_5 := ((c_1)^2 Q^n)^{b_1} \frac{c'}{a_1}$ and $a_6 := b_1$ under the assumption (4.34).

Next assume that $r/s \leq {a_1}^{-1/b_1}$. Then, by (4.37)

$$\left(\frac{|\eta|}{|\xi|}\right)^{b_1} \geq a_1((c_1)^2 Q^n)^{-b_1}.$$

Combining this with (4.33) implies

$$\eta - y \subseteq ((c_1)^2 Q^n)^{b_1} \frac{c}{a_1} \left(\frac{|\eta|}{|\xi|}\right)^{b_1} (\xi - x).$$

Again we have deduced the left-hand side inclusion of (3.10) albeit with $a'_5 := ((c_1)^2 Q^n)^{b_1} \frac{c}{a_1}$. By Lemma 3.3 we conclude that Ξ_ρ is a continuous ellipsoid cover.

Finally we prove the equivalence of ρ and ρ_Ξ. For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we set $\tilde{\theta}_{x,t} = \xi^r_x$, where $r = 2^{-t}$. We have just shown that

$$\Xi_\rho = \{\tilde{\theta}_{x,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

is a continuous ellipsoid cover.

Take any $x \neq y \in \mathbb{R}^n$. Let $r = 2\rho(x,\, y)$ and $t = -\log_2 r$. By the quasi-convex property of ρ, there exists a constant $Q \geq 1$ such that

$$\tilde{\theta}_{x,\,t} = \xi^r_x \subseteq B_\rho(x,\, r) \subseteq Q \cdot \xi^r_x = Q \cdot \tilde{\theta}_{x,\,t}.$$

By Lemma 3.5, there exists a constant $c > 0$ such that

$$x,\, y \in B_\rho(x,\, r) \subseteq Q \cdot \tilde{\theta}_{x,\,t} \subseteq \tilde{\theta}_{x,\,t-cQ}.$$

By (3.7), (4.31), and $2^{-t} = r = 2\rho(x,\, y)$, it follows that

$$\rho_\Xi(x, y) \leq |\tilde{\theta}_{x,\,t-cQ}| \leq a_2 2^{cQ} r = a_2 2^{cQ+1} \rho(x, y). \tag{4.38}$$

On the other hand, by the definition of ρ_Ξ, there exists an ellipsoid $\xi^{\tilde{r}}_z \in \Xi_\rho$, $z \in \mathbb{R}^n$, $\tilde{r} > 0$, such that $x,\, y \in \xi^{\tilde{r}}_z$ and $|\xi^{\tilde{r}}_z| \leq 2\rho_\Xi(x,\, y)$. Moreover, by the quasi-convexity of ρ,

$$\xi^{\tilde{r}}_z \subseteq B_\rho(z,\, \tilde{r}) \subseteq Q \cdot \xi^{\tilde{r}}_z.$$

Since $x,\, y \in B_\rho(z,\, \widetilde{r})$ and $\widetilde{r} \le c_1|B_\rho(z,\, \widetilde{r})|$ (ρ is 1-Ahlfors-regular) we have

$$\rho(x,y) \le \kappa[\rho(x,z)+\rho(z,y)] \le 2c_1\kappa|B_\rho(z,\,\widetilde{r})| \le 2c_1\kappa|Q\cdot\xi_z^{\widetilde{r}}| \le 4c_1\kappa Q^n\rho_\Xi(x,y). \tag{4.39}$$

Combining (4.38) with (4.39) yields equivalence of quasi-distances ρ and ρ_Ξ. □

5. Applications and examples

In this section we give applications and examples of quasi-distances illustrating our main result, Theorem 4.9. As a consequence of results about Hardy $H^p(\Theta)$ spaces with variable anisotropy associated with continuous ellipsoid cover Θ, which were introduced by Dekel, Petrushev, and Weissblat in [12], we deduce the following result.

Theorem 5.1. *Suppose that ρ is a quasi-convex and 1-Ahlfors-regular quasi-distance on $\mathbb{R}^n$. Then, $\mathbb{R}^n$ equipped with ρ and the Lebesgue measure is a space of homogeneous type for which Hardy space $H^p(\mathbb{R}^n,\rho)$ spaces exists for the entire range $0<p\le 1$. These spaces admit grand maximal function characterization, atomic decomposition, molecular decomposition, and their duals are Campanato spaces. Moreover, there exists a class of Calderón-Zygmund singular integral operators which are bounded on $H^p(\mathbb{R}^n,\rho)$ spaces for $0<p\le 1$.*

To wit Theorem 5.1 we define $H^p(\mathbb{R}^n,\rho)$ as the anisotropic Hardy space $H^p(\Xi_\rho)$, where Ξ_ρ is a continuous ellipsoid cover corresponding to quasi-distance ρ as in Theorem 4.9. Consequently, $H^p(\mathbb{R}^n,\rho)=H^p(\Xi_\rho)$ enjoys all properties of Hardy spaces with variable anisotropy shown in [1, 4, 12, 13].

Our first example involves a family of ellipses $\Theta_0 := \{\theta_{x,\,t} :\ x\in\mathbb{R}^2,\, t\in\mathbb{R}\}$ with

$$\theta_{x,\,t} := \left\{ z=(z_1,\,z_2)\in\mathbb{R}^2 :\ \frac{(z_1-x_1)^2}{\sigma_1^2}+\frac{(z_2-x_2)^2}{\sigma_2^2}\le 1\right\}$$

where semi-axes σ_1 and σ_2 are given by the following table:

t	x_2	σ_1	σ_2
$t\le 0$	$\mathbb{R}$	$2^{-\frac{t}{2}}$	$2^{-\frac{t}{2}}$
$t>0$	$\lvert x_2\rvert > 2^{-\frac{t}{3}}$	$2^{-\frac{t}{2}}$	$2^{-\frac{t}{2}}$
$t>0$	$2^{-\frac{t}{2}} < \lvert x_2\rvert \le 2^{-\frac{t}{3}}$	$2^{-\frac{5t}{6}}\frac{1}{\lvert x_2\rvert}$	$2^{-\frac{t}{6}}\lvert x_2\rvert$
$t>0$	$\lvert x_2\rvert \le 2^{-\frac{t}{2}}$	$2^{-\frac{t}{3}}$	$2^{-\frac{2t}{3}}$

We will show that Θ_0 is a continuous ellipsoid cover and give the formula of quasi-norm ρ_{Θ_0} induced by Θ_0. For this we need an elementary lemma.

Lemma 5.2. *Let $a_i,\, \beta_i>0$, $i=1,\,2,\,\dots,\,d$, where $d\ge 2$. Then the root $x>0$ of the equation $\sum_{i=1}^d a_i x^{\beta_i}=1$, satisfies $x\sim b := \min_{1\le i\le d} a_i^{-1/\beta_i}$. More precisely,*

$$\min_{1\le i\le d} d^{-1/\beta_i} b < x < b.$$

Proof. For $y > 0$ define $f(y) = \sum_{1\le i\le d} a_i y^{\beta_i}$. It is easy to see that f is strictly increasing and $f(x) = 1 < f(b)$, which implies that $x < b$. Since

$$f\bigg(\min_{1\le j\le d} d^{-1/\beta_j} b\bigg) \le \frac{1}{d}\sum_{i=1}^{d} a_i \min_{1\le j\le d}\frac{1}{a_j} < 1 = f(x),$$

we deduce that $\min_{1\le i\le d} d^{-1/\beta_i} b < x$. □

Proposition 5.3. Θ_0 *with is a continuous ellipsoid cover in the sense of Definition 3.1.*

Proof. It is obvious that Θ_0 satisfies Definition 3.1(i). We only need to show that any two intersecting ellipsoids $\theta_{x,\,t}$, $\theta_{y,\,t+s} \in \Theta_0$ satisfy Definition 3.1(ii), where $t \in \mathbb{R}$ and $s \ge 0$. We shall verify two typical cases while other cases are similar or trivial. Denote by σ_2 the vertical semi-axis of $\theta_{x,\,t}$ and by σ_2' the vertical semi-axis of $\theta_{y,\,t+s}$.

Case 1. Suppose that $2^{-t/2} < |x_2| \le 2^{-t/3}$ and $|y_2| \le 2^{-(t+s)/2}$, where $t > 0$ and $s \ge 0$. Then, we have

$$M_{x,\,t} = \mathrm{diag}(2^{-5t/6}/|x_2|,\, 2^{-t/6}|x_2|),\ M_{y,\,t+s} = \mathrm{diag}(2^{-(t+s)/3},\, 2^{-2(t+s)/3}).$$

By $\theta_{x,\,t} \cap \theta_{y,\,t+s} \ne \emptyset$, $|y_2| \le 2^{-(t+s)/2}$, $\sigma_2 = 2^{-t/6}|x_2|$, $\sigma_2' = 2^{-2(t+s)/3}$, $|x_2| \le 2^{-t/3}$, $t > 0$ and $s \ge 0$, we know that

$$|x_2| \le |y_2| + \sigma_2 + \sigma_2' \le 2^{-(t+s)/2} + 2^{-t/2} + 2^{-2(t+s)/3} \le 3\cdot 2^{-t/2}.$$

From this and $|x_2| > 2^{-t/2}$, it follows that

$$\|(M_{x,\,t})^{-1} M_{y,\,t+s}\| = \|\mathrm{diag}(2^{t/2-s/3}|x_2|,\, 2^{-t/2-2s/3}/|x_2|)\| \le \|\mathrm{diag}(3\cdot 2^{-s/3},\, 2^{-2s/3})\| \le 3\cdot 2^{-s/3}$$

and

$$\|(M_{y,\,t+s})^{-1} M_{x,\,t}\| = \|\mathrm{diag}(2^{-t/2+s/3}/|x_2|,\, 2^{t/2+2s/3}|x_2|)\| \le \|\mathrm{diag}(2^{s/3},\, 3\cdot 2^{2s/3}\| \le 3\cdot 2^{2s/3}.$$

Case 2. Suppose that $|x_2| \le 2^{-t/2}$ and $2^{-(t+s)/2} < |y_2| \le 2^{-(t+s)/3}$, where $t > 0$ and $s \ge 0$. Then, we have

$$M_{x,\,t} = \mathrm{diag}(2^{-t/3},\, 2^{-2t/3}),\ M_{y,\,t+s} = \mathrm{diag}(2^{-5(t+s)/6}/|y_2|,\, 2^{-(t+s)/6}|y_2|).$$

By $\theta_{x,\,t} \cap \theta_{y,\,t+s} \ne \emptyset$, $|x_2| \le 2^{-t/2}$, $\sigma_2 = 2^{-2t/3}$, $\sigma_2' = 2^{-(t+s)/6}|y_2|$, $|y_2| \le 2^{-(t+s)/3}$, $t > 0$ and $s \ge 0$, we know that

$$|y_2| \le |x_2| + \sigma_2 + \sigma_2' \le 2^{-t/2} + 2^{-2t/3} + 2^{-(t+s)/2} \le 3\cdot 2^{-t/2}.$$

From this and $|y_2| > 2^{-(t+s)/2}$, it follows that

$$\|(M_{x,\,t})^{-1} M_{y,\,t+s}\| = \|\mathrm{diag}(2^{-t/2-5s/6}/|y_2|,\, 2^{t/2-s/6}|y_2|)\| \le \|\mathrm{diag}(2^{-s/3},\, 3\cdot 2^{-s/6})\| \le 3\cdot 2^{-s/6}$$

and

$$\|(M_{y,\,t+s})^{-1} M_{x,\,t}\| = \|\mathrm{diag}(2^{t/2+5s/6}|y_2|,\, 2^{-t/2+s/6}/|y_2|)\| \le \|\mathrm{diag}(3\cdot 2^{5s/6},\, 2^{2s/3})\| \le 3\cdot 2^{5s/6}.$$

□

Proposition 5.4. *The quasi-distance ρ_{Θ_0} induced by the ellipsoid cover Θ_0 satisfies*

$$\rho_{\Theta_0}(x,\,y)\sim\begin{cases}|x-y|^2 & |x-y|\geq 1 \text{ or } |x-y|^{\frac{2}{3}}<|x_2|,\\ \left[(y_1-x_1)^2+\sqrt{(y_1-x_1)^4+4(y_2-x_2)^2}\right]^{\frac{3}{4}} & |x-y|<1 \;\&\; |x_2|\leq\varphi(x,\,y),\\ \max\{(x_1-y_1)^{\frac{6}{5}}|x_2|^{\frac{6}{5}},\,(x_2-y_2)^6|x_2|^{-6}\} & |x-y|<1\\ & \&\,\varphi(x,\,y)<|x_2|\leq|x-y|^{2/3},\end{cases}$$

where $\varphi(x,\,y):=2^{-\frac{3}{4}}\left[(y_1-x_1)^2+\sqrt{(y_1-x_1)^4+4(y_2-x_2)^2}\right]^{\frac{3}{4}}$.

Proof. By Proposition 3.10, it suffice to find the formula for

$$\rho_1(x,\,y):=\inf_{y\in\theta_{x,\,t}\in\Theta}|\theta_{x,\,t}|.$$

It is not hard to verify that ellipses in Θ_0 are nested, i.e., $\theta_{x,\,t_1}\subsetneq\theta_{x,\,t_2}$ for any $x\in\mathbb{R}^2$ and $t_1,\,t_2\in\mathbb{R}$ with $t_1>t_2$. Using this and the fact that ellipses in Θ_0 are closed, we know that $\rho_1(x,\,y)$ equals to the area of an ellipse $\theta_{x,\,t}$ for some $t\in\mathbb{R}$ such that y belongs to the boundary of $\theta_{x,\,t}$, i.e., $y\in\partial\theta_{x,\,t}$. Equivalently,

$$\frac{(y_1-x_1)^2}{\sigma_1^2}+\frac{(y_2-x_2)^2}{\sigma_2^2}=1. \tag{5.40}$$

We shall consider three cases.

Case 1. Suppose that $t\in\mathbb{R}$ and $\theta_{x,\,t}$ is a ball. Since $y\in\partial\theta_{x,\,t}$ we have

$$\frac{(y_1-x_1)^2}{2^{-t}}+\frac{(y_2-x_2)^2}{2^{-t}}=1\Longleftrightarrow|x-y|=2^{-\frac{t}{2}}. \tag{5.41}$$

By the definition of Θ_0, we know this happens if $x,\,y$ satisfy (5.41) for some $t\leq 0$ or for some $t>0$ and $|x_2|>2^{-t/3}$. Equivalently, we have either $|x-y|\geq 1$ or $|x-y|<1$ and $|x_2|>|x-y|^{\frac{2}{3}}$. In either of two subcases,

$$\rho_1(x,\,y)=|\theta_{x,\,t}|=\pi|x-y|^2.$$

Case 2. Suppose that $t>0$, $\sigma_1=2^{-t/3}$, and $\sigma_2=2^{-2t/3}$. Since $y\in\partial\theta_{x,\,t}$ we have

$$\frac{(y_1-x_1)^2}{2^{-\frac{2t}{3}}}+\frac{(y_2-x_2)^2}{2^{-\frac{4t}{3}}}=1. \tag{5.42}$$

This is equivalent to $|x-y|<1$ and $|x_2|\leq 2^{-\frac{t}{2}}$, where

$$2^{-\frac{t}{2}}=2^{-\frac{3}{4}}\left[(y_1-x_1)^2+\sqrt{(y_1-x_1)^4+4(y_2-x_2)^2}\right]^{\frac{3}{4}}=:\varphi(x,\,y).$$

Therefore, in this case,

$$\rho_1(x,\,y)=|\theta_{x,\,t}|=\pi 2^{-t}=\pi[\varphi(x,\,y)]^2.$$

Case 3. Suppose that $t>0$, $\sigma_1=2^{-5t/6}/|x_2|$, and $\sigma_2=2^{-t/6}|x_2|$. Since $y\in\partial\theta_{x,\,t}$ we have

$$\frac{(y_1-x_1)^2}{[2^{-\frac{5t}{6}}|x_2|^{-1}]^2}+\frac{(y_2-x_2)^2}{[2^{-\frac{t}{6}}|x_2|]^2}=1.$$

Let $a = (y_1 - x_1)^2|x_2|^2$, $b = (y_2 - x_2)^2/|x_2|^2$, and $z = 2^{t/3}$. Since Case 3 is complementary to Cases 1 and 2, we necessarily have $|x - y| < 1$, $\varphi(x,\, y) < |x_2| \le |x - y|^{2/3}$, and $az^5 + bz = 1$.

Since $a, b > 0$ and $z > 1$, by Lemma 5.2, we have

$$z = 2^{\frac{t}{3}} \sim \min\{(x_1 - y_1)^{-\frac{2}{5}}|x_2|^{-\frac{2}{5}},\, (x_2 - y_2)^{-2}|x_2|^2\}.$$

Thus, we have

$$\rho_1(x,\, y) = |\theta_{x,\,t}| = \pi 2^{-t} \sim \max\{(x_1 - y_1)^{\frac{6}{5}}|x_2|^{\frac{6}{5}},\, (x_2 - y_2)^6|x_2|^{-6}\}.$$

Combining Cases 1–3 with (3.19) shows Proposition 5.4. □

By Theorems 4.8 and 4.9 we deduce that ρ_{Θ_0} is quasi-convex and 1-Ahlfors-regular quasi-distance which satisfies the inner property. However, these properties are far from obvious from the formula for ρ_{Θ_0} in Proposition 5.4

Next we will give an example of a quasi-convex quasi-distance ρ, which is not 1-Ahlfors-regular, but which nevertheless generates a continuous ellipsoid cover. It will be convenient to relax the assumption of symmetry of quasi-distance in Definition 3.7 by the condition $\rho(x,\, y) \le C\rho(y,\, x)$ for any $x,\, y \in \mathbb{R}^n$, see [18, Section I.2.4]. This formally weaker condition implies that $\rho(x,\, y) \sim \rho(y,\, x)$. Hence, its symmetrization $[\rho(x,\, y) + \rho(y,\, x)]/2$ is a quasi-distance in the sense of Definition 3.7, albeit for (possibly) increased triangle constant κ.

The following example can be found in the monograph of Stein [18, Section I.2.6]. It is merely the simplest example of general class of balls and metrics studied by Nagel, Stein, and Wainger [17]. Let k be a non-negative integer and, for any $x \in \mathbb{R}^2$ and $\delta > 0$, let

$$B_k(x,\, \delta) := \{y \in \mathbb{R}^2 :\, |x_1 - y_1| < \delta,\, |x_2 - y_2| < \max\{\delta^{k+1},\, |x_1|^k\delta\}\}. \tag{5.43}$$

Then balls $\{B_k(x,\, \delta) :\, x \in \mathbb{R}^2,\, \delta \in (0,\, \infty)\}$ are a natural family of balls associated with the vector fields $X_1 := \partial/\partial x_1$ and $X_2 := x_1^k\partial/\partial x_2$. That is, $y \in B_k(x, \delta)$ if one can join x to y along a path whose velocity vector at any point is of the form $a_1X_1 + a_2X_2$, with $|a_1| \le 1$ and $|a_2| \le 1$, in elapsed time $\lesssim \delta$. The balls $B_k(x,\, \delta)$ can be equivalently defined as $B_{\rho_k}(x,\, \delta) := \{y \in \mathbb{R}^2 :\, \rho_k(y,\, x) < \delta\}$, where

$$\rho_k(y,\, x) = \begin{cases} \max\{|y_1 - x_1|,\, \min\{|y_2 - x_2|^{1/(k+1)},\, |y_2 - x_2|/|x_1|^k\}\} & \text{if } x_1 \ne 0, \\ \max\{|y_1 - x_1|,\, |y_2 - x_2|^{1/(k+1)}\} & \text{if } x_1 = 0. \end{cases} \tag{5.44}$$

Proposition 5.5. *Let k be a non-negative integer and let ρ_k be as in* (5.44)*. Then ρ_k is a quasi-distance which is quasi-convex and satisfies the inner property, but ρ_k is not* 1*-Ahlfors-regular.*

Proof. It is not difficult to check that ρ_k is equivalent to the metric associated with vector fields X_1 and X_2, see [17, Definition 1.1]. That is, the distance between x and y is the infimum of travel times between x and y along paths whose velocity vector at any point is of the form $a_1X_1 + a_2X_2$, with $|a_1| \le 1$ and $|a_2| \le 1$. By (5.43) any ball $B_{\rho_k}(x,\, \delta)$ is actually a rectangle. Hence, ρ_k is quasi-convex with $Q = \sqrt{2}$. Moreover, for any $\lambda \ge 1$, $\delta > 0$ and $x \in \mathbb{R}^2$, we have

$$\begin{aligned} \lambda\,(B_{\rho_k}(x,\, \delta) - x) &= \big\{(y_1,\, y_2) :\, |y_1| < \lambda\delta,\, |y_2| < \lambda \max\{\delta^{k+1},\, \delta|x_1|^k\}\big\} \\ &\subseteq \big\{(y_1,\, y_2) :\, |y_1| < \lambda\delta,\, |y_2| < \max\{(\lambda\delta)^{k+1},\, \lambda\delta|x_1|^k\}\big\} \\ &= B_{\rho_k}(x,\, \lambda\delta) - x. \end{aligned} \tag{5.45}$$

Hence, ρ_k satisfies the inner property as in Definition 4.5 with $a = 1$ and $b = 1$. By (5.43) we have

$$|B_{\rho_k}(x,\, \delta)| = 4\delta^2 \max\{\delta^k, |x_1|^k\}. \tag{5.46}$$

Hence, the Lebesgue measure is not 1-Ahlfors-regular with respect to ρ_k. □

In spite of Proposition 5.5, one can associate with ρ_k a continuous ellipsoid cover. By the quasi-convexity of ρ_k we can consider family of ellipsoids $\Xi_{\rho_k} = \{\xi_x^r : x \in \mathbb{R}^n, r > 0\}$ as in Definition 4.3

$$\xi_x^r \subseteq B_{\rho_k}(x, r) \subseteq Q \cdot \xi_x^r.$$

For any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ define

$$\theta_{x,\,t} = \xi_x^{r(t)}, \qquad \text{where } r(t) = \sup\{r > 0 : |\xi_x^r| \le 2^{-t}\}. \tag{5.47}$$

It follows from (5.43) and (5.47) that $\Theta = \{\theta_{x,\,t} : x \in \mathbb{R}^n, t \in \mathbb{R}\}$ satisfies property (i) in Definition 3.1. It takes considerably more effort to show that Θ satisfies the shape condition (ii) using Lemma 3.3. Consequently, the Hardy space $H^p(\mathbb{R}^n, \rho_k)$, which corresponds to ellipsoid cover Ξ_{ρ_k}, satisfies the conclusions of Theorem 5.1. We leave details to an interested reader.

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Department of Mathematics, University of Oregon, Eugene, OR 97403–1222, USA
Email address: mbownik@uoregon.edu

College of Mathematics and System Science, Xinjiang University, Urumqi, 830046, P. R. China
Email address: baodeli@xju.edu.cn

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, 9190401, Israel
Email address: tal.weissblat@mail.huji.ac.il