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NILPOTENT STRUCTURES AND COLLAPSING RICCI-FLAT METRICS ON THE K3 SURFACE

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1. Introduction

The main purpose of this paper is to describe a new mechanism by which Ricci-flat metrics on K3 surfaces can degenerate. It also suggests a new general phenomenon that could possibly occur in the context of other canonical metrics.

Recall in 1976, Yau's solution to the Calabi conjecture [Yau78] proved the existence of Kähler metrics with vanishing Ricci curvature, which are governed by the Riemannian analogue of the vacuum Einstein equation, on a compact Kähler manifold with zero first Chern class. These *Calabi-Yau metrics* led to the first known construction of compact Ricci-flat Riemannian manifolds which are not flat. Examples of such manifolds exist in abundance, and these metrics often appear in natural families parametrized by certain complex geometric data, namely, their *Kähler class* and their *complex structure*. As the complex geometric data degenerates, it is a natural question to understand the process of singularity formation of the corresponding Ricci-flat metrics.

There is a great deal known about the limiting process of Einstein metrics under a local volume non-collapsing assumption; see for example [And90, BKN89, Che03,

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CC97, CCT02, CD13, CN15, CN13, CT05, Tia90]. In general there is not much known in the collapsing case. However, in the case of Einstein 4-manifolds, in a pioneering work, Cheeger and Tian proved the first ϵ-regularity theorem without any non-collapsing assumption [CT06]. This result implies that there is a Gromov-Hausorff limit which is smooth away from finitely many points, and the sequence collapses with uniformly bounded curvature away from the singular set. But we note that this result does not give any information on the degeneration near the singular set.

To understand the degeneration of a sequence of metrics

$$(M_j^n, g_j, p_j) \xrightarrow{GH} (X_\infty^n, d_\infty, p_\infty)$$

more precisely near the singular set, one studies them at an infinitesimal scale: assuming that the curvature blows up around the points p_j, we choose rescaling factors $\lambda_j \to \infty$ such that

$$(M_j^n, \lambda_j^2 g_j, p_j) \xrightarrow{GH} (Y_\infty, \tilde{d}_\infty, \tilde{p}_\infty) \tag{1.1}$$

as $j \to \infty$ after passing to a subsequence. The limit in (1.1) can depend upon the choice of rescaling factors λ_j, and we will call any such limit a *bubble limit*.

In this paper we focus on the case of Kähler-Einstein metrics in complex dimension 2. In fact we will only consider Ricci-flat Kähler metrics on the K3 surface, i.e., the oriented smooth 4-manifold underlying a simply-connected compact complex surface with vanishing first Chern class. The Ricci-flat Kähler metrics in this case have holonomy $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ so they are in fact *hyperkähler*. The main result of this paper presents a new gluing construction of metrics of this type, relying crucially on their hyperkähler property. We can also identify all bubble limits in our construction, so our main result also gives some new understanding of the type of degeneration possible near the finite singular set in the theorem of [CT06].

1.1. **Gluing constructions of hyperkähler** K3 **surfaces.** There are many known gluing constructions of hyperkähler metrics on K3 surfaces in the literature, which we briefly review here.

1.1.1. *Kummer construction.* There are non-collapsing sequences of hyperkähler metrics on the K3 surface limiting to a flat orbifold $\mathbb{T}^4/\mathbb{Z}_2$. There are 16 singular points of convergence, at which Eguchi-Hanson metrics occur as bubble limits; see [LS94, Don12] and the references therein.

1.1.2. *Codimension*-1 *collapse.* In [Fos19] Foscolo constructed a family of hyperkähler metrics on the K3 surface that collapses to the flat orbifold $\mathbb{T}^3/\mathbb{Z}_2$. The collapse has bounded curvature away from finitely many points, and is given by shrinking the fibers of an S^1-fibration. In the simplest case, curvature blow-up occurs at the 8 singular points of $\mathbb{T}^3/\mathbb{Z}_2$, where the bubble limits are given by complete hyperkähler spaces with cubic volume growth, which in this case are ALF-D_2 spaces. Let us also point out that the results of [Fos19] have motivated the study of codimension-1 collapse of G_2-manifolds to 3-dimensional Calabi-Yau manifolds in [FHN].

1.1.3. *Codimension*-2 *collapse.* In [GW00], Gross and Wilson constructed a family of hyperkähler metrics on elliptic K3 surfaces with exactly 24 singular fibers of type I_1, which collapse to a singular metric d_∞ on a topological sphere $X_\infty^2 \approx S^2$, which is non-smooth at the 24 points corresponding to the singular fibers. Away from the singular fibers, the metric is modeled on the Green-Shapere-Vafa-Yau hyperkähler *semi-flat metrics* [GSVY90]. In a neighborhood of each singular fiber the metric is modeled on the *Ooguri-Vafa metric* (see [GW00, OV96]), which is an incomplete hyperkähler metric constructed using the Gibbons-Hawking ansatz which we will recall in Section 2.

1.1.4. *Codimension-*3 *collapse with torus fibers.* In [TY90], Tian and Yau constructed complete non-compact hyperkähler 4-manifolds with cylindrical ends by removing smooth fibers from rational elliptic surfaces, and were proved in [Hei12] to converge to their $\mathbb{R}\times\mathbb{T}^3$ flat asymptotic models at an exponential rate. Such spaces are known as ALH spaces in the literature. It is then possible to glue together two ALH spaces to obtain a family of hyperkähler metrics on the K3 surface which degenerates by developing a long neck modeled on $\mathbb{T}^3$ times an interval [CC20]. If one rescales these metrics to have diameter equal to 1, then the Gromov-Hausdorff limit is the unit interval and the bubble limits at each endpoint are the Tian-Yau asymptotically cylindrical metrics. Gluing of asymptotically cylindrical geometric structures is a very familiar construction in geometry; see for example [Flo91, KS01] for anti-self-dual metrics in dimension 4, and [Kov03, HHN15] for holonomy G_2 metrics in dimension 7.

1.2. **Main results.** The main result of this paper gives a new gluing construction in which a family of hyperkähler metrics on the K3 surface collapses to a unit interval and generically the collapse happens along a 3-dimensional *Heisenberg nilmanifold* (i.e., a nontrivial S^1-bundle over $\mathbb{T}^2$). Part of our motivation was an attempt to understand the hyperkähler metric degenerations corresponding to Type II complex structure degenerations of polarized complex K3 surfaces. A guiding example is when we have a family of quartic K3 surfaces Z_t in $\mathbb{C}P^3$ defined by the equation $tq + f_1 f_2 = 0$, where q is a general quartic and f_1 and f_2 are general quadrics. The general fiber is a smooth K3 surface while the central fiber is a union of two quadric surfaces X_1 and X_2, intersecting transversally along an elliptic curve defined by $f_1 = f_2 = 0$. We would like to understand the behavior of the Ricci-flat metrics on Z_t in the cohomology class $2\pi c_1(\mathcal{O}(1)|_{Z_t})$ as t tends to zero.

In 1987, R. Kobayashi proposed a conjectural mechanism for Ricci-flat metrics on K3 surfaces to degenerate into unions of Tian-Yau and Taub-NUT spaces; see Cases (i) and (ii) on p.223 in [Kob90]. Moreover, Kobayashi also proposed an identification of these limits with Type II polarized degenerations of complex K3 surfaces. In this paper, we focus mainly on the Riemannian geometric aspects of this problem, so our main result gives a verification of Kobayashi's expectation only at the level of hyperkähler structures. We do expect that some of our degenerations can be used to characterize some Type II polarized degenerations of complex K3 surfaces.

Given a del Pezzo surface M and a smooth anti-canonical curve $D \subset M$, Tian-Yau proved in [TY90] the existence of a hyperkähler metric on $M \setminus D$, with interesting asymptotic geometry at infinity. Namely, outside a compact set the manifold is diffeomorphic to $N \times [0, \infty)$, where N is an S^1-bundle over D of degree $d = c_1(X)^2$, and the metric is modeled on a doubly warped product so that as we move towards infinity, the S^1 fibers shrink in size while the base torus D expands. The volume growth rate of the hyperkähler metric is $4/3$ and the curvature decays quadratically. We call such a hyperkähler metric a *Tian-Yau metric* throughout this paper; for more precise details, we refer to Section 3. The proof in [TY90] uses the Calabi ansatz in a neighborhood of infinity and then solves a Monge-Ampère equation, so it is not a priori clear whether these metrics are unique or canonical in a suitable sense. Nevertheless they provide candidates for the bubble limits of the degeneration that we would like to understand.

In analogy with the cylindrical ALH gluing described above, one might expect that gluing together two Tian-Yau metrics would result in a hyperkähler K3 surface. However, an easy Euler characteristic argument shows that one cannot naively glue the ends of the two Tian-Yau metrics together to even match the topology of a K3 surface. Geometrically, even though the end of a Tian-Yau metric is topologically cylindrical, the metric itself is not. So we need to construct a *neck region* that approximates the Tian-Yau ends on both sides.

A key novel ingredient of this paper is the construction of such a transition region, specifically an incomplete hyperkähler 4-manifold that can be viewed as a doubly-periodic cousin of the *Ooguri-Vafa metric*. Recall that the Ooguri-Vafa metric is the metric arising from the Gibbons-Hawking ansatz applied to a harmonic function on $S^1 \times \mathbb{R}^2$ with a pole on $S^1 \times \{0\}$; or equivalently, a harmonic function on $\mathbb{R} \times \mathbb{R}^2$, periodic in the first variable, and with poles along $\mathbb{Z} \times \{0\} \subset \mathbb{R} \times \{0\}$. Our neck metric is instead constructed by applying the Gibbons-Hawking ansatz to a harmonic function on the flat cylinder $\mathbb{T}^2 \times \mathbb{R}$ with finitely many poles (which we call the *monopole points*) in $\mathbb{T}^2 \times \mathbb{R}$. This is equivalent to a harmonic function on $\mathbb{R}^2 \times \mathbb{R}$, doubly periodic in the first and second variables, and with poles on a rank 2 lattice. For more details of this construction we refer to Section 2. Here we point out that in analogy with the Ooguri-Vafa case, the resulting metric is *incomplete* because $\mathbb{T}^2 \times \mathbb{R}$ is parabolic, hence admits no globally positive harmonic functions; moreover, the two ends of this neck metric do indeed match up closely with the ends of the Tian-Yau metrics.

Our main theorem says that it is in fact possible to "glue together" two Tian-Yau metrics with a suitable neck region as above to construct families of Ricci-flat metrics on the K3 surface with a nontrivial nilpotent collapsing structure.

Theorem A. *Let b_+, b_- and m be positive integers satisfying*

$$1 \leq b_\pm \leq 9, \ 1 \leq m \leq b_+ + b_-.$$

Then there exists a family of hyperkähler metrics $\hat{h}_\beta$ on the K3 *surface which collapse to the standard metric on the closed interval* $[0,1]$*, i.e.,*

$$(\mathrm{K3}, \hat{h}_\beta) \xrightarrow{GH} ([0,1], dt^2), \ \beta \to \infty.$$

Moreover, for each sufficiently large $\beta \gg 1$, there exists a finite set $\mathcal{S} \equiv \{0, t_1, \ldots, t_m, 1\} \subset [0,1]$ and a continuous surjective map

$$F_\beta : \mathrm{K3} \to [0,1],$$

which is almost distance-preserving, i.e., for some constant $C_0 > 0$ independent of β,

$$\Big| |F_\beta(p) - F_\beta(q)| - d_{\hat{h}_\beta}(p,q) \Big| \leq \frac{C_0}{\beta}, \ \forall p, q \in \mathrm{K3},$$

such that the following properties hold.

(1) *(Regular collapsing regions) Denote by $T_\epsilon(\mathcal{S})$ the ϵ-tubular neighborhood of $\mathcal{S}$ and $\mathcal{R}_\epsilon \equiv [0,1] \setminus T_\epsilon(\mathcal{S})$. Then for every sufficiently small ϵ (depending*

on the minimal distance between the points in $\mathcal{S}$) and $k \in \mathbb{N}$, there exists $C_{k,\epsilon} > 0$ such that

$$\sup_{F_\beta^{-1}(\mathcal{R}_\epsilon)} |\nabla^k \mathrm{Rm}_{\hat{h}_\beta}| \le C_{k,\epsilon},$$

and for each $t \in \mathcal{R}_\epsilon$, $F_\beta^{-1}(t)$ is diffeomorphic to an S^1-fiber bundle over $\mathbb{T}^2$. Furthermore,

$$C_0^{-1}\beta^{-1} \le \mathrm{Diam}_{\hat{h}_\beta}(F_\beta^{-1}(t)) \le C_0\beta^{-1}, \quad C_0^{-1}\beta^{-2} \le \mathrm{Diam}_{\hat{h}_\beta}(S^1) \le C_0\beta^{-2}.$$

(2) *(Bubbling regions) The preimage $F_\beta^{-1}(T_\epsilon(\mathcal{S}))$ can be written as the union of different connected components $\mathcal{S}_\epsilon^- \cup \Big(\bigcup_{j=1}^m \mathcal{S}_\epsilon^j\Big) \cup \mathcal{S}_\epsilon^+$ such that the following properties hold.*
   (a) *For $j \in [1, m]$, there exists an $x_{\beta,j} \in \mathcal{S}_\epsilon^j$ such that $F_\beta(x_{\beta,j}) \to t_j$,*

   $$C_0^{-1}\beta^4 \le |\mathrm{Rm}_{\hat{h}_\beta}|(x_{\beta,j}) \le C_0\beta^4$$

   as $\beta \to \infty$, and rescalings of the metrics near $x_{\beta,j}$ converge to Taub-NUT metrics (see Example 2.1) in the pointed Gromov-Hausdorff sense. In fact, it is possible to have several distinct Taub-NUT bubbles coming out of the same component $\mathcal{S}_\epsilon^j$; see Theorem B for a more precise statement.
   (b) *There exist $x_{\beta,\pm} \in \mathcal{S}_\epsilon^\pm$ such that $F_\beta(x_{\beta,-}) \to 0$, $F_\beta(x_{\beta,+}) \to 1$,*

   $$C_0^{-1}\beta^3 \le |\mathrm{Rm}_{\hat{h}_\beta}|(x_{\beta,\pm}) \le C_0\beta^3$$

   as $\beta \to \infty$, and rescalings of the metrics near $x_{\beta,\pm}$ converge, in the pointed Gromov-Hausdorff sense, to Tian-Yau metrics on a del Pezzo surface of degree $b_\pm$, minus a smooth anti-canonical curve.

Remark 1.1. The Riemannian geometry of the regular collapsing regions is actually completely understood. For each $t \in \mathcal{R}_\epsilon$, $F_\beta^{-1}(t)$ is a 3-dimensional Heisenberg nilmanifold if the S^1-bundle is nontrivial, and is diffeomorphic to $\mathbb{T}^3$ otherwise. Furthermore, the universal cover of a regular preimage $F_\beta^{-1}(t_j + \epsilon, t_{j+1} - \epsilon)$ converges to a hyperkähler manifold $(\widetilde{U}_\infty, \tilde{g}_\infty)$ with a Heisenberg or Euclidean group of isometries according to whether the S^1-bundle is nontrivial or trivial. An explicit expression for $\tilde{g}_\infty$ in the Heisenberg case may be found in Section 2.2. In particular, our construction gives a concrete example of Lott's recent work classifying the regular regions in collapsing 4-manifolds with almost Ricci-flat metrics (see [Lot20] for more details).

For the precise definition of a 3-dimensional Heisenberg nilmanifold; see Section 2.1. These are S^1-bundles over $\mathbb{T}^2$, and thus they have a degree which is only well-defined up to sign. However, if one specifies a projection to an oriented $\mathbb{T}^2$, then the degree is a well-defined integer. We denote by Nil_b^3 a 3-dimensional nilmanifold of *degree* b, where we will always have a certain projection to an oriented $\mathbb{T}^2$ in mind. Let $t_0 = 0$, $t_{m+1} = 1$, and let d_j be the degree of a nilpotent fiber $\mathrm{Nil}_{d_j}^3$ on the interval $(t_j + \epsilon, t_{j+1} - \epsilon)$, $j = 0, \dots, m$, with $d_0 = b_-$ and $d_m = -b_+$. The following *Domain Wall Crossing Theorem* describes the possible jumps of the degrees of the nilmanifolds upon crossing the singular regions.

Theorem B. *Given any m-tuple of positive integers $(w_1,\dots,w_m)$ satisfying*

$$\sum_{j=1}^{m} w_j = b_- + b_+,$$

there exist examples in Theorem A with $d_j - d_{j+1} = w_{j+1}$, $j = 0,\dots,m-1$. Furthermore, near each singular point t_j, exactly w_j Taub-NUT bubbles occur.

Remark 1.2. This domain wall crossing phenomenon has been studied in the physics literature, namely, it arises in Type IIA massive superstring theory; see [Hul98].

1.3. **Outline of the paper.** Our proof adopts the description, originally due to Donaldson [Don06], of a hyperkähler metric in terms of a triple of three symplectic forms, a description which was also used, for example, in [CC20, FLS17, Fos19]. Let M^4 be an oriented 4-manifold with a volume form dvol_0. A triple of 2-forms $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$ is called *closed* if $d\omega_1 = d\omega_2 = d\omega_3 = 0$, and is called *definite* if the matrix $Q = (Q_{ij})$ defined by

$$\frac{1}{2}\omega_i \wedge \omega_j = Q_{ij}\,\mathrm{dvol}_0$$

is positive. Given a definite triple $\boldsymbol{\omega}$, the associated volume form is defined as

$$\mathrm{dvol}_{\boldsymbol{\omega}} = (\det(Q))^{\frac{1}{3}}\,\mathrm{dvol}_0,$$

which is independent of the choice of volume form dvol_0. We denote by

$$Q_{\boldsymbol{\omega}} \equiv (\det(Q))^{-\frac{1}{3}} Q$$

the normalized matrix with unit determinant. A definite triple $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$ is called a *hyperkähler triple* if it is closed and the renormalized coefficient matrix satisfies

$$Q_{\boldsymbol{\omega}} = \mathrm{Id}\,. \tag{1.2}$$

Note that (1.2) is equivalent to the equation

$$\frac{1}{2}\omega_i \wedge \omega_j = \frac{1}{6}\delta_{ij}(\omega_1^2+\omega_2^2+\omega_3^2), \quad 1 \le i \le j \le 3. \tag{1.3}$$

There is a well-known algebraic isomorphism of homogeneous spaces

$$\mathrm{SL}(4,\mathbb{R})/\,\mathrm{SO}(4) \cong \mathrm{SO}_0(3,3)/(\mathrm{SO}(3)\times\mathrm{SO}(3)) \tag{1.4}$$

(see for example [Sal89, Chapter 7]), so each definite triple $\boldsymbol{\omega}$ determines a Riemannian metric $g_{\boldsymbol{\omega}}$ such that each ω_j is self-dual with respect to $g_{\boldsymbol{\omega}}$ and $\mathrm{dvol}_{g_{\boldsymbol{\omega}}} = \mathrm{dvol}_{\boldsymbol{\omega}}$. If moreover $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$ is a hyperkähler triple, then $g_{\boldsymbol{\omega}}$ is a hyperkähler metric. Furthermore, $\omega_2 + \sqrt{-1}\omega_3$ is a holomorphic 2-form with respect to the complex structure determined by ω_1 which makes ω_1 a Kähler form.

Our proof will not directly construct a Ricci-flat metric on the K3 surface. Instead, we will construct a manifold $\mathcal{M}$ and a family of closed definite triples on $\mathcal{M}$, denoted by $\boldsymbol{\omega}_\beta^{\mathcal{M}}$, all of which depend upon a gluing parameter β. The associated Riemannian metric will be denoted by g_β. For β sufficiently large, we will then perturb the approximate triple to obtain a hyperkähler triple, which will then yield in particular a Ricci-flat Kähler metric. In order to carry out this perturbation, we will use the implicit function theorem (see Lemma 10.1). As we will see in Section 10.1, this requires finding a right inverse to the operator

$$\mathscr{L}_{g_\beta} = (\mathscr{D}_{g_\beta} \oplus \mathrm{Id})\otimes\mathbb{R}^3 : (\Omega^1(\mathcal{M})\oplus\mathcal{H}^+(\mathcal{M}))\otimes\mathbb{R}^3 \longrightarrow (\Omega^0(\mathcal{M})\oplus\Omega^2_+(\mathcal{M}))\otimes\mathbb{R}^3, \tag{1.5}$$

where $\mathcal{H}^+(\mathcal{M})$ denotes the space of self-dual harmonic 2-forms with respect to g_β, the operator $\mathscr{D}_{g_\beta}$ is defined by

$$\mathscr{D}_{g_\beta} \equiv d^* \oplus d^+ : \Omega^1(\mathcal{M}) \longrightarrow \Omega^0(\mathcal{M}) \oplus \Omega^2_+(\mathcal{M}),$$

where d^* is the divergence operator and d^+ is the self-dual part of the exterior derivative. We will refer to $\mathscr{L}_{g_\beta}$ as the *linearized operator*. We will require a bound for the right inverse of $\mathscr{L}_{g_\beta}$ in certain weighted Hölder norms which is independent of the gluing parameter β. We will next give an outline of the main steps required in order to achieve this bound.

In Section 2, we give some background on the Gibbons-Hawking ansatz. We also define the Heisenberg nilmanifolds and describe the *Calabi model space* used in the Tian-Yau construction from the Gibbons-Hawking point of view. Lastly, we construct a harmonic function whose associated Gibbons-Hawking ansatz defines the neck region $\mathcal{N}$.

The Calabi model space will be described in Section 3 from a complex geometric perspective, which will be used to obtain the precise asymptotic behavior of the complete hyperkähler Tian-Yau spaces. The main result is Proposition 3.4 which roughly states that a complete Tian-Yau space is exponentially asymptotic to a Calabi model space, up to any arbitrary order of derivatives.

In Section 4, we will establish a Liouville-type theorem for harmonic functions which says that any harmonic function of sufficiently small exponential growth on a complete hyperkähler Tian-Yau space has to be a constant. To prove this, the asymptotics proved in Section 3 will be crucial. These will allow us to reduce our problem to a question about harmonic functions on the Calabi model space, which will be solved using separation of variables. This step is already quite involved as it requires us to develop some new elliptic theory on a model space which is a doubly warped product rather than just a cylinder.

A 1-form ϕ is said to be *half-harmonic* if it is in the kernel of $\mathscr{D}$, i.e.,

$$d^*\phi = 0, \quad d^+\phi = 0. \tag{1.6}$$

In Section 5, we will prove a Liouville theorem for half-harmonic 1-forms on a complete hyperkähler Tian-Yau space (see Theorem 5.1). This will be crucial later in the proof of the key uniform estimate (Proposition 9.2) required for our gluing construction. An important observation here is that equation (1.6) is equivalent to the $(0,1)$-component of ϕ satisfying $\bar{\partial}^*\phi^{0,1} = \bar{\partial}\phi^{0,1} = 0$ for any choice of compatible integrable complex structure. This will allow us to invoke tools from complex geometry.

In Section 6 we will complete the construction of the neck region and construct a closed almost hyperkähler triple on a certain manifold $\mathcal{M}$. We will also describe some topological invariants of $\mathcal{M}$. Note that at this stage it would be difficult to show directly that $\mathcal{M}$ is actually *diffeomorphic* to K3, a fact which will follow easily once we have shown that it admits a hyperkähler metric.

Section 7 will focus on the rescaling geometry of the manifold $\mathcal{M}$. This leads to a decomposition of $\mathcal{M}$ (see Section 7.1 and 7.2) into 9 different regions, labeled I, II, III, $\mathrm{IV}_\pm$, $\mathrm{V}_\pm$, $\mathrm{VI}_\pm$, with each of these regions exhibiting different regularity and convergence behaviors. In particular, we will describe all of the rescaled Gromov-Hausdorff limits for base points in each of the various regions.

In Sections 8 and 9, we will set up the weighted analysis package and prove some technical results. The first step in Section 8 is to define weighted Hölder spaces which are consistent with the different collapsing behaviors in different regions of $\mathcal{M}$. The remainder of Section 8 will then consist of proving the weighted Schauder estimate in Proposition 8.2. In Section 9, we will prove a uniform injectivity estimate for the operator $\mathscr{D}_{g_\beta}$; see Proposition 9.2.

Existence of a bounded right inverse to the operator $\mathscr{L}_{g_\beta}$ for β sufficiently large will be proved in Section 10; see Proposition 10.2. In the remainder of Section 10, we will then complete the proofs of Theorems A and B.

2. The Gibbons-Hawking ansatz and the model space

We first recall the Gibbons-Hawking construction of 4-dimensional hyperkähler structures with an S^1 symmetry. Let (U, h_U) be a 3-dimensional parallelizable flat manifold. Then the metric $h_U = e_1^2 + e_2^2 + dz^2$, where e_1, e_2, dz are global parallel 1-forms. Let V be a positive harmonic function on U so that $*dV$ is a closed 2-form. We assume further that the de Rham class $[\frac{1}{2\pi} * dV]$ lies in $H^2(U, \mathbb{Z})$ so that $*dV$ is the curvature form of a unitary connection $-\sqrt{-1}\theta$ on a circle bundle $\pi : \mathfrak{M} \to U$. Then

$$g = V\pi^* h_U + V^{-1}\theta \otimes \theta$$

is a hyperkähler metric on $\mathfrak{M}$ invariant under the S^1-action. This is called the *Gibbons-Hawking ansatz*. The corresponding triple of symplectic forms is given by

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = (dz \wedge \theta + Ve_1 \wedge e_2, e_1 \wedge \theta + Ve_2 \wedge dz, e_2 \wedge \theta + Vdz \wedge e_1). \tag{2.1}$$

The triple $\boldsymbol{\omega}$ satisfies the hyperkähler triple equations (1.3).

By definition the metric g depends not only on the harmonic function V but also on the choice of connection θ with curvature form $*dV$. One can check that gauge equivalent connections lead to isometric metrics; so in essence g depends only on the gauge equivalence class of θ. Different gauge equivalence classes differ by tensoring with a flat connection, and the set of isomorphism classes of flat connections is given by $H^1(U, \mathbb{R})/H^1(U, \mathbb{Z})$.

Conversely, any 4-dimensional hyperkähler metric admitting a tri-holomorphic Killing symmetry is locally given by the Gibbons-Hawking ansatz. To see this we notice that in the formula above, we can intrinsically interpret V^{-1} as the norm-squared of the Killing field, and the projection map π as the hyperkähler moment map.

To get more interesting examples one often allows V to have isolated poles $\mathcal{P}_k \equiv \{p_1, \ldots, p_k\}$ such that near each p_j, V can be written as $\frac{1}{2r_j} + h_j$ where r_j is the distance function to p_j and h_j is a smooth harmonic function. Then the corresponding metric g is defined on a manifold $\mathfrak{M}$ admitting a projection $\pi : \mathfrak{M} \to U$, such that π is a circle bundle over $U \setminus \mathcal{P}_k$ and near each point p_j, π is modeled on the Hopf fibration

$$\pi_H : \mathbb{C}^2 \to \mathbb{R}^3,\ (z_1, z_2) \mapsto \big(|z_1|^2 - |z_2|^2, 2Re(z_1 z_2), 2Im(z_1 z_2)\big).$$

Example 2.1. Let $U = \mathbb{R}^3$. If $V = \sigma$ (a positive constant), then $(\mathfrak{M}, g)$ is a flat product $\mathbb{R}^3 \times S^1$. If $V = \frac{1}{2r}$, then $(\mathfrak{M}, g)$ is flat Euclidean space $\mathbb{R}^4$ and the map π is exactly the Hopf fibration. If $V_\sigma = \sigma + \frac{1}{2r}$ then $(\mathfrak{M}, g)$ is the *Taub-NUT space*. This is again diffeomorphic to $\mathbb{R}^4$ but has cubic volume growth and the length of the S^1 fibers approaches a positive constant at infinity. Notice that as σ varies, these metrics are isometric up to dilation. This is most easily seen using the above intrinsic description. We take the metric g_1 constructed using $V_1 = 1 + \frac{1}{2r}$

and rescale $g_\sigma = \sigma^{-1} g_1$. Then the length of the S^1 orbits becomes $(\sigma V_1)^{-1/2}$ and the hyperkähler moment map becomes $\pi_\sigma = \sigma^{-1}\pi$. Thus, g_σ can be written in Gibbons-Hawking form with potential σV_1.

By taking multiple poles, we similarly obtain other hyperkähler manifolds which are asymptotic to quotients of either $\mathbb{R}^4$ or Taub-NUT space by cyclic groups. These are referred to in the literature as ALE/ALF spaces of A_{k-1}-type, respectively. See [Min11] for a complete theory of ALF-A_{k-1} spaces.

Example 2.2. Let $U = S^1 \times \mathbb{R}^2$ and let V be a Green's function with exactly one pole on $S^1 \times \{0\}$. In [GW00] V is constructed by passing to the universal cover $\widetilde{U} = \mathbb{R}^3$, where the lifted function $\widetilde{V}$ is a periodic Green's function constructed using a Weierstraß series. V is only positive in a certain bounded open set in U. The corresponding hyperkähler metric on this bounded open set is called the *Ooguri-Vafa metric*. With one particular choice of a compatible complex structure, $\mathfrak{M}$ becomes a holomorphic elliptic fibration over a disc $\mathbb{D} \subset \mathbb{R}^2 = \mathbb{C}$, and the singular fiber has monodromy of type I_1. The Ooguri-Vafa metric plays a crucial role in the work of Gross-Wilson [GW00] on collapsing Calabi-Yau metrics on elliptic K3 surfaces with exactly 24 singular fibers of type I_1.

In the following subsections, we will consider Gibbons-Hawking spaces $(\mathfrak{M}, g)$ whose base is a large open subset of a flat cylinder $U \equiv \mathbb{T}^2_{x,y} \times \mathbb{R}_z$. In Section 2.2, we take V to be linear in z. This yields the model space at infinity of the Tian-Yau metrics [TY90] as well as of the gravitational instantons with $r^{4/3}$ volume growth and r^{-2} curvature decay from [Hei12]. In this situation, $\mathfrak{M}$ is diffeomorphic to the product of a line and a 3-dimensional Heisenberg nilmanifold. We begin by collecting together some useful basic facts about Heisenberg nilmanifolds in Section 2.1. In Section 2.3 we then consider a doubly-periodic analog of Example 2.2 over $\mathbb{T}^2$ times a bounded interval and show that this is asymptotic to the model space of Section 2.2 near the ends of the interval. Ultimately this new metric will serve as the neck region in our gluing construction.

2.1. **The Heisenberg nilmanifolds.**

The 3-dimensional Heisenberg group is

$$H(1,\mathbb{R}) \equiv \left\{ \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, t \in \mathbb{R} \right\}.$$

Fix $\epsilon > 0$ and $\tau \in \mathbb{C}$ with $Im(\tau) \neq 0$ and define a lattice $\Lambda \equiv \epsilon\mathbb{Z}\langle 1, \tau\rangle \subset \mathbb{R}^2_{x,y} = \mathbb{C}$. Let $\tau_1 = Re(\tau)$ and $\tau_2 = Im(\tau)$. Then $A = \mathrm{Area}(\mathbb{R}^2_{x,y}/\Lambda) = \epsilon^2\tau_2$. For $b \in \mathbb{Z}_+$, the *Heisenberg nilmanifold of degree* b, denoted by $\mathrm{Nil}^3_b(\epsilon, \tau)$, is the quotient of $H(1,\mathbb{R})$ by the left action generated by

$$\begin{aligned} (x, y, t) &\mapsto (x + \epsilon, y, t + \epsilon y), \\ (x, y, t) &\mapsto (x + \epsilon\tau_1, y + \epsilon\tau_2, t + \epsilon\tau_1 y), \\ (x, y, t) &\mapsto (x, y, t + Ab^{-1}). \end{aligned}$$

Note that the forms

$$dx, \quad dy, \quad \theta_b \equiv (2\pi b A^{-1})(dt - x dy) \tag{2.2}$$

are a basis of left-invariant 1-forms on Nil^3_b. Also, it is clear that Nil^3_b is the total space of a degree b circle fibration

$$S^1 \longrightarrow \mathrm{Nil}^3_b \xrightarrow{\pi} \mathbb{T}^2 \equiv \mathbb{R}^2_{x,y}/\Lambda. \tag{2.3}$$

The following result will be needed later in Proposition 6.6.

Proposition 2.3. *For Nil^3_b with $b \neq 0$, we have $b_1(\mathrm{Nil}^3_b) = b_2(\mathrm{Nil}^3_b) = 2$, and the de Rham cohomology group $H^1(\mathrm{Nil}^3_b)$ is generated by $\pi^* dx$ and $\pi^* dy$.*

Proof. The Gysin sequence associated to (2.3) yields

$$0 \to H^1(\mathbb{T}^2) \xrightarrow{\pi^*} H^1(\mathrm{Nil}^3_b) \to H^0(\mathbb{T}^2) \xrightarrow{\cup e} H^2(\mathbb{T}^2) \to \cdots \tag{2.4}$$

Since the Euler class e of the bundle is b times a generator of $H^2(\mathbb{T}^2)$, the mapping $\cup e : \mathbb{R} \cong H^0(\mathbb{T}^2) \to H^2(\mathbb{T}^2) \cong \mathbb{R}$ is just multiplication by b, hence is an isomorphism. Consequently, $\pi^* : H^1(\mathbb{T}^2) \to H^1(\mathrm{Nil}^3_b)$ is also an isomorphism. Since Nil^3_b is a compact orientable 3-manifold, Poincaré duality implies that $b_1 = b_2$. □

For $b \in \mathbb{Z}_+$, we define the Heisenberg nilmanifold Nil^3_{-b} to be the quotient of $H(1, \mathbb{R})$ by the action generated by

$$\begin{aligned}
&(x, y, t) \mapsto (x + \epsilon, y, t - \epsilon y),\\
&(x, y, t) \mapsto (x + \epsilon\tau_1, y + \epsilon\tau_2, t - \epsilon\tau_1 y),\\
&(x, y, t) \mapsto (x, y, t - Ab^{-1}).
\end{aligned}$$

Note that the generated action is conjugate to the previous action by the mapping $(x, y, t) \mapsto (-x, -y, -t)$, and the forms

$$dx, \quad dy, \quad \theta_{-b} \equiv (2\pi b A^{-1})(dt + x dy)$$

are a basis of left-invariant 1-forms on Nil^3_b.

2.2. **The Gibbons-Hawking model space.** Consider an oriented torus $\mathbb{T}^2$ with a flat metric of area A and let $U = \mathbb{T}^2_{x,y} \times \mathbb{R}_{z>0}$, where we have fixed a choice of an orthogonal frame $\{e_1, e_2\}$ on $\mathbb{T}^2$ such that $g_{\mathbb{T}^2} = A(e_1^2 + e_2^2)$. Let $V(z) = 2\pi b z \cdot A^{-1}$ for a positive integer $b > 0$. Fixing a connection form θ such that $d\theta = 2\pi b \cdot A^{-1}\mathrm{dvol}_{\mathbb{T}^2}$, the corresponding hyperkähler Gibbons-Hawking metric g is given by

$$g = \frac{2\pi b z}{A}\Big(g_{\mathbb{T}^2} + dz^2\Big) + \frac{A}{2\pi b z}\theta^2. \tag{2.5}$$

As before we denote by $\mathfrak{M}$ the underlying manifold of g. It has one complete end as $z \to \infty$ and one incomplete end as $z \to 0$. Moreover, for each $z_0 > 0$, the level set $\{z = z_0\}$ is a Heisenberg nilmanifold $\mathrm{Nil}^3_b(\epsilon, \tau)$ with a z_0-dependent left-invariant metric, where τ denotes the modulus of our flat 2-torus in the upper half-plane and $A = \epsilon^2\tau_2$ as in Section 2.1. Making the substitution $z = (3/2)s^{2/3}$ and then scaling appropriately, the Gibbons-Hawking metric g takes the form

$$ds^2 + s^{2/3} g_{\mathbb{T}^2} + s^{-2/3}\Big(\frac{A}{3\pi b}\theta\Big)^2.$$

In this form, it is easy to see that the volume growth is $\sim s^{4/3}$ and that $|\mathrm{Rm}| \sim s^{-2}$ as $s \to \infty$. One can also show using the Chern-Gauss-Bonnet formula that the L^2-norm of Rm is finite.

A very convenient property for our purposes is that if we do change the connection 1-form θ by a flat connection, then the associated Gibbons-Hawking metric (2.5) actually only changes by a diffeomorphism (although this diffeomorphism necessarily breaks the S^1-bundle structure on $\mathfrak{M}$). To see this, fix any choice of θ such that $d\theta = 2\pi b A^{-1}\,\mathrm{dvol}_{\mathbb{T}^2}$. Note that we can always arrange by parallel transport in the z-direction that the dz-component of θ vanishes. Then we only need to consider the case that θ gets changed by the pullback (under the projection $\mathfrak{M} \xrightarrow{\pi} U \to \mathbb{T}^2$) of a parallel 1-form η on $\mathbb{T}^2$. Write $\eta = v \lrcorner \,\mathrm{dvol}_{\mathbb{T}^2}$, where v is a parallel vector field on $\mathbb{T}^2$. Let $\hat{v}$ be the θ-horizontal lift of v to $\mathfrak{M}$ and let $\hat{f}_s$ be the 1-parameter group of diffeomorphisms generated by $\hat{v}$, which covers a 1-parameter group of translations on $\mathbb{T}^2$. Then

$$\frac{d}{ds}(\hat{f}_s^*\theta) = \hat{v} \lrcorner \hat{f}_s^*(d\theta) + d(\hat{v} \lrcorner \hat{f}_s^*\theta) = 2\pi b A^{-1}(\hat{v} \lrcorner \pi^*(\mathrm{dvol}_{\mathbb{T}^2})) = 2\pi b A^{-1}\pi^*\eta,$$

using the fact that $d\hat{f}_s|_x(\hat{v}|_x) = \hat{v}|_{\hat{f}_s(x)}$ for all $x \in \mathfrak{M}$. Thus,

$$\hat{f}_s^*\theta = \theta + 2\pi b s A^{-1}\pi^*\eta.$$

This shows that any two possible choices of θ with vanishing dz-component differ by the lift to $\mathfrak{M}$ of the translation action of $\mathbb{T}^2$ on itself. Consequently, the corresponding hyperkähler metrics (2.5) are clearly isometric as well. We note that with the particular choice $\theta = \theta_b$ from (2.2), these diffeomorphisms can be written down explicitly as follows: for all $p, q \in \mathbb{R}$, the mapping

$$\varphi(x, y, t, z) = (x - q, y + p, t + px, z)$$

descends to a diffeomorphism of $\mathrm{Nil}^3_b(\epsilon, \tau)_{x,y,t} \times \mathbb{R}_z$ which satisfies

$$\varphi^*\theta_b = \theta_b + 2\pi b A^{-1}(pdx + qdy).$$

Remark 2.4. View the flat torus $\mathbb{T}^2$ as an elliptic curve E of modulus τ with respect to the complex structure J defined by $Je_1 = e_2$. Then there is exactly one g-parallel complex structure J_0 on the total space $\mathfrak{M}$ that makes the projection map to E holomorphic. This choice of complex structure realizes $\mathfrak{M}$ as an open subset of the total space of a degree b holomorphic line bundle L over E (more precisely, as a tubular neighborhood of the zero section of L with the zero section removed). However, we can choose a different g-parallel complex structure J_1 on $\mathfrak{M}$ such that $J_1\theta = zdx$. With respect to J_1 we can view $\mathfrak{M}$ as a holomorphic elliptic fibration over a punctured disc in $\mathbb{C}$. The monodromy of this fibration is given by the matrix $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$; see [Sco83, pp. 469-470] for more details. This gives a different compactified model for $\mathfrak{M}$ where the compactifying divisor is a singular fiber of Kodaira type I_b. The J_1-Kähler form of our hyperkähler model metric is then given by an appropriate *semi-flat ansatz* [GSVY90, GW00], and provides the model at infinity for the gravitational instantons with volume growth $\sim r^{4/3}$ and curvature decay $\sim r^{-2}$ constructed in [Hei12]. We will pursue this observation further in [HSVZ], connecting the complete hyperkähler metrics of [TY90] and of [Hei12] by global hyperkähler rotations.

2.3. **Green's function on a flat cylinder.** The neck region in our gluing construction is given by a doubly-periodic analog of the Ooguri-Vafa metric. To construct this metric using the Gibbons-Hawking ansatz, we first need to construct a Green's function V_∞ on $(\mathbb{T}^2 \times \mathbb{R}, g_0)$, where $\mathbb{T}^2$ is any flat 2-torus. We also need to determine the asymptotics of V_∞ as $|z| \to \infty$ in order to ensure that the neck matches the Calabi model space from Section 2.2 on both ends.

It is known that the flat cylinder $\mathbb{T}^2 \times \mathbb{R}$ is *parabolic*, namely, it does not admit a positive Green's function. In fact, Cheng and Yau proved that any complete non-compact Riemannian manifold with $\mathrm{Vol}(B_R(p)) \leq CR^2$ must be parabolic; see Theorem 1 and Corollary 1 in [CY75].

We now construct a particular sign-changing Green's function V_∞. Equip $\mathbb{T}^2_{x,y} \times \mathbb{R}_z$ with the flat product metric g_0. Fix a point p on $\mathbb{T}^2 \times \mathbb{R}$ with $z(p) = 0$. For $R > 0$ let V_R denote the unique function on $\mathbb{T}^2 \times [-R, R]$ satisfying

$$\begin{gathered} -\Delta_{g_0} V_R = 2\pi\delta_p, \ z \in (-R, R), \\ V_R = 0, \ z = \pm R. \end{gathered}$$

The normalization of the right-hand side is precisely chosen in such a way that $V_R = \frac{1}{2r} + O(1)$ near p, where r denotes the g_0-distance to p. Thus, the 4-dimensional Gibbons-Hawking metric associated to V_R (or $V_R + C$ for any constant C) extends smoothly across p; cf. Example 2.1. For any $R > 0$, V_R is a positive Green's function that is vanishing on the boundary $\{z = \pm R\}$. If $R_2 > R_1$, then applying the maximum principle, $V_{R_2}(x) > V_{R_1}(x)$ for every $x \in B_{R_1}(p)$. Let us define

$$C_R \equiv \sup_{\partial B_1(p)} V_R.$$

Then it follows from the parabolicity of $\mathbb{T}^2 \times \mathbb{R}$ that $C_R \to \infty$ as $R \to \infty$. By [LT87, Theorem 1], $V_R(x, y, z) - C_R$ converges to a function $V_\infty(x, y, z)$ uniformly on compact subsets of $(\mathbb{T}^2 \times \mathbb{R}) \setminus \{p\}$ as $R \to \infty$. Moreover, $V_\infty \leq 0$ on the complement of $B_1(p)$ and

$$-\Delta_{g_0} V_\infty = 2\pi\delta_p \text{ on } \mathbb{T}^2 \times \mathbb{R}.$$

Notice also that V_∞ is symmetric in z by construction.

Theorem 2.5. *Let V_∞ be the sign-changing Green's function as defined above. Then there exists a constant $\underline{c}_0 \in \mathbb{R}$ such that for all $k \in \mathbb{N}$,*

$$|\nabla^k_{g_0}(V_\infty - k_\pm z - \underline{c}_0)| = O(e^{-\sqrt{\lambda_1}|z|}), \ z \to \pm\infty,$$

where

$$k_- = -k_+ = \frac{\pi}{\mathrm{Area}_{g_0}(\mathbb{T}^2)} > 0 \tag{2.6}$$

and $\lambda_1 > 0$ is the smallest positive eigenvalue of $-\Delta_{\mathbb{T}^2}$.

Proof. Consider the fiberwise average

$$\mathcal{V}_\infty(z) \equiv \frac{1}{\mathrm{Area}_{g_0}(\mathbb{T}^2)} \int_{\mathbb{T}^2 \times \{z\}} V_\infty(x, y, z)\, \mathrm{dvol}_{g_0}(x, y).$$

This is well-defined, smooth in z for $z \neq 0$, and continuous at $z = 0$. For $z \neq 0$, we have that

$$\begin{aligned} \mathcal{V}''_\infty(z) &= \frac{1}{\mathrm{Area}_{g_0}(\mathbb{T}^2)} \int_{\mathbb{T}^2 \times \{z\}} \Big(\frac{d^2 V_\infty}{dz^2}\Big)\, \mathrm{dvol}_{g_0} \\ &= -\frac{1}{\mathrm{Area}_{g_0}(\mathbb{T}^2)} \int_{\mathbb{T}^2 \times \{z\}} (\Delta_{\mathbb{T}^2} V_\infty)\, \mathrm{dvol}_{g_0} = 0. \end{aligned} \tag{2.7}$$

This implies that $\mathcal{V}_\infty(z)$ is a piecewise linear function. By construction, $\lim_{|z|\to+\infty} V_\infty(x,y,z) = -\infty$. Then for all $z > 0$,

$$\mathcal{V}'_\infty(z) = constant \equiv k_+ < 0. \tag{2.8}$$

Denote $D_0 \equiv \mathrm{Diam}_{g_0}(\mathbb{T}^2)$. Choose $R_0 > 10D_0$ large enough so that $V_\infty(x,y,z) \leq -1$ on the set $(\mathbb{T}^2 \times \mathbb{R}) \setminus B_{R_0}(p)$. Then for any fixed $q \in (\mathbb{T}^2 \times \mathbb{R}) \setminus B_{2R_0}(p)$ and $r \in (2D_0, 4D_0)$, we can apply the Harnack inequality to the harmonic function V_∞, which itself is negative in the geodesic ball $B_r(q) \subset (\mathbb{T}^2\times\mathbb{R})\setminus B_{R_0}(p)$. More precisely, passing to the universal cover and applying the standard Harnack inequality for positive harmonic functions on a fixed ball in $\mathbb{R}^3$, we see that there is a uniform constant $C_0 > 0$ depending only on $D_0 > 0$ such that for all $w_1, w_2 \in B_r(q)$,

$$\frac{1}{C_0} \leq \frac{V_\infty(w_1)}{V_\infty(w_2)} \leq C_0. \tag{2.9}$$

Since the fiber average $\mathcal{V}_\infty(z)$ of V_∞ is linear in z with slope $k_+ < 0$, (2.9) yields that

$$-C_2 z \leq V_\infty(x,y,z) \leq -C_1 z \tag{2.10}$$

for $z \gg 1$, where the constants C_1 and C_2 depend only on the constants C_0 and k_+.

We denote by $\Lambda_{\mathbb{T}^2} = \{\lambda_j\}_{j=1}^\infty$ the positive spectrum of $-\Delta_{\mathbb{T}^2}$ and expand V_∞ according to the eigenfunctions of $\Delta_{\mathbb{T}^2}$ along the torus fiber $\mathbb{T}^2 \times \{z\}$ for each fixed $z > 0$. This yields

$$V_\infty(x,y,z) = (k_+ z + c_+) + \sum_{j=1}^\infty f_j(z) h_j(x,y),$$

where $k_+ < 0$ is the constant of (2.8) and where

$$f_j''(z) = \lambda_j f_j(z),\ -\Delta_{\mathbb{T}^2} h_j = \lambda_j h_j,\ \int_{\mathbb{T}^2} |h_j|^2 = 1.$$

Immediately,

$$f_j(z) = c_j e^{-\sqrt{\lambda_j} z} + c_j^* e^{\sqrt{\lambda_j} z}.$$

Notice that

$$\int_{\mathbb{T}^2\times\{z\}} |V_\infty|^2 = (k_+ z + c_+)^2 + \sum_{j=1}^\infty |f_j(z)|^2.$$

By the linear growth property (2.10), we see that $c_j^* = 0$ for all $j \in \mathbb{Z}_+$. Therefore,

$$\int_{\mathbb{T}^2\times\{z\}} |V_\infty - (k_+ z + c_+)|^2 = \sum_{j=1}^\infty |c_j|^2 e^{-2\sqrt{\lambda_j} z} = O(e^{-2\sqrt{\lambda_1} z}) \text{ as } z \to +\infty, \tag{2.11}$$

where $\lambda_1 > 0$ is the minimum of $\Lambda_{\mathbb{T}^2}$. To see (2.11), note that the series converges for $z = 1$. Applying elliptic regularity to the harmonic function $\widehat{V}_\infty \equiv V_\infty - (k_+ z + c_+)$,

$$\|\widehat{V}_\infty\|_{W^{2,2}(B_{r/2}(q))} \leq C\|\widehat{V}_\infty\|_{L^2(B_r(q))} \leq Ce^{-2\sqrt{\lambda_1} z}$$

for all balls $B_r(q)$ as above, where C depends only on the diameter and on the injectivity radius of $\mathbb{T}^2$. By the 3-dimensional Sobolev embedding $W^{2,2} \hookrightarrow C^{0,\frac{1}{2}}$,

$$|V_\infty(x,y,z) - (k_+ z + c_+)| = O(e^{-\sqrt{\lambda_1} z}) \text{ as } z \to +\infty.$$

Standard elliptic regularity then shows that for any $k \in \mathbb{N}$,

$$|\nabla^k_{g_0}(V_\infty(x,y,z) - (k_+ z + c_+))| = O(e^{-\sqrt{\lambda_1} z}) \text{ as } z \to +\infty.$$

The same argument applies in the case $z \to -\infty$.

Now we prove the slope relation (2.6). Fix $R > 0$. Then by Green's formula,

$$\mathcal{V}'_\infty(R) - \mathcal{V}'_\infty(-R) = \int_{\mathbb{T}^2 \times [-R,R]} \Delta V_\infty.$$

Thus, by the definition of k_+ and the analogous definition of k_-,

$$k_+ \operatorname{Area}_{g_0}(\mathbb{T}^2) - k_- \operatorname{Area}_{g_0}(\mathbb{T}^2) = -2\pi.$$

It follows that

$$k_- - k_+ = \frac{2\pi}{\operatorname{Area}_{g_0}(\mathbb{T}^2)}.$$

Since V_∞ is symmetric in z, it holds that $k_- = -k_+$ and $c_- = c_+$ (we denote $\underline{c}_0 \equiv c_- = c_+$). □

It is straightforward to use superposition to extend the above construction to the case of multiple poles. Precisely, we have Corollary 2.6.

Corollary 2.6. *Let* $(\mathbb{T}^2 \times \mathbb{R}, g_0)$ *be a flat cylinder with a flat product metric* g_0*. Given a finite set* $\mathcal{P}_{m_0} \equiv \{p_1, \dots, p_{m_0}\} \subset \mathbb{T}^2 \times \mathbb{R}$ *such that* $\sum_{m=1}^{m_0} z(p_m) = 0$*, there is a sign-changing Green's function* V_∞ *with*

$$-\Delta_{g_0} V_\infty = 2\pi \sum_{k=1}^{m_0} \delta_{p_k}.$$

There also exist linear functions $L_\pm(z) = k_\pm z$ *with*

$$k_- = -k_+ = \frac{\pi m_0}{\operatorname{Area}_{g_0}(\mathbb{T}^2)} > 0$$

such that for all $k \in \mathbb{N}$*,*

$$|\nabla^k_{g_0}(V_\infty - L_\pm)| = O(e^{-\sqrt{\lambda_1}|z|}) \tag{2.12}$$

as $z \to \pm\infty$*, where* $\lambda_1 > 0$ *is the smallest positive eigenvalue of* $-\Delta_{\mathbb{T}^2}$*.*

Proof. By a translation in z, Theorem 2.5 produces a Green's function V_i with a pole at p_i and which is symmetric in z about $z(p_i)$. The condition $\sum_{m=1}^{m_0} z(p_m) = 0$ ensures that $V_\infty = \sum_{m=1}^{m_0} V_m$ will have the same constant term at both ends, which can then be removed by subtracting a constant. □

In Section 6, we will use V_∞ to construct a potential V_β which is positive in a large region, and such that $[\frac{1}{2\pi} * dV_\beta]$ is an *integral* class in $H^2(U, \mathbb{Z})$. This will be used to define our neck metric.

3. The asymptotic geometry of Tian-Yau spaces

In this section we review the Tian-Yau construction [TY90] of complete Ricci-flat Kähler metrics on the complement of a smooth anti-canonical divisor in a smooth Fano manifold. In complex dimension 2 these metrics are hyperkähler and will be used in our gluing construction in Section 6.

The Ricci-flat metrics constructed in [TY90] are asymptotic to the *Calabi model space* at infinity. We first give the definition of the latter. Let D be an $(n-1)$-dimensional compact Kähler manifold with trivial canonical bundle and let $L \to D$

be an ample line bundle. We fix a nowhere vanishing holomorphic $(n-1)$-form Ω_D on D with

$$\frac{1}{2}\int_D (\sqrt{-1})^{(n-1)^2}\Omega_D \wedge \overline{\Omega}_D = (2\pi c_1(L))^{n-1}. \tag{3.1}$$

By Yau's resolution of the Calabi conjecture [Yau78], there exists a unique Ricci-flat Kähler metric $\omega_D \in 2\pi c_1(L)$ satisfying the equation

$$\omega_D^{n-1} = \frac{1}{2}(\sqrt{-1})^{(n-1)^2}\Omega_D \wedge \overline{\Omega}_D.$$

Up to scaling there exists a unique hermitian metric h on L whose curvature form is $-\sqrt{-1}\omega_D$. We now fix a choice of h. Then the Calabi model space is the subset $\mathcal{C}$ of the total space of L consisting of all elements ξ with $0 < |\xi|_h < 1$, endowed with a nowhere vanishing holomorphic volume form $\Omega_{\mathcal{C}}$ and a Ricci-flat Kähler metric $\omega_{\mathcal{C}}$ which is incomplete as $|\xi|_h \to 1$ and complete as $|\xi|_h \to 0$. The holomorphic volume form $\Omega_{\mathcal{C}}$ is uniquely determined by the equation

$$Z \lrcorner\, \Omega_{\mathcal{C}} = p^*\Omega_D,$$

where $p : \mathcal{C} \to D$ is the bundle projection and Z is the holomorphic vector field generating the natural $\mathbb{C}^*$-action on the fibers of p. The metric $\omega_{\mathcal{C}}$ is given by the *Calabi ansatz*

$$\omega_{\mathcal{C}} = \frac{n}{n+1}\sqrt{-1}\partial\bar{\partial}(-\log|\xi|_h^2)^{\frac{n+1}{n}} \tag{3.2}$$

and satisfies the complex Monge-Ampère equation

$$\omega_{\mathcal{C}}^n = \frac{1}{2}(\sqrt{-1})^{n^2}\Omega_{\mathcal{C}} \wedge \overline{\Omega}_{\mathcal{C}},$$

hence is Ricci-flat. Define $z = (-\log|\xi|_h^2)^{1/n}$. It is easy to check that z is the $\omega_{\mathcal{C}}$-moment map for the natural S^1-action on L. Then the $\omega_{\mathcal{C}}$-distance function r to a fixed point in $\mathcal{C}$ satisfies

$$C^{-1}z^{\frac{n+1}{2}} \leq r \leq Cz^{\frac{n+1}{2}}$$

uniformly for all $z \geq 1$.

If $n = 2$ then $D = E$ is an elliptic curve and the Calabi model space is hyperkähler and agrees with the Gibbons-Hawking construction from Section 2.2. To see this, we choose $A = 2\pi \deg(L)$ and a flat Kähler form $\omega_E = Ae_1 \wedge e_2$ and a holomorphic 1-form $\Omega_E = A^{1/2}(e_1 + \sqrt{-1}e_2)$ on E such that $\omega_E = \frac{\sqrt{-1}}{2}\Omega_E \wedge \overline{\Omega}_E$. Then the Calabi ansatz produces a holomorphic 2-form $\Omega_{\mathcal{C}}$ and a Ricci-flat Kähler form $\omega_{\mathcal{C}}$ on $\mathcal{C}$ such that $\omega_{\mathcal{C}}^2 = \frac{1}{2}\Omega_{\mathcal{C}} \wedge \overline{\Omega}_{\mathcal{C}}$. In particular,

$$\boldsymbol{\omega}_{\mathcal{C}} \equiv (\omega_{\mathcal{C}}, Re(\Omega_{\mathcal{C}}), Im(\Omega_{\mathcal{C}})) \tag{3.3}$$

is a hyperkähler triple.

Proposition 3.1. *The hyperkähler structure $\boldsymbol{\omega}_{\mathcal{C}}$ is diffeomorphism equivalent to the one given by the Gibbons-Hawking ansatz with $V = z$ on $\mathrm{Nil}_b^3(\epsilon, \tau) \times (0, \infty)$ as in Section 2.2, where $b = \deg(L)$, τ is the modulus of E in the upper half-plane, $A = 2\pi b = \epsilon^2 Im(\tau)$, and $\theta = \theta_b$.*

Proof. The natural S^1-action on $\mathcal{C}$ obviously preserves $\boldsymbol{\omega}_{\mathcal{C}}$, so $\boldsymbol{\omega}_{\mathcal{C}}$ must be given by the Gibbons-Hawking construction. It suffices to determine the hyperkähler moment map $\pi = (\pi_1, \pi_2, \pi_3)$ and the potential V. We already know that the $\omega_{\mathcal{C}}$-moment map is given by $\pi_1 = (-\log|\xi|_h^2)^{1/2} = z$, and it is easy to check that the

$\Omega_{\mathcal{C}}$-moment map $\pi_2+\sqrt{-1}\pi_3$ equals the bundle projection $p : \mathcal{C} \to E$ composed with the Abel-Jacobi isomorphism $E \to \mathbb{C}/periods$, $x \mapsto \int_{x_0}^{x} \sqrt{-1}\Omega_E$, for an arbitrary but fixed basepoint $x_0 \in E$. Also, V^{-1} is the norm-squared of the Killing field, so that

$$V^{-1} = (-\log |\xi|_h^2)^{-1/2} = z^{-1}.$$

The Calabi construction provides us with an explicit realization $\mathfrak{M} = \mathcal{C}$ of the total space of the S^1-bundle and with a specific choice of connection form θ given by the Chern connection of L with respect to h. The diffeomorphism equivalence to the model $\mathrm{Nil}^3_b(\epsilon, \tau) \times (0, \infty)$ with connection form θ_b (after rotating $\Omega_{\mathcal{C}}$ to $e^{\sqrt{-1}\alpha}\Omega_{\mathcal{C}}$ if necessary) follows from the discussion in Section 2.2. $\square$

Remark 3.2. In [TY90] the volume growth and curvature decay rates of the n-dimensional Calabi metric $\omega_{\mathcal{C}}$ are estimated as $O(r^{\frac{2n}{n+1}})$ and $O(r^{-\frac{2}{n+1}})$, respectively. When $n = 2$, this suggests that $|\mathrm{Rm}|$ is borderline *not* in L^2. However, while the volume estimate is sharp for all n, the curvature estimate is sharp only for $n \geq 3$. For $n = 2$, the leading term in the asymptotic expansion of the curvature vanishes because the Calabi-Yau metric on an elliptic curve is flat, and the true curvature decay rate of $\omega_{\mathcal{C}}$ for $n = 2$ is $O(r^{-2})$. This was also pointed out by R. Kobayashi in [Kob90] but is perhaps most easily seen in the Gibbons-Hawking picture.

We now explain the Tian-Yau construction [TY90] of complete Ricci-flat Kähler metrics asymptotic to a Calabi ansatz metric at infinity. Let M be a smooth Fano manifold of complex dimension n, let $D \in |K_M^{-1}|$ be a smooth divisor, and let L denote the holomorphic normal bundle of D in M. Then D has trivial canonical bundle and L is ample, so in particular we can choose a holomorphic volume form Ω_D on D which satisfies (3.1). We fix a defining section S of D, so that S^{-1} can be viewed as a holomorphic n-form Ω_X on $X = M \setminus D$ with a simple pole along D. After scaling S by a nonzero complex constant, we may assume that Ω_D is the residue of Ω_X along D. (In practice this means that Ω_X is asymptotic to $\Omega_{\mathcal{C}}$ near D with respect to a suitable diffeomorphism between tubular neighborhoods of D in M and of the zero section in L.) Lastly, we fix a hermitian metric h_M on K_M^{-1} whose curvature form is strictly positive on M and restricts to the unique Ricci-flat Kähler form $\omega_D \in 2\pi c_1(L)$ on D. Then

$$\omega_X \equiv \frac{n}{n+1}\sqrt{-1}\partial\bar{\partial}(-\log |S|_{h_M}^2)^{\frac{n+1}{n}}$$

defines a Kähler form on a neighborhood of infinity in X. In fact, by multiplying h_M by a sufficiently small positive constant, we can arrange that ω_X is defined and strictly positive on all of X. As expected, ω_X is then complete and asymptotic to $\omega_{\mathcal{C}}$, where the hermitian metric h used in (3.2) is simply the restriction of h_M to $K_M^{-1}|_D = L$. In particular, ω_X is asymptotically Ricci-flat and the ω_X-distance function r_X to any fixed basepoint in X can be uniformly estimated by

$$C^{-1}(-\log |S|_{h_M}^2)^{\frac{n+1}{2n}} \leq r_X \leq C(-\log |S|_{h_M}^2)^{\frac{n+1}{2n}} \text{ as } |S|_{h_M} \to 0.$$

Theorem 3.3 is proved in [TY90] by solving a complex Monge-Ampère equation with reference metric ω_X. The exponential decay statement follows from Proposition 2.9 in [Hei12].

Theorem 3.3 ([TY90, Hei12]). *There is a smooth function ϕ on X such that $\omega_{TY} \equiv \omega_X + \sqrt{-1}\partial\bar{\partial}\phi$ is a complete Ricci-flat Kähler metric on X solving the Monge-Ampère equation*

$$\omega_{TY}^n = \frac{1}{2}(\sqrt{-1})^{n^2}\Omega_X \wedge \overline{\Omega}_X.$$

Moreover, there is a constant $\delta_0 = \delta_0(M, D) > 0$ such that for all integers $k \geq 0$,

$$|\nabla_{g_X}^k \phi|_{g_X} = O(e^{-\delta_0 r_X^{\frac{n}{n+1}}}) \text{ as } r_X \to \infty.$$

We remark that *polynomial* decay estimates were obtained in [KK10]. To make Theorem 3.3 useful for our gluing construction in this paper, we need to replace ω_X by the Calabi model metric $\omega_{\mathcal{C}}$ in (3.3). This amounts to estimating the convergence of ω_X to $\omega_{\mathcal{C}}$.

Proposition 3.4. *There is a diffeomorphism $\Phi : \mathcal{C} \setminus K' \to X \setminus K$, where $K \subset X$ is compact and $K' = \{|\xi|_h \geq \frac{1}{2}\}$, such that the following hold uniformly as $z \to +\infty$:*

(a) *We have the complex structure asymptotics*

$$|\nabla_{g_{\mathcal{C}}}^k(\Phi^* J_X - J_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-(\frac{1}{2}-\epsilon)z^n}) \text{ for all } k \geq 0, \epsilon > 0.$$

(b) *We have the holomorphic n-form asymptotics*

$$|\nabla_{g_{\mathcal{C}}}^k(\Phi^* \Omega_X - \Omega_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-(\frac{1}{2}-\epsilon)z^n}) \text{ for all } k \geq 0, \epsilon > 0.$$

(c) *There is some positive constant*

$$\underline{\delta} > 0 \tag{3.4}$$

such that for all $k \geq 0$ the Ricci-flat Kähler metric ω_{TY} satisfies the asymptotics

$$|\nabla_{g_{\mathcal{C}}}^k(\Phi^* \omega_{TY} - \omega_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-\underline{\delta} z^{n/2}}).$$

Proof. One can prove as in [Don96, Section 2] that (b) implies (a). The point is that Ω_X uniquely determines J_X because J_X is determined by knowing the subspace $\Lambda_{J_X}^{1,0} \subset \Lambda_{\mathbb{C}}^1 X$, and we have $\Lambda_{J_X}^{1,0} X = \ker T$, where $T : \Lambda_{\mathbb{C}}^1 X \to \Lambda_{\mathbb{C}}^{n+1} X$ is the $\mathbb{C}$-linear map defined by $T\alpha = \Omega_X \wedge \alpha$. See [CH13, Lemma 2.14] for details.

Item (b) can be proved by following the steps of a similar estimate in the asymptotically conical setting in [CH15, Section 2.2]; see also [Li20]. Fix any background hermitian metric g on M. Via g-orthogonal projection, the holomorphic normal bundle $L = N_D = T^{1,0}M|_D/T^{1,0}D$ is naturally isomorphic to the g-orthogonal complement $(T^{1,0}D)^\perp \subset T^{1,0}M$ as a C^∞ complex line bundle, and the g-normal exponential map defines a diffeomorphism from a neighborhood of the zero section in $(T^{1,0}D)^\perp$ to a neighborhood of D in M. Let Φ be the composition of these two maps. Then Φ is a diffeomorphism from a neighborhood of the zero section in L to a neighborhood of D in M, and the restriction of Φ to the zero section is Id_D.

Fix a point on D. Let $(z_1, \ldots, z_{n-1}, w)$ be local holomorphic coordinates on M centered at this point such that D is locally cut out by $w = 0$. Then $(z_1, \ldots, z_{n-1}, w)$ may also be viewed as local holomorphic coordinates on L corresponding to the coset of normal vectors $w(\frac{\partial}{\partial w} + T^{1,0}D) \in L$ based at the point $(z_1, \ldots, z_{n-1}, 0) \in D$. In these coordinates we may write

$$\Omega_X = \left(\frac{f(z)}{w} + g(z, w)\right) dz_1 \wedge \ldots \wedge dz_{n-1} \wedge dw, \tag{3.5}$$

$$\Omega_{\mathcal{C}} = \frac{f(z)}{w} dz_1 \wedge \ldots \wedge dz_{n-1} \wedge dw,$$

where f, g are local holomorphic functions with $f(z) \neq 0$ for all z. In order to compare $\Phi^*\Omega_X$ to $\Omega_{\mathcal{C}}$, we define new C^∞ complex coordinates $(z'_1, \ldots, z'_{n-1}, w)$ on M by

$$z'_i(z, w) = z_i - a_i(z)w, \text{ where } a_i(z) = \left.\frac{\partial(z_i \circ \Phi)}{\partial w}\right|_{(z,0)}. \tag{3.6}$$

Using the fact that $d\Phi$ is complex linear at $w = 0$ and that $\Phi_*(\frac{\partial}{\partial w}) = \frac{\partial}{\partial w} + \sum a_i(z) \frac{\partial}{\partial z^i}$ at $w = 0$, it is easy to check that these new coordinates satisfy

$$\Phi^* dz'_i = dz_i \text{ and } \Phi^* dw = dw \text{ at } w = 0.$$

Via a Taylor expansion, it follows directly from this that

$$\Phi^* z'_i = z_i + A_i w^2 + B_i w\overline{w} + C_i \overline{w}^2 \text{ and } \Phi^* w = w + Aw^2 + Bw\overline{w} + C\overline{w}^2 \tag{3.7}$$

with smooth functions A_i, B_i, C_i and A, B, C. We now express the coordinates (z, w) in (3.5) in terms of (z', w) using (3.6), and then use (3.7) to compare $\Phi^*\Omega_X$ to $\Omega_{\mathcal{C}}$. The first step yields

$$\Omega_X = \frac{f(z')}{w} dz'_1 \wedge \ldots \wedge dz'_{n-1} \wedge dw + \Upsilon \wedge dw,$$

where Υ extends to a smooth complex $(n-1)$-form on a neighborhood of D in M. Then

$$\begin{aligned} &\Phi^*\Omega_X - \Omega_{\mathcal{C}} \\ &= \frac{(A' + B'\frac{\overline{w}}{w} + C'\frac{\overline{w}^2}{w^2})(\Upsilon' \wedge dw) + (A'' + B''\frac{\overline{w}}{w} + C''\frac{\overline{w}^2}{w^2})(\Upsilon'' \wedge d\overline{w}) + (w\Theta' + \overline{w}\Theta'' + \frac{\overline{w}^2}{w}\Theta''')}{1 + Aw + B\overline{w} + C\frac{\overline{w}^2}{w}} \\ &\quad + \Phi^*\Upsilon \wedge (dw + w\phi' + \overline{w}\phi''), \end{aligned} \tag{3.8}$$

where $A', B', C', A'', B'', C'', \Upsilon', \Upsilon'', \Theta', \Theta'', \Theta''', \phi', \phi''$ extend to smooth complex functions, $(n-1)$-forms, n-forms, and 1-forms on a neighborhood of D in L, respectively. The reason for writing the right-hand side of (3.8) in this way is that a smooth complex n-form is small with respect to $g_{\mathcal{C}}$ if it either contains an explicit factor of w or $\overline{w}$ in front, or if it splits off a wedge factor of dw or $d\overline{w}$. Unfortunately the right-hand side of (3.8) is not smooth at the divisor but all non-smooth terms are due to factors of $\overline{w}/w$ which satisfy the same estimates as smooth functions.

It remains to prove appropriate estimates on $|\nabla^k_{g_{\mathcal{C}}} F|_{g_{\mathcal{C}}}$ for all $k \geq 0$, where F is either a smooth function on a neighborhood of D in L, or $F = \overline{w}/w$. To begin with, note that

$$|\nabla^k_{g_{\mathcal{C}}} z_i|_{g_{\mathcal{C}}} = O(1) \text{ for all } k \geq 0, \tag{3.9}$$

$$|w| = O(e^{-\frac{1}{2}z^n}),\ |\nabla^k_{g_{\mathcal{C}}} w|_{g_{\mathcal{C}}} = O(e^{-(\frac{1}{2}-\epsilon)z^n}) \text{ for all } k \geq 1, \epsilon > 0, \tag{3.10}$$

$$|w^{-1}| = O(e^{\frac{1}{2}z^n}),\ |\nabla^k_{g_{\mathcal{C}}} w^{-1}|_{g_{\mathcal{C}}} = O(e^{(\frac{1}{2}+\epsilon)z^n}) \text{ for all } k \geq 1, \epsilon > 0. \tag{3.11}$$

Here the bound $|z_i| = O(1)$ is clear and the bounds $|w^{\pm 1}| = O(e^{\mp\frac{1}{2}z^n})$ follow from the definition of the moment map $z = (-\log|\xi|^2_h)^{1/n}$ together with the fact that $|\xi|^2_h = |w|^2 e^{-\phi}$ with ϕ independent of $w, \overline{w}$. The higher derivative bounds in (3.9)–(3.11) follow from these pointwise bounds by using elliptic estimates for holomorphic functions on a Kähler manifold of C^∞ bounded geometry (these estimates apply here because $g_{\mathcal{C}}$ is Ricci-flat Kähler of bounded curvature). Note that the ϵ-terms in

(3.10)–(3.11) are necessary because the sup of $|w|$ over a $g_{\mathcal{C}}$-ball of radius 1 centered at a point with $w = w_0$ is $O(|w_0|^{1-\epsilon})$ for every $\epsilon > 0$ but is not $O(|w_0|)$, unlike on a cylinder $\mathbb{C}^*_w \times D$ with model metric $|d\log w|^2 + g_D$.

We now prove by induction that for all smooth functions F on a neighborhood of D in L,

$$|F| = O(1),\ |\nabla^k_{g_{\mathcal{C}}} F|_{g_{\mathcal{C}}} = O(e^{\epsilon z^n}) \text{ for all } k \geq 1, \epsilon > 0. \tag{3.12}$$

Indeed, the pointwise bound is clear, and for $k \geq 1$ we apply $\nabla^{k-1}_{g_{\mathcal{C}}}$ to the expansion

$$dF = \frac{\partial F}{\partial w}dw + \frac{\partial F}{\partial \overline{w}}d\overline{w} + \sum_{i=1}^{n-1}\frac{\partial F}{\partial z_i}dz_i + \sum_{i=1}^{n-1}\frac{\partial F}{\partial \overline{z}_i}d\overline{z}_i,$$

using the inductive hypothesis to control $\nabla^{k-1}_{g_{\mathcal{C}}}$ of the partial derivatives of F on the right-hand side and using (3.9)–(3.10) to control $\nabla^{k-1}_{g_{\mathcal{C}}}$ of $dz_i, d\overline{z}_i, dw, d\overline{w}$. This proves (3.12). By using (3.10)–(3.11) we can then prove in a similar manner that

$$\left|\frac{\overline{w}}{w}\right| = O(1),\ \left|\nabla^k_{g_{\mathcal{C}}}\left(\frac{\overline{w}}{w}\right)\right|_{g_{\mathcal{C}}} = O(e^{\epsilon z^n}) \text{ for all } k \geq 1, \epsilon > 0. \tag{3.13}$$

Taken together, (3.12) and (3.13) allow us to estimate all contributions to (3.8) in all C^k norms with respect to $g_{\mathcal{C}}$, proving item (b).

To prove item (c), notice that in local coordinates $\xi = (z_1, \ldots, z_{n-1}, w) \in L$ as above,

$$\Phi^*|S|^2_{h_M} = (1+G)|\xi|^2_h = (1+G)|w|^2 e^{-\phi}$$

with smooth real-valued locally defined functions G and ϕ, where G vanishes at $w = 0$ and ϕ does not depend on $w, \overline{w}$. Notice that $G = Fw + \overline{Fw}$ for some smooth complex-valued locally defined function F. This structure of the $\Phi^* J_X$-Kähler potential of $\Phi^*\omega_X$, together with (3.10), (3.12), and item (a), makes it possible to prove that for all $k \geq 0, \epsilon > 0$,

$$|\nabla^k_{g_{\mathcal{C}}}(\Phi^*\omega_X - \omega_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-(\frac{1}{2}-\epsilon)z^n}).$$

Similarly by Theorem 3.3 we get for some $\underline{\delta} > 0$ depending on δ_0 that for all $k \geq 0$,

$$|\nabla^k_{g_{\mathcal{C}}}(\Phi^*\omega_{TY} - \Phi^*\omega_X)|_{g_{\mathcal{C}}} = O(e^{-\underline{\delta} z^{n/2}}).$$

This completes the proof of item (c). $\square$

Remark 3.5. The decay of the Ricci-flat Kähler form ω_{TY} to its asymptotic model $\omega_{\mathcal{C}}$ is weaker than the decay of J_X and of Ω_X. The reason is that the latter is obtained by an explicit computation where the errors admit an expansion in terms of $|w| \sim e^{-\frac{1}{2}z^n}$. On the other hand, the decay rate of ω_{TY} depends on an analysis of the Tian-Yau solution of the Monge-Ampère equation, which is related to the fact that the decay rate of harmonic (not necessarily holomorphic) functions on the Calabi model space (see Section 4) is in general only $O(e^{-\delta z^{\frac{n}{2}}})$ for some $\delta > 0$.

When $n = 2$, the Tian-Yau metric is hyperkähler, and the corresponding hyperkähler triple $\boldsymbol{\omega}_{TY}$ is determined by ω_{TY} and Ω_X. Let $\boldsymbol{\omega}_{\mathcal{C}}$ be the Calabi hyperkähler triple defined in (3.3).

Corollary 3.6. *Under the same diffeomorphism Φ as in Proposition 3.4 we have that*

$$|\nabla^k_{g_{\mathcal{C}}}(\Phi^*\boldsymbol{\omega}_{TY} - \boldsymbol{\omega}_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-\underline{\delta} z}) \tag{3.14}$$

for all $k \geq 0$*, where* $\underline{\delta} > 0$ *is the same constant as in* (3.4).

In our gluing construction we need the following refinement of Corollary 3.6.

Lemma 3.7. *There exists a triple of* 1*-forms* $\boldsymbol{a}$ *with* $\partial_z \lrcorner \boldsymbol{a} = 0$ *such that*

$$\Phi^* \boldsymbol{\omega}_{TY} - \boldsymbol{\omega}_{\mathcal{C}} = d\boldsymbol{a} \tag{3.15}$$

and such that for all $k \geq 0$ *and all* $\epsilon > 0$*,*

$$|\nabla^k_{g_{\mathcal{C}}} \boldsymbol{a}|_{g_{\mathcal{C}}} = O(e^{-(\underline{\delta}-\epsilon)z}). \tag{3.16}$$

Proof. By Proposition 3.1, the Calabi space $\mathcal{C}$ is diffeomorphic to $\mathrm{Nil}^3_b \times (0,\infty)$ in such a way that the Calabi metric $g_{\mathcal{C}}$ becomes the Gibbons-Hawking metric (2.5). Ignoring this diffeomorphism, we have a 2-form triple $\boldsymbol{\phi}$ and a 1-form triple $\boldsymbol{\psi}$ such that $\partial_z \lrcorner \boldsymbol{\phi} = 0$, $\partial_z \lrcorner \boldsymbol{\psi} = 0$, and

$$\Phi^* \boldsymbol{\omega}_{TY} - \boldsymbol{\omega}_{\mathcal{C}} = \boldsymbol{\phi} + dz \wedge \boldsymbol{\psi}.$$

Since $\boldsymbol{\omega}_{TY}, \boldsymbol{\omega}_{\mathcal{C}}$ consist of closed forms, it follows that

$$d_{\mathrm{Nil}^3_b} \boldsymbol{\phi} = 0, \;\; \partial_z \boldsymbol{\phi} - d_{\mathrm{Nil}^3_b} \boldsymbol{\psi} = 0. \tag{3.17}$$

Now we define the 1-form triple

$$\boldsymbol{a} \equiv - \int_z^\infty \boldsymbol{\psi} \, dz.$$

Thanks to (3.14), this integral exists and satisfies (3.16). Note that this is not completely obvious because the 1-form basis $dx, dy, dt - xdy$ on Nil^3_b is not parallel with respect to $g_{\mathcal{C}}$. However, this effect is absorbed by the ϵ in (3.16) because all error terms are at worst polynomial in z. Property (3.15) now follows in a standard manner by using (3.17). ☐

4. Liouville theorem for harmonic functions

In this section and the next, we will set up some technical tools for the gluing construction. One of the crucial technical ingredients in analyzing the linearized operator is to establish a Liouville theorem on the complete non-compact hyperkähler manifolds of Tian-Yau that arise in our context.

Our main goal in this section is to prove a Liouville theorem for harmonic functions with a small enough exponential growth rate, on a complete Riemannian 4-manifold (X^4, g) with non-negative Ricci curvature which is asymptotically Calabi in the sense of the following Definition 4.1. This is a necessary step towards proving our Liouville theorem for half-harmonic 1-forms in Section 5.

Definition 4.1. Given some constant $\delta > 0$, a complete Riemannian manifold (X^4, g) is said to be δ-*asymptotically Calabi* if there exist a compact subset $K \subset X$ and a Calabi model space $(\mathcal{C}, g_{\mathcal{C}})$ as defined in Section 3, and a diffeomorphism $\Phi : \mathcal{C} \setminus K' \to X \setminus K$ with $K' = \{|\xi|_h \geq \frac{1}{2}\} \subset \mathcal{C}$ such that for all $k \geq 0$,

$$|\nabla^k_{g_{\mathcal{C}}}(\Phi^* g - g_{\mathcal{C}})|_{g_{\mathcal{C}}} = O(e^{-\delta z}) \text{ as } z \to \infty,$$

where $z = (-\log |\xi|_h^2)^{1/2}$ denotes the natural moment map coordinate on $(\mathcal{C}, g_{\mathcal{C}})$.

Example 4.2. According to Proposition 3.4, any complete hyperkähler Tian-Yau space (X^4, g) as in Theorem 3.3 is $\underline{\delta}$-asymptotically Calabi for some appropriate constant $\underline{\delta} > 0$.

The following is our main result in this section.

Theorem 4.3. *Let (X^4, g) be a complete Riemannian 4-manifold which is δ-asymptotically Calabi for some $\delta > 0$ and which has $\mathrm{Ric}_g \geq 0$. Then there exists an $\ell_0 \in (0,1)$ depending only upon (X^4, g) such that if u is a harmonic function on (X^4, g) with $u = O(e^{\ell_0 z})$ as $z \to \infty$, then u is a constant.*

The remainder of this section will be dedicated to the proof of this theorem.

4.1. **Separation of variables.** The first step is to analyze the Laplace operator on the Calabi model space using separation of variables. To recall the setup, we work with a Calabi model space $\mathcal{C}$ corresponding to a smooth divisor D defined as in Section 3. The Calabi metric is given by

$$\omega_{\mathcal{C}} = \frac{n}{n+1}\sqrt{-1}\partial\bar{\partial}(-\log|\xi|_h^2)^{\frac{n+1}{n}},$$

which is well-defined for $|\xi|_h < 1$. In this subsection, we will reduce the equations $\Delta_{\mathcal{C}} u = 0$ and $\Delta_{\mathcal{C}} u = v$ to some linear ODEs; see Proposition 4.4. A similar separation of variables was carried out in [KK10].

In order to carry out the separation of variables, we will study the local representation of the Laplace operator $\Delta_{\mathcal{C}}$ on $\mathcal{C}$. We choose local holomorphic coordinates $\underline{z} = \{z_i\}_{i=1}^{n-1}$ on the smooth divisor D and fix a local holomorphic trivialization e_0 of the line bundle L with $|e_0|^2 = e^{-\psi}$, where $\psi : D \to \mathbb{R}$ is a smooth function. So we get local holomorphic coordinates $(\underline{z}, w) \equiv (z_1, \ldots, z_{n-1}, w)$ on $\mathcal{C}$ by writing a point $\xi \in \mathcal{C}$ as $\xi = we_0(\underline{z})$. Then $|\xi|_h^2 = |w|^2 e^{-\psi}$. We may assume $\psi(0) = 1$, $d\psi(0) = 0$ and $\sqrt{-1}\partial\bar{\partial}\psi = \omega_D$. Let $\pi : \mathcal{C} \to D$ be the obvious projection map. Then we obtain

$$\omega_{\mathcal{C}} = (-\log|\xi|_h^2)^{\frac{1}{n}}\omega_D + \frac{1}{n}(-\log|\xi|_h^2)^{\frac{1}{n}-1} \cdot \sqrt{-1} \cdot \Big(\frac{dw}{w} - \partial\psi\Big) \wedge \Big(\frac{d\bar{w}}{\bar{w}} - \bar{\partial}\psi\Big).$$

For $u \in C^2(\mathcal{C}, \mathbb{R})$, the Laplacian at points of the fiber $\pi^{-1}(0)$ is given by

$$\Delta_{\mathcal{C}} u = (-\log|\xi|_h^2)^{-\frac{1}{n}}\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial z_i \partial \bar{z}_i} + n(-\log|\xi|_h^2)^{-\frac{1}{n}+1}|w|^2\frac{\partial^2 u}{\partial w \partial \bar{w}}.$$

Now denote $\varrho \equiv |\xi|_h$. Then we can write

$$w = \varrho e^{\frac{\psi}{2} + \sqrt{-1}\theta},$$

where ∂_θ generates the S^1-action on the total space of L. It is straightforward to check that

$$\frac{\partial\varrho}{\partial w} = \frac{\varrho}{2w}, \quad \frac{\partial\theta}{\partial w} = \frac{1}{2\sqrt{-1}w}, \quad \frac{\partial\varrho}{\partial z_i} = -\frac{1}{2}\varrho\partial_{z_i}\psi, \quad \frac{\partial\theta}{\partial z_i} = 0,$$
$$|w|^2\frac{\partial^2 u}{\partial w\partial\bar{w}} = \frac{1}{4}(\varrho^2 u_{\varrho\varrho} + \varrho u_\varrho + u_{\theta\theta}).$$

For a fixed $r_0 \in (0,1)$, the level set $Y^{2n-1} \equiv \{\varrho = r_0\}$ is equipped with the induced metric

$$h_0 = (-\log r_0^2)^{\frac{1}{n}} g_D + \frac{1}{n}(-\log r_0^2)^{\frac{1}{n}-1}(d\theta - \frac{1}{2}d^c\psi) \otimes (d\theta - \frac{1}{2}d^c\psi). \tag{4.1}$$

Now we consider a smooth function $\phi \in C^\infty(Y^{2n-1})$ with

$$\mathcal{L}_{\partial_\theta}\phi = \sqrt{-1} \cdot j \cdot \phi \tag{4.2}$$

for some integer j. Replacing ϕ by $\bar{\phi}$ if necessary, we may assume that $j \geq 0$. Then ϕ is induced by a smooth section $\hat{\phi}$ of $(L^*)^{\otimes j}$. More precisely, if we locally write $\hat{\phi}(\underline{z}) = \Phi(\underline{z})(e_0(\underline{z})^*)^{\otimes j}$, then

$$\phi(\underline{z}, w) = w^j \Phi(\underline{z})|_{\varrho = r_0} = r_0^j e^{j(\frac{\psi}{2} + \sqrt{-1}\theta)} \Phi(\underline{z}).$$

Now let $\hat{\phi}$ be a non-zero eigensection of the $\bar{\partial}$-Laplacian with eigenvalue $\hat{\lambda}$, i.e.,

$$\Delta_{\bar{\partial}} \hat{\phi} = \hat{\lambda} \hat{\phi}.$$

By the Kodaira-Nakano formula, $\Delta_{\bar{\partial}} = \Delta_{\partial} + j(n-1)$, so we have that

$$\hat{\lambda} \geq j(n-1).$$

By a direct calculation, we see that on $\pi^{-1}(0)$,

$$\sum_{i=1}^{n-1} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_i} = \Big(-\hat{\lambda} + \frac{j(n-1)}{2}\Big)\phi. \tag{4.3}$$

Moreover, by the local expression of h_0 given in (4.1), one can directly check that on $\pi^{-1}(0) \cap Y^{2n-1}$,

$$\Delta_{h_0} \phi = \Big((-\log r_0^2)^{-\frac{1}{n}} \Big(-\hat{\lambda} + \frac{j(n-1)}{2}\Big) - j^2 n (-\log r_0^2)^{-\frac{1}{n}+1}\Big)\phi. \tag{4.4}$$

Now suppose that a smooth function $u(\varrho, z)$ on the Calabi space $\mathcal{C}$ is of the form $u \equiv f(\varrho)\phi(y)$, where ϕ is a function on Y^{2n-1} satisfying (4.2) and (4.3). In polar coordinates, we obtain

$$\begin{aligned}\Delta_{\mathcal{C}} u = \phi(y) \cdot (-\log \varrho^2)^{-\frac{1}{n}} \\ \cdot \Big(\frac{n}{4}(-\log \varrho^2)(\varrho^2 f_{\varrho\varrho} + \varrho f_\varrho - j^2 f) - \frac{(n-1)}{2}\varrho f_\varrho - \Big(\hat{\lambda} - \frac{j(n-1)}{2}\Big) f\Big).\end{aligned}$$

Notice that this formula is now independent of the choice of local holomorphic coordinates. So u is harmonic if and only if

$$\frac{n}{4}(-\log \varrho^2)(\varrho^2 f_{\varrho\varrho} + \varrho f_\varrho - j^2 f) - \frac{(n-1)}{2}\varrho f_\varrho - \Big(\hat{\lambda} - \frac{j(n-1)}{2}\Big) f = 0.$$

Denote $z = (-\log \varrho^2)^{\frac{1}{n}}$. Then we obtain

$$f_{zz} - \Big(n\Big(\hat{\lambda} - \frac{j(n-1)}{2}\Big) + \frac{j^2 n^2}{4} z^n\Big) z^{n-2} f = 0.$$

As for the Poisson equation,

$$\Delta_{\mathcal{C}} u = v,$$

suppose that $v \equiv \zeta(\varrho) \cdot \phi(y)$. Then the same separation of variables gives the following ODE

$$f_{zz} - \Big(n\Big(\hat{\lambda} - \frac{j(n-1)}{2}\Big) + \frac{j^2 n^2}{4} z^n\Big) z^{n-2} f = z^{n-1} \cdot \zeta.$$

For our applications we focus on the case $n = 2$. So the corresponding ODE becomes

$$f_{zz} - (\lambda + j^2 z^2) f = z \cdot \zeta,$$

where $\lambda \equiv 2\hat{\lambda} - j \geq j$. We have assumed that $j \geq 0$ in the above discussion, but notice that the Laplace operator is a real operator, so the ODEs we get for j and

$-j$ are the same. Denote $z_0 \equiv (-\log r_0^2)^{\frac{1}{2}}$. Then each eigenvalue Λ of Δ_{h_0} can be represented by

$$\Lambda = \frac{\lambda}{2z_0} + 2z_0 \cdot j^2. \tag{4.5}$$

With the above computations, we are ready to set up the ODE system. We fix some $r_0 \in (0,1)$ and define (Y^3, h_0) to be the level set $\{r = r_0\}$ endowed with the induced Riemannian metric h_0. The representation (4.5) tells us that the eigenvalues of $-\Delta_{h_0}$ are given by linear combinations of λ and j^2.

We denote by $\{\Lambda_k\}_{k=0}^{\infty}$ the spectrum of Δ_{h_0} with $0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 < \cdots$ (allowing multiplicity), and let $\{\varphi_k\}_{k=0}^{\infty}$ be an L^2-orthonormal basis of the eigenfunctions which are homogeneous under the S^1-action such that

$$-\Delta_{h_0}\varphi_k = \Lambda_k \cdot \varphi_k, \quad \mathcal{L}_{\partial_\theta}\varphi_k = \sqrt{-1}j_k\varphi_k, \ j_k \in \mathbb{N}.$$

From above, for each $k \in \mathbb{Z}_+$, we have $\lambda_k \geq j_k$, and when $k = 0$, $\lambda_k = j_k = 0$, and each eigenvalue can be written in the form

$$\Lambda_k = \frac{\lambda_k}{2z_0} + 2z_0 \cdot j_k^2. \tag{4.6}$$

Let $u(z, \boldsymbol{y}) \in L^2(Y^3)$ for every z. Then we can write the L^2-expansion along Y^3 as follows:

$$u(z, \boldsymbol{y}) = \sum_{k=0}^{\infty} u_k(z) \cdot \varphi_k(\boldsymbol{y}), \quad \boldsymbol{y} \in Y^3. \tag{4.7}$$

Notice that it might seem more natural to write the Fourier series as a double sum (determined by the S^1-action), but the above expansion is much more convenient for our purposes, especially when we apply Weyl's law below.

Now we summarize our ODE reductions in Proposition 4.4.

Proposition 4.4. *Let $(\mathcal{C}, g_{\mathcal{C}})$ be the Calabi model space, and let u solve the Poisson equation $\Delta_{\mathcal{C}} u = v$ for some $v \in C^{K_0}(\mathcal{C})$ and $K_0 \in \mathbb{Z}_+$ sufficiently large. Let u, v have "fiberwise" expansions as in (4.7). Then for every $k \in \mathbb{N}$, the coefficient functions $u_k(z)$ and $v_k(z)$ satisfy*

$$\frac{d^2u_k(z)}{dz^2} - (j_k^2 z^2 + \lambda_k)u_k(z) = v_k(z) \cdot z, \quad z \geq 1.$$

4.2. **Uniform estimates for the fundamental solutions.** This subsection presents several technical lemmas which will be frequently used in the later sections. Lemma 4.6 will be used to study the asymptotics of harmonic functions in Section 4.3. Lemma 4.5 and Lemma 4.7 will be used in the uniform estimates of the Poisson equation in Section 4.4.

We consider solutions of the homogeneous ODE

$$\frac{d^2u(z)}{dz^2} - (j^2z^2 + \lambda)u(z) = 0.$$

In the case that $j = 0$ the ODE becomes

$$\frac{d^2u(z)}{dz^2} = \lambda \cdot u(z). \tag{4.8}$$

If $\lambda = 0$, the solutions to (4.8) are linear. If $\lambda > 0$, (4.8) has two linearly independent solutions $e^{\sqrt{\lambda}z}$ and $e^{-\sqrt{\lambda}z}$. All the required estimates in this case are straightforward. So we only focus on the case $j \in \mathbb{Z}_+$ which is much more technically involved.

In the case $j \in \mathbb{Z}_+$, we have already shown in Section 4.1 that λ and j satisfy the relation $\lambda \geq j \geq 1$. Hence for each pair of λ and j satisfying the above, we choose $h \geq 0$ such that

$$\lambda = (2h+1)j.$$

From now on, we focus on the differential equation (for every $j \in \mathbb{Z}_+$ and $h \geq 0$)

$$\frac{d^2u(z)}{dz^2} = j(jz^2 + 2h + 1)u(z). \tag{4.9}$$

We will simplify the above equation by the following transformations. Let

$$y = \sqrt{j}z, \ V(y) = u\Big(\frac{y}{\sqrt{j}}\Big).$$

Then $V(y)$ satisfies

$$\frac{d^2V(y)}{dy^2} = (y^2 + (2h+1))V(y).$$

Furthermore, make the transformation

$$V(y) = e^{-\frac{y^2}{2}}Q(y).$$

Then Q solves the differential equation

$$\frac{d^2Q(y)}{dy^2} - 2y\frac{dQ(y)}{dy} - 2(h+1)Q(y) = 0. \tag{4.10}$$

Notice that equation (4.10) is invariant under the change of variables $y \mapsto -y$. Given $y > 1$ and $h \geq 0$, we define the exponential integral $H_{-h-1}(y)$ as follows:

$$H_{-h-1}(y) \equiv \int_0^\infty e^{-t^2-2ty}t^h dt.$$

Here the subscript "$-h-1$" is consistent with the standard definition of the Hermite function. By straightforward calculations, for each given $h \geq 0$, $H_{-h-1}(y)$ and $H_{-h-1}(-y)$ are linearly independent solutions to (4.10). Moreover, they coincide with the usual *Hermite functions* up to a constant; see Section 10.5 of [Leb72]. Eventually, we obtain two solutions to (4.9), namely:

$$\mathcal{F}(z) \equiv e^{-\frac{jz^2}{2}}H_{-h-1}(-\sqrt{j}z) = e^{-\frac{y^2}{2}}\int_0^\infty e^{-t^2+2ty+h\log t}dt, \tag{4.11}$$

$$\mathcal{U}(z) \equiv e^{-\frac{jz^2}{2}}H_{-h-1}(\sqrt{j}z) = e^{-\frac{y^2}{2}}\int_0^\infty e^{-t^2-2ty+h\log t}dt. \tag{4.12}$$

Note that $\mathcal{F}, \mathcal{U}$ depend on parameters j, h, but for simplicity we omit the subscripts. The following elementary lemma is proved in Appendix A.

Lemma 4.5. *The Wronskian of $\mathcal{F}$ and $\mathcal{U}$ is a constant given by*

$$\mathcal{W}(\mathcal{F}, \mathcal{U}) = 2^{-h}\sqrt{j\pi}\Gamma(h+1) > 0.$$

In particular, $\mathcal{F}$ and $\mathcal{U}$ are linearly independent.

The regularity of the formal solutions obtained from the above separation of variables requires very precise uniform estimates for the fundamental solutions $\mathcal{F}$ and $\mathcal{U}$. We will use the *Laplace method*, which is inspired by [SS20] in a different context. Again we denote $y = \sqrt{j}z$ and define

$$F(t) \equiv -t^2 + 2ty + h \log t, \quad U(t) \equiv -t^2 - 2ty + h \log t. \tag{4.13}$$

Straightforward computations tell us that both F and U are strictly concave when $h \geq 0$. For a fixed y, let t_0 and s_0 be the unique (positive) critical points of F and U, respectively. It is straightforward to check that

$$t_0 = \frac{y}{2} + \sqrt{\frac{h^2}{2} + \frac{y^2}{4}} \quad \text{and} \quad s_0 = -\frac{y}{2} + \sqrt{\frac{h^2}{2} + \frac{y^2}{4}}. \tag{4.14}$$

Now we list some technical lemmas, where the proof can also be found in the appendix.

Lemma 4.6. *Let $\mathcal{F}(z)$, $\mathcal{U}(z)$ and $F(t)$, $U(t)$ be the functions defined in (4.11), (4.12) and (4.13), respectively. Then the following properties hold.*

(1) *(Uniform estimates) There exists a uniform constant $C_0 > 0$ independent of j and h such that for all $j \in \mathbb{Z}_+$ and $h \geq 0$*

$$C_0 \cdot e^{-\frac{jz^2}{2} + F(t_0(z))} \leq \mathcal{F}(z) \leq (1 + \sqrt{\pi}) e^{-\frac{jz^2}{2} + F(t_0(z))},$$
$$C_0 \cdot e^{-\frac{jz^2}{2} + U(s_0(z))} \leq \mathcal{U}(z) \leq (1 + \sqrt{\pi}) e^{-\frac{jz^2}{2} + U(s_0(z))}.$$

(2) *(Asymptotics) For fixed $j \in \mathbb{Z}_+$ and $h > 0$, we have the following asymptotic formulas:*

$$\lim_{z \to +\infty} \frac{\mathcal{F}(z)}{\sqrt{\pi} e^{\frac{jz^2}{2}} (\sqrt{j}z)^h} = 1, \tag{4.15}$$

$$\lim_{z \to +\infty} \frac{\mathcal{U}(z)}{e^{-\frac{jz^2}{2}} \Gamma(h+1)(2\sqrt{j}z)^{-h-1}} = 1. \tag{4.16}$$

Next, we define the functions

$$\widehat{F}(z) \equiv -\frac{jz^2}{2} + F(t_0(z)) \quad \text{and} \quad \widehat{U}(z) \equiv -\frac{jz^2}{2} + U(s_0(z)).$$

Lemma 4.7. *The functions $\widehat{F}(z)$ and $\widehat{U}(z)$ satisfy the following properties:*

(1) *(Uniform estimate) There is a uniform constant $C_0 > 0$ independent of $j \in \mathbb{Z}_+$ and $h \geq 0$ such that the following uniform estimate holds for all $z \geq 1$:*

$$0 < \frac{e^{\widehat{F}(z) + \widehat{U}(z)}}{\mathcal{W}(\mathcal{F}, \mathcal{U})} \leq C_0. \tag{4.17}$$

(2) *(Monotonicity) For every $\eta > 0$, $\widehat{F}(z) - \eta z$ is increasing in z and $\widehat{U}(z) + \eta z$ is decreasing in z for $z > 2\eta$.*

4.3. **Asymptotics of harmonic functions.** We will first prove two lemmas, which will be used to prove the main result of this subsection, Proposition 4.10.

Lemma 4.8. *Let φ_k satisfy $-\Delta_{h_0}\varphi_k = \Lambda_k \cdot \varphi_k$ with $\Lambda_k > 0$ and $\|\varphi_k\|_{L^2(Y^3)} = 1$. Then there exists $C > 0$ which depends only on the metric h_0 such that $\|\varphi_k\|_{L^\infty(Y^3)} \leq C \cdot \Lambda_k$.*

Proof. The eigenfunction φ_k satisfies the elliptic equation $-\Delta_{h_0}\varphi_k = \Lambda_k \cdot \varphi_k$, so it follows from standard elliptic regularity that there exists $C > 0$ depending only on the metric h_0 such that

$$\|\varphi_k\|_{W^{2,2}(Y^3)} \leq C \cdot (\Lambda_k + 1) \cdot \|\varphi_k\|_{L^2(Y^3)} \leq C \cdot \Lambda_k,$$

where $k \in \mathbb{Z}_+$. Applying the Sobolev embedding theorem,

$$\|\varphi_k\|_{C^{0,\frac{1}{2}}(Y^3)} \leq C\|\varphi_k\|_{W^{2,2}(Y^3)} \leq C \cdot \Lambda_k,$$

where $C > 0$ depends only on the metric h_0. The proof is complete. $\square$

Lemma 4.9. *Let $(\mathcal{C}, g_{\mathcal{C}})$ be the Calabi model space with a fixed fiber (Y^3, h_0). Let $K_0 \in \mathbb{Z}_+$ and let $v \in C^{2K_0}(\mathcal{C})$ have the fiberwise L^2-expansion*

$$v(z, \boldsymbol{y}) = \sum_{k=0}^{\infty} v_k(z) \cdot \varphi_k(\boldsymbol{y}).$$

Then for every $z \geq 1$ and $k \in \mathbb{Z}_+$,

$$|v_k(z)| \leq \frac{\mathrm{Vol}_{h_0}(Y^3)^{\frac{1}{2}}}{(\Lambda_k)^{K_0}} \cdot \|v\|_{C^{2K_0}(Y^3 \times \{z\})}.$$

Proof. For $k \in \mathbb{Z}_+$, since $-\Delta_{h_0}\varphi_k = \Lambda_k \cdot \varphi_k$ and $\|\varphi_k\|_{L^2(Y^3)} = 1$, we have that

$$\begin{aligned} |v_k(z)| &= \left| \int_{Y^3} v \cdot \varphi_k \right| = \left| \int_{Y^3} v \cdot \frac{(-\Delta_{h_0})^{K_0}\varphi_k}{(\Lambda_k)^{K_0}} \right| \\ &\leq \frac{1}{(\Lambda_k)^{K_0}} \int_{Y^3} |\Delta_{h_0}^{K_0} v| \cdot |\varphi_k| \leq \frac{\mathrm{Vol}_{h_0}(Y^3)^{1/2}}{(\Lambda_k)^{K_0}} \cdot \|v\|_{C^{2K_0}(Y^3 \times \{z\})}. \end{aligned}$$

$\square$

The main result in this subsection is the following.

Proposition 4.10. *Let u be a harmonic function on the Calabi model space $(\mathcal{C}, g_{\mathcal{C}})$. There exists a constant $\underline{\delta} > 0$, depending only on $(\mathcal{C}, g_{\mathcal{C}})$, such that if $u = O(e^{\delta z})$ as $z \to +\infty$ for some $\delta \in (0, \underline{\delta})$, then $u = L(z) + h(z, \boldsymbol{y})$, where:*

(1) *$L(z) = a_0 \cdot z + b_0$ for some constants $a_0, b_0 \in \mathbb{R}$,*
(2) *for every $k \in \mathbb{N}$, the harmonic function $h(z, \boldsymbol{y})$ satisfies*

$$|\nabla^k h(z, \boldsymbol{y})|_{g_{\mathcal{C}}} \leq C_k \cdot e^{-\underline{\delta} \cdot z/2}.$$

Proof. Since the harmonic function u is smooth, as computed in Section 4.1, separation of variables gives the following expansion along the fiber Y^3 which is actually $C^\ell_{loc}(\mathcal{C})$-convergent for any $\ell \in \mathbb{N}$:

$$u(z, \boldsymbol{y}) = a_0 \cdot z + b_0 + \sum_{k=1}^{\infty} u_k(z) \cdot \varphi_k(\boldsymbol{y}), \quad \boldsymbol{y} \in Y^3, \tag{4.18}$$

where for every $k \geq 1$, $u_k(z)$ satisfies the equation

$$\frac{d^2 u_k(z)}{dz^2} - (j_k^2 z^2 + \lambda_k) u_k(z) = 0 \tag{4.19}$$

for some $j_k \in \mathbb{N}$ and $\lambda_k \geq 0$. Let us denote

$$h(z, \boldsymbol{y}) \equiv \sum_{k=1}^{\infty} u_k(z) \cdot \varphi_k(\boldsymbol{y}), \quad \boldsymbol{y} \in Y^3.$$

Before discussing the asymptotic behavior of the harmonic function $h(z, \boldsymbol{y})$, let us give a more precise expression for each ODE solution u_k ($k \geq 1$) of (4.19) under the growth condition $u = O(e^{\delta z})$ for $0 < \delta < \underline{\delta}$, where the constant $\underline{\delta} > 0$ is chosen as follows:

$$\underline{\delta} \equiv \min \left\{ \sqrt{\lambda_k} \middle| k \in \mathbb{Z}_+ \right\} > 0. \tag{4.20}$$

For every $k \in \mathbb{Z}_+$, there are fundamental solutions $\mathcal{U}_k$, $\mathcal{F}_k$ of (4.19) such that every u_k satisfies

$$u_k(z) = C_k \cdot \mathcal{U}_k(z) + C_k^* \cdot \mathcal{F}_k(z), \quad C_k, C_k^* \in \mathbb{R}.$$

The growth condition $u = O(e^{\delta z})$ implies that

$$|u_k(z)| = \left| \int_{Y^3} u \cdot \varphi_k \right| \leq C_0 \cdot (\mathrm{Vol}_{h_0}(Y^3))^{1/2} \cdot e^{\delta z}, \quad \delta \in (0, \underline{\delta}). \tag{4.21}$$

There are two cases to analyze.

First, we consider the case $j_k = 0$. Then the linear equation (4.19) is reduced to

$$u_k''(z) - \lambda_k \cdot u_k(z) = 0.$$

By (4.6) and since $k \geq 1$, we have that $\lambda_k \geq 1$. Then fundamental solutions $\mathcal{F}_k$, $\mathcal{U}_k$ are chosen as

$$\mathcal{F}_k(z) \equiv e^{\sqrt{\lambda_k} \cdot z} \quad \text{and} \quad \mathcal{U}_k(z) \equiv e^{-\sqrt{\lambda_k} \cdot z}.$$

If we choose $\delta \in (0, \underline{\delta})$, then applying (4.21) again implies that $C_k^* = 0$ for each $k \in \mathbb{Z}_+$ which satisfies $j_k = 0$, and hence

$$u_k(z) = C_k e^{-\sqrt{\lambda_k} \cdot z}. \tag{4.22}$$

Next, we consider the case $k \in \mathbb{Z}_+$ such that $j_k \neq 0$. Now the fundamental solutions $\mathcal{F}_k, \mathcal{U}_k$ are defined by (4.11) and (4.12) respectively, where the parameters in the definition are uniquely fixed by k. Applying the asymptotics in Lemma 4.6 and the estimate (4.21), we have that $\mathcal{F}_k(z)$ grows faster than $u_k(z)$. So it follows that $C_k^* = 0$ for every $k \in \mathbb{Z}_+$ which satisfies $j_k \neq 0$.

Combining the above two cases, we conclude that if $\delta \in (0, \underline{\delta})$, then for every $k \in \mathbb{Z}_+$, there exists some constant $C_k \in \mathbb{R}$ such that $u_k(z) = C_k \cdot \mathcal{U}_k(z)$ and hence

$$h(z, \boldsymbol{y}) = \sum_{k=1}^{\infty} C_k \cdot \mathcal{U}_k(z) \cdot \varphi_k(\boldsymbol{y}).$$

By definition, in our situation $\mathcal{U}_k(z) > 0$, so there is no harm in assuming that $u_k(z_0) \neq 0$.

Now we are in a position to estimate the upper bound of the harmonic function $h(z, \boldsymbol{y})$. We still separate into two cases. First, we consider $k \in \mathbb{Z}_+$ with $j_k = 0$. For fixed $z_0 > 10^6$, we apply (4.22). Then for every sufficiently large $z \in (2z_0, +\infty)$,

$$\left|\frac{u_k(z)}{u_k(z_0)}\right| = \left|\frac{\mathcal{U}_k(z)}{\mathcal{U}_k(z_0)}\right| = e^{-\sqrt{\lambda_k}\cdot(z-z_0)} \leq e^{-\underline{\delta}(z-z_0)}, \tag{4.23}$$

and hence

$$\begin{aligned}\left|\sum_{k>0,\ j_k=0} u_k(z)\cdot\varphi_k(\boldsymbol{y})\right| &\leq \sum_{k>0,\ j_k=0}\left|\frac{u_k(z)}{u_k(z_0)}\right|\cdot|u_k(z_0)|\cdot|\varphi_k(\boldsymbol{y})| \\ &\leq Ce^{-\underline{\delta}z/2}\cdot\sum_{k>0,\ j_k=0}\frac{1}{(\Lambda_k)^{K_0-1}},\end{aligned}$$

where the last inequality follows from Lemma 4.8, Lemma 4.9 and (4.23). The positive integer K_0 is chosen sufficiently large so that the above series converges. Next, let $k \in \mathbb{Z}_+$ be such that $j_k \geq 1$. For fixed $z_0 > 10^6$ and $z \in (2z_0, \infty)$, applying the uniform estimates in Lemma 4.6 and the monotonicity in Lemma 4.7 (with $\eta = 2\underline{\delta}$), we find that

$$\left|\frac{u_k(z)}{u_k(z_0)}\right| = \left|\frac{\mathcal{U}_k(z)}{\mathcal{U}_k(z_0)}\right| \leq C_0\cdot e^{-\underline{\delta}\cdot z}.$$

Taking the sum and applying Lemma 4.8 again, we then have that

$$\begin{aligned}\left|\sum_{k>0,\ j_k\geq 1} u_k(z)\cdot\varphi_k(\boldsymbol{y})\right| &\leq \sum_{k>0,\ j_k\geq 1}\left|\frac{u_k(z)}{u_k(z_0)}\right|\cdot|u_k(z_0)|\cdot|\varphi_k(\boldsymbol{y})| \\ &\leq C_0\cdot e^{-\underline{\delta}\cdot z}\cdot\sum_{k>0,\ j_k\geq 1}\frac{1}{(\Lambda_k)^{K_0-1}}.\end{aligned}$$

The estimates in the above two cases imply that

$$\begin{aligned}|h(z,\boldsymbol{y})| &\leq C\left(e^{-\underline{\delta}\cdot z/2}\sum_{k>0,\ j_k=0}\frac{1}{(\Lambda_k)^{K_0-1}} + e^{-\underline{\delta}\cdot z}\sum_{k>0,\ j_k\geq 1}\frac{1}{(\Lambda_k)^{K_0-1}}\right) \\ &\leq Ce^{-\underline{\delta}\cdot z/2}\cdot\sum_{k=1}^{\infty}\frac{1}{(\Lambda_k)^{K_0-1}}. \qquad (4.24)\end{aligned}$$

Since the spectrum $\{\Lambda_k\}_{k=1}^{\infty}$ of Δ_{h_0} obeys Weyl's law on (Y^3, h_0), it follows that there is some constant $C_0 > 0$ depending only on h_0 such that for any sufficiently large $k \in \mathbb{Z}_+$,

$$C_0^{-1}k^{\frac{2}{3}} \leq |\Lambda_k| \leq C_0 k^{\frac{2}{3}}. \tag{4.25}$$

Now we fix $K_0 \geq 3$ and apply (4.25) to (4.24). Then

$$\sum_{k=1}^{\infty}\frac{1}{(\Lambda_k)^{K_0-1}} \leq C_0\sum_{k=1}^{\infty}\frac{1}{k^{\frac{4}{3}}} \leq C$$

and hence $|h(z,\boldsymbol{y})| \leq C\cdot e^{-\underline{\delta}\cdot z/2}$, where C depends on $\mathrm{Vol}_{h_0}(Y^3)$ and $\|u\|_{C^{K_0}(Y^3\times\{z_0\})}$ for some fixed $z_0 > 10^6$.

The higher order estimate of $h(z,\boldsymbol{y})$ follows from the standard Schauder estimate. Indeed, let us lift the harmonic function h to the local universal cover of $\mathcal{C}$ which is non-collapsed with uniformly bounded curvature. Let $\tilde{\boldsymbol{x}} = (\tilde{z}, \tilde{\boldsymbol{y}})$ be a lift of the point $\boldsymbol{x} = (\tilde{z}, \boldsymbol{y})$. It follows that for any $k \in \mathbb{Z}_+$ and $\alpha \in (0,1)$ on the local universal cover,

$$|h|_{C^{k,\alpha}(B_{r_0}(\tilde{\boldsymbol{x}}))} \le C_k \cdot |h|_{C^0(B_{2r_0}(\tilde{\boldsymbol{x}}))} \le C_k \cdot e^{-\underline{\delta}\cdot z/2},$$

where $r_0 > 0$ is some fixed constant independent of $\boldsymbol{x} \in \mathcal{C}$. Therefore, at the center $\boldsymbol{x} = (z, \boldsymbol{y})$ on the Calabi space $\mathcal{C}$, we have that

$$|\nabla^k h(z, \boldsymbol{y})| \le C_k \cdot |h|_{C^0(B_{2r_0}(\boldsymbol{x}))} \le C_k \cdot e^{-\underline{\delta}\cdot z/2}.$$

This completes the proof. □

4.4. **Regularity and asymptotics for the Poisson equation.** This subsection focuses on the solvability and uniform estimates for the Poisson equation near the end of a Calabi space. Our approach to this problem was inspired by the approach of [HS17] to the analogous problem on conical model spaces. First, we will prove a technical ODE lemma.

Lemma 4.11. *For each $k \in \mathbb{Z}_+$, let us consider the inhomogeneous ordinary differential equation*

$$\frac{d^2 u_k(z)}{dz^2} - (j_k^2 z^2 + \lambda_k) u_k(z) = v_k(z) \cdot z, \ \ z \ge 1, \tag{4.26}$$

where $\underline{\delta} > 0$ is the constant defined in (4.20). Assume that the function $v_k(z)$ satisfies the following property: there are constants

$$\eta_0 \in (-\underline{\delta}/2, \underline{\delta}/2) \setminus \{0\} \quad \text{and} \quad Q_k > 0$$

such that

$$|v_k(z)| \le Q_k \cdot e^{\eta_0 z}.$$

Denoting by $\mathcal{W}_k \equiv \mathcal{W}\big(\mathcal{F}_k(z), \mathcal{U}_k(z)\big)$ the Wronskian of $\mathcal{F}_k$ and $\mathcal{U}_k$, let $u_k(z)$ be the particular solution

$$u_k(z) \equiv \mathcal{W}_k^{-1}(\mathcal{D}_k(z) + \mathcal{G}_k(z)),$$

where

$$\mathcal{D}_k(z) \equiv \mathcal{F}_k(z) \int_z^\infty \mathcal{U}_k(r) \cdot \Big(v_k(r) \cdot r\Big) dr \quad \text{and} \quad \mathcal{G}_k(z) \equiv \mathcal{U}_k(z) \int_1^z \mathcal{F}_k(r) \cdot \Big(v_k(r) \cdot r\Big) dr.$$

Then there are constants $C_0 > 0$ and $\eta_0 < \eta < \eta_0 + \underline{\delta}/10$ which are independent of k such that the particular solution u_k satisfies the uniform estimate

$$|u_k(z)| \le C_0 \cdot Q_k \cdot e^{\eta z}.$$

Proof. It is elementary to verify that $u_k(z)$ is a particular solution of (4.26). We will prove that there exists some constant $\eta_0 < \eta < \eta_0 + \underline{\delta}/10$ such that

$$|\mathcal{W}_k^{-1}| \cdot |\mathcal{D}_k(z)| \le C_0 \cdot Q_k \cdot e^{\eta z}, \tag{4.27}$$

$$|\mathcal{W}_k^{-1}| \cdot |\mathcal{G}_k(z)| \le C_0 \cdot Q_k \cdot e^{\eta z}, \tag{4.28}$$

where the constant $C_0 > 0$ is independent of k. The rest of the lemma will follow from this.

First, we consider the case where $k \in \mathbb{Z}_+$ satisfies $j_k = 0$. The fundamental solutions are

$$\mathcal{F}_k(z) \equiv e^{\sqrt{\lambda_k}\cdot z} \quad \text{and} \quad \mathcal{U}_k(z) \equiv e^{-\sqrt{\lambda_k}\cdot z}.$$

Immediately, $\mathcal{W}_k = \mathcal{W}(\mathcal{F}_k(z), \mathcal{U}_k(z)) = 2\sqrt{\lambda_k}$ and hence for $\eta > \eta_0$,

$$\frac{|\mathcal{D}_k(z)|}{|\mathcal{W}_k|} \le \frac{\mathcal{F}_k(z)}{\mathcal{W}_k} \int_z^\infty \mathcal{U}_k(r) |v_k(r) \cdot r| dr \le \frac{Q_k \cdot e^{\sqrt{\lambda_k}\cdot z}}{\sqrt{\lambda_k}} \int_z^\infty e^{(-\sqrt{\lambda_k}+\eta)\cdot r} dr \le C_0 \cdot Q_k \cdot e^{\eta z}.$$

Similarly,

$$\frac{|\mathcal{G}_k(z)|}{|\mathcal{W}_k|} \leq \frac{\mathcal{U}_k(z)}{\mathcal{W}_k} \int_1^z \mathcal{F}_k(r)|v_k(r) \cdot r|dr \leq \frac{Q_k \cdot e^{-\sqrt{\lambda_k} \cdot z}}{\sqrt{\lambda_k}} \int_1^z e^{(\sqrt{\lambda_k}+\eta)\cdot r} dr \leq C_0 \cdot Q_k \cdot e^{\eta z}.$$

Next, consider the case $j_k \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$. From the uniform estimate in Lemma 4.6, we have

$$\left|\frac{\mathcal{D}_k(z)}{\mathcal{W}_k}\right| \leq \frac{C_0 e^{\widehat{F}_k(z)}}{\mathcal{W}_k} \int_z^\infty e^{\widehat{U}_k(r)}|v_k(r) \cdot r|dr \leq \frac{C_0 \cdot Q_k \cdot e^{\widehat{F}_k(z)}}{\mathcal{W}_k} \int_z^\infty e^{\widehat{U}_k(r)+\eta' r} dr,$$

where $\eta' > \eta_0$. We choose $\epsilon \in (\underline{\delta}/100, \underline{\delta}/10)$ and denote $\eta \equiv \eta' + \epsilon$. Applying the monotonicity in Lemma 4.7, we have that

$$\left|\frac{\mathcal{D}_k(z)}{\mathcal{W}_k}\right| \leq C_0 \cdot Q_k \cdot \frac{e^{\widehat{F}_k(z)}}{\mathcal{W}_k} \int_z^\infty e^{\widehat{U}_k(r)+\eta r} \cdot e^{-\epsilon r} dr \leq C_0 \cdot Q_k \cdot \frac{e^{\widehat{F}_k(z)+\widehat{U}_k(z)+\eta z}}{\mathcal{W}_k}.$$

The estimate (4.27) follows from this after applying the uniform estimate in Lemma 4.7. The estimate (4.28) is proved in a similar fashion. This completes the proof of the lemma. □

We now prove the solvability of the Poisson equation on the end of the Calabi model space.

Proposition 4.12. *Let $(\mathcal{C}, g_{\mathcal{C}})$ be the Calabi model space. There is a constant $\underline{\delta} > 0$ which depends only on $\mathcal{C}$ such that the following property holds: given any*

$$\eta_0 \in (-\underline{\delta}, \underline{\delta}) \setminus \{0\},$$

if $v \in C^{3K_0,\alpha}(\mathcal{C})$ for $K_0 \geq 3$ and $v(z, \boldsymbol{y}) = O(e^{\eta_0 z})$, then the equation

$$\Delta_{g_0} u = v \tag{4.29}$$

has a solution $u \in C^{3K_0+2,\alpha}(\mathcal{C})$ such that for any $\eta > \eta_0$,

$$|u(z, \boldsymbol{y})| + |\nabla u(z, \boldsymbol{y})| \leq C \cdot e^{\eta z}.$$

Proof. The basic strategy is to apply separation of variables to construct a solution to the equation (4.29). Given v and for any fixed $z \geq 1$, there is a fiberwise L^2-expansion

$$v(z, \boldsymbol{y}) = \sum_{k=0}^\infty v_k(z)\varphi_k(\boldsymbol{y}). \tag{4.30}$$

Now let $u_0(z)$ satisfy $u_0''(z) = v_0(z)$, and for each $k \in \mathbb{Z}_+$ let $u_k(z)$ be the particular solution in Lemma 4.11. Then we have the following formal solution $u(z, \boldsymbol{y})$ to the equation (4.29):

$$u(z, \boldsymbol{y}) = \sum_{k=0}^\infty u_k(z)\varphi_k(\boldsymbol{y}). \tag{4.31}$$

First, we will show that the above series converges in the C^0-topology and thus u is a C^0-function. Applying Lemma 4.8, Lemma 4.9 and Lemma 4.11, the L^2-expansion (4.31) yields

$$\begin{aligned}|u(z,\boldsymbol{y})| \leq & |u_0(z)\cdot\varphi_0(\boldsymbol{y})| + \sum_{k=1}^{\infty}|u_k(z)|\cdot|\varphi_k(\boldsymbol{y})| \\ \leq & C\cdot\Big(|u_0(z)| + \sum_{k=1}^{\infty}\frac{e^{\eta z}}{(\Lambda_k)^{K_0-1}}\Big) \leq C\cdot(|u_0(z)| + e^{\eta z}),\end{aligned}$$

where the last inequality follows from Weyl's law (see (4.25)) and we fix $K_0 \geq 3$. Moreover, integrating $v_0(z)$ twice gives $|u_0(z)| \leq C\cdot e^{\eta z}$. Therefore, $u \in C^0_{loc}(\mathcal{C})$ and $|u(z,\boldsymbol{y})| \leq C\cdot e^{\eta z}$.

Next, we will apply the standard elliptic regularity on the Calabi model space to show that $u \in C^2$ and thus u is a regular solution. We denote by

$$U_N(z,\boldsymbol{y}) \equiv \sum_{k=0}^{N} u_k(z)\varphi_k(\boldsymbol{y}) \quad\text{and}\quad V_N(z,\boldsymbol{y}) \equiv \sum_{k=0}^{N} v_k(z)\varphi_k(\boldsymbol{y}), \tag{4.32}$$

the partial sums of u and v respectively, which satisfy $\Delta_{g_{\mathcal{C}}} U_N = V_N$. For every $\boldsymbol{x} \equiv (z,\boldsymbol{y}) \in \mathcal{C}$, we will apply the elliptic regularity on the ball $B_2(\boldsymbol{x}) \subset \mathcal{C}$ to obtain the higher regularity of u. For this purpose, we first prove Claim 4.13.

Claim 4.13. For a fixed $\boldsymbol{x} = (z,\boldsymbol{y}) \in \mathcal{C}$, we have that $\|V_N - v\|_{C^0(B_2(\boldsymbol{x}))} \to 0$ as $N \to \infty$, where v and V_N are given in (4.30) and (4.32), respectively.

Proof of Claim 4.13. Let us compute the expansion of the error term $v - V_N$:

$$\begin{aligned}v - V_N = \sum_{k>N} v_k\varphi_k = & \sum_{k>N}\Big(\int_{Y^3} v\cdot\varphi_k \,\mathrm{dvol}_{h_0}\Big)\varphi_k \\ = & \sum_{k>N}\Big(\int_{Y^3} v\cdot\frac{(-\Delta_{h_0})^{K_0}\varphi_k}{(\Lambda_k)^{K_0}}\,\mathrm{dvol}_{h_0}\Big)\varphi_k.\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}\|v - V_N\|_{L^\infty(B_2(\boldsymbol{x}))} \leq & \sum_{k>N}\Big(\frac{1}{(\Lambda_k)^{K_0}}\int_{Y^3}|\Delta_{h_0}^{K_0} v|\cdot|\varphi_k|\,\mathrm{dvol}_{h_0}\Big)\|\varphi_k\|_{L^\infty(B_2(\boldsymbol{x}))} \\ \leq & V_0\cdot\|v\|_{C^{2K_0}(Y^3\times\{z\})}\cdot\sum_{k>N}\frac{1}{(\Lambda_k)^{K_0-1}},\end{aligned}$$

where $V_0 = \mathrm{Vol}_{h_0}(Y^3)$. For fixed $K_0 \geq 3$, applying Weyl's law as before gives us that

$$\|v - V_N\|_{L^\infty(B_2(\boldsymbol{x}))} \leq C\|v\|_{C^{2K_0}(Y^3\times\{z\})}\cdot\sum_{k>N}\frac{1}{k^{\frac{4}{3}}}.$$

The right-hand side limits to 0 as $N \to +\infty$, and this completes the proof of the claim. □

The proof of the higher order convergence then follows by exactly the same argument. In fact, we just need to replace $\|v\|_{C^{2K_0}}$ with the higher order norm $\|v\|_{C^{2K_0+m}}$ with $m \leq K_0$. Since $\Delta_{g_{\mathcal{C}}} U_N = V_N$, the standard $W^{2,p}$-estimate implies that for every $1 < p < \infty$, $\|U_N\|_{W^{2,p}(B_{3/2}(\boldsymbol{x}))} \leq C_{p,\boldsymbol{x}}$. By assumption, since $v \in$

$C^{3K_0}(\mathcal{C})$ with $K_0 \geq 3$, we have that $\|V_N\|_{C^2(B_2(\boldsymbol{x}))} \leq C_{\boldsymbol{x}}$. Hence the regularity of u will be improved as follows. For every $1 < p < \infty$,

$$\|U_N\|_{W^{4,p}(B_1(\boldsymbol{x}))} \leq C_{p,\boldsymbol{x}}(\|U_N\|_{W^{2,p}(B_{3/2}(\boldsymbol{x}))} + \|V_N\|_{W^{2,p}(B_{3/2}(\boldsymbol{x}))}) \leq C_{p,\boldsymbol{x}}.$$

Now taking $p > 4$ and applying the Sobolev embedding, we see that

$$\|U_N\|_{C^{3,\alpha}(B_1(\boldsymbol{x}))} \leq C_{p,\boldsymbol{x}}, \ \alpha \equiv 1 - 4p^{-1},$$

which implies that U_N converges to a smooth solution u as in the proof of Claim 4.13. Applying the standard Schauder estimate and bootstrapping, the statement of the proposition follows. □

4.5. **Proof of the Liouville theorem.** With the above technical preparations, we can now complete the proof of the main result of this section, Theorem 4.3. We need Lemma 4.14.

Lemma 4.14. *Let (M^n, g, p) be a complete non-compact manifold with* $\mathrm{Ric}_g \geq 0$. *Let ω be a harmonic 1-form on (M^n, g), i.e., $\Delta_H \omega = 0$, where Δ_H denotes the Hodge Laplacian. Assume that*

$$\lim_{d_g(x,p)\to+\infty} |\omega(x)| = 0. \tag{4.33}$$

Then $\omega \equiv 0$ on M^n.

Proof. Since ω is harmonic, by Bochner's formula, we have that $\Delta_g|\omega|^2 = 2|\nabla\omega|^2 + 2\,\mathrm{Ric}_g(\omega,\omega) \geq 0$. Combining (4.33) and the maximum principle, we have that $\omega \equiv 0$ on M^n. □

Proof of Theorem 4.3. Let u satisfy $\Delta_g u = 0$ on (X^4, g) and $u = O(e^{\ell_0 z})$ for some $\ell_0 \in (0,1)$ as $z \to \infty$. The main part of the proof is to determine an $\ell_0 > 0$ such that if the above growth condition of u holds, then u has at most linear growth, which enables us to apply Lemma 4.14.

By assumption, there is a diffeomorphism $\Phi : [10^2, +\infty) \times Y^3 \to X^4 \setminus K$ such that for all $k \in \mathbb{N}$, the following asymptotic behavior holds with respect to $g_{\mathcal{C}}$:

$$\|\Phi^* g - g_{\mathcal{C}}\|_{C^k(\mathcal{C})} \leq C_k e^{-\delta z}. \tag{4.34}$$

Claim 4.15. Assume that (X^4, g) is δ-asymptotically Calabi. Let $\hat{\delta} \in (0, \delta/10)$ be such that u satisfies

$$\Delta_g u = 0 \quad \text{and} \quad u = O(e^{\hat{\delta} z}) \quad \text{as } z \to +\infty.$$

Then for every fixed $k \in \mathbb{Z}_+$, there exists a constant $C_k(g) > 0$ such that for every $\boldsymbol{x} \in [T_0(k), +\infty) \times Y^3$ with $T_0(k) \geq 100^{k^3} > 0$, $\|\nabla^k_{g_{\mathcal{C}}} \Delta_{g_{\mathcal{C}}} u(\boldsymbol{x})\|_{g_{\mathcal{C}}} \leq C_k(g) \cdot e^{-\frac{\delta \cdot z_0}{2}}$, where $z \equiv z(\boldsymbol{x})$.

Proof. Near infinity of the Tian-Yau space (X^4, g), let us denote $\phi \equiv (\Delta_g - \Delta_{g_{\mathcal{C}}})u$. Then $\Delta_g u = 0$ implies that

$$\Delta_{g_{\mathcal{C}}} u + \phi = 0. \tag{4.35}$$

We will show that for each $k \in \mathbb{N}$ we have

$$\|\nabla^k_{g_{\mathcal{C}}} \phi(\boldsymbol{x})\| \leq C_k(g) \cdot e^{-\frac{\delta z}{2}}, \tag{4.36}$$

where $C_k(g) > 0$ depends only on $k \in \mathbb{N}$ and the curvature bound of the end $(X^4 \setminus K, g)$.

The higher order derivative estimate will be proved by using standard local covering arguments. Notice that the end $X^4 \setminus K$ is collapsing and has the following curvature bound for any $k \in \mathbb{N}$:

$$\sup_{X^4\setminus K} \|\nabla^k \operatorname{Rm}_g\|_g \leq Q_k(g)$$

for some $Q_k(g) > 0$. Then given any $\alpha \in (0,1)$, there exists a constant $r_0 \in (0,1)$ depending only on $Q_0(g)$ and α such that the following holds. Let $(\widetilde{B_{r_0}(\boldsymbol{x})}, \tilde{g}, \tilde{\boldsymbol{x}})$ be the Riemannian universal cover of $B_{r_0}(\boldsymbol{x})$, where $\tilde{\boldsymbol{x}}$ is a lift of $\boldsymbol{x}$ and $\boldsymbol{x} \in X^4 \setminus K$. Then the $C^{1,\alpha}$-harmonic radius of $(\widetilde{B_{r_0}(\boldsymbol{x})}, \tilde{\boldsymbol{x}})$ is uniformly bounded from below by $r_0/2$.

Using the standard Schauder estimate for the lifted function $\tilde{u}$ within the harmonic radius, we have that $\|\tilde{u}\|_{C^{2,\alpha}(B_{r_0/4}(\tilde{\boldsymbol{x}}))} \leq C \cdot \|\tilde{u}\|_{C^0(B_{r_0/2}(\tilde{\boldsymbol{x}}))} \leq C \cdot e^{\hat{\delta}\cdot z}$. Then $|\nabla_g^2 u(\boldsymbol{x})|_g \leq C \cdot e^{\hat{\delta}\cdot z}$. Next, by bootstrapping, we have that for every $k \in \mathbb{Z}_+$, $|\nabla_g^k u(\boldsymbol{x})|_g \leq C_k(g) \cdot e^{\hat{\delta}\cdot z}$. Combining this with (4.34), we find that

$$|\nabla^k \phi(\boldsymbol{x})| = |(\Delta_g - \Delta_{g_C}) u(\boldsymbol{x})| \leq C_k(g) \cdot e^{-\frac{\delta \cdot z}{2}}.$$

□

Next we choose the constant ℓ_0 by

$$\ell_0 \in \big(0, \min\big\{\delta \cdot 10^{-2}, \underline{\delta}\big\}\big), \tag{4.37}$$

where $\underline{\delta} > 0$ is the constant in Proposition 4.10. By assumption the harmonic function u satisfies the asymptotic behavior $u = O(e^{\ell_0 z})$. Then applying the above claim and Proposition 4.12 on $[T_0, +\infty) \times Y^3$, we see that there exists a solution to the equation

$$\Delta_{g_C} v = \phi \tag{4.38}$$

such that

$$|v(z,\boldsymbol{y})| + |\nabla_{g_C} v(z,\boldsymbol{y})|_{g_C} \leq C \cdot e^{-\ell z}$$

for some $\ell \in (-\delta/2, 0)$. Therefore, combining (4.35) and (4.38), we have

$$0 = \Delta_g u = \Delta_{g_C}(u + v)$$

with $u + v = O(e^{\ell_0 z})$. Since $0 < \ell_0 < 1$ has been specified in (4.37), we are now in a position to apply Proposition 4.10 to $u + v$, which shows that

$$u + v = az + b + h(z, \boldsymbol{y}),$$

where $h(z,\boldsymbol{y})$ satisfies $|\nabla_{g_C} h(z,\boldsymbol{y})|_{g_C} \leq C_k \cdot e^{-\underline{\delta}\cdot z/2}$ for every $k \in \mathbb{N}$.

The above asymptotics immediately imply that

$$|du|_g = |dv + dz + dh|_g \leq |dv|_g + |dz|_g + |dh|_g \to 0 \text{ as } z \to \infty. \tag{4.39}$$

Let Δ_H be the Hodge-Laplacian on (X^4, g). Since $\Delta_g u = 0$, it holds that

$$\Delta_H(du) = dd^*(du) = -d\Delta_g u = 0.$$

Since the complete space (X^4, g) satisfies $\operatorname{Ric}_g \geq 0$ and $|du|$ satisfies the decay property (4.39), Lemma 4.14 implies that $|du|_g \equiv 0$ on X^4. Therefore, u has to be a constant, as claimed. □

5. Liouville theorem for half-harmonic 1-forms

Recall from equation (1.6) that we call a 1-form γ on an oriented Riemannian 4-manifold *half-harmonic* if it satisfies $\mathscr{D}\gamma = 0$, i.e.,

$$d^+\gamma = 0, \quad d^*\gamma = 0. \tag{5.1}$$

If the manifold is compact, then such a form is automatically closed and co-closed, but this is no longer true on a non-compact manifold. The main result of this section is a Liouville theorem for half-harmonic 1-forms. Let $(X^4 = M \setminus D, \omega)$ be given by the Tian-Yau construction, where M is a smooth del Pezzo surface and D is a smooth anti-canonical divisor in M. By Proposition 3.4, we can fix a diffeomorphism $\Phi : \mathcal{C} \setminus K' \to X \setminus K$, where K and K' are compact subsets in X and $\mathcal{C}$ respectively, and X is then a $\underline{\delta}$-asymptotically Calabi space for some $\underline{\delta} > 0$. Let z denote the moment map coordinate on $\mathcal{C}$, as defined in Section 3.

Theorem 5.1. *There is some positive constant $\delta_h > 0$ which depends only on the geometry of X^4 such that if a half-harmonic 1-form γ on X satisfies*

$$|\gamma|_\omega = O(e^{\delta_h z}) \tag{5.2}$$

as $z \to \infty$, then $\gamma \equiv 0$.

Proof. We begin with an interpretation of (5.1) in terms of complex geometric data. Notice in the Tian-Yau construction we have a preferred complex structure J_{TY} on X induced from that on M. With respect to J_{TY} we can write $\gamma = \gamma^{1,0} + \gamma^{0,1}$ with $\gamma^{1,0} = \overline{\gamma^{0,1}}$. Then by the Kähler identities we have

$$d^+\gamma = 0 \Longleftrightarrow \begin{cases} \bar\partial\gamma^{0,1} = 0, \\ \sqrt{-1}(\bar\partial^*\gamma^{0,1} - \partial^*\gamma^{1,0}) = 0, \end{cases}$$

and

$$d^*\gamma = 0 \Longleftrightarrow \bar\partial^*\gamma^{0,1} + \partial^*\gamma^{1,0} = 0.$$

Thus, equation (5.1) is equivalent to

$$\bar\partial\gamma^{0,1} = \bar\partial^*\gamma^{0,1} = 0.$$

Theorem 5.1 will follow from Theorem 4.3 once we prove that there exists some small $\delta > 0$ and a smooth function $f = O(e^{\delta z})$ such that $\bar\partial f = \gamma^{0,1}$ (note that $\Delta f = \bar\partial^*\gamma^{0,1} = 0$).

We next give a brief outline of the proof of this fact. There are four steps.

In Step 1, we will construct a solution f to $\bar\partial f = \gamma$ such that $f = O(e^{\epsilon z^2})$ for all $\epsilon > 0$. This is done using a complex-geometric argument, which essentially amounts to an application of Hörmander's weighted L^2-estimates for the $\bar\partial$-operator. Interestingly it does not seem to be possible to obtain the required improvement $f = O(e^{\delta z})$ using only this type of method, owing to the fact that the function z^a is plurisubharmonic on the Calabi model space $(\mathcal{C}, g_{\mathcal{C}})$ only if $a \geq 2$. To overcome this problem we use the elliptic theory on $(\mathcal{C}, g_{\mathcal{C}})$ developed in Section 4. Thanks to the bound $f = O(e^{\epsilon z^2})$ for all $\epsilon > 0$ from Step 1 and the $O(e^{-(\frac{1}{2}-\epsilon)z^2})$ complex structure asymptotics of Proposition 3.4, it follows that $\bar\partial_{\mathcal{C}} f = O(e^{\delta z})$ on $(\mathcal{C}, g_{\mathcal{C}})$. In particular, since $\Delta = \mathrm{tr}(\sqrt{-1}\partial\bar\partial)$, the Poisson equation estimates of Proposition 4.12 imply that f can be decomposed into an $O(e^{\delta z})$ part f_1 and a $g_{\mathcal{C}}$-*harmonic* part f_2 which is $O(e^{\epsilon z^2})$ for all $\epsilon > 0$; see Step 2 for details. Observe that it would not be possible to compare $\Delta_{TY} f$ and $\Delta_{\mathcal{C}} f$ directly because g_{TY} and $g_{\mathcal{C}}$ are only asymptotic at rate $O(e^{-\delta z})$, which is too slow to beat the $O(e^{\epsilon z^2})$ growth of f from Step 1.

Step 3 analyzes the $g_{\mathcal{C}}$-harmonic part f_2 of f. It is clear from Section 4 that $f_2 = O(e^{Cz})$ for some large constant C. The required improvement $f_2 = O(e^{\delta z})$ comes from the first-order equation $\bar\partial_{\mathcal{C}} f_2 = O(e^{\delta z})$ satisfied by f_2 (in addition to $\Delta_{g_{\mathcal{C}}} f_2 = 0$). Technically this is done using separation of variables for the $\bar\partial_{\mathcal{C}}$-operator but the underlying idea can be easily explained: being of $O(e^{Cz})$ rather than $O(e^{Cz^2})$ growth, the leading terms of the harmonic function f_2 must be S^1-invariant, but on S^1-invariant functions the $\bar\partial_{\mathcal{C}}$-operator directly controls the radial derivative $\frac{\partial}{\partial z}$.

Step 4 concludes the proof by appealing to Theorem 4.3.

Step 1. In this step, we prove Proposition 5.2.

Proposition 5.2. *There is a smooth function f on X with $\bar\partial f = \gamma$ and $|f| = O(e^{\epsilon z^2})$ for all $\epsilon > 0$.*

Remark 5.3. We also have $\Delta_\omega f = 0$, but at this point we cannot apply Theorem 4.3 directly to conclude that f is a constant since this would require the stronger control $|f| = O(e^{\delta z})$.

Proof of Proposition 5.2. We work on the compact manifold M. Let S be a holomorphic section of K_M^{-1} with $S^{-1}(0) = D$ and let h be a smooth hermitian metric on K_M^{-1} whose curvature form ω_h is a Kähler form on M with positive Ricci curvature. By Theorem 3.3, near D we have that

$$C^{-1}\sqrt{-1}\partial\bar\partial(-\log|S|_h^2)^{3/2} \leq \omega_{TY} \leq C\sqrt{-1}\partial\bar\partial(-\log|S|_h^2)^{3/2}.$$

By a straightforward computation, this implies that

$$\omega_{TY} \leq C|S|_h^{-2}\omega_h, \tag{5.3}$$

and hence by hypothesis (5.2),

$$|\gamma|_{\omega_h} \leq C^{-1}|S|_h^{-1}|\gamma|_{\omega_{TY}} = O(|S|_h^{-1-\epsilon})$$

for any $\epsilon > 0$. Define $\alpha = \gamma \otimes S$. This is a section of $\Lambda_M^{0,1} \otimes K_M^{-1}$ which lies in $L^p_{\omega_h}(M, \Lambda_M^{0,1} \otimes K_M^{-1})$ for all $p > 1$. Since $\bar\partial\gamma = 0$, one can directly check that $\bar\partial\alpha = 0$ in the distributional sense. Now notice that $H^1(M, K_M^{-1}) = H^1(M, K_M \otimes L) = 0$ by the Kodaira vanishing theorem applied to the ample line bundle $L = K_M^{-2}$. Thus, we can define $\beta = \bar\partial^*\Delta_{\bar\partial}^{-1}\alpha$ with respect to ω_h. It follows from elliptic regularity that $\beta \in W^{1,p}_{\omega_h}(M, K_M^{-1})$ for all $p > 1$, so that $\beta \in C^\alpha_{\omega_h}(M, K_M^{-1})$ for all $\alpha < 1$. Moreover, by local regularity, we know that β is smooth away from D and $\bar\partial\beta = \alpha$. Let $f = \beta \otimes S^{-1}$. Then on X we have that $\bar\partial f = \gamma$. It follows that there is some constant $C > 0$ such that

$$f = O(|S|_h^{-1}) = O(e^{Cz^2}). \tag{5.4}$$

Lemma 5.4 allows us to improve (5.4) to the growth order $e^{\epsilon z^2}$ for any $\epsilon > 0$. The key point is that the estimate (5.3) can be improved to almost $O(1)$ in directions tangential to D.

Lemma 5.4. *Denote $\beta_0 = \beta|_D$. Then $\bar\partial\beta_0 = 0$, i.e., β_0 is a holomorphic section of $K_M^{-1}|_D$.*

Proof. We choose a finite cover $D = \bigcup_{k=1}^{N_0} O_k$ such that for each k there exists a local holomorphic coordinate system (w_1, w_2) on some domain $U_k \subset M$ such that $U_k \cap D = O_k = \{w_2 = 0\}$. We will show that $\bar{\partial}\beta_0 = 0$ in every $O_k \subset D$ in the distributional sense. Let ψ be a smooth section of $\Lambda_D^{0,1} \otimes (K_M^{-1}|_D)$ with compact support in O_k. It suffices to show that $\langle \beta_0, \bar{\partial}^*\psi\rangle_{O_k} = 0$. To this end, write $\psi(w_1) = \sigma(w_1) d\bar{w}_1 \otimes (dw_1 \wedge dw_2)^{-1}$ for some smooth function $\sigma \in C_0^\infty(O_k, \mathbb{C})$ and use this to define the trivial extension $\hat{\psi}(w_1, w_2) = \sigma(w_1) d\bar{w}_1 \otimes (dw_1 \wedge dw_2)^{-1}$ for all $(w_1, w_2) \in U_k$. Denote by $O_k(\tau)$ the slice $\{w_2 = \tau\}$ in U_k, which is a complex submanifold of M, and equip $O_k(\tau)$ with the restriction of the Kähler metric ω_h from M. Notice that $\hat{\psi}$ restricts to a smooth section of $\Lambda^{0,1}_{O_k(\tau)} \otimes (K_M^{-1}|_{O_k(\tau)})$ with compact support in $O_k(\tau)$. Since $\bar{\partial}\beta = \alpha$ and $\beta \in W^{1,p} \cap C^\alpha$ for any $p > 1$, it follows that

$$\langle \beta_0, \bar{\partial}^*\psi\rangle_{O_k} = \lim_{\tau\to 0} \langle \beta, \bar{\partial}^*\hat{\psi}\rangle_{O_k(\tau)} = \lim_{\tau\to 0}\langle \bar{\partial}\beta, \hat{\psi}\rangle_{O_k(\tau)} = \lim_{\tau\to 0}\langle \alpha, \hat{\psi}\rangle_{O_k(\tau)}. \tag{5.5}$$

Notice that

$$|\gamma(\partial_{\bar{w}_1})| \le |\gamma|_{\omega_{TY}} |\partial_{\bar{w}_1}|_{\omega_{TY}} \le |\gamma|_{\omega_{TY}} (-\log |S|_h^2)^{\frac{1}{4}} = O(|S|_h^{-\epsilon}).$$

Since $\alpha = \gamma \otimes S$, it then follows that $|\alpha(\partial_{\bar{w}_1})| = O(|S|_h^{1-\epsilon}) \to 0$ uniformly as $w \to 0$. Using (5.5) and noticing that $\partial_{\bar{w}_1}$ is tangential to $O_k(\tau)$, it follows that

$$\langle \beta_0, \bar{\partial}^*\psi\rangle_{O_k} = 0,$$

as desired. By standard elliptic regularity, β_0 is a holomorphic section. $\square$

Since M is Fano, we have that $H^1(M, \mathcal{O}_M) = 0$, so by a standard exact sequence ([GH94, p.139]) the restriction map $H^0(M, K_M^{-1}) \to H^0(D, K_M^{-1}|_D)$ is surjective. This means that we can find some $\beta_1 \in H^0(M, K_M^{-1})$ such that $\beta_1|_D = \beta_0|_D$. Let $f = (\beta - \beta_1) \otimes S^{-1}$. Then we still have $\bar{\partial} f = \gamma$ on X but now since $\beta - \beta_1 = 0$ on D and $\beta \in C^\alpha_{\omega_h}(M, K_M^{-1})$ for all $\alpha < 1$, we finally obtain Proposition 5.2. $\square$

Step 2. Let $f = u + \sqrt{-1}v$ be the smooth function constructed in Proposition 5.2 with $\Delta_{\omega_{TY}} u = \Delta_{\omega_{TY}} v = 0$. In this step, we reduce the problem to a question on the Calabi model space through the diffeomorphism $\Phi : (\mathcal{C} \setminus K', \omega_{\mathcal{C}}, J_{\mathcal{C}}) \to (X \setminus K, \omega_{TY}, J_{TY})$ of Proposition 3.4. The main point is to obtain a decomposition $u = u_1 + u_2$ and $v = v_1 + v_2$ such that $u_1 = O(e^{\delta z})$, $v_1 = O(e^{\delta z})$, and $\Delta_{\omega_{\mathcal{C}}} u_2 = \Delta_{\omega_{\mathcal{C}}} v_2 = 0$.

The idea of the proof of Step 2 is as follows. First, we will estimate $\Delta_{\omega_{\mathcal{C}}} u$ and $\Delta_{\omega_{\mathcal{C}}} v$ and all of their derivatives. Specifically, we will prove that they have slow exponential growth rates (as shown in (5.8)). Then applying Proposition 4.12, we can construct solutions to the Poisson equations

$$\Delta_{\omega_{\mathcal{C}}} u_1 = \Delta_{\omega_{\mathcal{C}}} u, \ \Delta_{\omega_{\mathcal{C}}} v_1 = \Delta_{\omega_{\mathcal{C}}} v,$$

such that $u_1 = O(e^{\delta z})$ and $v_1 = O(e^{\delta z})$. This completes the desired decomposition of u and v.

To obtain derivative estimates for $\Delta_{\omega_{\mathcal{C}}} u$ and $\Delta_{\omega_{\mathcal{C}}} v$, we will prove derivative estimates for $du + J_{\mathcal{C}} dv$. To start with, by the assumption on γ and the first order equation given by Step 1, we have that

$$du + J_{TY} dv = Re(\gamma) = O(e^{\delta_h z}). \tag{5.6}$$

Then applying the asymptotic estimate for J_{TY} in Proposition 3.4, we can convert the above growth control to a corresponding estimate for $du + J_{\mathcal{C}}dv$. In fact, applying item (b) of Proposition 3.4, for any $\epsilon > 0$ and for any $k \in \mathbb{N}$,

$$|\nabla^k_{\omega_{\mathcal{C}}}(\Phi^* J_{TY} - J_{\mathcal{C}})|_{\omega_{\mathcal{C}}} = O(e^{(-\frac{1}{2}+\epsilon)z^2}).$$

We also need derivative estimates for u and v with respect to the model metric $\omega_{\mathcal{C}}$. Notice that by Step 1, $f = u + \sqrt{-1}v = O(e^{\epsilon z^2})$ for any $\epsilon > 0$, which implies that

$$|u| + |v| \leq 2|f| = O(e^{\epsilon z^2}).$$

Since u and v satisfy $\Delta_{\omega_{TY}} u = \Delta_{\omega_{TY}} v = 0$, by applying local elliptic estimates on local universal covers as in the proof of Claim 4.15, we have that for all $\epsilon > 0$ and $k \geq 1$,

$$|\nabla^k_{\omega_{TY}} u|_{\omega_{TY}} = O(e^{\epsilon z^2}), \quad |\nabla^k_{\omega_{TY}} v|_{\omega_{TY}} = O(e^{\epsilon z^2}).$$

In terms of the Calabi model metric we then have

$$|\nabla^k_{\omega_{\mathcal{C}}} u|_{\omega_{\mathcal{C}}} = O(e^{\frac{\epsilon z^2}{2}}), \quad |\nabla^k_{\omega_{\mathcal{C}}} v|_{\omega_{\mathcal{C}}} = O(e^{\frac{\epsilon z^2}{2}}).$$

Similarly, since γ is half-harmonic, we have that for all $k \geq 0$,

$$|\nabla^k_{\omega_{\mathcal{C}}}(du + J_{TY}dv)|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}).$$

Applying these to (5.6), we get for $k \in \mathbb{N}$,

$$\begin{aligned} |\nabla^k_{\omega_{\mathcal{C}}}(du + J_{\mathcal{C}}dv)|_{\omega_{\mathcal{C}}} &= \Big|\nabla^k_{\omega_{\mathcal{C}}}(du + J_{TY}dv) + \nabla^k_{\omega_{\mathcal{C}}}\Big((J_{\mathcal{C}} - \Phi^* J_{TY})dv\Big)\Big|_{\omega_{\mathcal{C}}} \\ &= O(e^{\delta_h z}) + O(e^{-\frac{z^2}{4}}) = O(e^{\delta_h z}). \end{aligned}$$

Now we proceed to prove the derivative estimates for $\Delta_{\omega_{\mathcal{C}}} u$ and $\Delta_{\omega_{\mathcal{C}}} v$ by using the system

$$du + J_{TY}dv = Re(\gamma). \tag{5.7}$$

The advantage of the above equation is that $\Delta_{\omega_{TY}} = \mathrm{Tr}_{\omega_{TY}}(d \circ J_{TY} \circ d)$ so that the behavior of $\Delta_{\omega_{TY}}$ will follow from the asymptotics of J_{TY}. In fact, taking the differential of (5.7) yields

$$d \circ J_{TY} \circ du = dJ_{TY} Re(\gamma), \; d \circ J_{TY} \circ dv = dRe(\gamma).$$

Then using item (a) of Proposition 3.4, similar to the above we have for all $k \in \mathbb{Z}_+$ that

$$|\nabla^k_{\omega_{\mathcal{C}}} dJ_{\mathcal{C}}du|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}), \quad |\nabla^k_{\omega_{\mathcal{C}}} dJ_{\mathcal{C}}dv|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}).$$

Taking the trace, we then obtain that

$$|\nabla^k_{\omega_{\mathcal{C}}} \Delta_{\omega_{\mathcal{C}}} u|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}), \quad |\nabla^k_{\omega_{\mathcal{C}}} \Delta_{\omega_{\mathcal{C}}} v|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}). \tag{5.8}$$

Applying the linear theory for $\Delta_{\omega_{\mathcal{C}}}$ in Proposition 4.12, for $\delta_h \ll \underline{\delta}$, we can choose u_1 and v_1 solving the equations

$$\Delta_{\omega_{\mathcal{C}}} u_1 = \Delta_{\omega_{\mathcal{C}}} u \text{ and } \Delta_{\omega_{\mathcal{C}}} v_1 = \Delta_{\omega_{\mathcal{C}}} v,$$

such that u_1 and v_1 satisfy

$$|\nabla^k_{\omega_{\mathcal{C}}} u_1|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}), \; |\nabla^k_{\omega_{\mathcal{C}}} v_1|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z}). \tag{5.9}$$

So we have finished the proof of the decomposition $u = u_1 + u_2$ and $v = v_1 + v_2$ such that

$$\Delta_{\omega_{\mathcal{C}}} u_2 = \Delta_{\omega_{\mathcal{C}}} v_2 = 0.$$

We also obtain that

$$|du_2 + J_{\mathcal{C}} dv_2|_{\omega_{\mathcal{C}}} = O(e^{\delta_h z})$$

and

$$|u_2| = O(e^{\epsilon z^2}), \ |v_2| = O(e^{\epsilon z^2}). \tag{5.10}$$

Step 3. Now we estimate the harmonic functions u_2 and v_2 with respect to the model metric $\omega_{\mathcal{C}}$ using separation of variables for $\bar{\partial}_{\mathcal{C}}$. The goal is to improve the growth order of u_2 and v_2 from $O(e^{\epsilon z^2})$ for all $\epsilon > 0$ to $O(z)$ using the fact that they also satisfy a first-order equation.

Proposition 5.5. *We have that* $|u_2| = O(z)$ *and* $|v_2| = O(z)$ *as* $z \to +\infty$.

Proof of Proposition 5.5. Let $\psi \equiv du_2 + J_{\mathcal{C}} dv_2 = O(e^{\delta_h z})$. For a fixed fiber (Y^3, h_0), let $\{\Lambda_k\}_{k=0}^{\infty}$ be the spectrum of Δ_{h_0} with $\Lambda_0 = 0$, and $\{\varphi_k\}_{k=0}^{\infty}$ be the corresponding eigenfunctions such that $-\Delta_{h_0}\varphi_k = \Lambda_k \cdot \varphi_k$ and $\mathcal{L}_{\partial_\theta}\varphi_k = \sqrt{-1} j_k \varphi_k$. As in Section 4.1, we can write the expansions

$$u_2 = \sum_{k=0}^{\infty} f_k(z)\varphi_k \quad \text{and} \quad v_2 = \sum_{k=0}^{\infty} g_k(z)\varphi_k,$$

which imply that

$$du_2 = \sum_{k=0}^{\infty} \Big(f_k'(z) \cdot \varphi_k \cdot dz + f_k(z) \cdot d\varphi_k \Big)$$

and

$$dv_2 = \sum_{k=0}^{\infty} \Big(g_k'(z)\varphi_k \cdot dz + g_k(z) \cdot d\varphi_k \Big).$$

On $\mathcal{C}$ by the definition in Section 4.1, we have $J_{\mathcal{C}}(zdz) = d\theta$. So it follows that

$$\sum_{k=0}^{\infty} \big(f_k'(z) - \sqrt{-1} j_k \cdot z \cdot g_k(z)\big)\varphi_k = \psi(\partial_z),$$

$$\sum_{k=0}^{\infty} \big(z^{-1} \cdot g_k'(z) + \sqrt{-1} j_k \cdot f_k(z)\big)\varphi_k = \psi(\partial_\theta).$$

This implies that for each $k \in \mathbb{N}$,

$$f_k'(z) - \sqrt{-1} j_k \cdot z \cdot g_k(z) = O(e^{\delta_h z}), \quad z^{-1} \cdot g_k'(z) + \sqrt{-1} j_k \cdot f_k(z) = O(e^{\delta_h z}). \tag{5.11}$$

There are three different cases for $k \in \mathbb{N}$ to consider.

(1) If $j_k \in \mathbb{Z}_+$, then each of $f_k(z)$ and $g_k(z)$ is given by a linear combination of fundamental solutions $\mathcal{F}_k(z)$ and $\mathcal{U}_k(z)$ as defined in (4.11) and (4.12), respectively. Using the asymptotics in Lemma 4.6 and (5.10), we have that $f_k(z) = C_k \cdot \mathcal{U}_k(z)$ and $g_k(z) = C_k' \cdot \mathcal{U}_k(z)$, where $\mathcal{U}_k(z) = O(e^{-\frac{j_k z^2}{2}})$.

(2) If $j_k = 0$, $\lambda_k \neq 0$, then each of $f_k(z)$ and $g_k(z)$ is given by $C_k^* \cdot e^{\sqrt{\lambda_k} z} + C_k \cdot e^{-\sqrt{\lambda_k} z}$. Let $\underline{\delta} > 0$ be the constant in Proposition 4.10. Using (5.11) and $\psi = O(e^{\delta_h z})$, we conclude that, if $\delta_h < \underline{\delta}$, then both f_k and g_k must be proportional to $e^{-\sqrt{\lambda_k} z}$.

(3) If $k = 0$, then $f_0(z) = \kappa_0 \cdot z + \mu_0$ and $g_0(z) = \kappa_0' \cdot z + \mu_0'$ for κ_0, κ_0', μ_0, $\mu_0' \in \mathbb{R}$.

Applying the same argument as in the proof of Proposition 4.10, we have that the sum of the above decaying terms still decays exponentially. That is, there are constants κ_0, κ_0', μ_0, μ_0' such that

$$u_2(z, \boldsymbol{y}) = \kappa_0 \cdot z + \mu_0 + \underline{w}(z, \boldsymbol{y}), \quad v_2(z, \boldsymbol{y}) = \kappa_0 \cdot z + \mu_0 + \underline{w}^*(z, \boldsymbol{y}),$$

where $\underline{w}(z, \boldsymbol{y}) = O(e^{-\epsilon z})$ and $\underline{w}^*(z, \boldsymbol{y}) = O(e^{-\epsilon z})$ for some $\epsilon > 0$. This completes the proof. □

Step 4. We now complete the proof of Theorem 5.1. By Proposition 5.5 and (5.9),

$$|u| = O(e^{\delta_h z}), \ |v| = O(e^{\delta_h z}).$$

If $\delta_h > 0$ is chosen such that $\delta_h < \ell_0$ with $\ell_0 > 0$ in Theorem 4.3, then we conclude that u and v must be constant so that $\gamma = 0$. □

6. Construction of the approximate hyperkähler triple

In this section, by gluing two hyperkähler Tian-Yau spaces with a neck region that satisfies an appropriate topological balancing condition, we will obtain a closed oriented 4-manifold $\mathcal{M}$ such that $\mathcal{M}$ has the same homological invariants as the K3 surface; see Proposition 6.6. Moreover, we will construct a triple of symplectic forms $\boldsymbol{\omega}_\beta^{\mathcal{M}}$ on $\mathcal{M}$, depending on a parameter $\beta > 0$, which is approximately hyperkähler with smaller and smaller errors as $\beta \to \infty$. For β sufficiently large, this triple will be perturbed to a hyperkähler triple $\boldsymbol{\omega}_\beta^{\mathrm{HK}}$ in Section 10.

6.1. **The Tian-Yau pieces.** First, we briefly describe the geometry of the Tian-Yau building blocks. Fix $b_\pm \in \{1, \ldots, 9\}$ as in Theorem A. Let $(X^4_{b_-}, g_{b_-})$ and $(X^4_{b_+}, g_{b_+})$ be two hyperkähler Tian-Yau spaces obtained by removing two anti-canonical elliptic curves from two del Pezzo surfaces of degree b_- and b_+, respectively. Denote by $\mathrm{Nil}^3_{b_\pm} = \mathrm{Nil}^3_{b_\pm}(\epsilon_\pm, \tau_\pm)$ the corresponding Heisenberg nilmanifolds with $\deg(\mathrm{Nil}^3_{b_\pm}) = b_\pm$ defined in Section 2.1. It follows from Proposition 3.1 and Corollary 3.6 that there exist "coordinate systems" on the ends of the Tian-Yau spaces $(X^4_{b_\pm}, g_{b_\pm})$,

$$\Phi^{TY}_\pm : [\zeta_0^\pm, \infty) \times \mathrm{Nil}^3_{b_\pm} \to X^4_{b_\pm}, \tag{6.1}$$

such that

$$(\Phi^{TY}_\pm)^* \boldsymbol{\omega}^\pm = \boldsymbol{\omega}^{\mathcal{C},\pm} + O(e^{-\underline{\delta}_\pm z_\pm}), \quad z_\pm \to \infty,$$

where $\boldsymbol{\omega}^{\mathcal{C},\pm}$ is the Calabi model hyperkähler triple defined by applying the Gibbons-Hawking ansatz to the flat product space $\mathbb{T}^2_\pm \times [\zeta_0^\pm, \infty)$, where $\mathbb{T}^2_\pm \equiv \mathbb{C}/\epsilon_\pm\langle 1, \tau_\pm\rangle$, with the harmonic function

$$V_\pm(z_\pm) = 2\pi b_\pm z_\pm A_\pm^{-1},$$

and the choice of connection 1-form $\theta_{b_\pm} \equiv (2\pi b_\pm A_\pm^{-1})(dt - xdy)$ on $\mathrm{Nil}^3_{b_\pm}$ as given as in (2.2). Note also that the metric in these coordinates admits an expansion

$$(\Phi^{TY}_\pm)^* g_{b_\pm} = V_\pm(g_{\mathbb{T}^2} + dz_\pm^2) + V_\pm^{-1}\theta^2_{b_\pm} + O(e^{-\underline{\delta}_\pm z_\pm}), \quad z_\pm \to \infty. \tag{6.2}$$

In our construction, we will always make the following assumptions.

- The elliptic curves removed from the del Pezzo surfaces satisfy $\tau_+ = \tau_-$.

- $A = \epsilon_{\pm}^2 Im(\tau_{\pm}) = 1$.

Note the latter assumption can always be arranged by scaling the Tian-Yau metrics. For parameters $T_- > 0$ and $T_+ > 0$, define the following compact regions in $X^4_{b_\pm}$:

$$X^4_{b_\pm}(T_\pm) \equiv X^4_{b_\pm} \Big\backslash \Phi^{TY}_{\pm}\Big((T_\pm, +\infty) \times \mathrm{Nil}^3_{b_\pm}\Big). \tag{6.3}$$

6.2. **The neck region.** Consider the flat cylinder $(\mathbb{T}^2 \times \mathbb{R}, g_0)$ and choose a finite set of monopoles $\mathcal{P}_{m_0} \equiv \{p_1, \dots, p_{m_0}\} \subset \mathbb{T}^2 \times \mathbb{R}$ such that $\sum_{m=1}^{m_0} z(p_m) = 0$, where

$$m_0 = b_- + b_+.$$

Let V_∞ be the Green's function from Corollary 2.6, which satisfies

$$-\Delta_{g_0} V_\infty = 2\pi \sum_{m=1}^{m_0} \delta_{p_m}.$$

By Corollary 2.6, we can write

$$V_\infty(x,y,z) = \begin{cases} \pi(b_- + b_+)z + h_-, & z \ll -1, \\ -\pi(b_- + b_+)z + h_+, & z \gg 1, \end{cases}$$

and

$$|\nabla^k h_\pm(x,y,z)|_{g_0} = O(e^{-\sqrt{\lambda_1}|z|}) \tag{6.4}$$

as $|z| \to \infty$ for all $k \geq 0$, where λ_1 is the smallest positive eigenvalue of $-\Delta_{\mathbb{T}^2}$.

Next, for β sufficiently large, we define a new harmonic function on $\mathbb{T}^2 \times \mathbb{R}$ by

$$V_\beta(x,y,z) \equiv V_\infty(x,y,z) + kz + \beta, \tag{6.5}$$

where

$$k = \pi(b_- - b_+).$$

Then near the two ends of the neck region, the harmonic function V_β can be written as

$$V_\beta(x,y,z) = \begin{cases} 2\pi b_- z + \beta + h_-, & z \ll -1, \\ -2\pi b_+ z + \beta + h_+, & z \gg 1. \end{cases} \tag{6.6}$$

Using this potential, we next define the neck metric through the Gibbons-Hawking ansatz. First, we note that $H_2((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, \mathbb{Z})$ has dimension $m_0 + 1$ with generators being small spheres around the monopole points and any torus of the form $\mathbb{T}^2 \times \{z'\}$, where z' is any value of z for which there are no monopole points. It is easy to see that the 2-form $\frac{1}{2\pi} * dV_\beta$ attains integer values on these cycles, which implies that the cohomology class $[\frac{1}{2\pi} * dV_\beta]$ is integral. We then denote by $\mathcal{N}^4_{m_0}$ the total space of the S^1-bundle over $(\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}$ with Euler class $[\frac{1}{2\pi} * dV_\beta]$ union finitely many points $\tilde{\mathcal{P}}_{m_0} = \{\tilde{p}_1, \dots, \tilde{p}_{m_0}\}$ with $\pi(\tilde{p}_i) = p_i$, and extend the bundle projection to a mapping $\pi : \mathcal{N}^4_{m_0} \to \mathbb{T}^2 \times \mathbb{R}$. Choose a connection 1-form θ on $\mathcal{N}^4_{m_0} \setminus \tilde{\mathcal{P}}_{m_0}$ such that

$$d\theta = *dV_\beta,$$

and note that θ is independent of the gluing parameter β. Applying the Gibbons-Hawking ansatz to V_β, we obtain a triple $\boldsymbol{\omega}^{\mathcal{N}} = (\omega_1^{\mathcal{N}}, \omega_2^{\mathcal{N}}, \omega_3^{\mathcal{N}})$ of smooth closed 2-forms over $\mathcal{N}^4_{m_0}$ (see (2.1)) which induces an incomplete hyperkähler metric over the open subset of $\mathbb{T}^2 \times \mathbb{R}$ where $V_\beta > 0$.

For real parameters T_1, T_2, we define

$$\mathcal{N}^4_{m_0}(T_1, T_2) \equiv \pi^{-1}\{(x, y, z) \in \mathbb{T}^2 \times \mathbb{R} \mid T_1 < z < T_2\},$$

and let $T_\pm$ be defined by

$$4\pi b_\pm \cdot T_\pm = \beta. \tag{6.7}$$

The reason for this choice of $T_\pm$ is due to a matching condition which we will explain in the next subsection. Denote

$$U_- \equiv (-T_-, -T_- + 1) \subset (-\infty, \infty), \quad U_+ \equiv (T_+ - 1, T_+) \subset (-\infty, \infty). \tag{6.8}$$

Lemma 6.1 shows that there is a "gauge transformation" which makes the connection form exponentially close to the Calabi model connection form for $z \in U_\pm$.

Lemma 6.1. *For β sufficiently large, there is a diffeomorphism*

$$\Phi^{\mathcal{N}}_{\beta,\pm} : U_\pm \times \mathrm{Nil}^3_{\mp b_\pm} \to \mathcal{N}^4_{m_0}(U_\pm) \tag{6.9}$$

such that

$$(\Phi^{\mathcal{N}}_{\beta,\pm})^* dx = dx, \quad (\Phi^{\mathcal{N}}_{\beta,\pm})^* dy = dy, \quad (\Phi^{\mathcal{N}}_{\beta,\pm})^* z = z, \tag{6.10}$$

and $(\Phi^{\mathcal{N}}_{\beta,\pm})^*\theta - \theta_{b_\pm} = \pi^* a_\pm$ for a 1-form $a_\pm$ on $\mathbb{T}^2 \times U_\pm$, with

$$\left|\nabla^k_{g_0} a_\pm\right|_{g_0} = O(e^{-\delta\beta}) \quad \text{as} \quad \beta \to \infty, \tag{6.11}$$

for some $\delta > 0$ independent of β and for all $k \leq 4$.

Proof. We only prove the negative case since the positive case is similar. We first view both sides of (6.9) as principal S^1-bundles over $\mathbb{T}^2 \times U_-$, endowed with unitary connections θ_{b_-} and θ respectively. It is easy to see by construction that they have the same Euler number b_-. So by general theory we can find a bundle isomorphism $H : U_- \times \mathrm{Nil}^3_{b_-} \to \mathcal{N}^4_{m_0}(U_-)$. In particular H covers the identity map on $\mathbb{T}^2 \times U_-$. Now both $H^*\theta$ and θ_{b_-} are unitary connection 1-forms on the same $U(1)$ bundle. The difference of the curvatures is

$$H^* d\theta - d\theta_{b_-} = d\theta - d\theta_{b_-} = *d(V_\beta - V_- - \beta) = *dh_-.$$

Notice by (6.4) and definition of T_- that for some $\delta > 0$ we have on $\mathbb{T}^2 \times U_-$,

$$|\nabla^k(*dh_-)|_{g_0} = O(e^{-\delta\beta}), \quad k \leq 4.$$

Basic Hodge theory then implies that by modifying H by a gauge transformation if necessary, we may assume that $\theta - \theta_{b_-} = \theta_f + \pi^* a_-$, where a_- satisfies (6.11) and θ_f is a flat connection. As already discussed in Section 2.2, θ_f can be removed by composition with the horizontal lift of a rotation on $\mathbb{T}^2$. □

Next, let $\boldsymbol{\omega}^{\mathcal{C},\pm}_\beta$ denote the hyperkähler triple on $U_\pm \times \mathrm{Nil}^3_{\mp b_\pm}$ obtained by applying the Gibbons-Hawking ansatz to $\mathbb{T}^2 \times U_\pm$ with the harmonic function $V_\pm(z) + \beta$ and the choice of the connection 1-form $\theta_{b_\pm}$. Let $g^{\mathcal{C},\pm}_\beta$ denote the associated Riemannian metric. In the following we will need to compare the hyperkähler triples $\boldsymbol{\omega}^{\mathcal{N}}$ on the neck with the model hyperkähler triple $\boldsymbol{\omega}^{\mathcal{C},\pm}_\beta$ for $z \in U_\pm$.

Proposition 6.2. *There exist smooth triples of 1-forms $\boldsymbol{\alpha}^\pm$ on $U_\pm \times \mathrm{Nil}^3_{\mp b_\pm}$ such that*

$$(\Phi^{\mathcal{N}}_{\beta,\pm})^* \boldsymbol{\omega}^{\mathcal{N}} - \boldsymbol{\omega}^{\mathcal{C},\pm}_\beta = d\boldsymbol{\alpha}^\pm, \tag{6.12}$$

with

$$|\nabla^k \boldsymbol{\alpha}^{\pm}|_{g_\beta^{\mathcal{C},\pm}} = O(e^{-\delta\beta}), \quad \beta \to \infty,$$

for some $\delta > 0$ independent of β and for all $k \leq 4$.

Proof. Again we only deal with the negative case. We only construct α_1^-, since the argument for the others is similar. By Lemma 6.1, we have

$$(\Phi_{\beta,-}^{\mathcal{N}})^* \omega_1^{\mathcal{N}} - (\omega_\beta^{\mathcal{C},-})_1 = \pi^*(h_- dx \wedge dy + dz \wedge a_-),$$

where

$$|\nabla^k h_-|_{g_0} = O(e^{-\delta\beta}), \quad |\nabla^k a_-|_{g_0} = O(e^{-\delta\beta}),$$

for some $\delta > 0$ and all $k \leq 4$, as $\beta \to \infty$. Using (2.7) and (2.12), we clearly have

$$\int_{\mathbb{T}^2} h_-(x, y, z)\, \mathrm{dvol}_{\mathbb{T}^2} = 0.$$

Since $\omega_1^{\mathcal{N}}$ and $(\omega_\beta^{\mathcal{C},-})_1$ are both closed, the 2-form

$$\lambda = h_- dx \wedge dy + dz \wedge a_-$$

is also closed. Next we want to find a primitive for λ. First, from basic Hodge theory on $\mathbb{T}^2$ we can find a smooth 1-form ζ, with no dz component, such that $d\zeta - h_- dx \wedge dy = dz \wedge \eta$ for a 1-form η, and such that $|\nabla^k \zeta|_{g_0} = O(e^{-\delta\beta})$ and $|\nabla^k \eta|_{g_0} = O(e^{-\delta\beta})$ for all $k \leq 4$ as $\beta \to \infty$. Then $\lambda - d\zeta = dz \wedge (a_- - \eta)$. Define

$$\xi \equiv \int_{-T_-}^{z} (\partial_z \lrcorner (\lambda - d\zeta))\, dz.$$

Then one can check that $d\xi = \lambda - d\zeta$, and that $|\nabla^k \xi|_{g_0} = O(e^{-\delta\beta})$ for all $k \leq 4$ as $\beta \to \infty$. The 1-form

$$\alpha_1^- = \pi^*(\zeta + \xi)$$

satisfies $d\alpha_1^- = (\Phi_{\beta,-}^{\mathcal{N}})^* \omega_1^{\mathcal{N}} - (\omega_\beta^{\mathcal{N},-})_1$, and $|\nabla^k \alpha_1^-|_{\pi^* g_0} = O(e^{-\delta\beta})$ for all $k \leq 4$ as $\beta \to \infty$. To finish, we need to compare $|\nabla^k \alpha_1^-|_{\pi^* g_0}$ with $|\nabla^k \alpha_1^-|_{g_\beta^{\mathcal{C},-}}$. A computation using the explicit formula for the model metric $g_\beta^{\mathcal{C},-}$ shows that this introduces error terms which are at most polynomial in β. This completes the proof. ☐

6.3. **The attaching maps and constraints.** We next define the attaching maps which will be used to construct the manifold $\mathcal{M}$. Define the *negative damage zone* by

$$DZ_- \equiv \Phi_-^{TY}\Big((T_-, T_- + 1) \times \mathrm{Nil}_{b_-}^3 \Big) \subset X_{b_-}^4$$

using the coordinates given by (6.1) on the end of $X_{b_-}^4$. Define

$$\mathcal{Z}_- : \mathbb{R} \times \mathrm{Nil}_{b_-}^3 \to \mathbb{R} \times \mathrm{Nil}_{b_-}^3, \quad \mathcal{Z}_-(z_-, p) = (z_- - 2T_-, p).$$

Define the *negative attaching map* $\Psi_- : DZ_- \to \mathcal{N}_{m_0}^4$ by

$$\Psi_- = \Phi_{\beta,-}^{\mathcal{N}} \circ \mathcal{Z}_- \circ (\Phi_-^{TY})^{-1}. \tag{6.13}$$

Similarly, define the *positive damage zone* by

$$DZ_+ \equiv \Phi_-^{TY}\Big((T_+, T_+ + 1) \times \mathrm{Nil}_{b_+}^3 \Big) \subset X_{b_+}^4,$$

using the coordinates given by (6.1) on the end of $X^4_{b_+}$. Define

$$\mathcal{Z}_+ : \mathbb{R} \times \mathrm{Nil}^3_{b_+} \to \mathbb{R} \times \mathrm{Nil}^3_{-b_+}, \quad \mathcal{Z}_+(z_+, p) = (2T_+ - z_+, \psi_+(p)),$$

where $\psi_+ : \mathrm{Nil}^3_{b_+} \to \mathrm{Nil}^3_{-b_+}$ is the orientation-reversing diffeomorphism given by

$$\psi(x, y, t) = (-x, -y, -t).$$

Define the *positive attaching map* $\Psi_+ : DZ_+ \to \mathcal{N}^4_{m_0}$ by

$$\Psi_+ = \Phi^{\mathcal{N}}_{\beta,+} \circ \mathcal{Z}_+ \circ (\Phi^{TY}_+)^{-1}. \tag{6.14}$$

We obtain the manifold $\mathcal{M}$ by gluing the pieces together using the attaching maps:

$$\mathcal{M} \equiv X^4_{b_-}(T_- + 1) \bigcup_{\Psi_-} \mathcal{N}^4_{m_0}(-T_-, T_+) \bigcup_{\Psi_+} X^4_{b_+}(T_+ + 1).$$

The manifold $\mathcal{M}$ carries an orientation compatible with both Tian-Yau pieces, and we will fix this orientation in the following.

Next, we want the potentials to agree up to the constant term in the damage zones after identifying the corresponding regions using the attaching maps. On DZ_-, modulo exponentially decaying errors, we have that

$$\Psi^*_- V_\beta = 2\pi b_-(z_- - 2T_-) + \beta = 2\pi b_- z_- - 2\pi b_-(2T_-) + \beta,$$

which we want to be equal to the leading terms of V_-. This requires that

$$0 = -2\pi b_-(2T_-) + \beta. \tag{6.15}$$

Similarly, on the other damage zone DZ_+, modulo exponentially decaying errors, we have that

$$\Psi^*_+ V_\beta = -2\pi b_+(2T_+ - z_+) + \beta = 2\pi b_+ z_+ - 2\pi b_+(2T_+) + \beta,$$

which we want to be equal to the leading terms of V_+, so we must have

$$0 = -2\pi b_+(2T_+) + \beta. \tag{6.16}$$

So we see that the gluing procedure requires the matching condition

$$4\pi b_\pm \cdot T_\pm = \beta. \tag{6.17}$$

This explains our choice of $T_\pm$ in (6.7).

Remark 6.3. From now on, we will view β as the only independent parameter in our gluing construction, with all other parameters determined by β.

6.4. **Definite triples and topology of $\mathcal{M}$.** All of the pieces in our gluing construction have hyperkähler triples

$$\begin{aligned} \boldsymbol{\omega}^- &\equiv (\omega_1^-, \omega_2^-, \omega_3^-) \ \text{ on } X^4_{b_-}, \\ \boldsymbol{\omega}^{\mathcal{N}} &\equiv (\omega_1^{\mathcal{N}}, \omega_2^{\mathcal{N}}, \omega_3^{\mathcal{N}}) \ \text{ on } \mathcal{N}^4_{m_0}(-T_-, T_+), \\ \boldsymbol{\omega}^+ &\equiv (\omega_1^+, \omega_2^+, \omega_3^+) \ \text{ on } X^4_{b_+}. \end{aligned}$$

Next, we will glue these triples in the damage zones DZ_- and DZ_+ to define a triple of symplectic forms on $\mathcal{M}$. The first step is Proposition 6.4.

Proposition 6.4. *There exist smooth triples of 1-forms $\boldsymbol{a}^{\pm} \in \Omega^1(DZ_{\pm}) \otimes \mathbb{R}^3$ satisfying*

$$\boldsymbol{\omega}^{\pm} + d\boldsymbol{a}^{\pm} = \Psi_{\pm}^{*}\boldsymbol{\omega}^{\mathcal{N}} \text{ in } DZ_{\pm},$$

such that for $k \leq 4$,

$$|\nabla^k \boldsymbol{a}^{\pm}| \leq C \cdot e^{-\delta\beta} \text{ in } DZ_{\pm},$$

where $\delta > 0$ and C are constants independent of β.

Proof. The mapping $\mathcal{Z}_{\pm}$ satisfies

$$\mathcal{Z}_{\pm}^{*}\boldsymbol{\omega}_{\beta}^{\mathcal{C},\pm} = \boldsymbol{\omega}^{\mathcal{C},\pm},$$

so pulling back equation (6.12) under the mapping $\mathcal{Z}_{\pm}$ yields

$$\mathcal{Z}_{\pm}^{*}(\Phi_{\beta,\pm}^{\mathcal{N}})^{*}\boldsymbol{\omega}^{\mathcal{N}} - \boldsymbol{\omega}^{\mathcal{C},\pm} = d(\mathcal{Z}_{\pm}^{*}\boldsymbol{\alpha}^{\pm}). \tag{6.18}$$

Consequently, the neck metric is close to the Calabi model metric in the damage zone after pulling back by the attaching map. The proposition then follows from Lemma 3.7 and Proposition 6.2. □

Let $\phi_{\pm}$ be cutoff functions such that

$$\phi_{\pm} = \begin{cases} 0 & \text{for } z_{\pm} \leq T_{\pm}, \\ 1 & \text{for } z_{\pm} \geq T_{\pm} + 1. \end{cases}$$

Then we define the closed triple

$$\boldsymbol{\omega}_{\beta}^{\mathcal{M}} = \begin{cases} \boldsymbol{\omega}^{-} & \text{on } X_{b_-}^4(T_-), \\ \boldsymbol{\omega}^{-} + d\big(\phi_{-}\boldsymbol{a}^{-}\big) & \text{on } X_{b_-}^4(T_-, T_- + 1), \\ \boldsymbol{\omega}^{\mathcal{N}} & \text{on } \mathcal{N}_{m_0}^4(-T_- + 1, T_+ - 1), \\ \boldsymbol{\omega}^{+} + d\big(\phi_{+}\boldsymbol{a}^{+}\big) & \text{on } X_{b_+}^4(T_+, T_+ + 1), \\ \boldsymbol{\omega}^{+} & \text{on } X_{b_+}^4(T_+). \end{cases} \tag{6.19}$$

The next result, which follows immediately from Proposition 6.4, shows that $\boldsymbol{\omega}^{\mathcal{M}}$ is very close to being a hyperkähler triple for β sufficiently large.

Corollary 6.5. *There exists a constant $C > 0$ independent of the gluing parameter $\beta > 0$ such that*

$$\|Q_{\boldsymbol{\omega}_{\beta}^{\mathcal{M}}} - \mathrm{Id}\,\|_{C^1(\mathcal{M})} \leq Ce^{-\delta_q\beta},$$

where $\delta_q > 0$ is a constant independent of β, and the norm is measured with respect to g_β, the Riemannian metric associated to $\boldsymbol{\omega}_{\beta}^{\mathcal{M}}$. Consequently, the triple $\boldsymbol{\omega}_{\beta}^{\mathcal{M}}$ is a closed, definite triple for β sufficiently large.

We next analyze some topological properties of the manifold $\mathcal{M}$. Proposition 6.6 shows that the Betti numbers do agree with those of the K3 surface.

Proposition 6.6. *The compact oriented manifold $\mathcal{M}$ has the following topological properties:*

$$b_1(\mathcal{M}) = 0,\ \chi(\mathcal{M}) = 24,\ b_2^{+}(\mathcal{M}) = 3,\ b_2^{-}(\mathcal{M}) = 19.$$

Proof. We write the manifold $\mathcal{M}$ as the union of open sets $U \cup V$, where

$$U = \mathcal{N}^4_{m_0}(-T_-, T_+),\ V = X_{b_-}(T_- + 1) \sqcup X_{b_+}(T_- + 1),$$

with $X_{b_\pm} = Q_{b_\pm} \setminus \mathbb{T}^2$, $m_0 = b_- + b_+$, and $Q_{b_\pm}$ a del Pezzo surface of degree $b_\pm$. Clearly, $U \cap V$ deformation retracts onto $\mathrm{Nil}^3_{b_-} \sqcup \mathrm{Nil}^3_{b_+}$.

Next, we claim that the de Rham cohomology $H^1(X_{b_\pm}) = 0$. To see this, we use the long exact sequence of a pair in de Rham cohomology

$$\cdots \to H^k_c(Q_{b_\pm} \setminus \mathbb{T}^2) \to H^k(Q_{b_\pm}) \to H^k(\mathbb{T}^2) \xrightarrow{\phi} H^{k+1}_c(Q_{b_\pm} \setminus \mathbb{T}^2) \to \cdots; \tag{6.20}$$

see [Spi79, Chapter 11]. Since $H^3(Q_{b_\pm}) = 0$, (6.20) yields an exact sequence

$$\cdots \to H^2(Q_{b_\pm}) \xrightarrow{i^*} H^2(\mathbb{T}^2) \to H^3_c(Q_{b_\pm} \setminus \mathbb{T}^2) \to 0.$$

Here the mapping $i^* : H^2(Q_{b_\pm}) \to H^2(\mathbb{T}^2)$ is just the pullback under inclusion, which is dual to the mapping on homology $i_* : H_2(\mathbb{T}^2, \mathbb{R}) \to H_2(Q_{b_\pm}, \mathbb{R})$. Since $\mathbb{T}^2$ is a complex submanifold of a Kähler manifold, this latter mapping is injective, so the mapping i^* is surjective, and by Poincaré duality we conclude that

$$H^1(Q_{b_\pm} \setminus \mathbb{T}^2) \cong H^3_c(Q_{b_\pm} \setminus \mathbb{T}^2) = 0.$$

Since we have just showed that $H^1(X_{b_\pm}) = 0$, the Mayer-Vietoris sequence in cohomology for the pair $\{U, V\}$ yields an exact sequence

$$0 \to H^1(\mathcal{M}) \to H^1(\mathcal{N}_{m_0}) \xrightarrow{i^*} H^1(\mathrm{Nil}^3_{b_-} \sqcup \mathrm{Nil}^3_{b_+}) \cong H^1(\mathrm{Nil}^3_{b_-}) \oplus H^1(\mathrm{Nil}^3_{b_+}). \tag{6.21}$$

The mapping i^* is the pullback under inclusion of the two nilmanifold fibers of the neck at each end. We claim that this mapping is injective. To see this, let $\mathcal{P}_{m_0} \equiv \{p_1, \ldots, p_{m_0}\}$ denote the monopole points in $B = \mathbb{T}^2 \times (-T_-, T_+)$, where $m_0 = b_- + b_+$. Then there are $\tilde{p}_j \in \mathcal{N}^4_{m_0}$ such that $\mathcal{N}^4_{m_0} \setminus \widetilde{\mathcal{P}}_{m_0}$ is a circle bundle over $B \setminus \mathcal{P}_{m_0}$, i.e.,

$$S^1 \longrightarrow \mathcal{N}^4_{m_0} \setminus \widetilde{\mathcal{P}}_{m_0} \xrightarrow{\pi} B \setminus \mathcal{P}_{m_0}. \tag{6.22}$$

The Gysin sequence of (6.22) begins with

$$0 \to H^1(B \setminus \mathcal{P}_{m_0}) \xrightarrow{\pi^*} H^1(\mathcal{N}^4_{m_0} \setminus \widetilde{\mathcal{P}}_{m_0}) \to \cdots \tag{6.23}$$

It is easy to see that the inclusion induces isomorphisms $H^1(B \setminus \mathcal{P}_{m_0}) \cong H^1(B) \cong \mathbb{R} \oplus \mathbb{R}$ and $H^1(\mathcal{N}^4_{m_0} \setminus \widetilde{\mathcal{P}}_{m_0}) \cong H^1(\mathcal{N}^4_{m_0})$. Then (6.23) becomes

$$0 \to \mathrm{span}\{dx, dy\} \xrightarrow{\pi^*} H^1(\mathcal{N}^4_{m_0}) \to \cdots$$

Together with Proposition 2.3 and the exact sequence (6.21), we conclude that $i^*\pi^* dx$ and $i^*\pi^* dy$ are both nontrivial and linearly independent in $H^1(\mathrm{Nil}^3_{b_-} \sqcup \mathrm{Nil}^3_{b_+})$, so that i^* is injective as claimed. Then (6.21) implies that $b_1(\mathcal{M}) = 0$. Since $\mathcal{M}$ is a compact orientable 4-manifold, Poincaré duality also implies that $b_3(\mathcal{M}) = 0$.

Next, it follows from the fibration (6.22) that $\chi(\mathcal{N}^4_{m_0} \setminus \widetilde{\mathcal{P}}_{m_0}) = 0$, and therefore

$$\chi(\mathcal{N}^4_{m_0}) = \#\text{ of monopole points} = m_0 = b_- + b_+.$$

For a Tian-Yau space, we have that

$$\chi(X^4_b) = \chi(Q_b \setminus \mathbb{T}^2) = \chi(Q_b) - \chi(\mathbb{T}^2) = \chi(Q_b),$$

where Q_b is a degree b del Pezzo surface, and so

$$\chi(X_b^4) = \chi(Q_b \setminus \mathbb{T}^2) = 12 - b.$$

Note also that $\chi(\mathrm{Nil}^3_{b_-}) = \chi(\mathrm{Nil}^3_{b_+}) = 0$ since it is an orientable 3-manifold. Then we have that

$$\chi(\mathcal{M}) = \chi(X^4_{b_-}) + \chi(\mathcal{N}) + \chi(X^4_{b_+}) = (12 - b_-) + (b_+ + b_-) + (12 - b_+) = 24.$$

Since we have shown above that $b_1(\mathcal{M}) = b_3(\mathcal{M}) = 0$, this proves that $b_2(\mathcal{M}) = 22$.

Finally, as we constructed in (6.19) the approximate definite triple $\boldsymbol{\omega}^{\mathcal{M}}_\beta \equiv (\omega_1, \omega_2, \omega_3)$, which are self-dual 2-forms forming a basis of Λ^2_+ at every point, the bundle $\Lambda^2_+(\mathcal{M})$ is a trivial rank 3 bundle. Also, ω_1 being non-zero everywhere means that there is an almost complex structure ($\omega_1/|\omega_1|$ is a unit norm self-dual 2-form, which is equivalent to an orthogonal almost complex structure). By Corollary 6.5, for $\beta \gg 1$, the rank 2 subbundle $V \subset \Lambda^2_0$ given by the orthogonal complement of $\omega_1/|\omega_1|$ is trivial. This implies that $c_1(T\mathcal{M}, J) = 0$, and the Hirzebruch signature theorem implies that

$$2\chi(\mathcal{M}) + 3\tau(\mathcal{M}) = \int_{\mathcal{M}} c_1^2 = 0,$$

from which it follows that $\tau(\mathcal{M}) = -16$. Therefore, $b_2^+(\mathcal{M}) = 3$ and $b_2^-(\mathcal{M}) = 19$. □

7. Geometry and regularity of the approximate metric

In this section, we will give a detailed analysis of the geometry of $(\mathcal{M}, g_\beta)$.

7.1. **Subdivision and regularity estimate on the glued space.** Since the arguments in the next sections are very tedious and involved, in this subsection we will list some fixed constants and make necessary conventions which will be frequently used in the later proofs.

7.1.1. *Tian-Yau spaces and their asymptotic rates.* To start with, for two positive integers

$$b_-, b_+ \in \{1, 2, \ldots, 9\},$$

let $(X^4_{b_-}, g_{b_-}, q_-)$ and $(X^4_{b_+}, g_{b_+}, q_+)$ be fixed hyperkähler Tian-Yau spaces with reference points $q_- \in X^4_{b_-}$ and $q_+ \in X^4_{b_+}$ such that their degrees are b_- and b_+, respectively. See Section 3 for the definition of a Tian-Yau space and the natural coordinate z outside a large compact subset. As we introduced in Section 6.1, on $(X^4_{b_-}, g_{b_-}, q_-)$ and $(X^4_{b_+}, g_{b_+}, q_+)$, there are diffeomorphisms

$$\Phi_\pm : [\zeta_0^\pm, +\infty) \times \mathrm{Nil}^3_{b_\pm} \to X^4_{b_\pm} \setminus K_\pm$$

between the Gibbons-Hawking space (as a circle bundle over a flat cylinder $\mathbb{T}^2 \times \mathbb{R}$) and the Tian-Yau space outside a compact subset. We define the definite constants $D_0^\pm$ by

$$D_0^\pm \equiv \text{the distance between } q_\pm \text{ and the level set } \{\boldsymbol{x} \in X^4_{b_\pm} | z_\pm(\boldsymbol{x}) = \zeta_0^\pm\}.$$

Proposition 3.4 shows that there are some positive constants

$$\underline{\delta}_1 > 0,\ \underline{\delta}_2 > 0 \tag{7.1}$$

such that for any $k \in \mathbb{N}$

$$\left|\nabla^k_{g^-_{\mathcal{C}}}\Big((\Phi^-)^*\omega^-_{TY} - \omega^-_{\mathcal{C}}\Big)\right|_{g^-_{\mathcal{C}}} = O(e^{-\underline{\delta}_1 z_-})$$

and

$$\left|\nabla^k_{g^+_{\mathcal{C}}}\Big((\Phi^+)^*\omega^+_{TY} - \omega^+_{\mathcal{C}}\Big)\right|_{g^+_{\mathcal{C}}} = O(e^{-\underline{\delta}_2 z_+}),$$

where $\omega^{\pm}_{TY}, \omega^{\pm}_{\mathcal{C}}$ are the Kähler forms on the Tian-Yau spaces $X^4_{b_\pm}$ and on the Calabi model spaces, respectively.

7.1.2. *Some notation regarding the neck region.* Next we fix some parameters in the neck region for ease of our discussions in the sections to come.

Let $\mathcal{P}_{m_0} \equiv \{p_1, \ldots, p_{m_0}\}$ be the set of monopole points on the flat cylinder $(\mathbb{T}^2_{x,y} \times \mathbb{R}_z, g_0)$ with $\sum_{m=1}^{m_0} z(p_m) = 0$, where g_0 is the flat product metric with coordinates (x, y, z) such that there are definite constants $\iota_0 > 0$ and $T_0 > 0$ such that for all $k \neq l$, we have that

$$\iota_0 \leq d_{g_0}(p_k, p_l) \leq T_0.$$

For any $p_m \in \mathcal{P}_{m_0}$, we define the associated distance function

$$d_m(\boldsymbol{x}) \equiv d_{g_\beta}(p_m, \boldsymbol{x}),\ \ \boldsymbol{x} \in (\mathcal{M}, g_\beta), \tag{7.2}$$

where the metric g_β is determined by the approximate hyperkähler triple (6.19). In our proof, the following notation will also be needed. Given $\beta \gg 1$, by Theorem 2.5, the defining Green's function V_β of the Gibbons-Hawking metric on the neck satisfies the following property: there are constants

$$\underline{\epsilon}_1 > 0,\ \ \underline{\epsilon}_2 > 0, \tag{7.3}$$

such that for any $k \in \mathbb{N}$ we have that

$$\begin{aligned} \left|\nabla^k_{g_0}\Big(V_\beta - (2\pi b_- \cdot z + \beta)\Big)\right| &= O(e^{\underline{\epsilon}_1 z}), \qquad z \to -\infty, \\ \left|\nabla^k_{g_0}\Big(V_\beta - (-2\pi b_+ \cdot z + \beta)\Big)\right| &= O(e^{-\underline{\epsilon}_2 z}), \qquad z \to +\infty. \end{aligned}$$

Moreover, by the definition of V_β in (6.5), the estimate

$$C^{-1} \cdot \beta \leq V_\beta(\boldsymbol{x}) \leq C \cdot \beta \tag{7.4}$$

holds away from the monopole points in the neck region $\mathcal{N}^4_{m_0}(-T_-, T_+)$. Thus, there are uniform constants $\iota_0' > 0,\ T_0' > 0$, such that

$$\iota_0' \cdot (\beta)^{\frac{1}{2}} \leq d_{g_\beta}(p_m, p_l) \leq T_0' \cdot (\beta)^{\frac{1}{2}}, \quad \forall 1 \leq m < l \leq m_0.$$

The following functions defined on $\mathcal{N}^4_{m_0}$ as well as on the Tian-Yau pieces naturally arise from the construction of the model metric and are crucial in analyzing the rescaled limits and defining the weight functions in the next section.

(1) We define the following functions on the neck:

$$\begin{aligned} L_-(\boldsymbol{x}) &\equiv (2\pi b_- \cdot z(\boldsymbol{x}) + \beta)^{\frac{1}{2}}, \quad -T_- \leq z(\boldsymbol{x}) < 0, \\ L_+(\boldsymbol{x}) &\equiv (-2\pi b_+ \cdot z(\boldsymbol{x}) + \beta)^{\frac{1}{2}}, \quad 0 \leq z(\boldsymbol{x}) < T_+. \end{aligned}$$

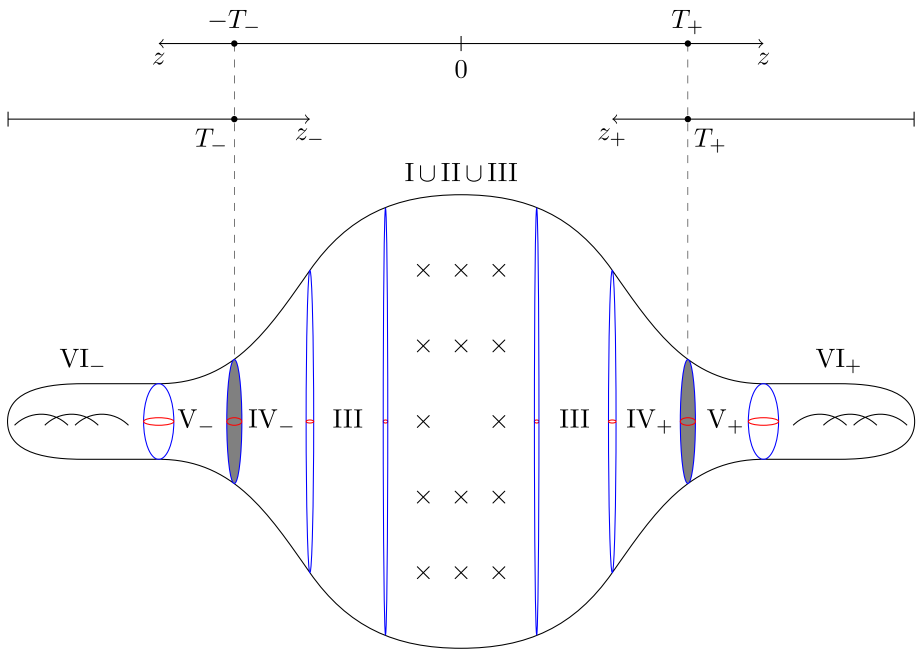

FIGURE 7.1. The red circles represent the S^1 fibers and the blue curves represent the base $\mathbb{T}^2$s of the nilmanifolds. The $\times$s are the monopole points in the neck region $\mathcal{N}$. The gray regions are in the "damage zones" between IV$_\pm$ and V$_\pm$.

(2) For $\boldsymbol{x} \in \mathcal{M}$ located in the end region of $X^4_{b_\pm}$ and satisfying $\zeta_0^\pm \leq z_\pm(\boldsymbol{x}) \leq T_\pm$, we define

$$\underline{L}_\pm(\boldsymbol{x}) \equiv (2\pi b_\pm \cdot z_\pm(\boldsymbol{x}))^{\frac{1}{2}},$$

where $\zeta_0^\pm$ was defined in (6.1) and will be used throughout.

7.1.3. *Subdivision of the manifold $\mathcal{M}$.* The manifold $(\mathcal{M}, g_\beta)$ will be divided into 9 regions depending on the different collapsing behaviors of g_β (see Figure 7.1):

$$\begin{aligned}
&\mathrm{I}: \ d_m(\boldsymbol{x}) \leq \beta^{-\frac{1}{2}} \text{ for some } 1 \leq m \leq m_0,\\
&\mathrm{II}: \ 2\beta^{-\frac{1}{2}} \leq d_m(\boldsymbol{x}) \leq (\iota_0'/4)\cdot(\beta)^{\frac{1}{2}} \text{ for some } 1 \leq m \leq m_0,\\
&\mathrm{III}: \ z(\boldsymbol{x}) \in [-m_0T_0, m_0T_0] \text{ and } d_m(\boldsymbol{x}) \geq (\iota_0'/2)\cdot(\beta)^{\frac{1}{2}} \text{ for all } 1 \leq m \leq m_0,\\
&\mathrm{IV}_-: \ z(\boldsymbol{x}) \in [-T_-/2, -2m_0T_0],\\
&\mathrm{IV}_+: \ z(\boldsymbol{x}) \in [2m_0T_0, T_+/2],\\
&\mathrm{V}_\pm: \ \boldsymbol{x} \in X^4_{b_\pm} \text{ and } 2\zeta_0^\pm \leq z_\pm(\boldsymbol{x}) \leq T_\pm,\\
&\mathrm{VI}_\pm: \ \boldsymbol{x} \in \overline{B_{D_0^\pm}(q_\pm)} \subset X^4_{b_\pm}.
\end{aligned}$$

We note that for $\boldsymbol{x} \in \mathrm{IV}_\pm$, we have that

$$T_0' \cdot (\beta)^{\frac{1}{2}} \leq d_m(\boldsymbol{x}) \leq R_\pm \text{ for all } 1 \leq m \leq m_0,$$

where

$$R_- \equiv \sup\Big\{d_{g_\beta}(\boldsymbol{x}, p_1)\Big| - T_- \leq z(\boldsymbol{x}) \leq 0\Big\}$$

and

$$R_+ \equiv \sup\Big\{d_{g_\beta}(\boldsymbol{x}, p_1)\Big| 0 \leq z(\boldsymbol{x}) \leq T_+\Big\}.$$

Recall (7.4) and that $T_\pm = O(\beta)$. Hence the uniform estimate $C^{-1} \cdot \beta^{\frac{3}{2}} \leq R_\pm \leq C \cdot \beta^{\frac{3}{2}}$ holds.

Remark 7.1. Notice that the 9 regions do not completely cover the manifold $\mathcal{M}$. However, each gap region has the same geometric behavior as the regions adjacent to it in the above subdivision. In particular, the curvature estimates and the rescaled geometries in each gap region will be the same as in the adjacent regions, so we will ignore these gap regions in what follows.

Next we prove uniform curvature estimates on $\mathcal{M}$ which will be crucial in showing that certain rescalings of the approximate metric have bounded curvature.

Lemma 7.2. *The following uniform curvature estimates hold for* $(\mathcal{M}, g_\beta)$*.*

(1) *Let us denote* $r(\boldsymbol{x}) \equiv d_{g_0}(\boldsymbol{x}, \mathcal{P}_{m_0})$ *on* $(\mathbb{T}^2 \times \mathbb{R}, g_0)$*. Then there exists a constant* $C > 0$ *such that for each* $1 \leq m \leq m_0$ *and for every* $\boldsymbol{x} \in B_{r_0}(p_m)$ *with* $r_0 \equiv \frac{1}{2}\operatorname{InjRad}_{g_0}(\mathbb{T}^2)$*, the following curvature estimates hold:*

$$|\operatorname{Rm}|(\boldsymbol{x}) \leq \begin{cases} C\beta, & 0 \leq r(\boldsymbol{x}) < \beta^{-1}, \\ \frac{C}{\beta^2 r(\boldsymbol{x})^3}, & \beta^{-1} \leq r(\boldsymbol{x}) < r_0. \end{cases} \tag{7.5}$$

In terms of the intrinsic distance function with respect to the Riemannian metric g_β*,*

$$|\operatorname{Rm}|(\boldsymbol{x}) \leq \frac{C}{\beta^{\frac{1}{2}} d_m(\boldsymbol{x})^3}, \quad \beta^{-1} \leq r(\boldsymbol{x}) < r_0, \tag{7.6}$$

where d_m *is defined in* (7.2)*.*

(2) *If* $\boldsymbol{x}$ *is in the neck region but has some definite distance away from the monopoles, the following curvature estimates hold for some uniform constant* $C > 0$*:*

$$|\operatorname{Rm}|(\boldsymbol{x}) \leq \frac{C}{\beta^2 \cdot |z(\boldsymbol{x})|}, \quad |z(\boldsymbol{x})| \geq \frac{r_0}{10} \text{ and } -T_- \leq z(\boldsymbol{x}) \leq T_+. \tag{7.7}$$

(3) *For* $\boldsymbol{x} \in X_{b_\pm}(T_\pm) \subset \mathcal{M}$ *(recall* (6.3)*), there is a constant* $C > 0$ *such that*

$$|\operatorname{Rm}|(\boldsymbol{x}) \leq \begin{cases} C, & z_\pm(\boldsymbol{x}) < \zeta_0^\pm, \\ \frac{C}{d_{g_\beta}(\boldsymbol{x}, q_\pm)^2}, & z_\pm(\boldsymbol{x}) \geq \zeta_0^\pm. \end{cases} \tag{7.8}$$

Remark 7.3. The curvature estimates in Lemma 7.2 are sharp in the following sense. The second estimate in (7.5) corresponds to the curvature behavior of the Taub-NUT metric, which is exactly of inverse cubic decay. The curvature estimate in (7.8) is sharp as well because the curvatures have precisely quadratic decay along the end of a complete Tian-Yau space.

Proof. The proof only requires straightforward calculations, so we only present a sketch. We use the following formula for the pointwise norm squared of the curvature of a Gibbons-Hawking metric:

$$|\operatorname{Rm}|^2 = \frac{1}{2}V_\beta^{-1}\Delta^2(V_\beta^{-1}); \tag{7.9}$$

see [GW00]. We just need to consider the case of one monopole point located at the origin. The case of several monopole points follows easily from this case. Let $r_0 \equiv \frac{1}{2}\operatorname{InjRad}_{g_0}(\mathbb{T}^2)$. Then we have the expansion

$$V_\beta(\boldsymbol{x}) = \frac{1}{2r(\boldsymbol{x})} + \beta + h(\boldsymbol{x}),\ x \in B_{r_0}(0^3), \tag{7.10}$$

where h is a bounded harmonic function.

First, we estimate the curvature in the case $r(\boldsymbol{x}) < \beta^{-1}$. By (7.10), it follows that

$$V_\beta^{-1} = \frac{2r}{1+2r\beta+2rh} \le Cr(1-2r\beta+4r^2\beta^2+8r^3\beta^3)$$

for $r < \beta^{-1}$, and then

$$|V_\beta^{-1}\Delta^2(V_\beta^{-1})| \le Cr\beta^3.$$

The first estimate in (7.5) follows from this.

Next, let $\boldsymbol{x} \in B_{r_0}(0^3)$ satisfy $r(\boldsymbol{x}) \ge \beta^{-1}$. Substituting (7.10) into (7.9), a similar expansion formula proves the second estimate in (7.5).

Now we relate the intrinsic distance function $d_m(\boldsymbol{x})$ and the Euclidean radial function $r(\boldsymbol{x})$. By directly estimating the integral of $\sqrt{V_\beta}$, we have that

$$\frac{1}{C'}\cdot\sqrt{r(\boldsymbol{x})} \le d_m(\boldsymbol{x}) \le C'\sqrt{r(\boldsymbol{x})},\ r(\boldsymbol{x}) < \beta^{-1},$$
$$\frac{1}{C'}\cdot\beta^{\frac{1}{2}}\cdot r(\boldsymbol{x}) \le d_m(\boldsymbol{x}) \le C'\cdot\beta^{\frac{1}{2}}\cdot r(\boldsymbol{x}),\ r(\boldsymbol{x}) \ge \beta^{-1},$$

where $C' > 0$ is some universal constant. The curvature estimate in (7.6) immediately follows from this.

From now on, we consider the case that $\boldsymbol{x}$ is in the neck region and satisfies $|z(\boldsymbol{x})| \ge \frac{r_0}{10}$. In this case, V_β has the expansion (6.6). Note that since the cutoff function and its derivatives up to third order are uniformly bounded, the curvature of the glued metric is of order β^{-3} in the damage zone regions. The curvature estimate in (7.7) follows from this observation and (7.9).

Next, we recall from Section 2.2 that for the model spaces, the defining harmonic functions are $V_-(\boldsymbol{x}) = \frac{2\pi b_- z(\boldsymbol{x})}{A}$ and $V_+(\boldsymbol{x}) = \frac{2\pi b_+ z(\boldsymbol{x})}{A}$, so (7.9) implies the first estimate in (7.8). Hence the complete end of the model space has precisely inverse quadratic curvature decay. It follows from Proposition 3.4 that the Tian-Yau metric does also. □

7.2. **Rescaled geometries.** In this subsection, we will analyze the geometry of each region defined in Section 7.1.3, which can be viewed as geometric preparation for defining our weighted Hölder spaces. In our context, we will discuss a sequence $(\mathcal{M}, g_j)$ with $g_j \equiv g_{\beta_j}$ for the gluing parameter sequence $\beta_j \to \infty$. In the remaining part of this section, we will rescale as follows: for every $\boldsymbol{x}_j \in \mathcal{M}$, we will choose the rescaling factors $\lambda_j > 0$ and the corresponding rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$ so that

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty).$$

The choice of λ_j will be consistent with the definition of the weight function. We will also need to improve the Gromov-Hausdorff convergence to convergence with some higher regularity. To this end, we will select subdomains $U_j \subset \mathcal{M}$ which are of almost full measure such that the above rescaling has uniformly bounded curvature in U_j.

Region I.

We will first prove that, after an appropriate rescaling, the blow-up limit around each monopole point is the Taub-NUT space. Let $\mathbb{T}^2 \times \mathbb{R}$ be equipped with a flat product metric g_0. Given a constant $\beta > 0$, let V_β be the harmonic function defined in (6.5) that satisfies

$$-\Delta_{g_0} V_\beta = 2\pi \sum_{m=1}^{m_0} \delta_{p_m}$$

and has the following expansion:

$$V_\beta(\boldsymbol{x}) = \frac{1}{2|\boldsymbol{x} - p_m|} + h_m(\boldsymbol{x}) + \beta \quad \text{as } \boldsymbol{x} \in B_{r_0}^{g_0}(p_m) \subset \mathbb{T}^2 \times \mathbb{R},$$

where each h_m is a bounded harmonic function on $B_{r_0}^{g_0}(p_m)$, $r_0 \equiv \frac{1}{2}\min\{d_0, i_0\}$, and

$$d_0 \equiv \min_{1 \le m < l \le m_0} d_{g_0}(p_m, p_l), \quad i_0 \equiv \mathrm{InjRad}_{g_0}(\mathbb{T}^2 \times \mathbb{R}).$$

Let $(\mathcal{N}^4_{m_0}, g_\beta)$ be the Gibbons-Hawking space equipped with the hyperkähler metric

$$g_\beta \equiv V_\beta \cdot g_{\mathbb{T}^2 \times \mathbb{R}} + V_\beta^{-1} \cdot \theta^2,$$

where θ is a connection 1-form satisfying the monopole equation $d\theta = *dV_\beta$. Denote by π the bundle projection. Given any positive constant $\sigma > 0$, we define the rescaled metric as follows:

$$\lambda_{\sigma,\beta} \equiv \sigma \cdot \beta^{\frac{1}{2}}, \quad \tilde{g}_{\sigma,\beta} \equiv (\lambda_{\sigma,\beta})^2 \cdot g_\beta.$$

Then we have Lemma 7.4.

Lemma 7.4. *Let $p_m \in \mathcal{P}_{m_0}$ be a monopole point and let $\pi^{-1}(p_m) = \boldsymbol{p_m} \in \mathcal{N}^4_{m_0}$. For every fixed positive constant $\sigma > 0$, we have the following C^∞-convergence:*

$$(\mathcal{N}^4_{m_0}, \tilde{g}_{\sigma,\beta}, \boldsymbol{p}_m) \xrightarrow{C^\infty} (\mathbb{R}^4, \tilde{g}_{\sigma,\infty}, \tilde{p}_{m,\infty}) \text{ as } \beta \to +\infty,$$

where $(\mathbb{R}^4, \tilde{g}_{\sigma,\infty}, \tilde{p}_{m,\infty})$ is a Ricci-flat Taub-NUT space with

$$\tilde{g}_{\sigma,\infty} = G_\sigma \cdot g_{\mathbb{R}^3} + (G_\sigma)^{-1}\theta^2 \quad \text{and} \quad G_\sigma(p) = \frac{1}{2d_0(p, 0^3)} + \frac{1}{\sigma^2},$$

where d_0 is the distance function in the Euclidean space $\mathbb{R}^3$.

Proof. Let $B_{r_0}^{g_0}(p_m) \subset \mathbb{T}^2 \times \mathbb{R}$ be the ball defined as above with $\pi^{-1}(p_m) = \boldsymbol{p}_m$. We choose a standard coordinate system $\{x, y, z\}$ in $B_{r_0}^{g_0}(p_m)$ such that $p_m = (0, 0, 0)$. Then in these coordinates,

$$V_\beta(x, y, z) = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} + h(x, y, z) + \beta,$$

where h is a bounded harmonic function on $B^{g_0}_{r_0}(p_m)$. Let us rescale the coordinates by

$$\tilde{x}_\beta \equiv \gamma_\beta \cdot x,\ \tilde{y}_\beta \equiv \gamma_\beta \cdot y,\ \tilde{z}_\beta \equiv \gamma_\beta \cdot z,$$

where $\gamma_\beta \equiv \sigma^2 \cdot \beta$. So the rescaled metrics $\tilde{g}_{\sigma,\beta}$ converge to

$$\tilde{g}_{\sigma,\infty} = G_\sigma \cdot g_{\mathbb{R}^3} + (G_\sigma)^{-1}\theta^2 \quad \text{and} \quad G_\sigma(p) = \frac{1}{2d_0(p,0^3)} + \frac{1}{\sigma^2},$$

where d_0 is the distance function in the Euclidean space $\mathbb{R}^3$. This tells us that $\tilde{g}_{\sigma,\infty}$ is a Taub-NUT metric, and the proof is complete. $\square$

Returning to the analysis of Region I, in this case we choose $\lambda_j \equiv (\beta_j)^{\frac{1}{2}}$ and consider the rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$. Applying Lemma 7.4, the rescaled spaces converge to the standard Taub-NUT space, i.e.,

$$(\mathcal{M}, \tilde{g}_j, p_m) \xrightarrow{GH} (\mathbb{R}^4, \tilde{g}_\infty, p_{m,\infty}),$$

where the length of the S^1-fiber at infinity equals 1. By the regularity theory of harmonic functions, the above convergence can be improved to C^ℓ everywhere for any $\ell \in \mathbb{Z}_+$.

Region II.

We will now analyze the convergence of appropriately rescaled spaces for every fixed reference point $\boldsymbol{x}_j$ in Region II. To understand the geometries of the rescaled limits, there are three cases to consider, each of which depends on the distance of a reference point $\boldsymbol{x}_j$ to the monopoles. Namely,

(a) There is a uniform constant $\sigma_0 > 0$ such that $2\beta_j^{-\frac{1}{2}} \leq d_m(\boldsymbol{x}_j) \leq \frac{1}{\sigma_0} \cdot \beta_j^{-\frac{1}{2}}$.

(b) The distance to a monopole $d_m(\boldsymbol{x}_j)$ satisfies

$$\frac{d_m(\boldsymbol{x}_j)}{\beta_j^{-\frac{1}{2}}} \to \infty \quad \text{and} \quad \frac{d_m(\boldsymbol{x}_j)}{\beta_j^{\frac{1}{2}}} \to 0.$$

(c) There is some uniform constant $C_0 > 0$ such that

$$0 < C_0 \cdot \beta_j^{\frac{1}{2}} \leq d_m(\boldsymbol{x}_j) \leq \frac{\iota_0'}{4} \cdot \beta_j^{\frac{1}{2}}.$$

In this region, the rescaled metric $\tilde{g}_j = \lambda_j^2 \cdot g_j$ is chosen as

$$\lambda_j \equiv \begin{cases} \Big(d_m(\boldsymbol{x}_j)\Big)^{-1}, & \text{in Case (a) and Case (b)}, \\ (d_j)^{-1}, \quad d_j \equiv \min_{1\leq m\leq m_0} d_m(\boldsymbol{x}_j), & \text{in Case (c)}. \end{cases} \tag{7.11}$$

Now we proceed to describe the rescaled limits in each of the above cases. Applying Lemma 7.4 to Case (a), the rescaled spaces converge to a Ricci-flat Taub-NUT space with a monopole $p_{m,\infty}$, i.e.,

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{R}^4, \tilde{g}_\infty, \boldsymbol{x}_\infty),$$

where $d_{\tilde{g}_\infty}(\boldsymbol{x}_\infty, p_{m,\infty}) = 1$ and the S^1-fiber at infinity has length at least $\sigma_0 > 0$. Moreover, the above convergence is C^∞ everywhere.

In Case (b), we have the convergence

$$\Big(\mathcal{M} \setminus B^{g_j}_{\beta_j^{-\frac{1}{2}}}(p_m), \tilde{g}_j, \boldsymbol{x}_j\Big) \xrightarrow{GH} (\mathbb{R}^3 \setminus \{0^3\}, g_{\mathbb{R}^3}, \boldsymbol{x}_\infty),$$

where $g_{\mathbb{R}^3}$ is the standard Euclidean metric on $\mathbb{R}^3$. To prove this, note the following. In terms of the rescaled metrics $\tilde{g}_j$, the diameters of the fibers converge in the following way:

$$\mathrm{Diam}_{\tilde{g}_j}(S^1) \leq C \cdot \beta_j^{-\frac{1}{2}} \cdot \frac{1}{d_m(\boldsymbol{x}_j)} \to 0. \tag{7.12}$$

Now denote $\gamma_j \equiv \frac{\beta_j^{\frac{1}{2}}}{d_m(\boldsymbol{x}_j)}$ and define the rescaled coordinates

$$x_j \equiv \gamma_j \cdot x,\ y_j \equiv \gamma_j \cdot y,\ z_j \equiv \gamma_j \cdot z.$$

Then one can check that the metric tensor $\tilde{g}_j$ in terms of the rescaled coordinates converges to the Euclidean metric $dx_\infty^2 + dy_\infty^2 + dz_\infty^2$, where (x_j, y_j, z_j) converges to $(x_\infty, y_\infty, z_\infty)$ with $|dx_\infty| = |dy_\infty| = |dz_\infty| = 1$. Therefore, by (7.12), the rescaled Gromov-Hausdorff limit is the punctured Euclidean space $\mathbb{R}^3 \setminus \{0^3\}$. Moreover, applying Lemma 7.2, it follows that the sequence converges with uniformly bounded curvature away from the origin.

In Case (c), we will prove that the rescaled limit is a punctured flat cylinder. That is, let $s_j > 0$ be a sequence of numbers such that

$$s_j \to 0, \quad \frac{1}{s_j \beta_j} \to 0.$$

Then we claim that

$$\Big(\mathcal{M} \setminus \pi^{-1}\Big(\bigcup_{m=1}^{m_0} B_{s_j}^{g_0}(p_m)\Big), \tilde{g}_j, \boldsymbol{x}_j\Big) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty\Big),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$ and $\mathcal{P}_{m_0} \equiv \{p_1, \ldots, p_{m_0}\} \subset \mathbb{T}^2 \times \mathbb{R}$ is the set of all monopole points.

To see this, we will carefully look at the convergence in a sequence of punctured domains with unbounded diameter. We denote $U(a,b) \equiv \{\boldsymbol{x} \in \mathcal{M} | a \leq z(\boldsymbol{x}) \leq b\}$. Let $\xi_j > 0$ be a sequence with $\xi_j/\beta_j \to 0$ and choose a sequence of punctured domains

$$\mathring{U}_j \equiv U(z(\boldsymbol{x}_j) - \xi_j, z(\boldsymbol{x}_j) + \xi_j) \setminus \pi^{-1}\Big(\bigcup_{m=1}^{m_0} B_{s_j}^{g_0}(p_m)\Big),$$

where $B_{s_j}^{g_0}(p_m)$ are balls of radii s_j in the flat product metric g_0 on $\mathbb{T}^2 \times \mathbb{R}$. It is straightforward to see that

$$\mathrm{Diam}_{\tilde{g}_j}(\mathring{U}_j) \approx C \cdot \xi_j \to \infty, \tag{7.13}$$

$$\mathrm{Diam}_{\tilde{g}_j}\Big(\pi^{-1}(B_{s_j}^{g_0}(p_m))\Big) \approx C \cdot s_j \to 0. \tag{7.14}$$

The above arguments show that the limit $\mathring{U}_\infty$ is a complete space minus m_0 points.

On the other hand, we will show that the metrics $\tilde{g}_j$ converge to a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. In fact, for every $\boldsymbol{y} \in \mathring{U}_j$, there is a bounded harmonic function h_j such that V_{β_j} satisfies

$$|V_{\beta_j}(\boldsymbol{y}) - (h_j(\boldsymbol{y}) + 2\pi b_- \cdot z(\boldsymbol{y}) + \beta_j)| \leq \frac{1}{2s_j}.$$

In Case (c), let us define $\gamma_j \equiv \beta_j^{-\frac{1}{2}} \cdot d_j$, where d_j is defined in (7.11). By assumption, there are constants $C_0 > 0$, $\iota_0' > 0$, such that for every $j \in \mathbb{Z}_+$, we have that

$C_0 \le \gamma_j \le \frac{\iota_0'}{4}$. By the definition of γ_j and λ_j, we have that $\lambda_j = \gamma_j^{-1} \cdot \beta_j^{-\frac{1}{2}}$. Since $\frac{1}{s_j \beta_j} \to 0$, the following holds for some uniform constant $C > 0$:

(7.15)
$$\left|\lambda_j^2 V_{\beta_j}(\boldsymbol{y}) - \frac{1}{\gamma_j^2}\right| = \frac{|V_{\beta_j}(\boldsymbol{y}) - \beta_j|}{\gamma_j^2 \beta_j} \le \frac{C + C \cdot \xi_j + \frac{1}{2s_j}}{\gamma_j^2 \beta_j} = \frac{\frac{C}{\beta_j} + \frac{C\xi_j}{\beta_j} + \frac{1}{2s_j\beta_j}}{\gamma_j^2} \to 0.$$

Therefore, applying (7.13), (7.14) and (7.15), we have that

$$(\mathring{U}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty \Big), \tag{7.16}$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$ and $\mathcal{P}_{m_0}$ has m_0 points. As in Case (b), applying Lemma 7.2, we see that away from the monopole points, curvatures are uniformly bounded in (7.16).

Region III.

For every fixed $\boldsymbol{x}_j$ in Region III, we define $\lambda_j \equiv \beta_j^{-\frac{1}{2}}$ and set $\tilde{g}_j \equiv \lambda_j^2 g_j$. Similar to Case (c) of Region II, let $s_j > 0$ be a sequence of numbers such that $s_j \to 0$ and $\frac{1}{s_j\beta_j} \to 0$. Then applying the arguments in Case (c) of Region II, we have

$$\Big(\mathcal{M} \setminus \pi^{-1}\Big(\bigcup_{m=1}^{m_0} B_{s_j}^{g_0}(p_m)\Big), \tilde{g}_j, \boldsymbol{x}_j\Big) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty \Big),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. Applying Lemma 7.2, it follows that the sequence converges with uniformly bounded curvature away from the monopole points.

Region IV$_-$.

For fixed $\boldsymbol{x}_j$ in Region IV$_-$, we choose the following rescaling factor:

$$\lambda_j \equiv (L_-(\boldsymbol{x}_j))^{-1},$$

and consider the corresponding rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$. To start with, let us estimate from below the rescaled distance from $\boldsymbol{x}_j$ to a monopole. For every $\boldsymbol{x}_j$ in Region VI$_-$, by the definition of this region, we have that

$$d_{\tilde{g}_j}(p_m, \boldsymbol{x}_j) \ge \frac{10T_0' \cdot \beta_j^{\frac{1}{2}}}{L_-(\boldsymbol{x}_j)} = \frac{10T_0' \cdot \beta_j^{\frac{1}{2}}}{(2\pi b_- \cdot z(\boldsymbol{x}_j) + \beta_- + \beta_j)^{\frac{1}{2}}}.$$

If $\beta_j > 0$ is sufficiently large, then immediately

$$d_{\tilde{g}_j}(p_m, \boldsymbol{x}_j) \ge 5T_0' > 0.$$

Now we consider the following cases.

(a) There is a constant $C_0 > 10T_0'$ independent of j such that

$$5T_0' \le d_{\tilde{g}_j}(p_m, \boldsymbol{x}_j) \equiv \lambda_j \cdot d_m(\boldsymbol{x}_j) \le C_0$$

for each $1 \le m \le m_0$.

(b) The reference points $\boldsymbol{x}_j$ in Region IV$_-$ satisfy

$$d_{\tilde{g}_j}(p_m, \boldsymbol{x}_j) \equiv \lambda_j \cdot d_m(\boldsymbol{x}_j) \to \infty.$$

In Case (a), we have the convergence

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty \Big),$$

where g_0 is a flat product metric and the set $\mathcal{P}_{m_0}$ contains m_0 points. To see this, first notice that there is some constant $C > 0$ such that $|z(\boldsymbol{x}_j)| \leq C$. Let us denote

$$U(a,b) \equiv \{\boldsymbol{x} \in \mathcal{M} | a \leq z(\boldsymbol{x}) \leq b\},$$

and let us also choose a sequence $\xi_j > 0$ satisfying $\xi_j \to \infty$ and $\frac{\xi_j}{\beta_j} \to 0$. For every fixed $\boldsymbol{x}_j$ in Region IV$_-$, we choose a punctured domain

$$\mathring{U}_j \equiv U(z(\boldsymbol{x}_j) - \xi_j, z(\boldsymbol{x}_j) + \xi_j) \setminus \pi^{-1}\Big(\bigcup_{m=1}^{m_0} B_{s_j}^{g_0}(p_m)\Big), \tag{7.17}$$

where $B_{s_j}^{g_0}(p_m)$ are balls of radii s_j in the flat product metric g_0 on $\mathbb{T}^2 \times \mathbb{R}$ and $s_j > 0$ is a sequence of numbers satisfying $s_j \to 0$ and $\frac{1}{s_j\beta_j} \to 0$. Then it is straightforward to see that

$$\operatorname{Diam}_{\tilde{g}_j}(\mathring{U}_j) \approx C \cdot \xi_j \to \infty. \tag{7.18}$$

Hence the limit space $\mathring{U}_\infty$ has two ends. Moreover,

$$\operatorname{Diam}_{\tilde{g}_j}\Big(\pi^{-1}(B_{s_j}^{g_0}(p_m))\Big) \approx C \cdot s_j \to 0. \tag{7.19}$$

Therefore, the limit space $\mathring{U}_\infty$ is a complete space minus m_0 points.

Next, we will show that $\tilde{g}_j$ converges to a flat product metric g_0 on $\mathbb{T}^2 \times \mathbb{R}$. To this end, it suffices to show that for every $\boldsymbol{y} \in \mathring{U}_j$, $\frac{V_{\beta_j}(\boldsymbol{y})}{(L_-(\boldsymbol{x}_j))^2} \to 1$ as $j \to +\infty$. In fact, for every $\boldsymbol{y} \in \mathring{U}_j$, there is a bounded harmonic function h_j such that the Green's function V_{β_j} satisfies

$$|V_{\beta_j}(\boldsymbol{y}) - (h_j(\boldsymbol{y}) + 2\pi b_- \cdot z(\boldsymbol{y}) + \beta_j)| \leq \frac{1}{2s_j}.$$

Since $\frac{1}{s_j\beta_j} \to 0$ and $\frac{\xi_j}{\beta_j} \to 0$, the following holds for some uniform constant $C > 0$:

$$\Big|\frac{V_{\beta_j}(\boldsymbol{y})}{(L_-(\boldsymbol{x}_j))^2} - 1\Big| = \frac{|V_{\beta_j}(\boldsymbol{y}) - (L_-(\boldsymbol{x}_j))^2|}{|2\pi b_- \cdot z(\boldsymbol{x}_j) + \beta_- + \beta_j|} \leq \frac{C + C\xi_j + \frac{1}{2s_j}}{\beta_j - C} \to 0. \tag{7.20}$$

Therefore, applying (7.18), (7.19) and (7.20), we have

$$(\mathring{U}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty\Big),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$ and $\mathcal{P}_{m_0}$ comprises m_0 points. Moreover, by Lemma 7.2, it follows that the sequence converges with uniformly bounded curvature away from the monopoles.

In Case (b), it holds that

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{T}^2 \times \mathbb{R}, g_0, \boldsymbol{x}_\infty),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. The proof of this is similar to the previous case. Here we choose the domain

$$U_j \equiv U(z(\boldsymbol{x}_j) - \xi_j, z(\boldsymbol{x}_j) + \xi_j),$$

where the sequence of numbers $\xi_j > 0$ satisfies $\xi_j \to \infty$ and $\frac{\xi_j}{\beta_j} \to 0$. Then the same arguments show that

$$(U_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{T}^2 \times \mathbb{R}, g_0, \boldsymbol{x}_\infty),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. By Lemma 7.2, it follows that the sequence converges with uniformly bounded curvature in any compact subset of bounded diameter containing $\boldsymbol{x}_j$.

Region IV_+**.**

For every fixed reference point $\boldsymbol{x}_j$ in Region IV_+, we choose the rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$ with $\lambda_j \equiv (L_+(\boldsymbol{x}_j))^{-1}$. The rescaled limits are the same as those in Region IV_-.

Region V_-**.**

Recall that Region V_- is a large region in the Tian-Yau space $(X^4_{b_-}, g_{b_-}, q_-)$ with $q_- \in X^4_{b_-}$. For every fixed reference point $\boldsymbol{x}_j$ in Region V_-, we choose the rescaled metric

$$\lambda_j \equiv (\underline{L}_-(\boldsymbol{x}_j))^{-1}, \quad \tilde{g}_j = \lambda_j^2 g_j,$$

where g_j in Region V_- coincides with g_{b-} and we have defined $\underline{L}_-(\boldsymbol{y}) \equiv (2\pi b_- \cdot z_-(\boldsymbol{y}))^{\frac{1}{2}}$. We need to analyze the following cases.

(a) $z_-(\boldsymbol{x}_j) \to \infty$.

(b) There is some constant $C_0 > 0$ independent of the index j such that $10\zeta_0^- \leq z_-(\boldsymbol{x}_j) \leq C_0$.

In Case (a), we have the convergence

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{T}^2 \times \mathbb{R}, g_0, \boldsymbol{x}_\infty),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. To see this, we denote $\zeta_j \equiv z_-(\boldsymbol{x}_j) \to \infty$. Let $\xi_j > 0$ satisfy

$$\xi_j \to \infty \quad \text{and} \quad \frac{\xi_j}{\zeta_j} \to 0.$$

Then we choose an unbounded annular domain

$$U_j^- \equiv U^-(\zeta_j - \xi_j, \zeta_j + \xi_j) = \{\boldsymbol{y} \in \mathcal{M} | \zeta_j - \xi_j \leq z_-(\boldsymbol{y}) \leq \zeta_j + \xi_j\}.$$

We will show that

$$(U_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{T}^2 \times \mathbb{R}, g_0, \boldsymbol{x}_\infty),$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. Applying the same arguments as before, we have

$$\mathrm{Diam}_{\tilde{g}_j}(U_j) \to \infty,$$

so that the limit space U_∞ has two ends. It follows that U_∞ is complete. In addition, we need to show that $\tilde{g}_j$ converges to a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$. To prove this, let us recall the asymptotics of the Tian-Yau metric g_{b_-} (for instance, see (6.2)):

$$\left|(\Phi_-^{TY})^* g_{b_-} - \Big(V_-(\boldsymbol{y}) \cdot g_0 + V_-^{-1}(\boldsymbol{y}) \cdot \theta_{b_-}^2\Big)\right| \leq C \cdot e^{-\underline{\delta}_- \cdot z_-(\boldsymbol{y})}, \quad z_-(\boldsymbol{y}) \to +\infty,$$

where g_0 is a flat product metric on $\mathbb{T}^2 \times \mathbb{R}$ and where $V_-(\boldsymbol{y}) \equiv 2\pi b_- \cdot z_-(\boldsymbol{y})$. Then straightforward calculations imply that

$$\left|\underline{L}_-(\boldsymbol{x}_j)^{-2} \cdot (\Phi_-^{TY})^* g_{b_-} - g_0\right| \to 0.$$

Indeed, it suffices to verify that

$$\left|\frac{V_-(\boldsymbol{y})}{(\underline{L}_-(\boldsymbol{x}_j))^2} - 1\right| = \frac{|V_-(\boldsymbol{y}) - (\underline{L}_-(\boldsymbol{x}_j))^2|}{2\pi b_- \cdot \zeta_j} = \frac{\xi_j}{\zeta_j} \to 0.$$

By Lemma 7.2, it follows that the sequence converges with uniformly bounded curvature in any compact subset containing $\boldsymbol{x}_j$ of bounded diameter. This completes the analysis of Case (a).

In Case (b), since $d(q_-, \boldsymbol{x}_j) \leq C_0$, there is some constant $C_0' > 0$ (depending only on the constant $C_0 > 0$ and the geometric data of g_{b_-}) such that

$$\frac{1}{C_0'} \leq \underline{L}_-(\boldsymbol{x}_j) \leq C_0'.$$

Therefore, the limit space $(\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$ is a complete Ricci-flat Tian-Yau space which is a simple rescaling of $(X^4_{b_-}, g_{b-}, q_-)$. The convergence in this case is moreover smooth on compact subsets.

Region V_+**.**

For every fixed reference point $\boldsymbol{x}_j$ in Region V_+, we choose the rescaled metric $\tilde{g}_j = \lambda_j^2 g_j$ with $\lambda_j \equiv (\underline{L}_+(\boldsymbol{x}_j))^{-1}$. So the rescaling geometries are the same as those of Region V_-.

Region VI_-**.**

We choose $\lambda_j \equiv 1$ and the limit is $(X^4_{b_-}, g_{b_-}, q_-)$, a complete Tian-Yau space.

Region VI_+**.**

We choose $\lambda_j \equiv 1$ and the limit is $(X^4_{b_+}, g_{b_+}, q_+)$, again a complete Tian-Yau space.

8. Weighted Schauder estimate

The main part of this section is to establish the appropriate weighted analysis consistent with the different rescaled geometries of the manifold $(\mathcal{M}, g_\beta)$. The main result in this section is the weighted Schauder estimate in Proposition 8.2.

Let δ and ν be fixed constants to be determined later. Fix the gluing parameter $\beta > 0$. The weight functions $\rho_{\delta,\nu}^{(k+\alpha)}$ with $k \in \mathbb{N}$ and $0 \leq \alpha < 1$ are defined as follows (the notations are defined in Section 7.1):

$$\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{x}) \equiv \begin{cases} e^{\delta\cdot(2T_-)} \cdot (\beta^{-\frac{1}{2}})^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{I}, \\ e^{\delta\cdot(2T_-)} \cdot (d_m(\boldsymbol{x}))^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{II}, \\ e^{\delta\cdot(2T_-)} \cdot (\beta^{\frac{1}{2}})^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{III}, \\ e^{\delta\cdot(z(\boldsymbol{x})+2T_-)} \cdot (L_-(\boldsymbol{x}))^{\nu+k+\alpha} & \boldsymbol{x} \in \mathrm{IV}_-, \\ e^{\delta\cdot(z(\boldsymbol{x})+2T_-)} \cdot (L_+(\boldsymbol{x}))^{\nu+k+\alpha} & \boldsymbol{x} \in \mathrm{IV}_+, \\ e^{\delta\cdot(z_-(\boldsymbol{x}))} \cdot (\underline{L}_-(\boldsymbol{x}))^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{V}_-, \\ e^{\delta\cdot(-z_+(\boldsymbol{x})+2T_-+2T_+)} \cdot (\underline{L}_+(\boldsymbol{x}))^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{V}_+, \\ e^{\delta\cdot\zeta_0^-} \cdot \Big(\underline{L}_-(\zeta_0^-)\Big)^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{VI}_-, \\ e^{\delta(-\zeta_0^+ +2T_- +2T_+)} \cdot \Big(\underline{L}_+(\zeta_0^+)\Big)^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathrm{VI}_+ . \end{cases} \tag{8.1}$$

Each weight function matches up across gluing regions. To see this, we refer readers to the definition of the attaching maps in (6.13) and (6.14) showing the relationship between z, z_-, z_+ in the damage zones. The above regions do not entirely cover $\mathcal{M}$, so we extend each weight function to a smooth function on the entire manifold $\mathcal{M}$ by using appropriate cutoff functions in each gap region. By Remark 7.1, the rescaling geometry in each gap region is the same as that of the adjacent regions; therefore these gap regions can be ignored in the weighted analysis.

Before defining the weighted Hölder norms, we make a remark on the rescaled geometries. For any fixed $x \in \mathcal{M}$, the computations in Section 7.2 imply that

$$\sup_{B_{1/\lambda_x}(x)} |\operatorname{Rm}|_{\hat{g}_\beta} \leq C_0 \cdot \lambda_x^2$$

for some uniform constant $C_0 > 0$, where $\lambda_x > 0$ is the metric rescaling factor associated to x as in the discussions in Section 7.2. So it is standard that there is a uniform constant $s_0 > 0$ such that

$$\operatorname{InjRad}_{\hat{g}_\beta}(\hat{x}) \geq \frac{s_0}{2\lambda_x},$$

where $\hat{x}$ is a lift of x in the universal cover of $B_{s_0/\lambda_x}(x)$. For every $x \in \mathcal{M}$, let us set $r_x \equiv \frac{s_0}{\lambda_x}$. We are now ready to give the definition of the weighted Hölder spaces.

Definition 8.1. Given $\delta, \nu \in \mathbb{R}$, $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the weighted Hölder space $C^{k,\alpha}_{\delta,\nu}(\mathcal{M})$ is defined as the space of all $C^{k,\alpha}$-tensor fields (of some fixed type, for example, functions, 1-forms, symmetric 2-tensors, etc.) equipped with the following weighted norm with respect to the Riemannian metric g_β. For a tensor field ω on $\mathcal{M}$ we define the following.

(1) The weighted $C^{k,\alpha}$-seminorm is

$$[\omega]_{C^{k,\alpha}_{\delta,\nu}} \equiv \sup_{x\in\mathcal{M}} \sup \Big\{ \rho^{(k+\alpha)}_{\delta,\nu}(x) \cdot \frac{|\nabla^k \hat{\omega}(\hat{x}) - \nabla^k \hat{\omega}(\hat{y})|}{(d_{\hat{g}_\beta}(\hat{x},\hat{y}))^\alpha} \;\Big|\; \hat{y} \in B_{r_x/2}(\hat{x}) \Big\},$$

where $\hat{x}$ denotes a lift of x to the universal cover of $B_{r_x}(x)$, the difference of the two covariant derivatives is defined in terms of parallel translation in $B_{r_x/2}(\hat{x})$ along the unique geodesic connecting $\hat{x}$ and $\hat{y}$, and $\hat{\omega}, \hat{g}_\beta$, are the lifts of ω, g_β, respectively.

(2) The weighted $C^{k,\alpha}$-norm is

$$\|\omega\|_{C^{k,\alpha}_{\delta,\nu}(\mathcal{M})} \equiv \sum_{j=0}^{k} \Big\| \rho^{(j)}_{\delta,\nu} \cdot \nabla^j \omega \Big\|_{C^0(\mathcal{M})} + [\omega]_{C^{k,\alpha}_{\delta,\nu}(\mathcal{M})}.$$

Proposition 8.2 (The weighted Schauder estimate)**.** *Consider $(\mathcal{M}, g_\beta)$ with a sufficiently large gluing parameter $\beta > 0$. Let $\mathscr{D}_{g_\beta} \equiv d^+ \oplus d^*$. Then there exists a uniform constant $C > 0$ (independent of β) such that for every $\omega \in \Omega^1(\mathcal{M})$, it holds that*

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} \leq C\Big(\|\mathscr{D}_{g_\beta}\omega\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} + \|\omega\|_{C^0_{\delta,\nu}(\mathcal{M})} \Big). \tag{8.2}$$

Proof. We will argue by contradiction and suppose that no such uniform constant $C > 0$ exists. This means that there exist the following sequences:

(1) a sequence of numbers $\beta_j \to \infty$,

(2) a sequence of gluing metrics $(\mathcal{M}, g_j)$ with weight functions $\rho^{(k+\alpha)}_{j,\delta,\nu}$ (for simplicity, we will denote these by $\rho^{(k+\alpha)}_{\delta,\nu}$ because there is no ambiguity),

(3) a sequence of differential 1-forms $\omega_j \in \Omega^1(\mathcal{M})$ such that as $j \to \infty$,

$$\|\omega_j\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M},g_j)} = 1, \tag{8.3}$$

$$\|\mathscr{D}_{g_j}\omega_j\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M},g_j)} + \|\omega_j\|_{C^0_{\delta,\nu}(\mathcal{M},g_j)} \to 0. \tag{8.4}$$

Our main goal is to prove a local version of the weighted Schauder estimate claimed in (8.2). Precisely, it suffices to show that there is some uniform constant $C > 0$ (independent of j) such that for every $\omega \in \Omega^1(\mathcal{M})$ and for every $\boldsymbol{x}_j \in \mathcal{M}$, there is some $r_j > 0$ (depending on the location of $\boldsymbol{x}_j$) such that the following estimate holds in $B_{2r_j}(\boldsymbol{x}_j) \subset (\mathcal{M}, g_j)$:

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu}(B_{r_j}(\boldsymbol{x}_j))} \leq C\Big(\|\mathscr{D}_{g_j}\omega\|_{C^{0,\alpha}_{\delta,\nu+1}(B_{2r_j}(\boldsymbol{x}_j))} + \|\omega\|_{C^0_{\delta,\nu}(B_{2r_j}(\boldsymbol{x}_j))}\Big). \tag{8.5}$$

Once (8.5) is established, a contradiction almost immediately arises, which completes the proof. Indeed, (8.3) implies that either $\|\rho^{(1)}_{\delta,\nu} \cdot \nabla\omega_j\|_{C^0(\mathcal{M},g_j)} \geq \frac{1}{2}$ or $[\omega_j]_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M},g_j)} \geq \frac{1}{2}$. We assume that $\|\rho^{(1)}_{\delta,\nu} \cdot \nabla\omega_j\|_{C^0(\mathcal{M},g_j)} \geq \frac{1}{2}$ as the argument for the other case is identical. Hence, by definition, there exists some $\boldsymbol{x}_j \in \mathcal{M}_j$ with

$$|\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}_j) \cdot \nabla\omega_j(\boldsymbol{x}_j)| \geq \frac{1}{2}. \tag{8.6}$$

By (8.5), there is some $r_j > 0$ which depends on $\boldsymbol{x}_j$ such that

$$\|\omega_j\|_{C^{1,\alpha}_{\delta,\nu}(B_{r_j}(\boldsymbol{x}_j))} \leq C\Big(\|\mathscr{D}_{g_j}\omega_j\|_{C^{0,\alpha}_{\delta,\nu+1}(B_{2r_j}(\boldsymbol{x}_j))} + \|\omega_j\|_{C^0_{\delta,\nu}(B_{2r_j}(\boldsymbol{x}_j))}\Big) \to 0.$$

The above estimate implies that

$$|\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}_j) \cdot \nabla\omega_j(\boldsymbol{x}_j)| \leq \|\omega_j\|_{C^1_{\delta,\nu}(B_{r_j}(\boldsymbol{x}_j))} \leq \|\omega_j\|_{C^{1,\alpha}_{\delta,\nu}(B_{r_j}(\boldsymbol{x}_j))} \to 0. \tag{8.7}$$

However, (8.7) contradicts (8.6). The proof is done.

So the main part of the proof of the proposition is to establish (8.5). In our proof, the primary strategy is to rescale the metric g_j by setting $\tilde{g}_j \equiv \lambda_j^2 g_j$. Recall that in the previous section, we showed that, for every reference point $\boldsymbol{x}_j \in \mathcal{M}$, after an appropriate rescaling, there is a subdomain U_j containing $\boldsymbol{x}_j$ that has uniformly bounded geometry away from at most finitely many singular points. Then the standard Schauder estimate in the rescaled space is available. That is, there is some uniform constant $\tilde{r}_0 > 0$ such that for every $\tilde{\omega} \in \Omega^1(\mathcal{M})$,

$$\|\omega\|_{C^{1,\alpha}(B^{\tilde{g}_j}_{\tilde{r}_0}(\boldsymbol{x}_j))} \leq C\Big(\|\mathscr{D}_{\tilde{g}_j}\omega\|_{C^{0,\alpha}(B^{\tilde{g}_j}_{2\tilde{r}_0}(\boldsymbol{x}_j))} + \|\omega\|_{C^0(B^{\tilde{g}_j}_{2\tilde{r}_0}(\boldsymbol{x}_j))}\Big), \tag{8.8}$$

where the geodesic balls $B^{\tilde{g}_j}_{2\tilde{r}_0}(\boldsymbol{x}_j) \subset U_j$ converge to a ball in a smooth space keeping curvatures uniformly bounded. One can obtain the estimate (8.8) by lifting everything to the local universal covers, whose $C^{1,\alpha}$-harmonic radius is uniformly bounded from below. Once we obtain (8.8), we will get the weighted estimate (8.5) after an appropriate rescaling.

In the following arguments, for every fixed $\boldsymbol{x}_j \in \mathcal{M}$, we will choose the corresponding rescaled metrics $\tilde{g}_j = \lambda_j^2 g_j$ defined as in Section 7.2.

Region I.

We prove (8.5) around the monopole $p_m \in \mathcal{P}_{m_0}$. Let $\lambda_j = \beta_j^{\frac{1}{2}}$ and choose the rescaled metric $\tilde{g}_j \equiv \lambda_j^2 g_j$. Then

$$\Big(\mathcal{M}, \tilde{g}_j, p_m\Big) \xrightarrow{C^\infty} \Big(\mathbb{R}^4, \tilde{g}_\infty, p_{m,\infty}\Big) \text{ as } \beta_j \to \infty,$$

where $\tilde{g}_\infty$ is the standard Taub-NUT metric such that the length of the S^1-fiber at infinity equals 1. Since the above convergence is C^∞, the rescaled sequence $(\mathcal{M}, \tilde{g}_j, p_m)$ has bounded geometry and thus the standard Schauder estimate holds in the geodesic ball $B_2^{\tilde{g}_j}(p_m)$ with respect to the rescaled metric $\tilde{g}_j$. Precisely, there is a uniform constant such that for every $\omega \in \Omega^1(\mathcal{M})$,

$$\|\omega\|_{C^{1,\alpha}(B_1^{\tilde{g}_j}(p_m))} \le C\Big(\|\mathscr{D}_{\tilde{g}_j}\omega\|_{C^{0,\alpha}(B_2^{\tilde{g}_j}(p_m))} + \|\omega\|_{C^0(B_2^{\tilde{g}_j}(p_m))}\Big).$$

Now we rescale back to the original metric g_j. With respect to the original metric, the above Schauder estimate is equivalent to the following:

$$\begin{aligned}\|\rho_{\delta,\nu}^{(1)} \cdot \nabla\omega\|_{C^0(B_{r_j}(p_m))} + \|\rho_{\delta,\nu}^{(1+\alpha)} \cdot \nabla\omega\|_{C^\alpha(B_{r_j}(p_m))}&\\ \le C\Big(\|\rho_{\delta,\nu+1}^{(\alpha)} \cdot \mathscr{D}_{g_j}\omega\|_{C^\alpha(B_{2r_j}(p_m))} + \|\rho_{\delta,\nu}^{(0)} \cdot \omega\|_{C^0(B_{2r_j}(p_m))}\Big),&\end{aligned}$$

where $r_j = \lambda_j^{-1}$. Therefore, by the definition of the weighted Hölder norm,

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu}(B_{r_j}(p_m))} \le C\Big(\|\mathscr{D}_{g_j}\omega\|_{C^{0,\alpha}_{\delta,\nu+1}(B_{2r_j}(p_m))} + \|\omega\|_{C^0_{\delta,\nu}(B_{2r_j}(p_m))}\Big).$$

So the estimate (8.5) has been proved in Region I.

Region II.

We will prove (8.5) for every $\boldsymbol{x}_j$ in Region II. We break down this region in Case (a), (b), and (c) (with different rescaling geometries) as for Region II in Section 7.2.

In each of the above cases, the rescaled spaces $(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j)$ have uniformly bounded curvatures and converge to a smooth limit space. This enables us to obtain the standard Schauder estimate in any ball of a definite radius in the rescaled spaces. Specifically, let $\boldsymbol{x}_j$ be a fixed point in Region II. Then the standard Schauder estimate in $B_{1/6}^{\tilde{g}_j}(\boldsymbol{x}_j)$ states that for any $\omega \in \Omega^1(\mathcal{M})$,

$$\|\omega\|_{C^{1,\alpha}(B_{1/6}^{\tilde{g}_j}(\boldsymbol{x}_j))} \le C\Big(\|\mathscr{D}_{\tilde{g}_j}\omega\|_{C^\alpha(B_{1/3}^{\tilde{g}_j}(\boldsymbol{x}_j))} + \|\omega\|_{C^0(B_{1/3}^{\tilde{g}_j}(\boldsymbol{x}_j))}\Big).$$

Then rescaling to the original metrics g_j, we find that

$$\begin{aligned}\|\rho_{\delta,\nu}^{(1)}(\boldsymbol{x}_j) \cdot \nabla\omega\|_{C^0(B_{r_j}(\boldsymbol{x}_j))} + \|\rho_{\delta,\nu}^{(1+\alpha)}(\boldsymbol{x}_j) \cdot \nabla\omega\|_{C^\alpha(B_{r_j}(\boldsymbol{x}_j))}&\\ \le C\Big(\|\rho_{\delta,\nu+1}^{(\alpha)}(\boldsymbol{x}_j) \cdot \mathscr{D}_{g_j}\omega\|_{C^\alpha(B_{2r_j}(\boldsymbol{x}_j))} + \|\rho_{\delta,\nu}^{(0)}(\boldsymbol{x}_j) \cdot \omega\|_{C^0(B_{2r_j}(\boldsymbol{x}_j))}\Big).&\end{aligned}$$

The above $r_j > 0$ is defined as follows:

$$r_j \equiv \begin{cases} \frac{1}{6}\lambda_j^{-1} = \frac{1}{6}d_m(\boldsymbol{x}_j), & \text{in Case (a) and Case (b)},\\ \frac{1}{6}d_j, & \text{in Case (c)},\end{cases}$$

where $d_j \equiv \min_{1\le m\le m_0} d_{g_j}(p_m, \boldsymbol{x}_j)$. We need to show that the values $\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})$ for all $\boldsymbol{y} \in B_{r_j}(\boldsymbol{x}_j)$ are equivalent. To see this, note by the triangle inequality that for every $\boldsymbol{y} \in B_{r_j}(\boldsymbol{x}_j)$,

$$\left(\frac{5}{6}\right)^{\nu+k+\alpha} \le \frac{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})}{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{x}_j)} \le \left(\frac{7}{6}\right)^{\nu+k+\alpha}.$$

By the definition of the weighted norm, (8.5) now follows.

Region III.

The rescaled metrics are $\tilde{g}_j \equiv \beta_j^{-1} \cdot g_j$, and so coincide with Case (c) of Region II.

Regions IV_- **and** IV_+**.**

We only prove the estimate (8.5) for every fixed reference point $\boldsymbol{x}_j$ in Region IV_-. The proof for the other region is identical.

For a fixed $\boldsymbol{x}_j$ in Region IV$_-$, we define $\tilde{g}_j = \lambda_j^2 g_j$ with $\lambda_j \equiv (L_-(\boldsymbol{x}_j))^{-1}$. In Section 7.2, the rescaled limits were separated into Case (a) and Case (b). We follow the notation as stated there. Notice that, in each of the above cases, the rescaled spaces have uniformly bounded curvature, allowing us to state the standard Schauder estimate in the following way.

In Case (a), by assumption, we can pick some definite constant

$$\tilde{r}_0 \equiv \frac{T_0'}{2} \leq \frac{d_{\tilde{g}_j}(p_m, \boldsymbol{x}_j)}{10}$$

such that for every $\omega \in \Omega^1(\mathcal{M})$,

$$\|\omega\|_{C^{1,\alpha}(B^{\tilde{g}_j}_{\tilde{r}_0}(\boldsymbol{x}_j))} \leq C\Big(\|\mathscr{D}_{\tilde{g}_j}\omega\|_{C^{\alpha}(B^{\tilde{g}_j}_{2\tilde{r}_0}(\boldsymbol{x}_j))} + \|\omega\|_{C^{0}(B^{\tilde{g}_j}_{2\tilde{r}_0}(\boldsymbol{x}_j))}\Big).$$

In the above estimate, the constant $C > 0$ depends only on $T_0' > 0$ and the flat product metric g_0 (in particular, C does not depend on C_0). Let us rescale back to the original metric g_j and take $r_j \equiv (L_-(\boldsymbol{x}_j)) \cdot \tilde{r}_0 > 0$. Then

$$\begin{aligned}\|\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^0(B_{r_j}(\boldsymbol{x}_j))} &+ \|\rho^{(1+\alpha)}_{\delta,\nu}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^\alpha(B_{r_j}(\boldsymbol{x}_j))}\\ &\leq C\Big(\|\rho^{(\alpha)}_{\delta,\nu+1}(\boldsymbol{x}_j)\cdot\mathscr{D}_{g_j}\omega\|_{C^\alpha(B_{2r_j}(\boldsymbol{x}_j))} + \|\rho^{(0)}_{\delta,\nu}(\boldsymbol{x}_j)\cdot\omega\|_{C^0(B_{2r_j}(\boldsymbol{x}_j))}\Big).\end{aligned}$$

All that remains to show is that the values of the weight function $\rho^{(k+\alpha)}_{\delta,\nu}$ are equivalent for every $\boldsymbol{y} \in B_{2r_j}(\boldsymbol{x}_j)$. To see this, denote $\zeta_j \equiv z(\boldsymbol{x}_j)$. Then, by straightforward computations, there is a uniform constant $C_1 > 0$ such that for every $\boldsymbol{y} \in B_{2r_j}(\boldsymbol{x}_j)$, we have that $|z(\boldsymbol{y}) - \zeta_j| \leq C_1$. Moreover, we can show that there is some uniform constant $C_2 > 0$ such that $\frac{\zeta_j}{\beta_j} \leq C_2$ for any $j \in \mathbb{Z}_+$. By the definition of the weight function in Region IV$_-$, we also have that

$$\frac{\rho^{(k+\alpha)}_{\delta,\nu}(\boldsymbol{y})}{\rho^{(k+\alpha)}_{\delta,\nu}(\boldsymbol{x}_j)} = e^{\delta\cdot(z(\boldsymbol{y})-\zeta_j)}\cdot\frac{(L_-(\boldsymbol{y}))^{\nu+k+\alpha}}{(L_-(\boldsymbol{x}_j))^{\nu+k+\alpha}}.$$

The above estimates then imply that there is a uniform constant $C_3 > 0$ such that

$$\frac{1}{C_3} \leq \left|\frac{\rho^{(k+\alpha)}_{\delta,\nu}(\boldsymbol{y})}{\rho^{(k+\alpha)}_{\delta,\nu}(\boldsymbol{x}_j)}\right| \leq C_3.$$

This completes the proof of (8.5) in Case (a). The proof of the estimate in Case (b) follows similarly.

Regions V$_-$ and V$_+$.

We only need to prove the estimate (8.5) for the reference point $\boldsymbol{x}_j$ in Region V$_-$ because the estimate in Region V$_+$ is identical. As in the discussion in Section 7.2, there are Case (a) and Case (b) to be considered.

First, we prove the weighted Schauder estimate (8.5) in Case (a). Let $\lambda_j \equiv (\underline{L}(\boldsymbol{x}_j))^{-1}$ and $\tilde{g}_j \equiv \lambda_j^2 g_j$. Then we have shown in Section 7.2 that

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{T}^2 \times \mathbb{R}, g_0, \boldsymbol{x}_\infty).$$

Moreover, the curvatures are uniformly bounded in the above convergence, which implies the standard Schauder estimate for every $\omega \in \Omega^1(\mathcal{M})$, namely

$$\|\omega\|_{C^{1,\alpha}(B^{\tilde{g}_j}_1(\boldsymbol{x}_j))} \leq C\Big(\|\mathscr{D}_{\tilde{g}_j}\omega\|_{C^{\alpha}(B^{\tilde{g}_j}_2(\boldsymbol{x}_j))} + \|\omega\|_{C^{0}(B^{\tilde{g}_j}_2(\boldsymbol{x}_j))}\Big).$$

Rescaling back to the original metrics g_j, we find that

$$\begin{aligned}&\|\rho_{\delta,\nu}^{(1)}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^0(B_{r_j}(\boldsymbol{x}_j))}+\|\rho_{\delta,\nu}^{(1+\alpha)}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^\alpha(B_{r_j}(\boldsymbol{x}_j))}\\&\qquad\le C\Big(\|\rho_{\delta,\nu+1}^{(\alpha)}(\boldsymbol{x}_j)\cdot\mathscr{D}_{g_j}\omega\|_{C^\alpha(B_{2r_j}(\boldsymbol{x}_j))}+\|\rho_{\delta,\nu}^{(0)}(\boldsymbol{x}_j)\cdot\omega\|_{C^0(B_{2r_j}(\boldsymbol{x}_j))}\Big).\end{aligned}$$

Now the last step is to show that all the values $\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})$ are equivalent for every $\boldsymbol{y}\in B_{2r_j}(\boldsymbol{x}_j)$. To see this, denote $\zeta_j\equiv z_-(\boldsymbol{x}_j)$ and $\xi_j\equiv z_-(\boldsymbol{y})$. Then straightforward computations imply that $|\xi_j-\zeta_j|\le C_0$ for some uniform constant $C_0>0$. By the definition of the weight function in Region V_-, we see that

$$\frac{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})}{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{x}_j)}=e^{\delta\cdot(\xi_j-\zeta_j)}\cdot\frac{(L_-(\boldsymbol{y}))^{\nu+k+\alpha}}{(L_-(\boldsymbol{x}_j))^{\nu+k+\alpha}}.$$

Therefore

$$\frac{1}{C_1}\le\frac{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})}{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{x}_j)}\le C_1,$$

which completes the proof of Case (a).

Next we prove Case (b). We showed in Section 7.2 that the limit space $(\mathcal{M}_\infty,\tilde g_\infty,\boldsymbol{x}_\infty)$ is a finite rescaling of $(X^4_{b_-},g_{b-},q_-)$. Moreover, the Gromov-Hausdorff convergence is strengthened as $(\mathcal{M},\tilde g_j,\boldsymbol{x}_j)\xrightarrow{C^k}(\mathcal{M}_\infty,\tilde g_\infty,\boldsymbol{x}_\infty)$ for any $k\in\mathbb{Z}_+$ which implies the following Schauder estimate:

$$\|\omega\|_{C^{1,\alpha}(B_1^{\tilde g_j}(\boldsymbol{x}_j))}\le C\Big(\|\mathscr{D}_{\tilde g_j}\omega\|_{C^\alpha(B_2^{\tilde g_j}(\boldsymbol{x}_j))}+\|\omega\|_{C^0(B_2^{\tilde g_j}(\boldsymbol{x}_j))}\Big),$$

where $C>0$ is independent of j. Now we rescale the above estimate to the original metric whilst also rescaling ω as in Case (a), which gives

$$\begin{aligned}&\|\rho_{\delta,\nu}^{(1)}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^0(B_{r_j}(\boldsymbol{x}_j))}+\|\rho_{\delta,\nu}^{(1+\alpha)}(\boldsymbol{x}_j)\cdot\nabla\omega\|_{C^\alpha(B_{r_j}(\boldsymbol{x}_j))}\\&\qquad\le C\Big(\|\rho_{\delta,\nu+1}^{(\alpha)}(\boldsymbol{x}_j)\cdot\mathscr{D}_{g_j}\omega\|_{C^\alpha(B_{2r_j}(\boldsymbol{x}_j))}+\|\rho_{\delta,\nu}^{(0)}(\boldsymbol{x}_j)\cdot\omega\|_{C^0(B_{2r_j}(\boldsymbol{x}_j))}\Big),\end{aligned}$$

where $r_j\equiv\underline{L}_-(\boldsymbol{x}_j)$. Similar to Case (a), there is some constant $C_2>0$ which is independent of the index j and the constant C_0 such that

$$\frac{1}{C_2}\le\frac{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{y})}{\rho_{\delta,\nu}^{(k+\alpha)}(\boldsymbol{x}_j)}\le C_2.$$

By definition, the weighted Schauder estimate (8.5) immediately follows from this.

Regions VI_- and VI_+.

First, we consider the case where the fixed reference point $\boldsymbol{x}_j$ is in Region VI_-. Then the weighted estimate (8.5) becomes the standard Schauder estimate on $B_1(\boldsymbol{x}_j)\subset X^4_{b_-}$. Indeed, as the weight function in this region is uniformly bounded, (8.5) is equivalent to the standard one.

The proof in Region VI_+ proceeds identically. ☐

9. The injectivity estimate for $\mathscr{D}_g$

In this section, we will prove the uniform injectivity of the linearized operator which is a crucial technical ingredient in proving the existence of a hyperkähler triple. For ease of exposition, we state a standard Liouville theorem on the flat cylinder $\mathbb{T}^2 \times \mathbb{R}$.

Lemma 9.1. *Let $\mathbb{T}^2 \times \mathbb{R}$ be equipped with a flat product metric g_0 and let λ_0 be the lowest positive eigenvalue of $-\Delta_{g_0}$. If u satisfies $\Delta_{g_0} u = 0$ and $|u(z)| = O(e^{\lambda z})$ for some $\lambda \in (0, \sqrt{\lambda_0})$, then $u \equiv 0$.*

Now we state the main technical result of this subsection.

Proposition 9.2 (The injectivity estimate for $\mathscr{D}_g$)**.** *Consider $(\mathcal{M}, g_\beta)$ with sufficiently large gluing parameter $\beta > 0$. Assume that the parameters δ and ν satisfy*

(1) $0 < \delta < \frac{1}{10^3} \min\{\underline{\delta}_1, \underline{\delta}_2, \underline{\epsilon}_1, \underline{\epsilon}_2, \lambda_0, \delta_h, \delta_q\}$,
(2) $\nu \in (0, 1)$,

where the constants $\underline{\delta}_1, \underline{\delta}_2, \underline{\epsilon}_1, \underline{\epsilon}_2$ have been specified in (7.1) and (7.3) respectively, $\lambda_0 > 0$ is as in Lemma 9.1, $\delta_h > 0$ is as in Theorem 5.1 and δ_q is as in Corollary 6.5. Then for every $\alpha \in (0, 1)$, there exists a uniform constant $C = C(\alpha, \delta, \nu) > 0$ which is independent of β such that for every $\omega \in \Omega^1(\mathcal{M})$,

$$\|\omega\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} \leq C \cdot \|\mathscr{D}_{g_\beta}\omega\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})}.$$

Proof. By Proposition 8.2, it suffices to show that there exists a uniform constant $C > 0$ such that

$$\|\omega\|_{C^{0}_{\delta,\nu}(\mathcal{M})} \leq C \cdot \|\mathscr{D}_{g_\beta}\omega\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})}$$

for all $\omega \in \Omega^1(\mathcal{M})$. We argue by contradiction and suppose that no such uniform constant exists. Then we have the following:

(1) a sequence of spaces $(\mathcal{M}_j, g_j)$ with gluing parameters $\beta_j \to \infty$,
(2) a sequence of 1-forms $\omega_j \in \Omega^1(\mathcal{M}_j)$ such that as $j \to \infty$,

$$\|\omega_j\|_{C^0_{\delta,\nu}(\mathcal{M}_j, g_j)} = 1 \quad \text{and} \quad \|\mathscr{D}_{g_j}\omega_j\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M}_j, g_j)} \to 0,$$

(3) a sequence of points $\boldsymbol{x}_j \in \mathcal{M}_j$ satisfying $|\rho^{(0)}_{j,\delta,\nu}(\boldsymbol{x}_j) \cdot \omega_j(\boldsymbol{x}_j)| = 1$, where $\rho^{(0)}_{j,\delta,\nu}$ is a sequence of weight functions in $(\mathcal{M}_j, g_j)$.

Now we are in a position to rescale the above sequences to produce a contradiction. To start with, let g_j be as above, and denote the rescaling factors as follows.

(1) **Rescaling of the metrics.** Let $\tilde{g}_j = \lambda_j^2 \cdot g_j$ with λ_j to be determined later. Then with respect to the fixed reference point $\boldsymbol{x}_j \in \mathcal{M}_j$ as in (3) above, we have the convergence

$$(\mathcal{M}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathcal{M}_\infty, \tilde{d}_\infty, \boldsymbol{x}_\infty).$$

(2) **Rescaling of the 1-forms.** The 1-forms ω_j will be rescaled by $\tilde{\omega}_j \equiv \kappa_j \cdot \omega_j$ for a sequence $\kappa_j > 0$ to be determined later.
(3) **Rescaling of the weight functions.** We denote by $\rho^{(k+\alpha)}_{j,\delta,\nu}$ and $\rho^{(k+\alpha)}_{\infty,\delta,\nu}$ the weight functions on $\mathcal{M}_j$ and $\mathcal{M}_\infty$, respectively. Fix $k = 0, 1$ and $\alpha \in (0, 1)$, and rescale the weight function $\rho^{(k+\alpha)}_{j,\delta,\nu}$ by $\tilde{\rho}^{(k+\alpha)}_{j,\delta,\nu} = \tau_j^{(k+\alpha)} \cdot \rho^{(k+\alpha)}_{j,\delta,\nu}$ with $\tau_j^{(k+\alpha)}$ to be determined later.

The above rescaling factors are chosen to satisfy the following scale-invariance properties:

$$1 \leq \tau_j^{(0)} \cdot \kappa_j \cdot \lambda_j^{-1} \leq 10, \quad 1 \leq \tau_j^{(1)} \cdot \kappa_j \cdot \lambda_j^{-2} \leq 10, \quad 1 \leq \tau_j^{(1+\alpha)} \cdot \kappa_j \cdot \lambda_j^{-2-\alpha} \leq 10.$$

In this way, we will obtain a sequence of 1-forms $\tilde{\omega}_j \in \Omega^1(\mathcal{M}_j)$ with the following properties:

$$\|\tilde{\omega}_j\|_{C^0_{\delta,\nu}(\mathcal{M},\tilde{g}_j)} = 1, \quad |\tilde{\rho}^{(0)}_{j,\delta,\nu}(\boldsymbol{x}_j) \cdot \tilde{\omega}_j(\boldsymbol{x}_j)| = 1, \quad \|\mathscr{D}_{\tilde{g}_j}\tilde{\omega}_j\|_{C^\alpha_{\delta,\nu+1}(\mathcal{M},\tilde{g}_j)} \to 0.$$

The basic strategy of the proof is to combine compactness arguments and Liouville theorems. More precisely, if $(\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$ is non-collapsed, we apply Proposition 8.2 to obtain a limiting 1-form $\tilde{\omega}_\infty \in (\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$ such that

$$\|\tilde{\omega}_\infty\|_{C^0_{\delta,\nu}(\mathcal{M}_\infty,\tilde{g}_\infty)} = 1, \quad |\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty) \cdot \tilde{\omega}_\infty(\boldsymbol{x}_\infty)| = 1, \quad \mathscr{D}_{\tilde{g}_\infty}\tilde{\omega}_\infty \equiv 0. \tag{9.1}$$

Then we apply our Liouville theorems to show that the above limiting 1-form $\tilde{\omega}_\infty$ with controlled weighted norm vanishes on $\mathcal{M}_\infty$, which gives a contradiction. Next, for a collapsed limit $(\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$, to understand the limiting behavior of the operators $\mathscr{D}_{\tilde{g}_j}$ and the contradicting 1-forms $\tilde{\omega}_j$, we will lift everything to an appropriately chosen non-collapsed (local) normal cover such that $C^{1,\alpha}$-compactness still applies on such a covering space. On the other hand, we will show that there is a representation $\tilde{\omega}_j = f_j^x\theta_j^x + f_j^y\theta_j^x + f_j^z\theta_j^z + f_j^t\theta_j^t$ such that $(f_j^x, f_j^y, f_j^z, f_j^t)$ converges to a 4-tuple of harmonic functions $(f_\infty^x, f_\infty^y, f_\infty^z, f_\infty^t)$. In addition, we will also show that at least one of f_∞^x, f_∞^y, f_∞^z and f_∞^t has a positive weighted Hölder norm at $\boldsymbol{x}_\infty$. The desired contradiction then arises from various versions of Liouville theorems for harmonic functions in the different collapsed regions.

We will now produce the desired contradiction in each of the regions discussed in Section 7.2. Specifically, we will explain how to choose λ_j, κ_j, τ_j, and apply a Liouville theorem in each region.

Region I.

Assume that the reference point $\boldsymbol{x}_j$ is in Region I. Then the rescaled limit is the Taub-NUT space $(\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$ with a limiting monopole $p_{m,\infty}$. We choose the rescaling factors as follows:

$$\lambda_j = \beta_j^{\frac{1}{2}}, \quad \tau_j^{(k+\alpha)} = e^{-\delta\cdot 2T_-} \cdot (\beta_j^{\frac{1}{2}})^{\nu+k+\alpha}, \quad \kappa_j = e^{\delta\cdot 2T_-} \cdot (\beta_j^{-\frac{1}{2}})^{\nu-1}.$$

Then we have that $d_{\tilde{g}_\infty}(p_{m,\infty}, \boldsymbol{x}_\infty) \leq C$ and the rescaled weight function in the limit space is given by

$$\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in B_1(p_{m,\infty}), \\ (d_{\tilde{g}_\infty}(\boldsymbol{x}, p_{m,\infty}))^{\nu+k+\alpha}, & \boldsymbol{x} \in \mathcal{M}_\infty \setminus B_2(p_{m,\infty}). \end{cases}$$

The limiting 1-form $\tilde{\omega}_\infty \in \Omega^1(\mathcal{M}_\infty)$ then satisfies

$$\mathscr{D}_{\tilde{g}_\infty}\tilde{\omega}_\infty \equiv 0, \quad |\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty) \cdot \tilde{\omega}_\infty(\boldsymbol{x}_\infty)| = 1, \quad \|\tilde{\omega}_\infty\|_{C^0_{\delta,\nu}(\mathcal{M}_\infty)} = 1.$$

Notice that the above norm bound implies that for all $\boldsymbol{x} \in \mathcal{M}_\infty \setminus B_2(p_{m,\infty})$,

$$|\tilde{\omega}_\infty(\boldsymbol{x})| \leq (d_{\tilde{g}_\infty}(\boldsymbol{x}, p_{m,\infty}))^{-\nu}.$$

Since $\tilde{\omega}_\infty$ is in the kernel of $\mathscr{D}_{\tilde{g}_\infty}$, $\tilde{\omega}_\infty$ is harmonic with respect to the Taub-NUT metric $\tilde{g}_\infty$. Applying Lemma 4.14 and using $\nu > 0$, we conclude that $\tilde{\omega}_\infty \equiv 0$.

Region II.

Now we discuss the case that the reference points $\boldsymbol{x}_j$ are in Region II. In Section 7.2, the rescaled geometries were separated into Case (a), Case (b), and Case (c). We work with the same division in what follows.

We start with our analysis in Case (a). By Lemma 7.4, the rescaled limit in Case (a) is a Ricci-flat Taub-NUT space $(\mathcal{M}_\infty, \tilde{g}_\infty, \boldsymbol{x}_\infty)$ such that the S^1-fiber at infinity has length at least $\sigma_0 > 0$. The rescaling factors in this case are

$$\lambda_j = (d_m(\boldsymbol{x}_j))^{-1},\ \tau_j^{(k+\alpha)} = e^{-\delta\cdot 2T_-}\cdot(d_m(\boldsymbol{x}_j))^{-\nu-k-\alpha},\ \kappa_j = e^{\delta\cdot 2T_-}\cdot(d_m(\boldsymbol{x}_j))^{\nu-1}.$$

In the rescaled limit space, the limiting reference point $\boldsymbol{x}_\infty$ satisfies $d_{\tilde{g}_\infty}(\boldsymbol{x}_\infty, p_{m,\infty}) = 1$. Moreover, the rescaled weight function in the limit space is given by

$$\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x}) = (d_{\tilde{g}_\infty}(p_{m,\infty}, \boldsymbol{x}))^{-\nu-k-\alpha}, \qquad \boldsymbol{x}\in\mathcal{M}_\infty\setminus B_2(p_{m,\infty}),$$

and the limiting 1-form $\tilde{\omega}_\infty \in \Omega^1(\mathcal{M}_\infty)$ satisfies

$$\mathscr{D}_{\tilde{g}_\infty}\tilde{\omega}_\infty \equiv 0, \quad |\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty)\cdot\tilde{\omega}_\infty(\boldsymbol{x}_\infty)| = 1, \quad \|\tilde{\omega}_\infty\|_{C^0_{\delta,\nu}(\mathcal{M}_\infty)} \le 1.$$

The above weighted norm bound implies that for every $\boldsymbol{x}\in\mathcal{M}_\infty\setminus B_2(p_{m,\infty})$, the limiting 1-form $\tilde{\omega}_\infty$ satisfies the pointwise estimate

$$|\tilde{\omega}_\infty(\boldsymbol{x})| \le \Big(d_{\tilde{g}_\infty}(p_{m,\infty}, \boldsymbol{x})\Big)^{-\nu}.$$

Applying Lemma 4.14, we deduce that $\tilde{\omega}_\infty \equiv 0$ on $\mathcal{M}_\infty$, which completes the proof of Case (a).

In Case (b), the rescaled limit is $\mathbb{R}^3\setminus\{0^3\}$. Let us choose the rescaling factors as follows:

$$\lambda_j = (d_m(\boldsymbol{x}_j))^{-1},\ \tau_j^{(k+\alpha)} = e^{-\delta\cdot 2T_-}\cdot(d_m(\boldsymbol{x}_j))^{-\nu-k-\alpha},\ \kappa_j = e^{\delta\cdot 2T_-}\cdot(d_m(\boldsymbol{x}_j))^{\nu-1}.$$

In terms of the above rescaled metric, the reference point $\boldsymbol{x}_\infty$ satisfies $d_{g_0}(\boldsymbol{x}_\infty, 0^3) = 1$. Moreover, the rescaled weight function in the limit space is given by

$$\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x}) = (d_{g_0}(0^3, \boldsymbol{x}))^{-\nu-k-\alpha}, \qquad \boldsymbol{x}\in\mathbb{R}^3\setminus\{0^3\}.$$

We will now analyze the limiting behavior of the operator $\mathscr{D}_{\tilde{g}_j}$ under the collapsing sequence $(\mathcal{M}, g_j, \boldsymbol{x}_j)$. Our basic strategy is to reduce the convergence of the 1-form $\tilde{\omega}_j$ to the convergence of the coefficient functions. Let

$$\tilde{\omega}_j = f_j^x\cdot\theta_j^x + f_j^y\cdot\theta_j^y + f_j^z\cdot\theta_j^z + f_j^t\cdot\theta_j^t,$$

where

$$\theta_j^x \equiv \lambda_j\cdot\beta_j^{\frac12}\cdot dx,\ \theta_j^y \equiv \lambda_j\cdot\beta_j^{\frac12}\cdot dy,\ \theta_j^z \equiv \lambda_j\cdot\beta_j^{\frac12}\cdot dz,\ \theta_j^t \equiv \lambda_j\cdot\beta_j^{\frac12}\cdot dt.$$

By straightforward computations,

$$|\theta_j^x|_{\tilde{g}_j} = |\theta_j^y|_{\tilde{g}_j} = |\theta_j^z|_{\tilde{g}_j} \to 1,\ |\theta_j^t|_{\tilde{g}_j} \to 0. \tag{9.2}$$

Now let us construct the limits of the above coefficient functions. We start with the Gromov-Hausdorff convergence

$$(\mathcal{M}, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} (\mathbb{R}^3, g_0, \boldsymbol{x}_\infty)$$

with $|\boldsymbol{x}_\infty| = 1$. For any fixed $R > 10$, let $A^{g_0}_{\frac1R,R}(0^3)$ be an annulus in $\mathbb{R}^3$ with respect to the Euclidean metric g_0. The first step is to obtain the limits of the coefficient functions f_j^x, f_j^y, f_j^z, f_j^t with controlled weighted norms in the flat annulus $A^{g_0}_{\frac1R,R}(0^3)$ under the Gromov-Hausdorff convergence. Letting $R\to\infty$, we will apply Arzelà-Ascoli to obtain global limiting functions.

First, fixing any $R > 0$, we consider a Euclidean annulus $A^{g_0}_{\frac{1}{R},R}(0^3) \subset \mathbb{R}^3$. We claim that there are limiting functions $f^x_{\infty,R}$, $f^y_{\infty,R}$, $f^z_{\infty,R}$ and $f^t_{\infty,R}$ on $A^{g_0}_{\frac{1}{R},R}(0^3) \subset \mathbb{R}^3$. Indeed, for any fixed $R > 0$, there exist $\bar{s}_0(R) > 0$ and $N_0(R) > 0$ such that $\{B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})\}_{k=1}^N$ with $N \leq N_0$ is a finite collection of Euclidean balls covering $A^{g_0}_{\frac{1}{R},R}(0^3)$ which satisfies

(1) $A^{g_0}_{\frac{1}{R},R}(0^3) \subset \bigcup_{s=1}^N B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}) \subset A^{g_0}_{\frac{1}{3R},3R}(0^3)$,
(2) $\frac{\bar{s}_0}{3} \leq d_{g_0}(\boldsymbol{y}_{\infty,k}, \boldsymbol{y}_{\infty,k'}) \leq \bar{s}_0$ for all $1 \leq k < k' \leq N$.

We will verify that there exists a subsequence (still denoted by j) such that the above finite cover satisfies the following compatibility:

(C1) $(f^x_j, f^y_j, f^z_j, f^t_j)$ converges to harmonic functions $(f^x_{\infty,k}, f^y_{\infty,k}, f^z_{\infty,k}, f^t_{\infty,k})$ on every $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})$.
(C2) The above locally defined limiting functions can be patched together in the sense that if $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}) \cap B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k'}) \neq \emptyset$, then

$$f^x_{\infty,k}(\boldsymbol{y}_\infty) = f^x_{\infty,k'}(\boldsymbol{y}_\infty),\ f^y_{\infty,k}(\boldsymbol{y}_\infty) = f^y_{\infty,k'}(\boldsymbol{y}_\infty),$$
$$f^z_{\infty,k}(\boldsymbol{y}_\infty) = f^z_{\infty,k'}(\boldsymbol{y}_\infty),\ f^t_{\infty,k}(\boldsymbol{y}_\infty) = f^t_{\infty,k'}(\boldsymbol{y}_\infty),$$

holds for all $\boldsymbol{y}_\infty \in B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}) \cap B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k'})$.

The above compatibility properties immediately imply that there are well-defined harmonic limiting functions $f^x_{\infty,R}$, $f^y_{\infty,R}$, $f^z_{\infty,R}$ and $f^t_{\infty,R}$ on $A_{\frac{1}{R},R}(0^3)$.

To show property (C1), by taking some subsequence, it suffices to show that for each ball $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})$ in the above finite cover, there is some subsequence in the original sequence $\{j\}$ such that the coefficient functions $f^x_{j,k}$ converge to a harmonic function $f^x_{\infty,k}$. For this purpose, we need to locally unwrap the collapsed fibers and discuss the convergence of the coefficient functions $f^x_{j,k}$ on non-collapsed local universal covers.

We take a sequence of geodesic balls $B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})$ with

$$(B_{2\bar{s}_0}(\boldsymbol{y}_{j,k}), \tilde{g}_j) \xrightarrow{GH} (B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}), g_0).$$

Denote by ℓ_j $(\to 0)$ the length of the collapsing S^1-fiber at $\boldsymbol{y}_j$ and define

$$\Gamma_j = \Gamma_{\epsilon_j}(\boldsymbol{y}_{j,k}) \equiv \mathrm{Image}[\pi_1(B_{\epsilon_j}(\boldsymbol{y}_j)) \to \pi_1(B_{2\bar{s}_0}(\boldsymbol{y}_j))],$$

where $\epsilon_j > 0$ is chosen such that $2\ell_j \leq \epsilon_j \leq 4\ell_j$. In our situation, $\pi_1(B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})) = \Gamma_j$ and Γ_j is isomorphic to $\mathbb{Z}$. Now let

$$\mathrm{pr}_j : (\widehat{B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})}, \hat{g}_j, \hat{\boldsymbol{y}}_{j,k}) \longrightarrow (B_{2\bar{s}_0}(\boldsymbol{y}_{j,k}), \tilde{g}_j, \boldsymbol{y}_{j,k})$$

be the universal covering map with $B_{2\bar{s}_0}(\boldsymbol{y}_{j,k}) = \widehat{B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})}/\Gamma_j$. On the universal covers, we have the following diagram of equivariant Gromov-Hausdorff convergence

$$\begin{tikzcd}
\Big(\widehat{B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})}, \hat{g}_j, \Gamma_j, \hat{\boldsymbol{y}}_{j,k}\Big) \arrow[r, "eqGH"] \arrow[d, "\mathrm{pr}_j"'] & \Big(\widehat{Y}_k, \hat{g}_\infty, \Gamma_\infty, \hat{\boldsymbol{y}}_{\infty,k}\Big) \arrow[d, "\mathrm{pr}_\infty"] \\
\Big(B_{2\bar{s}_0}(\boldsymbol{y}_{j,k}), \tilde{g}_j, \boldsymbol{y}_{j,k}\Big) \arrow[r, "GH"] & \Big(B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}), g_0, \boldsymbol{y}_{\infty,k}\Big)
\end{tikzcd}$$

which satisfies the following properties:

(e1) the universal covers $(\widehat{B_{2\bar{s}_0}(\boldsymbol{y}_{j,k})}, \hat{g}_j, \hat{\boldsymbol{y}}_{j,k})$ are non-collapsed and have uniformly bounded curvature,
(e2) the limiting Lie group Γ_∞ is diffeomorphic to $\mathbb{R}$ and acts isometrically on the limit space $(\widehat{Y}_k, \hat{g}_\infty, \hat{\boldsymbol{y}}_{\infty,k})$,
(e3) the universal covering maps pr_j converge to a Riemannian submersion

$$\mathrm{pr}_\infty : (\widehat{Y}_k, \hat{g}_\infty, \hat{\boldsymbol{y}}_{\infty,k}) \longrightarrow (B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}), g_0, \boldsymbol{y}_{\infty,k})$$

with $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}) = \widehat{Y}_k/\Gamma_\infty$,
(e4) for every $\hat{\boldsymbol{z}}_\infty \in \widehat{Y}_k$, the orbit $\Gamma_\infty \cdot \hat{z}_\infty$ is a geodesic in $\widehat{Y}_k$ and isometric to $(\mathbb{R}, dt^2)$. In particular, $(\widehat{Y}_k, \hat{g}_\infty, \hat{\boldsymbol{y}}_\infty)$ is isometric to $B_{2\bar{s}_0}(0^3) \times \mathbb{R}$ in the Euclidean space $\mathbb{R}^4$.

Indeed, property (e1) follows from Lemma 7.2. Properties (e2) and (e3) follow from the definition of equivariant convergence.

Proof of Property (e4). Define the rescaled coordinate functions

$$x_j \equiv \gamma_j \cdot x,\ y_j \equiv \gamma_j \cdot y,\ z_j \equiv \gamma_j \cdot z,$$

where $\gamma_j = \lambda_j \cdot \beta^{\frac{1}{2}}$. By direct computation, it is easy to see that

$$\Delta_{\tilde{g}_j} x_j = \Delta_{\tilde{g}_j} y_j = \Delta_{\tilde{g}_j} z_j = 0,$$

and by (9.2), away from the monopole points, we have that

$$|\nabla_{\tilde{g}_j} x_j|_{\tilde{g}_j} = |\nabla_{\tilde{g}_j} y_j|_{\tilde{g}_j} = |\nabla_{\tilde{g}_j} z_j|_{\tilde{g}_j} \to 1.$$

To see that the orbits are totally geodesic, we will prove that

$$|\nabla^2_{\tilde{g}_j} x_j|_{\tilde{g}_j} = |\nabla^2_{\tilde{g}_j} y_j|_{\tilde{g}_j} = |\nabla^2_{\tilde{g}_j} z_j|_{\tilde{g}_j} \to 0. \tag{9.3}$$

It suffices to verify this for x_j as the proofs for y_j and z_j are identical to this case. First, Bochner's formula gives that

$$\frac{1}{2}\Delta_{\tilde{g}_j} |\nabla x_j|^2_{\tilde{g}_j} = |\nabla^2 x_j|^2_{\tilde{g}_j}. \tag{9.4}$$

Due to Cheeger-Colding (see [CC96]), there exist cutoff functions $\varphi_j : \mathcal{M} \to [0,1]$ with

$$\varphi_j(x) = \begin{cases} 1, & x \in B_R(p_j), \\ 0, & x \in \mathcal{M} \setminus B_{2R}(p_j), \end{cases}$$

and an absolute constant $C_0 > 0$ such that $R|\nabla_{\tilde{g}_j}\varphi_j|_{\tilde{g}_j} + R^2|\Delta_{\tilde{g}_j}\varphi_j| \le C_0$. Integrating (9.4) over $B_{4R}(p_j)$ yields

$$\begin{aligned}
&⨍_{B_{4R}(p_j)} \varphi_j |\nabla^2 x_j|^2_{\tilde{g}_j}\, \mathrm{dvol}_{\tilde{g}_j} \\
&= \frac{1}{\mathrm{Vol}_{\tilde{g}_j}(B_{4R}(p_j))} \int_{B_{4R}(p_j)} \varphi_j |\nabla^2 x_j|^2_{\tilde{g}_j}\, \mathrm{dvol}_{\tilde{g}_j} \\
&= \frac{1}{\mathrm{Vol}_{\tilde{g}_j}(B_{4R}(p_j))} \int_{B_{4R}(p_j)} \frac{1}{2}\varphi_j \Delta_{\tilde{g}_j}(|\nabla x_j|^2_{\tilde{g}_j} - 1)\, \mathrm{dvol}_{\tilde{g}_j} \\
&= \frac{1}{2\,\mathrm{Vol}_{\tilde{g}_j}(B_{4R}(p_j))} \int_{B_{4R}(p_j)} (\Delta_{\tilde{g}_j}\varphi_j) \cdot (|\nabla x_j|^2_{\tilde{g}_j} - 1)\, \mathrm{dvol}_{\tilde{g}_j} \to 0
\end{aligned}$$

as $j \to \infty$. Therefore, by volume comparison,

$$\fint_{B_R(p_j)} |\nabla^2 x_j|^2_{\tilde{g}_j} \, \mathrm{dvol}_{\tilde{g}_j} \to 0 \tag{9.5}$$

as $j \to \infty$. Let $p_\infty \in \mathbb{R}^3 \setminus \{0^3\}$ with $B_{2\bar{s}_0}(p_\infty) \subset \mathbb{R}^3 \setminus \{0^3\}$ and choose a sequence of geodesic balls $B_{2\bar{s}_0}(p_j)$ such that $(B_{2\bar{s}_0}(p_j), \tilde{g}_j) \xrightarrow{GH} (B_{2\bar{s}_0}(p_\infty), g_0)$. By Lemma 7.2, the curvatures on $B_{\bar{s}_0}(p_j)$ are uniformly bounded by $C \cdot \bar{s}_0^{-2}$, where $C > 0$ is an absolute constant. On the other hand, since $\Delta_{\tilde{g}_j} x_j = 0$, (9.5) can be strengthened to

$$\sup_{B_{\bar{s}_0}(p_j)} |\nabla^2 x_j|^2_{\tilde{g}_j} \to 0.$$

This completes the proof of (9.3).

To finish the proof of Property (e4), note that by (9.3), the second fundamental form of each Γ_∞-orbit is vanishing. In other words, each Γ_∞-orbit is a geodesic in $\widehat{Y}_k$. Combined with the facts that the limiting projection pr_∞ is a Riemannian submersion and $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})$ is a Euclidean ball, we conclude that $\widehat{Y}_k \equiv B_{2\bar{s}_0}(0^3) \times \mathbb{R}$ and $\hat{g}_\infty$ is isometric to the Euclidean metric. $\square$

We will apply the above equivariant convergence to construct harmonic functions $f^x_{\infty,k}$, $f^y_{\infty,k}$, $f^z_{\infty,k}$ and $f^t_{\infty,k}$ in $B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})$. We only show the construction of $f^x_{\infty,k}$. Notice that Proposition 8.2 implies that the Γ_j-invariant lifted functions $\hat{f}^x_j$ satisfy the uniform weighted Schauder estimate

$$\|\hat{f}^x_j\|_{C^{1,\alpha}_{\delta,\nu}(\widehat{B_{2\bar{s}_0}(y_j)})} \leq C$$

with respect to the lifted weight function. Applying Arzelà-Ascoli, after passing to a subsequence, there is a limiting function $\hat{f}^x_{\infty,k}$ that satisfies

$$\|\hat{f}^x_{\infty,k}\|_{C^{1,\alpha}_{\delta,\nu}(\widehat{Y}_k)} \leq C.$$

Combined with the above equivariant convergence, we find that the limit function $\hat{f}^x_{\infty,k}$ is Γ_∞-invariant and descends to a function $f^x_{\infty,k}$ in $B_{2\bar{s}_0}(\boldsymbol{y}_\infty)$. Now we prove that $f^x_{\infty,k}$ is a harmonic function on $B_{2\bar{s}_0}(\boldsymbol{y}_\infty)$. As the lifted 1-forms $\hat{\omega}_j$ also satisfy

$$\|\hat{\omega}_j\|_{C^{1,\alpha}_{\delta,\nu}(\widehat{B_{2\bar{s}_0}(y_j)})} \leq C,$$

there is a limiting 1-form $\hat{\omega}_{\infty,k}$ that satisfies $\|\hat{\omega}_{\infty,k}\|_{C^{1,\alpha}_{\delta,\nu}(\widehat{Y}_k)} \leq C$. The contradiction assumption then implies that

$$\mathscr{D}_{\hat{g}_\infty} \hat{\omega}_{\infty,k} \equiv 0 \text{ in } \widehat{Y}^k.$$

The standard elliptic regularity theory for $\mathscr{D}_{\hat{g}_\infty}$ now shows that the 1-form $\hat{\omega}_{\infty,k}$ is C^∞ which in turn implies that $\hat{f}^x_{\infty,k} \in C^\infty(\widehat{Y}_k)$. Lemma 9.3 will be used throughout what follows. Its proof is left to the reader.

Lemma 9.3. *Let (M^4, g) be an oriented Riemannian 4-manifold and let $\omega \in \Omega^1(M^4)$ satisfy $\mathscr{D}_g \omega = 0$, where $\mathscr{D}_g \equiv d^+ \oplus d^*$. Then $\Delta_H \omega = 0$, where Δ_H is the Hodge Laplacian on (M^4, g).*

Lemma 9.3 implies that $\Delta_{\hat{g}_\infty}(\hat{f}^x_{\infty,k}) = 0$. Applying Property (e4), $\Delta_{g_0}(f^x_{\infty,k}) = \Delta_{\hat{g}_\infty}(\hat{f}^x_{\infty,k}) = 0$. The construction of the harmonic limiting functions $f^y_{\infty,k}$, $f^z_{\infty,k}$ and $f^t_{\infty,k}$ is verbatim.

We are now ready to prove (C2). To this end, we take the union of disjoint balls

$$B_\infty \equiv B_{2\bar{s}_0}(y_{\infty,k}) \cup B_{2\bar{s}_0}(y_{\infty,k'}), \quad B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k}) \cap B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k'}) \neq \emptyset.$$

Let $B_j \equiv B_{2\bar{s}_0}(\boldsymbol{y}_{j,k}) \cup B_{2\bar{s}_0}(\boldsymbol{y}_{j,k'})$. Then we have that $(B_j, \tilde{g}_j) \xrightarrow{GH} (B_\infty, g_0)$. By the same arguments as above, B_j has uniformly bounded curvature and the universal covering space $(\widehat{B}_j, \hat{g}_j)$ is non-collapsed. Moreover, the equivariant convergence with properties (e1)–(e5) as above still holds in this case. By passing to some subsequence, the lifted coefficient functions $\hat{f}_j^x$ are $C^{1,\alpha'}$-converging to some invariant limiting function $\hat{f}^x_{\infty,k,k'}$ on $\widehat{B}_\infty$ for any $\alpha' \in (0,\alpha)$ satisfying

$$\hat{f}^x_{\infty,k,k'}|_{\widehat{Y}_k} = \hat{f}^x_{\infty,k}.$$

Therefore, $\hat{f}^x_{\infty,k,k'}$ descends to a function $f^x_{\infty,k,k'}$ on B_∞ with

$$f^x_{\infty,k,k'}|_{B_{2\bar{s}_0}(\boldsymbol{y}_{\infty,k})} = f^x_{\infty,k}.$$

In addition, $f^x_{\infty,k,k'}$ is harmonic on B_∞, so we have managed to extend the local harmonic limiting function $f^x_{\infty,k}$ to the union B_∞. Repeating the above arguments, we can extend the limiting functions to the whole annulus $A^{g_0}_{\frac{1}{R},R}(0^3)$.

The above construction gives a control of the harmonic functions $(f^x_{\infty,R}, f^y_{\infty,R}, f^z_{\infty,R}, f^t_{\infty,R})$ on the flat annulus $\mathcal{A}_R \equiv A^{g_0}_{\frac{1}{R},R}(0^3)$ with respect to the weighted norm:

(9.6)
$$|\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty)| \cdot \Big(|f^x_{\infty,R}(\boldsymbol{x}_\infty)| + |f^y_{\infty,R}(\boldsymbol{x}_\infty)| + |f^z_{\infty,R}(\boldsymbol{x}_\infty)| + |f^t_{\infty,R}(\boldsymbol{x}_\infty)|\Big) \geq \frac{1}{30},$$
$$\|f^x_{\infty,R}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{A}_R)} + \|f^y_{\infty,R}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{A}_R)} + \|f^z_{\infty,R}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{A}_R)} + \|f^t_{\infty,R}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{A}_R)} \leq C.$$

We are now able to define a global harmonic 4-tuple on the punctured Euclidean space $\mathbb{R}^3 \setminus \{0^3\}$ by applying standard exhaustion arguments. Let $R \to +\infty$. By applying (9.6) and Arzelà-Ascoli, there exists a global 4-tuple of harmonic functions $(f^x_\infty, f^y_\infty, f^z_\infty, f^t_\infty)$ on $\mathbb{R}^3 \setminus \{0^3\}$ such that

$$|\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty)| \cdot \Big(|f^x_\infty(\boldsymbol{x}_\infty)| + |f^y_\infty(\boldsymbol{x}_\infty)| + |f^z_\infty(\boldsymbol{x}_\infty)| + |f^t_\infty(\boldsymbol{x}_\infty)|\Big) \geq \frac{1}{30},$$
$$\|f^x_\infty\|_{C^{1,\alpha}_{\delta,\nu}(\mathbb{R}^3\setminus\{0^3\})} + \|f^y_\infty\|_{C^{1,\alpha}_{\delta,\nu}(\mathbb{R}^3\setminus\{0^3\})} + \|f^z_\infty\|_{C^{1,\alpha}_{\delta,\nu}(\mathbb{R}^3\setminus\{0^3\})} + \|f^t_\infty\|_{C^{1,\alpha}_{\delta,\nu}(\mathbb{R}^3\setminus\{0^3\})} \leq C.$$

The weighted norm bound implies that the limiting functions have the following controlled behavior:

$$(|f^x_\infty| + |f^y_\infty| + |f^z_\infty| + |f^t_\infty|)(\boldsymbol{x}) \leq C\Big(d_{g_0}(\boldsymbol{x}, 0^3)\Big)^{-\nu}, \ \forall \boldsymbol{x} \in \mathbb{R}^3 \setminus \{0^3\}.$$

By the standard removable singularity theorem, the 4-tuple of harmonic functions $(f^x_\infty, f^y_\infty, f^z_\infty, f^t_\infty)$ extend to the entire Euclidean space $\mathbb{R}^3$. Using $0 < \nu < 1$ and applying the standard Liouville theorem for harmonic functions, we conclude that $f^x_\infty = f^y_\infty = f^z_\infty = f^t_\infty \equiv 0$. So a contradiction arises, which completes the proof of Case (b).

Now we consider Case (c). We have shown in Section 7.2 that the rescaled limit in Case (c) is a punctured flat cylinder $(\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}$. More precisely, we chose a sequence of punctured unbounded domains $\mathring{U}_j$ containing $\boldsymbol{x}_j$ such that

$$(\mathring{U}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} \Big((\mathbb{T}^2 \times \mathbb{R}) \setminus \mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty\Big).$$

Then the curvatures are uniformly bounded away from the singular points, that is, points in $\mathcal{P}_{m_0}$.

Let us denote $d_j \equiv \min_{1\leq m\leq m_0}\{d_m(p_m, \boldsymbol{x}_j)\}$ and choose the following rescaling factors

$$\lambda_j = d_j^{-1}, \quad \tau_j^{(k+\alpha)} = e^{-\delta\cdot 2T_-}\cdot (d_j^{-1})^{\nu+k+\alpha}, \quad \kappa_j = e^{\delta\cdot 2T_-}\cdot (d_j)^{\nu-1}.$$

So the rescaled weight function in the limit space satisfies

$$\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x}) = \begin{cases} (d_{g_0}(p_m,\boldsymbol{x}))^{\nu+k+\alpha}, & \boldsymbol{x}\in B^{g_0}_{\iota_0''}(p_m) \text{ for some } 1\leq m\leq m_0,\\ Q_{\nu,k,\alpha}, & \boldsymbol{x}\in \bigcap_{m=1}^{m_0} A_m^{g_0}(2\iota_0'', T_0''),\\ e^{\delta z(\boldsymbol{x})}, & \boldsymbol{x}\in (\mathbb{T}^2\times\mathbb{R})\setminus \bigcup_{m=1}^{m_0} B^{g_0}_{T_0''}(p_m),\end{cases}$$

where $\iota_0'' \in [1, \frac{\iota_0'}{C_0}]$ is some constant of definite size and $Q_{\nu,k,\alpha}$ depends only on ν, k and α.

Similar to Case (b), to apply the Liouville theorem on the collapsed limit, we will construct a globally defined 4-tuple of harmonic functions $(f^x_\infty, f^y_\infty, f^z_\infty, f^t_\infty)$ on the limit space.

Fix $R > 0$, denote by $T_R(S)$ the R-tubular neighborhood of a compact set S. Applying Lemma 7.2, we have the following curvature estimate on the sequence of annuli $T^{\tilde{g}_j}_{3R}(\mathcal{P}_{m_0}) \setminus T^{\tilde{g}_j}_{\frac{1}{R}}(\mathcal{P}_{m_0})$:

$$\|\operatorname{Rm}_{\tilde{g}_j}\|_{L^\infty\left(T^{\tilde{g}_j}_{3R}(\mathcal{P}_{m_0})\setminus T^{\tilde{g}_j}_{\frac{1}{R}}(\mathcal{P}_{m_0})\right)} \leq K_0\cdot R^2,$$

where $K_0 > 0$ is an absolute constant. Applying the same arguments as in Case (b), one can construct a 4-tuple of harmonic functions $(f^x_\infty, f^y_\infty, f^z_\infty, f^t_\infty)$ on $(\mathbb{T}^2\times\mathbb{R})\setminus\mathcal{P}_{m_0}$ which satisfy the following weighted estimates on $(\mathbb{T}^2\times\mathbb{R})\setminus\mathcal{P}_{m_0}$:

$$\begin{aligned} |\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty)|\cdot\Big(|f^x_\infty(\boldsymbol{x}_\infty)| + |f^y_\infty(\boldsymbol{x}_\infty)| + |f^z_\infty(\boldsymbol{x}_\infty)| + |f^t_\infty(\boldsymbol{x}_\infty)|\Big) &\geq \frac{1}{30},\\ \|f^x_\infty\|_{C^{1,\alpha}_{\delta,\nu}} + \|f^y_\infty\|_{C^{1,\alpha}_{\delta,\nu}} + \|f^z_\infty\|_{C^{1,\alpha}_{\delta,\nu}} + \|f^t_\infty\|_{C^{1,\alpha}_{\delta,\nu}} &\leq C. \end{aligned} \tag{9.7}$$

This implies that

$$\begin{aligned} &|f^x_\infty(\boldsymbol{x})| + |f^y_\infty(\boldsymbol{x})| + |f^z_\infty(\boldsymbol{x})| + |f^t_\infty(\boldsymbol{x})| \leq C\Big(d_{g_0}(\boldsymbol{x}, p_m)\Big)^{-\nu},\ \boldsymbol{x}\in B^{g_0}_{\iota_0''}(p_m),\\ &|f^x_\infty(\boldsymbol{x})| + |f^y_\infty(\boldsymbol{x})| + |f^z_\infty(\boldsymbol{x})| + |f^t_\infty(\boldsymbol{x})| \leq Ce^{-\delta z(\boldsymbol{x})},\ |z(\boldsymbol{x})|\geq Z_0, \end{aligned} \tag{9.8}$$

for some sufficiently large $Z_0 > 0$, since $\Delta_{g_0} f^x_\infty = \Delta_{g_0} f^y_\infty = \Delta_{g_0} f^z_\infty = \Delta_{g_0} f^t_\infty \equiv 0$ on the punctured cylinder $(\mathbb{T}^2\times\mathbb{R})\setminus\mathcal{P}_{m_0}$. Since we have required that $0 < \nu < 1$, it is standard to verify that the singularities in $\mathcal{P}_{m_0}$ are removable. It follows that the harmonic functions f^x_∞, f^y_∞, f^z_∞ and f^t_∞ extend to the entire flat cylinder $\mathbb{T}^2\times\mathbb{R}$ and satisfy the asymptotic behavior as in (9.8). Applying Lemma 9.1 to the coefficient functions with the growth condition (9.8), we conclude that $f^x_\infty = f^y_\infty = f^z_\infty = f^t_\infty \equiv 0$ on $\mathbb{T}^2\times\mathbb{R}$. This contradicts (9.7), so the proof of Case (c) is complete.

Region III.

The proof for Region III is identical to Case (c) of Region II.

Regions IV_- **and Region** IV_+**.**

We only focus on the case that the reference points $\boldsymbol{x}_j$ are located in Region IV_-. The proof for Region IV_+ is verbatim. Region IV_- has two different types of rescaling geometries of Section 7.2, which are given by Case (a) and Case (b).

For fixed reference points $\boldsymbol{x}_j$ located in Case (a), we have the following convergence:

$$(\mathring{U}_j, \tilde{g}_j, \boldsymbol{x}_j) \xrightarrow{GH} \Big((\mathbb{T}^2\times\mathbb{R})\setminus\mathcal{P}_{m_0}, g_0, \boldsymbol{x}_\infty\Big),$$

where g_0 is a flat product metric on $\mathbb{T}^2\times\mathbb{R}$ and $\mathring{U}_j$ is defined in (7.17). We choose the rescaling factors as follows:

$$\lambda_j=(L_-(\boldsymbol{x}_j))^{-1},\ \tau_j^{(k+\alpha)}=e^{-\delta(2T_-)}\cdot(L_-(\boldsymbol{x}_j))^{-\nu-k-\alpha},\ \kappa_j=e^{\delta(2T_-)}\cdot(L_-(\boldsymbol{x}_j))^{\nu-1}.$$

The limiting weight function is then

$$\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}=\begin{cases}(d_{g_0}(p_m,\boldsymbol{x}))^{\nu+k+\alpha}, & \boldsymbol{x}\in B^{g_0}_{\iota_0''}(p_m)\text{ for some }1\le m\le m_0,\\ Q_{\nu,k,\alpha}, & \boldsymbol{x}\in\bigcap_{m=1}^{m_0}A^{g_0}_m(2\iota_0'',T_0''),\\ e^{\delta z(\boldsymbol{x})}, & \boldsymbol{x}\in(\mathbb{T}^2\times\mathbb{R})\setminus\bigcup_{m=1}^{m_0}B^{g_0}_{T_0''}(p_m),\end{cases}$$

where $\iota_0''\in[1,\frac{\iota_0}{C_0}]$ is some constant of definite size and $Q_{\nu,k,\alpha}$ depends only on ν, k and α. The rest of the proof is identical to the proof of Case (c) in Region II.

Next we prove Case (b). We showed in Section 7.2 that in this case we have the following convergence:

$$(U_j,\tilde{g}_j,\boldsymbol{x}_j)\xrightarrow{GH}(\mathbb{T}^2\times\mathbb{R},g_0,\boldsymbol{x}_\infty),$$

where g_0 is a flat product metric on $\mathbb{T}^2\times\mathbb{R}$. The rescaling factors are chosen as

$$\lambda_j=(L_-(\boldsymbol{x}_j))^{-1},\quad \tau_j^{(k+\alpha)}=e^{-\delta(2T_--z_j)}\cdot(L_-(\boldsymbol{x}_j))^{-\nu-k-\alpha},$$
$$\kappa_j=e^{\delta(2T_-+z_j)}\cdot(L_-(\boldsymbol{x}_j))^{\nu-1},$$

where $z_j\equiv z(\boldsymbol{x}_j)$. We also translate the z-coordinate by $\tilde{z}(\boldsymbol{x})=z(\boldsymbol{x})-z_j$, which gives the limiting weight function $\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x})=e^{\delta\cdot\tilde{z}(\boldsymbol{x})}$ for every $\boldsymbol{x}\in\mathbb{T}^2\times\mathbb{R}$. With z replaced by $\tilde{z}$, the remainder of the contradiction argument is the same as Case (c) of Region II.

Regions V_- and V_+.

First, we assume that the reference points $\boldsymbol{x}_j$ are located in Region V_+. As discussed in Section 7.2, it is natural to subdivide Region V_+ into Case (a) and Case (b). In this region, we choose the corresponding rescaling factors

$$\lambda_j=\frac{1}{\underline{L}_+(\boldsymbol{x}_j)},\quad \tau_j^{(k+\alpha)}=\frac{e^{-2\delta\cdot(T_-+T_+-\frac{z_j}{2})}}{(\underline{L}_+(\boldsymbol{x}_j))^{\nu+k+\alpha}},\quad \kappa_j=\frac{e^{2\delta\cdot(T_-+T_+-\frac{z_j}{2})}}{(\underline{L}_+(\boldsymbol{x}_j))^{1-\nu}},$$

where $z_j\equiv z_+(\boldsymbol{x}_j)$.

First, the rescaled limit in Case (a) is the flat cylinder $\mathbb{T}^2\times\mathbb{R}$ and we have the convergence

$$(U_j,\tilde{g}_j,\boldsymbol{x}_j)\xrightarrow{GH}(\mathbb{T}^2\times\mathbb{R},g_0,\boldsymbol{x}_\infty).$$

Hence, under the z-coordinate translation $\tilde{z}_+(\boldsymbol{x})=z_+(\boldsymbol{x})-z_j$, the limiting weight function is $\tilde{\rho}^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x})=e^{-\delta\cdot\tilde{z}_+(\boldsymbol{x})}$. With z_+ replaced by $\tilde{z}_+$, the remaining arguments are exactly the same as those for Case (b) of Region $\mathrm{IV}_\pm$. This completes the proof of this case.

Next, we consider Case (b) of Region V_+. If the $\boldsymbol{x}_j$ satisfy $10\zeta_0^+\le z_+(\boldsymbol{x}_j)\le C_0$, then we have the convergence $(\mathcal{M},\tilde{g}_j,\boldsymbol{x}_j)\xrightarrow{C^\ell}(\mathcal{M}_\infty,\tilde{g}_\infty,\boldsymbol{x}_\infty)$ for any $\ell\in\mathbb{Z}_+$, where the limit space is a fixed rescaling of $(X^4_{b_+},g_{b_+},q_+)$. Also passing to a subsequence, we have $z_+(\boldsymbol{x}_j) \to \mathfrak{c}_0 \in [10\zeta_0^+, C_0]$. Hence, using the same translation $\tilde{z}_+(\boldsymbol{x}) = z_+(\boldsymbol{x}) - z_j$, the limiting weight function is

$$\rho^{(k+\alpha)}_{\infty,\delta,\nu}(\boldsymbol{x}) = e^{-\delta\cdot\tilde{z}_+(\boldsymbol{x})} \cdot (1 + \mathfrak{c}_0^{-1} \cdot \tilde{z}_+(\boldsymbol{x}))^{\frac{\nu}{2}},\ \boldsymbol{x} \in X^4_{b_+} \setminus B_{2D_0^+}(q_+).$$

On the other hand, the limiting 1-form $\tilde{\omega}_\infty \in \Omega^1(X^4_{b_+})$ satisfies the following:

$$\mathscr{D}_{g_{b_+}}\tilde{\omega}_\infty \equiv 0, \quad |\tilde{\rho}^{(0)}_{\infty,\delta,\nu}(\boldsymbol{x}_\infty) \cdot \tilde{\omega}_\infty(\boldsymbol{x}_\infty)| = 1, \quad \|\tilde{\omega}_\infty\|_{C^0_{\delta,\nu}(X^4_{b_+})} = 1.$$

Notice that $\nu \in (0,1)$ and $z_j \gg 1$. We also have the C^0-bound

$$|\tilde{\omega}_\infty(\boldsymbol{x})| \leq C_2 \cdot e^{\delta\cdot\tilde{z}_+(\boldsymbol{x})},\ \boldsymbol{x} \in X^4_{b_+} \setminus B_{2D_0^+}(q_+).$$

If $\delta \in (0, \delta_h)$, applying Theorem 5.1 then gives that we have that $\tilde{\omega}_\infty \equiv 0$ on $X^4_{b_+}$.

The proof is the same if the $\boldsymbol{x}_j$ are in Region V_-. The only difference appears in Case (b), where the C^0-bound of the limiting harmonic 1-form $\tilde{\omega}_\infty$ becomes

$$|\tilde{\omega}_\infty(\boldsymbol{x})| \leq C_1 \cdot e^{-\delta\cdot z_-(\boldsymbol{x})} \cdot (z_-(\boldsymbol{x}))^{-\frac{\nu}{2}},\ \boldsymbol{x} \in X^4_{b_-} \setminus B_{2D_0^-}(q_-).$$

We are now able to apply a much simpler Liouville theorem (Lemma 4.14) to the above decaying harmonic 1-form $\tilde{\omega}_\infty$ from which the contradiction arises.

Regions VI_- and VI_+.

If the $\boldsymbol{x}_j$ are located in Region VI_-, the proof is identical to Case (b) of Region V_-. If $\boldsymbol{x}_j$ are located in Region VI_+, the proof is the same as Case (b) of Region V_+.

Combining all of the above regions, the proof of Proposition 9.2 is complete. □

10. Perturbation to genuine hyperkähler metrics

Theorem 10.5 is the main existence theorem of a hyperkähler triple and will be proved in Section 10.1. Then in Section 10.2 we will prove the main theorems introduced in Section 1.2.

10.1. **The existence of a hyperkähler triple.**

We begin with some general remarks about perturbing a closed, definite triple $\boldsymbol{\omega}$ to a genuine hyperkähler triple. This material is from [Fos19], with some minor changes. We seek a triple of closed 2-forms $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ such that $\underline{\boldsymbol{\omega}} \equiv \boldsymbol{\omega} + \boldsymbol{\theta}$ is an actual hyperkähler triple on $\mathcal{M}$, which is the system

$$\frac{1}{2}(\omega_i + \theta_i) \wedge (\omega_j + \theta_j) = \delta_{ij}\,\mathrm{dvol}_{\boldsymbol{\omega}+\boldsymbol{\theta}},$$

which is equivalent to

$$\frac{1}{2}(\omega_i \wedge \omega_j + \omega_i \wedge \theta_j + \omega_j \wedge \theta_i + \theta_i \wedge \theta_j) = \frac{1}{6}\delta_{ij}\sum_{j=1}^{3}\Big(\omega_j^2 + \theta_j^2 + 2\omega_j \wedge \theta_j\Big). \tag{10.1}$$

Now split $\boldsymbol{\theta}$ into its self-dual and anti-self-dual parts with respect to $g_{\boldsymbol{\omega}}$, writing $\boldsymbol{\theta} = \boldsymbol{\theta}^+ + \boldsymbol{\theta}^-$. We define a matrix $A = (A_{ij})$ by

$$\theta_i^+ = \sum_{j=1}^{3} A_{ij}\omega_j \tag{10.2}$$

and define also a matrix $S_{\boldsymbol{\theta}^-} = (S_{ij})$ by

$$\frac{1}{2}\theta_i^- \wedge \theta_j^- = S_{ij}\,\mathrm{dvol}_{\boldsymbol{\omega}},\ 1 \leq i \leq j \leq 3.$$

For any 3×3 real matrix B, we denote by

$$\mathrm{tf}(B) = B - \frac{1}{3}\,\mathrm{Tr}(B)\,\mathrm{Id}$$

the trace-free part of B. Then we can write (10.1) as the matrix equation

$$\mathrm{tf}(Q_{\boldsymbol{\omega}} A^T + Q_{\boldsymbol{\omega}} A + A Q_{\boldsymbol{\omega}} A^T) = \mathrm{tf}(-Q_{\boldsymbol{\omega}} - S_{\boldsymbol{\theta}^-}). \tag{10.3}$$

For simplicity, in our context, we always identify a triple of self-dual 2-forms with a 3×3-matrix as in (10.2). Then observe that a solution of the following gauge-fixed system

$$d^+\boldsymbol{\eta} + \boldsymbol{\xi} = \mathfrak{F}_0\Big(\,\mathrm{tf}(-Q_{\boldsymbol{\omega}} - S_{d^-\boldsymbol{\eta}})\Big),\ d^*\boldsymbol{\eta} = 0, \tag{10.4}$$

is also a solution of (10.3). Here $\mathfrak{F}_0$ denotes the local inverse near zero of the local diffeomorphism $\mathfrak{G}_0 : \mathscr{S}_0(\mathbb{R}^3) \to \mathscr{S}_0(\mathbb{R}^3)$ on the space of trace-free symmetric 3×3-matrices $\mathscr{S}_0(\mathbb{R}^3)$ defined by

$$\mathfrak{G}_0(A) = \mathrm{tf}(Q_{\boldsymbol{\omega}} A^T + A Q_{\boldsymbol{\omega}} + A Q_{\boldsymbol{\omega}} A^T).$$

Moreover, $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ with $\eta_j \in \Omega^1(\mathcal{M})$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ with $\xi_i \in \mathcal{H}^+(\mathcal{M})$ (the space of self-dual harmonic 2-forms with respect to $g_{\boldsymbol{\omega}}$), and $d^{\pm}\boldsymbol{\eta}$ is the self-dual or anti-self-dual part of $d\boldsymbol{\eta} = \boldsymbol{\theta} - \boldsymbol{\xi}$ with respect to $g_{\boldsymbol{\omega}}$, respectively.

The linearization of the elliptic system (10.4) at $\boldsymbol{\eta} = 0$ is precisely the operator (1.5) mentioned in the Introduction:

$$\mathscr{L} = (\mathscr{D} \oplus \mathrm{Id}) \otimes \mathbb{R}^3 : (\Omega^1(\mathcal{M}) \oplus \mathcal{H}^+(\mathcal{M})) \otimes \mathbb{R}^3 \longrightarrow (\Omega^0(\mathcal{M}) \oplus \Omega^2_+(\mathcal{M})) \otimes \mathbb{R}^3,$$

where

$$\mathscr{D} \equiv d^* + d^+ : \Omega^1(\mathcal{M}) \longrightarrow (\Omega^0(\mathcal{M}) \oplus \Omega^2_+(\mathcal{M})).$$

For any sufficiently large gluing parameter $\beta \gg 1$, denote by $\boldsymbol{\omega}^{\mathcal{M}}_\beta = (\omega_1, \omega_2, \omega_3)$ the approximate definite triple on $\mathcal{M}$ which was constructed in Section 6. To prove the existence of a hyperkähler triple, we will solve the gauge-fixed elliptic system (10.4) based at $\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathcal{M}}_\beta$. We will use the following version of the implicit function theorem; see for example [RS05, Theorem 4.4.2].

Lemma 10.1. *Let $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$ be a C^1-map between two Banach spaces such that $\mathscr{F}(x) - \mathscr{F}(0) = \mathscr{L}(x) + \mathscr{N}(x)$, where the operator $\mathscr{L} : \mathfrak{A} \to \mathfrak{B}$ is linear and $\mathscr{N}(0) = 0$. Assume that*

(1) *$\mathscr{L}$ is an isomorphism with $\|\mathscr{L}^{-1}\| \leq C_1$,*
(2) *there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that*
   (a) *$\|\mathscr{N}(x) - \mathscr{N}(y)\|_{\mathfrak{B}} \leq C_2 \cdot (\|x\|_{\mathfrak{A}} + \|y\|_{\mathfrak{A}}) \cdot \|x - y\|_{\mathfrak{A}}$ for all $x, y \in B_r(0) \subset \mathfrak{A}$,*
   (b) *$\|\mathscr{F}(0)\|_{\mathfrak{B}} \leq \frac{r}{2C_1}$.*

Then there exists a unique solution to $\mathscr{F}(x) = 0$ in $\mathfrak{A}$ such that

$$\|x\|_{\mathfrak{A}} \leq 2C_1 \cdot \|\mathscr{F}(0)\|_{\mathfrak{B}}.$$

To apply the above implicit function theorem, we need to verify the above properties in our context. Recall that $\boldsymbol{\omega}^{\mathcal{M}}_\beta \equiv (\omega_1, \omega_2, \omega_3)$ is the closed, definite triple on $\mathcal{M}$ constructed in Section 6 which induces a Riemannian metric g_β such that the triple $\boldsymbol{\omega}^{\mathcal{M}}$ is self-dual with respect to g. Define the following Banach spaces:

$$\mathfrak{A} \equiv \Big(C^{1,\alpha}_{\delta,\nu}(\mathring{\Omega}^1(\mathcal{M})) \oplus \mathcal{H}^+(\mathcal{M})\Big) \otimes \mathbb{R}^3, \quad \mathfrak{B} \equiv \Big(C^{0,\alpha}_{\delta,\nu+1}(\Lambda^+(\mathcal{M}))\Big) \otimes \mathbb{R}^3,$$

where $\mathcal{H}^+(\mathcal{M})$ is the space of self-dual harmonic 2-forms on $\mathcal{M}$, $\Lambda^+(\mathcal{M})$ is the space of self-dual 2-forms on $\mathcal{M}$, and $\mathring{\Omega}^1(\mathcal{M}) \equiv \{\eta \in \Omega^1(\mathcal{M}) | d^*\eta = 0\}$. Notice that Proposition 6.6 implies that

$$\dim(\mathcal{H}^+(\mathcal{M})) = b_2^+(\mathcal{M}) = 3.$$

Clearly the closed self-dual 2-forms $\boldsymbol{\omega}_\beta^{\mathcal{M}} \equiv (\omega_1, \omega_2, \omega_3)$ form a basis of $\mathcal{H}^+(\mathcal{M})$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be equipped with the following weighted Hölder norms: for $(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+) \in \mathfrak{A}$ and $\boldsymbol{\xi}^+ \in \mathfrak{B}$, let us set

$$\|(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+)\|_{\mathfrak{A}} \equiv \|\boldsymbol{\eta}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} + \|\bar{\boldsymbol{\xi}}^+\|_{L^2(\mathcal{M})}, \quad \|\boldsymbol{\xi}^+\|_{\mathfrak{B}} \equiv \|\boldsymbol{\xi}^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})}.$$

The operator $\mathscr{F} : \mathfrak{A} \to \mathfrak{B}$ is defined by

$$\mathscr{F}(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+) \equiv d^+\boldsymbol{\eta} + \bar{\boldsymbol{\xi}}^+ - \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta - S_{d^-\boldsymbol{\eta}})\Big),$$

which is given by the system (10.4). The corresponding linearization is

$$\mathscr{L}_{g_\beta} \equiv (d^+ \oplus \mathrm{Id}) \otimes \mathbb{R}^3 : \mathfrak{A} \longrightarrow \mathfrak{B}.$$

So the nonlinear part is given by

$$\mathscr{N}(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+) \equiv \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta)\Big) - \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta - S_{d^-\boldsymbol{\eta}})\Big).$$

First, we will check Property (1) in Lemma 10.1.

Proposition 10.2. *Let the gluing parameter $\beta \gg 1$ be sufficiently large. Then there exists a constant $C > 0$ independent of β such that for every $\boldsymbol{\xi}^+ \equiv (\xi_1^+, \xi_2^+, \xi_3^+) \in \mathfrak{B}$, there exists a unique pair*

$$(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+) \equiv \Big((\eta_1, \eta_2, \eta_3), (\bar{\xi}_1^+, \bar{\xi}_2^+, \bar{\xi}_3^+)\Big) \in \mathfrak{A},$$

which solves $\mathscr{L}_{g_\beta}(\boldsymbol{\eta}, \bar{\boldsymbol{\xi}}^+) = \boldsymbol{\xi}^+$ and satisfies the estimate

$$\|\boldsymbol{\eta}\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} + \|\bar{\boldsymbol{\xi}}^+\|_{L^2(\mathcal{M})} \le C e^{10\delta\cdot\beta} \cdot \|\boldsymbol{\xi}^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})}, \tag{10.5}$$

where δ and ν are the constants in Proposition 9.2.

Proof. First, we prove that $\mathscr{L}_{g_\beta}$ is surjective. It follows from Hodge theory that

$$\Omega^2_+(\mathcal{M}) = \mathcal{H}^+(\mathcal{M}) \oplus d^+(\mathring{\Omega}^1(\mathcal{M})).$$

This clearly implies that the operator $\mathscr{L}_{g_\beta} = (d^+ \oplus \mathrm{Id}) \otimes \mathbb{R}^3 : \mathfrak{A} \longrightarrow \mathfrak{B}$ is surjective.

The remainder of the proof is a contradiction argument. We will argue on the level of forms which will imply the result for triples. If (10.5) does not hold for a uniform constant, then there exist sequences $\beta_j \to \infty$ and $\eta_j \in \Omega^1(\mathcal{M})$, $\bar{\xi}_j^+ \in \mathcal{H}_+(\mathcal{M})$ with $\langle d^+\eta_j, \bar{\xi}_j^+\rangle_{L^2(\mathcal{M})} = 0$, such that

$$e^{10\delta\cdot\beta_j}\|d^+\eta_j + \bar{\xi}_j^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \to 0 \quad \text{as } j \to \infty, \tag{10.6}$$

$$\|\eta_j\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} + \|\bar{\xi}_j^+\|_{L^2(\mathcal{M})} = 1. \tag{10.7}$$

Pairing $\bar{\xi}_j^+$ and $d^+\eta_j + \bar{\xi}_j^+$ gives $\|\bar{\xi}_j^+\|^2_{L^2(\mathcal{M})} = \langle d^+\eta_j + \bar{\xi}_j^+, \bar{\xi}_j^+\rangle_{L^2(\mathcal{M})}$. Using (10.6), we find that

$$\begin{aligned}\|\bar{\xi}_j^+\|^2_{L^2(\mathcal{M})} &\le \epsilon_j e^{-10\delta\cdot\beta_j}\int_{\mathcal{M}} |\bar{\xi}_j^+|(\rho^{(0+\alpha)}_{\delta,\nu+1})^{-1}\,\mathrm{dvol}_{g_{\beta_j}} \\ &\le \epsilon_j e^{-10\delta\cdot\beta_j}\|\bar{\xi}_j^+\|_{L^2(\mathcal{M})}\Big\{\int_{\mathcal{M}}(\rho^{(0+\alpha)}_{\delta,\nu+1})^{-2}\,\mathrm{dvol}_{g_{\beta_j}}\Big\}^{\frac{1}{2}},\end{aligned}$$

where $\epsilon_j \to 0$ as $j \to \infty$. It is easy to check that

$$\int_{\mathcal{M}} (\rho^{(0+\alpha)}_{\delta,\nu+1})^{-2} \,\mathrm{dvol}_{g_{\beta_j}} < C,$$

where C is independent of β_j, so that as $j \to \infty$,

$$e^{10\delta\cdot\beta_j}\|\bar{\xi}_j^+\|_{L^2(\mathcal{M})} \to 0. \tag{10.8}$$

Next, since each triple $\boldsymbol{\omega}^{\mathcal{M}}_{\beta_j}$ is harmonic and spans $\mathcal{H}_+(\mathcal{M})$ at every point, we can write

$$\bar{\xi}^+ = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3. \tag{10.9}$$

Recall by the definition of the triple $\boldsymbol{\omega}^{\mathcal{M}}_{\beta_j}$, for every $1 \le p, q \le 3$,

$$\frac{1}{2}\int_{\mathcal{M}} \omega_p \wedge \omega_q = \int_{\mathcal{M}} Q_{pq}\,\mathrm{dvol}_{\boldsymbol{\omega}^{\mathcal{M}}_{\beta_j}}.$$

Hence for any self-dual harmonic form $\bar{\xi}^+ \in \mathcal{H}_+(\mathcal{M})$,

$$\|\bar{\xi}^+\|^2_{L^2(\mathcal{M})} = 2\sum_{p,q=1}^{3} \lambda_p\lambda_q \int_{\mathcal{M}} Q_{pq}\,\mathrm{dvol}_{\boldsymbol{\omega}^{\mathcal{M}}_{\beta_j}}.$$

Thus, applying the volume estimate $C^{-1}\beta_j^2 \le \mathrm{Vol}_{g_{\beta_j}}(\mathcal{M}) \le C\beta_j^2$ and Proposition 6.4, we have the estimate

$$C^{-1}\beta_j^2(\lambda_{1,j}^2 + \lambda_{2,j}^2 + \lambda_{3,j}^2) \le \|\bar{\xi}_j^+\|^2_{L^2(\mathcal{M})}.$$

The above and (10.8) imply that $\beta_j \cdot \lambda_{k,j} \cdot e^{10\delta\cdot\beta_j} \to 0$ as $j \to \infty$ for $k = 1, 2, 3$. We then have that

$$\begin{aligned}\|\bar{\xi}_j^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} &= \|\lambda_{1,j}\omega_1 + \lambda_{2,j}\omega_2 + \lambda_{3,j}\omega_3\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \\ &\le \lambda_{1,j}\|\omega_1\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} + \lambda_{2,j}\|\omega_2\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} + \lambda_{3,j}\|\omega_3\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})}.\end{aligned}$$

Since $\|\omega_k\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \le Ce^{5\delta\cdot\beta_j}$ for $1 \le k \le 3$, the above implies that

$$\|\bar{\xi}_j^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \le C\epsilon_j\beta_j^{-1}e^{-5\delta\cdot\beta_j}$$

for some sequence $\epsilon_j \to 0$ as $j \to \infty$, so we have proved that $\|\bar{\xi}_j^+\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \to 0$ as $j \to \infty$. Hence we have proved that our sequence satisfies

$$\|d^+\eta_j\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \to 0 \quad \text{and} \quad \|\eta_j\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} \to 1$$

as $j \to \infty$, which contradicts Proposition 9.2. $\square$

In Lemma 10.3, we will verify Property (2) of Lemma 10.1.

Lemma 10.3. *Consider $(\mathcal{M}, g_\beta)$ with sufficiently large $\beta \gg 1$. Let δ and ν be the constants in Proposition 9.2. Then there are constants $r_0 > 0$ and $C > 0$ (independent of β) such that for every $\boldsymbol{v}_1 \equiv (\boldsymbol{\eta}_1, \bar{\boldsymbol{\xi}}_1^+) \in B_r(0) \subset \mathfrak{A}$, $\boldsymbol{v}_2 \equiv (\boldsymbol{\eta}_2, \bar{\boldsymbol{\xi}}_2^+) \in B_r(0) \subset \mathfrak{A}$ with $r < r_0$, we have that*

$$\|\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2)\|_{\mathfrak{B}} \le C(\|\boldsymbol{v}_1\|_{\mathfrak{A}} + \|\boldsymbol{v}_2\|_{\mathfrak{A}}) \cdot \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathfrak{A}}. \tag{10.10}$$

Proof. By definition, for any $\boldsymbol{v} \equiv (\boldsymbol{\omega}, \bar{\boldsymbol{\xi}}^+)$,

$$\mathscr{N}(\boldsymbol{v}) \equiv \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta)\Big) - \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta - S_{d^-\boldsymbol{\eta}})\Big),$$

and hence

$$\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2) = \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta - S_{d^-\boldsymbol{\eta}_2})\Big) - \mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta - S_{d^-\boldsymbol{\eta}_1})\Big).$$

Since $\mathfrak{F}_0 : \mathscr{S}_0(\mathbb{R}^3) \to \mathscr{S}_0(\mathbb{R}^3)$ is a smooth map on the space of trace-free symmetric (3×3)-matrices $\mathscr{S}_0(\mathbb{R}^3)$, there is some universal constant $C > 0$ such that

$$\begin{aligned}|\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2)| &\leq C|d^-\boldsymbol{\eta}_1 * d^-\boldsymbol{\eta}_1 - d^-\boldsymbol{\eta}_2 * d^-\boldsymbol{\eta}_2| \\ &\leq C(|d^-\boldsymbol{\eta}_1| + |d^-\boldsymbol{\eta}_2|) \cdot |d^-(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)|.\end{aligned}$$

Multiplying by the weight function gives us

$$\begin{aligned}&\rho^{(0)}_{\delta,\nu+1}(\boldsymbol{x}) \cdot |\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2)| \\ &\qquad \leq C\Big(\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}) \cdot (|d^-\boldsymbol{\eta}_1| + |d^-\boldsymbol{\eta}_2|)\Big) \cdot \Big(\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}) \cdot |d^-(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2)|\Big),\end{aligned}$$

where we have used the pointwise estimate $\rho^{(0)}_{\delta,\nu+1}(\boldsymbol{x}) \leq C \cdot (\rho^{(1)}_{\delta,\nu}(\boldsymbol{x}))^2$. Taking sup norms leads to

$$\begin{aligned}&\|\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2)\|_{C^0_{\delta,\nu+1}(\mathcal{M})} \\ &\qquad \leq C\Big(\|\boldsymbol{v}_1\|_{C^1_{\delta,\nu}(\mathcal{M})} + \|\boldsymbol{v}_2\|_{C^1_{\delta,\nu}(\mathcal{M})}\Big) \cdot \Big(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{C^1_{\delta,\nu}(\mathcal{M})}\Big).\end{aligned}$$

By similar computations, we also have the Hölder seminorm estimate

$$\begin{aligned}&\Big[\mathscr{N}(\boldsymbol{v}_1) - \mathscr{N}(\boldsymbol{v}_2)\Big]_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \\ &\qquad \leq C\Big(\|\boldsymbol{v}_1\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})} + \|\boldsymbol{v}_2\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})}\Big) \cdot \Big(\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{C^{1,\alpha}_{\delta,\nu}(\mathcal{M})}\Big).\end{aligned}$$

So we obtain the estimate (10.10) for the nonlinear errors. □

Proposition 10.4. *Consider $(\mathcal{M}, g_\beta)$ with sufficiently large gluing parameter $\beta \gg 1$. Let δ and ν be chosen as in Proposition 9.2. Then there exists a constant $C > 0$ independent of β such that*

$$\|\mathscr{F}(0)\|_{\mathfrak{B}} \leq Ce^{-\frac{\delta_q \beta}{2}},$$

where $\delta_q > 0$ is the constant in Corollary 6.5.

Proof. In our context, it holds that $\mathscr{F}(0) = -\mathfrak{F}_0\Big(\operatorname{tf}(-Q_\beta)\Big)$. Since $\mathfrak{F}_0 : \mathscr{S}_0(\mathbb{R}^3) \to \mathscr{S}_0(\mathbb{R}^3)$ is a smooth map on the space of trace-free symmetric (3×3)-matrices and $\mathfrak{F}_0(0) = 0$, we have that

$$\|\mathscr{F}(0)\|_{\mathfrak{B}} \leq C\|\operatorname{tf}(Q_\beta)\|_{\mathfrak{B}}.$$

Since the weight function in the damage zones (inside Region $\mathrm{IV}_\pm$) satisfies $|\rho^{(\alpha)}_{\delta,\nu}(\boldsymbol{x})| \ll e^{\frac{\delta_q \cdot \beta}{10}}$, the desired error estimate follows from the estimate in Corollary 6.5. □

Now we are ready to prove the existence of a hyperkähler triple on $\mathcal{M}$.

Theorem 10.5. *Consider $(\mathcal{M}, g_\beta)$ with sufficiently large gluing parameter $\beta \gg 1$. Denote by $\boldsymbol{\omega}_\beta^{\mathcal{M}}$ the definite triple constructed by Proposition 6.4. Let δ and ν be the constants in Proposition 9.2. Then there exists a hyperkähler triple $\boldsymbol{\omega}_\beta^{\mathrm{HK}}$ such that*

$$\|\boldsymbol{\omega}_\beta^{\mathcal{M}} - \boldsymbol{\omega}_\beta^{\mathrm{HK}}\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \leq Ce^{-\delta_0\beta}, \tag{10.11}$$

where $C > 0$ and $\delta_0 > 0$ are independent of β. In particular, $\mathcal{M}$ is diffeomorphic to the K3 *surface.*

Proof. It suffices to verify the conditions of Lemma 10.1. In fact, Proposition 10.2, Lemma 10.3 and Proposition 10.4 verify Property (1), Property (2a) and Property (2b) in Lemma 10.1, respectively. So applying the implicit function theorem given by Lemma 10.1, the existence of the hyperkähler triple $\boldsymbol{\omega}_\beta^{\mathrm{HK}}$ follows. The error estimate (10.11) follows directly from the implicit function theorem.

Since the hyperkähler triple $\boldsymbol{\omega}_\beta^{\mathrm{HK}}$ determines a hyperkähler metric on $\mathcal{M}$, and since $\chi(\mathcal{M}) = 24$ by Proposition 6.6, $\mathcal{M}$ must be diffeomorphic to the K3 surface. □

10.2. **Completion of main proofs.** In this subsection, we prove Theorems A and B. Let us begin with the following ϵ-regularity theorem for collapsed Einstein manifolds due to Naber and the fourth author of this paper; see [NZ16] for more details.

Theorem 10.6. *Let (M^n, g, p) satisfy $\mathrm{Ric}_g \equiv \lambda g$ and $|\lambda| \leq n-1$. Given a manifold (Z^k, z^k) with $k = \dim(Z^k) < n$, there are uniform constants $\delta_0 > 0$, $w_0 > 0$ and $C_0 > 0$ which depend only on n and the geometry of $B_1(z^k)$ such that the following property holds: if*

$$d_{GH}(B_2(p), B_2(z^k)) < \delta_0,$$

then the group $\Gamma_{\delta_0}(p) \equiv \mathrm{Image}[\pi_1(B_{\delta_0}(p)) \to \pi_1(B_2(p))]$ has a nilpotent subgroup $\mathcal{N}$ of index bounded by w_0 such that $\mathrm{rank}(\mathcal{N}) \leq n-k$.

Furthermore, if $\mathrm{rank}(\mathcal{N}) = n-k$, then $\sup_{B_1(p)} |\mathrm{Rm}| \leq C_0$. Conversely, if $\sup_{B_3(p)} |\mathrm{Rm}| \leq C_0$, then $\mathrm{rank}(\mathcal{N}) = n-k$.

Proof of Theorem A. First, we consider the simpler case that there is only one cluster of monopoles, i.e., $m = 1$. Without loss of generality, one can assume that all the monopoles in the neck region are located on the same torus fiber of $\mathbb{T}^2 \times \mathbb{R}$.

We start the proof by describing the hyperkähler metrics $\hat{h}_\beta$ and the continuous map $F_\beta : \mathrm{K3} \to [0,1]$. Given any sufficiently large parameter $\beta \gg 1$, denote by g_β the approximate metric which is almost Ricci-flat and determined by the approximate triple constructed in Section 6 with

$$C^{-1}\beta^{\frac{3}{2}} \leq \mathrm{Diam}_{g_\beta}(\mathcal{M}) \leq C\beta^{\frac{3}{2}}.$$

By Theorem 10.5, there exists a hyperkähler triple $\boldsymbol{\omega}_\beta^{\mathrm{HK}}$ which has the associated hyperkähler metric $\hat{g}_\beta$. The estimate (10.11), and the fact that the weight function is bounded below by a positive constant (independent of β), imply that

$$\|\boldsymbol{\omega}_\beta^{\mathcal{M}} - \boldsymbol{\omega}_\beta^{\mathrm{HK}}\|_{C^0(\mathcal{M})} \leq Ce^{-\delta_0\beta}.$$

The isomorphism (1.4) then implies that

$$\|\hat{g}_\beta - g_\beta\|_{C^0(\mathcal{M})} \leq Ce^{-\delta_0\beta}. \tag{10.12}$$

Next, using the definition of the weighted Hölder norms, the estimates (10.11) and (10.12), together with the isomorphism (1.4), we obtain the bound

$$\|\hat{g}_\beta - g_\beta\|_{C^{0,\alpha}_{\delta,\nu+1}(\mathcal{M})} \leq Ce^{-\delta_0\beta} \tag{10.13}$$

for some $C > 0$ and $\delta_0 > 0$ independent of β. Let $\hat{h}_\beta$ be the rescaling of the hyperkähler metric $\hat{g}_\beta$ with $\mathrm{Diam}_{\hat{h}_\beta}(\mathcal{M}) = 1$. Denote by h_β the rescaling of g_β with $\mathrm{Diam}_{h_\beta}(\mathcal{M}) = 1$. Then

$$\|\hat{h}_\beta - h_\beta\|_{C^0(\mathcal{M})} \leq Ce^{-\frac{\delta_0\beta}{2}}.$$

Now we are ready to define the map $F_\beta : \mathcal{M} \to [0,1]$. First, recalling the notation in Section 6, we extend the function z on the neck region to a function $\tilde{z}$ on $\mathcal{M}$ as follows:

$$\tilde{z}(\boldsymbol{x}) = \begin{cases} \zeta_0^- - 2T_-, & \boldsymbol{x} \in X^4_{b_-} \setminus \{z_- \geq \zeta_0^-\}, \\ z_-(\boldsymbol{x}) - 2T_-, & \boldsymbol{x} \in X^4_{b_-} \cap \{\zeta_0^- \leq z_- \leq T_-\}, \\ z(\boldsymbol{x}), & \boldsymbol{x} \in \mathcal{N}(T_-, T_+), \\ 2T_+ - z_+(\boldsymbol{x}), & \boldsymbol{x} \in X^4_{b_+} \cap \{\zeta_0^+ \leq z_+ \leq T_+\}, \\ 2T_+ - \zeta_0^+, & \boldsymbol{x} \in X^4_{b_+} \setminus \{z_+ \geq \zeta_0^+\}, \end{cases}$$

and then define

$$F_\beta(\boldsymbol{x}) = \frac{\tilde{z}(\boldsymbol{x}) - \zeta_0^- + 2T_-}{2(T_+ + T_-) - \zeta_0^- - \zeta_0^+}.$$

It then follows from the gluing construction that there is some point $t_1 \in (0,1)$ such that $F_\beta^{-1}(t_1)$ is a singular S^1-bundle over $\mathbb{T}^2$ with exactly $(b_- + b_+)$ vanishing circles. In fact, the vanishing circles occur at the monopoles of the neck region $\mathcal{N}^4_{m_0}$ constructed in Section 6.2, which itself is a Gibbons-Hawking space over $\mathbb{T}^2 \times \mathbb{R}$. Moreover, for each $t \in (0, t_1) \cup (t_1, 1)$, the fiber $F_\beta^{-1}(t)$ is diffeomorphic to a Heisenberg nilmanifold with

$$\deg(F_\beta^{-1}(t)) = \begin{cases} b_-, & t \in (0, t_1), \\ b_+, & t \in (t_1, 1). \end{cases}$$

By Section 6, there is some uniform constant $C_0 > 0$ such that for each regular fiber,

$$C_0^{-1}\beta^{-1} \leq \mathrm{Diam}_{\hat{h}_\beta}(F_\beta^{-1}(t)) \leq C_0\beta^{-1}, \quad C_0^{-1}\beta^{-2} \leq \mathrm{Diam}_{\hat{h}_\beta}(S^1) \leq C_0\beta^{-2}.$$

With these diameter estimates, we are ready to prove the uniform curvature estimates by applying Theorem 10.6. Fix any $\epsilon \in (0, 10^{-2})$ and choose $\beta > 0$ sufficiently large such that

$$\mathrm{Diam}_{\hat{h}_\beta}(F_\beta^{-1}(t)) < \frac{\delta_0 \cdot \epsilon}{10},$$

where $\delta_0 > 0$ is the dimensional constant from Theorem 10.6. Now, for a ball around each regular point $B_\epsilon(x) \subset F_\beta^{-1}([0,1] \setminus T_{2\epsilon}(\mathcal{S}))$ with $\mathcal{S} \equiv \{0, t_1, 1\}$,

$$\Gamma_{\delta_0\epsilon}(x) \equiv \mathrm{Image}[\pi_1(B_{\delta_0\epsilon}(x)) \to B_\epsilon(x)] \cong \pi_1(\mathrm{Nil}^3),$$

hence $\mathrm{rank}(\Gamma_{\delta_0\epsilon}(x)) = 3$. Then by Theorem 10.6, we know that $\sup_{B_{\epsilon/2}(x)} |\mathrm{Rm}_{\hat{h}_\beta}| \leq C_{0,\epsilon}$, where $C_{0,\epsilon} > 0$ depends only on ϵ and is independent of β. The higher order curvature estimates can be proved by considering a local universal cover and applying the standard regularity theory for non-collapsing Einstein metrics. This completes the proof of item (1) of Theorem A.

Now we proceed to prove item (2). We still apply Theorem 10.6 to determine the curvature blow-up behavior around the singular fiber. In fact, if $x \in T_{\epsilon/2}(F_\beta^{-1}(t_1))$, it suffices to show that

$$\sup_{B_{\epsilon/2}(x)} |\operatorname{Rm}_{\hat{h}_\beta}| \to \infty \quad \text{as } \beta \to \infty. \tag{10.14}$$

Notice that

$$\Gamma_{\epsilon/2}(x) \equiv \operatorname{Image}[\pi_1(B_{\epsilon/2}(x)) \to B_{1/10}(x)] \cong \mathbb{Z} \oplus \mathbb{Z},$$

hence $\operatorname{rank}(\Gamma_{\epsilon/2}(x)) = 2 < 3$. Theorem 10.6 therefore implies (10.14).

The next part is to prove the classification of the bubble limits of item (2) of the statement of the theorem. Fixing the gluing parameter $\beta \gg 1$, we analyze the curvature behavior of the approximate metric g_β in the gluing construction scaled such that

$$C^{-1}\beta^{\frac{3}{2}} \leq \operatorname{Diam}_{g_\beta}(\mathcal{M}, g) \leq C\beta^{\frac{3}{2}}.$$

There are two cases to analyze.

First, let the reference point $\boldsymbol{x}_\beta$ be a point of maximal curvature of a Tian-Yau piece. It follows directly from the construction that, as $\beta \to +\infty$, the curvature $|\operatorname{Rm}_{g_\beta}|(\boldsymbol{x}_\beta)$ is uniformly bounded but does not go to 0. So $(\mathcal{M}, g_\beta, \boldsymbol{x}_\beta)$ converges to a complete hyperkähler Tian-Yau space $(X^4, g_{TY}, \boldsymbol{x}_\infty)$ in the pointed C^k-topology for any $k \in \mathbb{Z}_+$. We will show that $(\mathcal{M}, \hat{g}_\beta, \boldsymbol{x}_\beta)$ also converges to the same Tian-Yau space $(X^4, g_{TY}, \boldsymbol{x}_\infty)$ in the pointed C^k-topology for any $k \in \mathbb{Z}_+$. In fact, the estimate (10.13) implies that $(\mathcal{M}, \hat{g}_\beta, x_\beta)$ converges to the same Tian-Yau space $(X^4, g_{TY}, \boldsymbol{x}_\infty)$ in the pointed $C^{0,\alpha}$-topology. The stronger convergence follows from the regularity result for non-collapsed Einstein metrics in [AC92]. Since the rescaling factor $\beta^{\frac{3}{2}}$ is much smaller than exponential, the bubble limit of $(\mathcal{M}, \hat{h}_\beta)$ around $\boldsymbol{x}_\beta$ is a complete hyperkähler Tian-Yau space.

Next, we consider the case in which the reference point $\boldsymbol{x}_\beta$ is very close to one of monopoles, i.e., $\boldsymbol{x}_\beta \in B_{\beta^{-\frac{1}{2}}}(p_m)$ in terms of the metric $\hat{h}_\beta$, where $p_m \in \mathcal{P}_{b_-+b_+} \equiv \{p_1, \ldots, p_{b_-+b_+}\}$. Applying Lemma 7.4, we have that

$$(\mathcal{M}, \beta \cdot g_\beta, \boldsymbol{x}_\beta) \longrightarrow (\mathbb{R}^4, g_{TN}, \boldsymbol{x}_\infty),$$

where g_{TN} is the Taub-NUT metric and the convergence is with respect to the pointed C^k-topology for any $k \in \mathbb{Z}_+$. Applying the error estimate (10.13) and the same arguments as above, we see that $(\mathcal{M}, \beta \cdot h_\beta, \boldsymbol{x}_\beta)$ converges to $(\mathbb{R}^4, g_{TN}, \boldsymbol{x}_\infty)$ in the pointed C^k-topology for any $k \in \mathbb{Z}_+$. This implies that in terms of the hyperkähler metric $\hat{h}_\beta$, we have the pointed C^k-convergence for any $k \in \mathbb{Z}_+$,

$$(\mathcal{M}, \beta^4 \cdot \hat{h}_\beta, \boldsymbol{x}_\beta) \longrightarrow (\mathbb{R}^4, g_{TN}, \boldsymbol{x}_\infty).$$

So the proof of (2) is done.

The above completes the proof in the case of 1 singular point of convergence in the interior of the interval. We are now in a position to generalize the gluing construction in Section 6 to produce multiple singular points of convergence in the interior of the interval.

First, we fix two hyperkähler Tian-Yau spaces $(X^4_{b_-}, g_{b_-}, p_-)$ and $(X^4_{b_+}, g_{b_+}, p_+)$ with $b_-, b_+ \in \{1, \ldots, 9\}$. Let $\{w_j\}_{j=1}^m$ be positive integers satisfying

$$w_1 + \cdots + w_m = b_- + b_+.$$

For each $1 \le j \le m$, we choose the neck region $\mathcal{N}^4_{w_j}$ to be a Gibbons-Hawking space over a finite flat cylinder $(\mathbb{T}^2 \times [-T_j, T_{j+1}], g_0)$ with w_j monopoles. As in the construction of Section 7, each pair of monopoles in $\mathcal{N}^4_{w_j}$ has definite and bounded distance. Now let $G_j : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ be a global sign-changing Green's function which satisfies

$$-\Delta_{g_0} G_j = 2\pi \sum_{s=1}^{w_j} \delta_{p_s}.$$

Then there are constants $k_j^- > 0$ and $k_j^+ < 0$ such that

$$\begin{aligned} &|\nabla^k_{g_0}(G_j - k_j^- z)| \le C_k e^{\lambda_1 z}, \quad z < -100 \cdot \beta, \\ &|\nabla^k_{g_0}(G_j - k_j^+ z)| \le C_k e^{-\lambda_1 z}, \quad z > 100 \cdot \beta, \\ &k_j^- = -k_j^+ = \pi w_j. \end{aligned}$$

Note that the first step of the gluing is to modify the above Green's function by adding a linear function: let $V_j \equiv G_j + (\ell_j z + \beta)$ be such that two adjacent neck regions have compatible slopes, that is,

$$k^-_{j+1} + \ell_{j+1} = k^+_j + \ell_j \quad \text{and} \quad k^-_1 + \ell_1 = 2\pi b_-.$$

Immediately, we have that $k^+_1 + \ell_1 = 2\pi(b_- - w_1)$. Eventually, one can check that at the right end of the last neck region $\mathcal{N}^4_{w_m}$,

$$k^+_m + \ell_m = 2\pi\Big(b_- - \sum_{j=1}^m w_j\Big) = -2\pi b_+.$$

Applying the construction in Section 6, we obtain a manifold

$$\mathcal{M} = \tag{10.15}$$
$$X^4_{b_-}(T_1) \bigcup_{\Psi_1} \mathcal{N}^4_{w_1}(-T_1 - 1, T_2) \bigcup_{\Psi_2} \ldots \bigcup_{\Psi_m} \mathcal{N}^4_{w_m}(-T_m - 1, T_{m+1}) \bigcup_{\Psi_{m+1}} X^4_{b_+}(T_{m+1} + 1),$$

where the attaching maps $\Psi_1, \ldots \Psi_m$ are chosen analogously to Ψ_-, and Ψ_{m+1} is chosen analogously to Ψ_+. Furthermore, there is an approximate hyperkähler triple $\boldsymbol{\omega}^{\mathcal{M}}$ on $\mathcal{M}$ which is hyperkähler away from the damage zones and satisfies the conclusions of Proposition 6.4. The weight function on $\mathcal{M}$ is defined in an analogous way to (8.1), and the arguments in the previous sections are easily modified to prove the existence of a hyperkähler metric $\hat{g}_\beta$ close to g_β.

To satisfy the matching conditions analogous to those in Section 6.3, the parameters T_j are then all chosen proportional to β so that the diameter of the neck region $\mathcal{N}^4_{w_j}(-T_j - 1, T_{j+1})$ in the metric $\hat{g}_\beta$ is proportional to $\beta^{3/2}$. Thus, for the sequence of unit diameter hyperkähler metrics $\hat{h}_\beta$, these neck regions limit to nontrivial intervals, therefore there are exactly m distinct singular points of convergence $t_j \in (0,1), j = 1 \ldots m$, in the interior of the interval. The analysis of the regular collapsing regions and the bubbling regions is the same as above. □

Proof of Theorem B. This is a consequence of the above construction. Indeed, let

$$\mathcal{N}^4_{w_1}(-T_1-1,T_2),\dots,\mathcal{N}^4_{w_m}(-T_m-1,T_{m+1})$$

be the neck regions in (10.15) such that for each $1\le j\le m$, the neck region $\mathcal{N}^4_{w_j}(-T_j-1,T_{j+1})$ has exactly w_j monopoles with the same z-coordinate. Notice that the degree of the nilmanifold fiber is determined by the ending slope of the Green's function. Corollary 2.6 implies that the degree of the nilpotent fibers will jump by w_j when crossing a singular fiber in $\mathcal{N}^4_{w_j}(-T_j-1,T_{j+1})$. One can also see that there are w_j Taub-NUT bubbles at each singular point $t_j\in(0,1), j=1\dots m$. □

Remark 10.7. If we take each collection of w_j monopole points in $\mathcal{N}^4_{w_j}(-T_j-1,T_{j+1})$ to have distance exactly proportional to β^{-1} from each other (in the flat metric on $\mathbb{T}^2\times\mathbb{R}$), then the corresponding bubble limit will be a multi-Taub-NUT ALF-A_{w_j-1} metric instead of w_j distinct Taub-NUT bubbles. It is also possible to obtain nontrivial bubble trees. For example, if the pairwise distance of the monopole points is proportional to β^{-2}, then there will be a first bubble an ALF orbifold with a cyclic orbifold point of order w_j, and a deepest bubble an ALE-A_{w_j-1} metric.

Appendix A. Uniform estimates for Hermite functions

In this appendix, we give the proofs of Lemma 4.5, Lemma 4.6, and Lemma 4.7 in Section 4.2.

Proof of Lemma 4.5. Since $\frac{d}{dz}\mathcal{W}(\mathcal{F}(z),\mathcal{U}(z))=\mathcal{F}''(z)\mathcal{U}(z)-\mathcal{F}(z)\mathcal{U}''(z)=0$, we obtain

$$\mathcal{W}(\mathcal{F}(z),\mathcal{U}(z))=\mathcal{W}(\mathcal{F}(0),\mathcal{U}(0))=2\mathcal{F}'(0)\mathcal{U}(0).$$

Moreover, direct computation gives that

$$\mathcal{F}'(0)=2\sqrt{j}\cdot\int_0^\infty e^{-t^2}t^{h+1}dt=\sqrt{j}\cdot\Gamma\Big(\frac{h}{2}+1\Big)$$

and $\mathcal{U}(0)=\frac{1}{2}\Gamma\Big(\frac{h}{2}+\frac{1}{2}\Big)$. Applying the Legendre duplication formula (see (1.2.3) in [Leb72]),

$$\frac{\Gamma(t)\Gamma(t+\frac{1}{2})}{\Gamma(2t)}=\frac{\sqrt{\pi}}{2^{2t-1}},\ t>0,$$

we have that $\mathcal{W}(\mathcal{F}(z),\mathcal{U}(z))=2^{-h}\cdot\sqrt{j\pi}\cdot\Gamma(h+1)>0$. □

Proof of Lemma 4.6. First, we prove the uniform estimates in item (1) of Lemma 4.6. By the definition of $\mathcal{F}$ and $\mathcal{U}$, it suffices to prove that

$$C_0\cdot e^{F(t_0)}\le\int_0^\infty e^{F(t)}dt\le(1+\sqrt{\pi})e^{F(t_0)},\tag{A.1}$$

$$C_0\cdot e^{U(s_0)}\le\int_0^\infty e^{U(t)}dt\le(1+\sqrt{\pi})e^{U(s_0)},\tag{A.2}$$

where $C_0>0$ is a uniform constant independent of h and j. We only prove the first inequality. The second can be proved in exactly the same way.

To prove the integral estimates in (A.1), let us write

$$\int_0^\infty e^{F(t)}dt=\int_0^{2t_0}e^{F(t)}dt+\int_{2t_0}^\infty e^{F(t)}dt.\tag{A.3}$$

In our computations, we will make a change of variable $t = t_0 \cdot (1+\xi)$ such that $\xi \in (-1,1)$. Applying Taylor's theorem to $F(t) - F(t_0)$, we have that

$$F(t) - F(t_0) = F(t_0(1+\xi)) - F(t_0) = \frac{F''(\theta)}{2} \cdot t_0^2 \cdot \xi^2, \tag{A.4}$$

where θ is some number between t and t_0. Notice that

$$F''(t) = -2 - \frac{h}{t^2}, \quad h \geq 0. \tag{A.5}$$

Since θ is between t_0 and $t \in [0, 2t_0]$, (A.5) implies that $F''(\theta) \leq F''(2t_0) < 0$. Combining this with (A.4) then yields the following upper bound for the first term in (A.3):

$$\int_0^{2t_0} e^{F(t)} dt = e^{F(t_0)} \int_0^{2t_0} e^{F(t)-F(t_0)} dt \leq e^{F(t_0)} \cdot t_0 \cdot \int_{-1}^{1} e^{\frac{F''(2t_0)}{2} \cdot t_0^2 \cdot \xi^2} d\xi.$$

Immediately $F''(2t_0) = -\Big(2 + \frac{h}{4t_0^2}\Big)$, so we have that

$$\int_0^{2t_0} e^{F(t)} dt \leq e^{F(t_0)} \cdot t_0 \cdot \int_{-1}^{1} e^{-(t_0^2 + \frac{h}{8}) \cdot \xi^2} d\xi \leq \frac{\sqrt{\pi} \cdot t_0}{\sqrt{t_0^2 + \frac{h}{8}}} \cdot e^{F(t_0)} \leq \sqrt{\pi} \cdot e^{F(t_0)}.$$

Next, we estimate the second term in (A.3). The property $F''(t) < 0$ implies that $F'(t)$ is monotone decreasing, so we have that $F'(t) \leq F'(2t_0)$ for any $t \geq 2t_0$. Applying Taylor's theorem again, we derive that $F(t) \leq F(2t_0) + F'(2t_0) \cdot (t - 2t_0)$, which in turn implies that

$$\int_{2t_0}^{\infty} e^{F(t)} dt \leq e^{F(2t_0)} \int_{2t_0}^{\infty} e^{F'(2t_0) \cdot (t-2t_0)} dt = \frac{e^{F(2t_0)}}{-F'(2t_0)}.$$

Since $F'(t) < 0$ for all $t > t_0$, we have that $F(2t_0) \leq F(t_0)$. Also notice that $F'(2t_0) = -\frac{4t_0^2 + h}{2t_0} < 0$ with $0 < t_0 < +\infty$. Hence,

$$\int_{2t_0}^{\infty} e^{F(t)} dt \leq \frac{2t_0}{4t_0^2 + h} \cdot e^{F(t_0)} \leq e^{F(t_0)}.$$

Combining the above, we deduce that

$$\int_0^{\infty} e^{F(t)} dt \leq (1 + \sqrt{\pi}) e^{F(t_0)}.$$

The lower bound estimate in (A.1) also follows from Laplace's method. Indeed,

$$\int_0^{\infty} e^{F(t)} dt \geq \int_{t_0}^{2t_0} e^{F(t)} dt \geq e^{F(t_0)} \int_{t_0(y)}^{2t_0(y)} e^{\frac{F''(t_0)}{2}(t-t_0)^2} dt \geq C_0 \cdot e^{F(t_0)}.$$

So the proof of (A.1) is done. The proof of (A.2) is identical.

Now we prove the asymptotics in item (2) of Lemma 4.6. For simplicity, we will calculate the asymptotic behavior in y. For fixed h and j, as $y \to \infty$, it is straightforward to verify that

$$t_0 = y + \frac{h^2}{2y} + O(y^{-3}) \quad \text{and} \quad s_0 = \frac{h^2}{2y} + O(y^{-3}). \tag{A.6}$$

These asymptotics imply that

$$F(t_0) = y^2 + h \log y + O(y^{-2}) \quad \text{and} \quad U(s_0) = -h + h \log h - h \log(2y) + O(y^{-2}). \tag{A.7}$$

First, we prove the asymptotics for $\mathcal{F}$. As in the proof of item (1), we have that

$$\begin{aligned}\int_0^\infty e^{F(t)}dt &= e^{F(t_0)}\int_0^{2t_0} e^{F(t)-F(t_0)}dt + \int_{2t_0}^\infty e^{F(t)}dt \\ &= t_0 e^{F(t_0)}\int_{-1}^1 e^{-t_0^2\epsilon^2+h(\log(1+\epsilon)-\epsilon)}d\epsilon + \int_{2t_0}^\infty e^{F(t)}dt.\end{aligned}$$

Notice that

$$\lim_{t_0\to\infty} t_0\int_{-1}^1 e^{-t_0^2\epsilon^2+h(\log(1+\epsilon)-\epsilon)}d\epsilon = \sqrt{\pi} \quad \text{and} \quad \lim_{t_0\to\infty} e^{-F(t_0)}\int_{2t_0}^\infty e^{F(t)}dt = 0.$$

Moreover, by (A.6), $\lim_{y\to+\infty} t_0(y)\to\infty$. It follows that

$$\lim_{z\to\infty}\frac{\mathcal{F}(z)}{\sqrt{\pi}e^{-\frac{jz^2}{2}+F(t_0(z))}} = 1.$$

Combining the above limit and (A.7), the proof of (4.15) is complete.

In the case $j\in\mathbb{Z}_+$ and $h>0$, we will prove the asymptotic behavior of $\mathcal{U}$. We write

$$\begin{aligned}\int_0^\infty e^{U(t)}dt &= s_0 e^{U(s_0)}\int_{-1}^\infty e^{-s_0^2\epsilon^2+h(\log(1+\epsilon)-\epsilon)}d\epsilon \\ &= s_0 e^{U(s_0)}\int_{-1}^\infty e^{-s_0^2\epsilon^2}\cdot(1+\epsilon)^h e^{-h\epsilon}d\epsilon.\end{aligned}$$

We claim that

$$\lim_{s_0\to 0}\int_{-1}^\infty (e^{-s_0^2\epsilon^2}-1)\cdot(1+\epsilon)^h e^{-h\epsilon}d\epsilon = 0.$$

In fact, it is straightforward to see that for any $s_0>0$, we have that $-1\le e^{-s_0^2\epsilon^2}-1\le 0$. Also for any fixed $h>0$,

$$\int_{-1}^\infty (1+\epsilon)^h e^{-h\epsilon}d\epsilon < \infty.$$

Applying the dominated convergence theorem yields

$$\lim_{s_0\to 0}\int_{-1}^\infty (e^{-s_0^2\epsilon^2}-1)\cdot(1+\epsilon)^h e^{-h\epsilon}d\epsilon = 0.$$

This completes the proof of the claim.

Next, by the definition of the gamma function,

$$\int_{-1}^\infty (1+\epsilon)^h e^{-h\epsilon}d\epsilon = e^h\int_0^\infty e^{-hs}s^h ds = e^h h^{-h-1}\Gamma(h+1).$$

Therefore,

$$\lim_{s_0\to 0}\int_{-1}^\infty e^{-s_0^2\epsilon^2+h(\log(1+\epsilon)-\epsilon)}d\epsilon = e^h h^{-h-1}\Gamma(h+1).$$

Since $\lim_{y\to+\infty} s_0 = 0$ and U satisfies (A.7), we eventually obtain (4.16). □

Proof of Lemma 4.7. First, we prove item (1). Let us begin by discussing the case $h=0$ and $j\in\mathbb{Z}_+$. Direct computations give that $t_0=y$ and $s_0=0$. So, by definition, we have that $\widehat{F}(z)=\frac{jz^2}{2}$ and $\widehat{U}(z)=-\frac{jz^2}{2}$. (4.17) now follows.

Next, we prove the case $j \in \mathbb{Z}_+$ and $h > 0$. Let us denote $a \equiv 2y$. By elementary calculations,

$$t_0 - s_0 = \frac{a}{2}, \quad t_0^2 + s_0^2 = \frac{a^2}{4} + h, \quad t_0 s_0 = \frac{h}{2}.$$

So we have that

$$\begin{aligned} F(t_0) + U(s_0) &= -(t_0^2 + s_0^2) + a(t_0 - s_0) + h\log(t_0 s_0) \\ &= \frac{a^2}{4} - h + h\log\left(\frac{h}{2}\right) = jz^2 - h + h\log\left(\frac{h}{2}\right). \end{aligned}$$

Therefore $e^{\widehat{F}(z)+\widehat{U}(z)} \leq e^{-h+h\log(\frac{h}{2})}$. Applying Lemma 4.5 then leads to

$$\frac{e^{\widehat{F}(z)+\widehat{U}(z)}}{\mathcal{W}(z)} \leq \frac{e^{-h+h\log(\frac{h}{2})}}{\sqrt{j\pi}2^{-h}\Gamma(h+1)} \leq C_0.$$

Now we prove the monotonicity formulas in item (2). Let $y = \sqrt{j}z$ and $a = 2y$. Then, by definition,

$$\begin{aligned} \widehat{F} &= -\frac{a^2}{8} - (t_0(a))^2 + at_0(a) + h\log(t_0(a)), \\ \widehat{U} &= -\frac{a^2}{8} - (s_0(a))^2 - as_0(a) + h\log(s_0(a)). \end{aligned}$$

We show that $\widehat{F}$ is increasing in a and $\widehat{U}$ is decreasing in a. To see this, just observe that

$$\frac{d\widehat{F}}{da} = -\frac{a}{4} + t_0(a) + \Big(-2t_0(a) + a + \frac{h}{t_0(a)}\Big)t_0'(a) = \sqrt{\frac{h}{2} + \frac{a^2}{16}} \geq \frac{a}{4},$$

so that $\widehat{F}(z) - \eta z$ is increasing for $z > 2\eta$. Similarly, the monotonicity of $\widehat{U}(z) + \eta z$ follows from the bound

$$\frac{d\widehat{U}}{da} = -\frac{a}{4} - s_0(a) + \Big(-2s_0(a) - a + \frac{h}{s_0(a)}\Big)s_0'(a) = -\sqrt{\frac{h}{2} + \frac{a^2}{16}} \leq -\frac{a}{4}.$$

□

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