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Paper

A global Lyapunov function for the coherent Ising machine

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Abstract: Two classical Ising machine schemes, the Oscillator Ising Machine (OIM) and the Bistable Latch Ising Machine (BLIM), have been shown to feature global Lyapunov functions, *i.e.*, continuous "energy-like" functions whose local minima are naturally found by the physics of these schemes. We show that the Coherent Ising Machine (CIM), an optical scheme that predated OIM and BLIM, also has a global Lyapunov function that approximates the Ising Hamiltonian at stable equilibrium points. Our result sharpens understanding of CIM operation, revealing that its mechanism for breaking out of local minima is a purely probabilistic classical one, similar to Gibbs sampling.

Key Words: global Lyapunov function, coherent Ising machine, oscillator Ising machine, bistable latch Ising machine, Gibbs sampling.

1. Introduction

Hardware Ising machines have emerged as a promising means to solve classically difficult (*e.g.*, NP-complete) computational problems. The premise of Ising machines is that specialized hardware implementing the Ising computational model [1] can solve difficult combinatorial problems more effectively than classical algorithms (such as semidefinite programming and simulated annealing) run on digital computers. Ising machines first came into prominence with D-Wave's adiabatic quantum annealer [2, 3] and the Coherent Ising Machine (CIM) [4–6]. While both have been portrayed as "genuinely quantum" Ising machines, only D-Wave's machine has so far been shown to feature quantum advantage [7, 8].

Following D-Wave and CIM, two other types of Ising Machines based on *purely classical* mechanisms were developed in the author's group, namely Oscillator Ising Machines (OIMs [9–11]) and Bistable Latch Ising Machines (BLIMs [12]). Both these schemes work on nonlinear dynamical principles. In particular, they have been shown to feature *global Lyapunov functions*, the presence of which helps explain their operation as Ising machines. Specifically, in both these schemes, the Lyapunov function has been shown to closely approximate the Ising Hamiltonian (which defines the problem being solved) at stable equilibrium points, if operating parameters are set appropriately. The very fact that a global Lyapunov function exists automatically implies its local minimization by natural dynamics; this central property of these Ising machines constitutes a rigorous underpinning for explaining their ability to find near-global minima.

In this work, we show that the **Coherent Ising Machine also has a global Lyapunov function** and present an analytical expression for it. We also show that, like for OIM and BLIM, the Lyapunov function has a finite lower bound, and that it approximates the Ising Hamiltonian at stable equilibrium points. Based on these results, we argue that CIM's global minimization mechanism is a purely probabilistic one, similar to Gibbs

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sampling used in simulated annealing. This supports arguments that in spite of low-level quantum principles that may be involved in the optical underpinnings of CIM, its operation as an Ising machine is governed by purely classical mechanisms. Our results put CIM in the same class of Ising machines that OIM and BLIM belong to, in spite of very different underlying physics and engineering implementations. We propose the term **Lyapunov Ising Machine (LIM)** for such Ising schemes with global Lyapunov functions. This common basis will, we hope, lead to a more holistic understanding of Ising machine schemes and their similarities and differences.

2. A global Lyapunov function that approximates the Ising Hamiltonian

2.1 The c-number equations for CIM

We start with the c-number equations for CIM [5, Equations 22–25], referring the reader to [5] for a discussion of the steps involved in obtaining these stochastic differential equations:

$$\begin{aligned} dc_i &= \left[(-1+p-c_i^2-s_i^2)c_i+\sum_j J_{ij}\,\tilde{c}_j\right]dt+\frac{1}{A_s}\sqrt{c_i^2+s_i^2+\frac{1}{2}}\,dW_{1i},\\ ds_i &= (-1-p-c_i^2-s_i^2)s_i\,dt+\frac{1}{A_s}\sqrt{c_i^2+s_i^2+\frac{1}{2}}\,dW_{2i}. \end{aligned} \tag{1}$$

In the above, i, j range from 1 to n, the total number of spins. A_s and $p = 2$ are parameters, while W_{1i} and W_{2i} represent independent white Gaussian noise processes. c_i and s_i represent the in-phase and quadrature components of each DOPO[1] pulse (*i.e.*, spin). The term $\sum_j J_{ij}\tilde{c}_j$ in the first equation represents coupling between spins, where J_{ij} is the coupling weight between spins i and j.[2] $\tilde{c}_i$ is a constant-shifted version of c_i, given by

$$\tilde{c}_i \triangleq c_i-\sqrt{\frac{1-T}{T}}\,\frac{f_i}{A_s}, \tag{2}$$

where T and f_i are constants.

To devise our (deterministic) Lyapunov result, we set the noise terms to zero, resulting in the following deterministic version of the c-number equations:

$$\begin{aligned} \frac{dc_i}{dt} &= \Delta_i+(-1+p-c_i^2-s_i^2)c_i+\sum_j J_{ij}\,c_j,\\ \frac{ds_i}{dt} &= (-1-p-c_i^2-s_i^2)s_i. \end{aligned} \tag{3}$$

Δ_i in (3) is given by

$$\Delta_i \triangleq -\sum_j J_{ij}\sqrt{\frac{1-T}{T}}\,\frac{f_i}{A_s}. \tag{4}$$

For convenience in derivations below, we rewrite (3) as

$$\begin{aligned} \frac{dc_i}{dt} &= \Delta_i+(-1+p+\gamma_i-c_i^2-s_i^2)c_i-\sum_j J_{ij}(c_i-c_j),\\ \frac{ds_i}{dt} &= (-1-p-c_i^2-s_i^2)s_i, \end{aligned} \tag{5}$$

where

$$\gamma_i \triangleq \sum_j J_{ij}. \tag{6}$$

2.2 A global Lyapunov function for the deterministic c-number equations

Given a set of deterministic differential equations like (5), a global Lyapunov function is a scalar function of the unknowns that is provably non-increasing for every solution of the differential equations. We now show that the scalar function

[1] Degenerate Optical Parametric Oscillator.

[2] Note that $J_{ij} = J_{ji}$, and $J_{ii} = 0$, in all Ising models. [5] uses the notation $\tilde{\zeta}_{ij}$ for the coupling coefficients.

$$L(c_1,\cdots,c_n,s_1,\cdots,s_n) \triangleq -\sum_{i=1}^{n}\left[\Delta_i c_i + \frac{-1+p+\gamma_i}{2}c_i^2 + \frac{-1-p}{2}s_i^2 - \frac{c_i^4+s_i^4}{4} - \frac{c_i^2 s_i^2}{2} - \frac{1}{4}\sum_{j=1}^{n} J_{ij}(c_i-c_j)^2\right] \quad (7)$$

is a global Lyapunov function for (5).

To prove that $\frac{dL}{dt} \leq 0$, we start with

$$\frac{dL}{dt} = \sum_{k=1}^{n}\left[\frac{\partial L}{\partial c_k}\frac{dc_k}{dt} + \frac{\partial L}{\partial s_k}\frac{ds_k}{dt}\right]. \quad (8)$$

Differentiating (7) with respect to c_k results in

$$\frac{\partial L}{\partial c_k} = -\left[\Delta_k + (-1+p+\gamma_k)c_k - c_k^3 - c_k s_k^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}(c_i-c_j)\left(\delta_{ik}-\delta_{jk}\right)\right], \quad (9)$$

where δ_{ik} is the Kronecker delta function. The summation in the last term of (9) can be simplified as follows:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}(c_i-c_j)\left(\delta_{ik}-\delta_{jk}\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}(c_i-c_j)\delta_{ik} - \sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}(c_i-c_j)\delta_{jk} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} J_{ij}(c_i-c_j)\delta_{ik} - \sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}(c_i-c_j)\delta_{jk} \\
&= \sum_{j=1}^{n} J_{kj}(c_k-c_j) - \sum_{i=1}^{n} J_{ik}(c_i-c_k) = \sum_{j=1}^{n} J_{kj}(c_k-c_j) + \sum_{i=1}^{n} J_{ki}(c_k-c_i) \\
&= \sum_{j=1}^{n} J_{kj}(c_k-c_j) + \sum_{j=1}^{n} J_{kj}(c_k-c_j) = 2\sum_{j=1}^{n} J_{kj}(c_k-c_j).
\end{aligned} \quad (10)$$

Using (10) in (9) yields

$$\begin{aligned}
\frac{\partial L}{\partial c_k} &= -\left[\Delta_k + (-1+p+\gamma_k)c_k - c_k^3 - c_k s_k^2 - \sum_{j=1}^{n} J_{kj}(c_k-c_j)\right] \\
&= -\left[\Delta_k + \left(-1+p+\gamma_k - c_k^2 - s_k^2\right)c_k - \sum_{j=1}^{n} J_{kj}(c_k-c_j)\right] \\
&= -\frac{dc_k}{dt} \quad \text{(using (5))}.
\end{aligned} \quad (11)$$

Similarly, differentiating (7) with respect to s_k results in

$$\frac{\partial L}{\partial s_k} = -\left[(-1-p)s_k - s_k^3 - c_k^2 s_k\right] = -\left[(-1-p) - s_k^2 - c_k^2\right]s_k = -\frac{ds_k}{dt}. \quad (12)$$

Putting (12) and (11) in (8) results in

$$\frac{dL}{dt} = -\sum_{k=1}^{n}\left[\left(\frac{dc_k}{dt}\right)^2 + \left(\frac{ds_k}{dt}\right)^2\right] \leq 0, \quad (13)$$

proving that $L(\cdots)$ in (7) is a global Lyapunov function for (5).

2.3 The Lyapunov function (7) is bounded from below

Observe that the expression for $L(\cdot)$ in (7) is a multivariate polynomial, with the highest-degree terms (*i.e.*, $\frac{c_i^4+s_i^4}{4} + \frac{c_i^2 s_i^2}{2}$) being quartic and positive. Since these terms dominate all lower-degree terms as c_i and s_i grow large, $L(\cdot)$ has a finite lower bound. Together with (13), this property implies that the c-number equations (5) will settle to equilibrium points that are finite local minima of the Lyapunov function.

2.4 Equilibrium points and stability of uncoupled DOPO equations

To establish a connection between the Lyapunov function and Ising problems, it is useful to first examine each uncoupled DOPO for equilibrium points and stability. Uncoupled DOPOs are described by (5) with J_{ij} set to 0, *i.e.*,

$$\begin{aligned} \frac{dc_i}{dt} &= (-1+p-c_i^2-s_i^2)c_i, \\ \frac{ds_i}{dt} &= (-1-p-c_i^2-s_i^2)s_i. \end{aligned} \tag{14}$$

Equations for the equilibrium points, obtained by making c_i and s_i constant *w.r.t.* time, are given by

$$\begin{aligned} (-1+p-c_i^2-s_i^2)c_i &= 0, \\ (-1-p-c_i^2-s_i^2)s_i &= 0. \end{aligned} \tag{15}$$

Only three real-valued solutions of the above equilibrium equations are possible; they are

$$c_i = 0,\, s_i = 0; \tag{16}$$

$$c_i = +1,\, s_i = 0; \text{ and} \tag{17}$$

$$c_i = -1,\, s_i = 0. \tag{18}$$

Stability can be assessed by linearizing (14) around each equilibrium point; the linearized equations are

$$\frac{d}{dt}\begin{bmatrix}\delta c_i \\ \delta s_i\end{bmatrix} = \underbrace{\begin{bmatrix} -1+p-3c_i^2-s_i^2 & -2c_is_i \\ -2c_is_i & -1-p-c_i^2-3s_i^2 \end{bmatrix}}_{A} \begin{bmatrix}\delta c_i \\ \delta s_i\end{bmatrix}, \tag{19}$$

where $[c_i, s_i]^T$ is an equilibrium point and $[\delta c_i, \delta s_i]^T$ is a small deviation from it. The eigenvalues of the matrix A determine the stability of the equilibrium point. For the three equilibria in (16) to (18), this matrix has the values[3]

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}, \text{ and } \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}, \tag{20}$$

respectively. The first matrix has a positive eigenvalue, while the second and third have only negative eigenvalues; thus, the equilibria $s_i = 0, c_i = \pm 1$ ((17) and (18)) are stable, while (16), *i.e.*, $c_i = s_i = 0$, is unstable. In other words, each DOPO is bistable in the in-phase component c_i.

2.5 Equilibrium points only perturbed under weak coupling

That each of the two bistable equilibria (*i.e.*, (17) and (18)) is in fact stable immediately implies that if the coupling terms J_{ij} in (5) are nonzero but sufficiently small (thus constituting only a small perturbation to (14)), then a) the equilibria are perturbed only slightly (from (17) and (18)), and b) the perturbed equilibria retain the stability properties of the unperturbed ones. In other words, the equilibria and stability properties of the uncoupled DOPOs do not alter significantly in the presence of small amounts of coupling.

2.6 The Lyapunov function approximates the Ising Hamiltonian at stable equilibria

Recall that the (discrete) Ising Hamiltonian is given by [11]

$$H = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} J_{ij}\, s_i s_j, \tag{21}$$

where $\{s_i\}$ are binary spins taking values in ± 1.[4]

At bistable equilibrium points (*i.e.*, small perturbations from (17) and (18)), the Lyapunov function (7) simplifies to

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[3] recalling that $p = 2$.

[4] So-called "external magnetic field" terms of the form $\sum_i B_i s_i$ can easily be incorporated in (21) [13, Equations 2 and 1].

$$
\begin{aligned}
L(c_1,\cdots,c_n,s_1,\cdots,s_n) &\simeq -\sum_{i=1}^{n}\left[\Delta_i c_i + \frac{-1+p+\gamma_i}{2} + -\frac{1}{4} - \frac{1}{4}\sum_{j=1}^{n} J_{ij}(c_i-c_j)^2\right] \\
&= -\sum_{i=1}^{n}\left[\Delta_i c_i + \frac{-1+p+\gamma_i}{2} + -\frac{1}{4} - \frac{1}{4}\sum_{j=1}^{n} J_{ij}(1+1-2c_ic_j)\right] \\
&= -\sum_{i=1}^{n}\left[\Delta_i c_i + \frac{-1+p}{2} + -\frac{1}{4} + \frac{1}{2}\sum_{j=1}^{n} J_{ij}\,c_ic_j\right].
\end{aligned} \tag{22}
$$

Observe that if $c_i \simeq 1$ is identified with spin value $s_i = +1$, and $c_i \simeq -1$ with $s_i = -1$, the last term of (22) approximates the Ising Hamiltonian (21). If $\Delta_i = 0$, the Lyapunov function (22) approximates a constant-shifted version of the Ising Hamiltonian, with the constant shift being $-\frac{1}{4}$. Constant shifts to the Hamiltonian do not alter the structure of Ising problems at all—*e.g.*, the locations and relative values of not only ground states, but also local minima and other features, remain the same. Given that the Lyapunov function is naturally minimized (locally), this close correspondence between the Lyapunov function and the Hamiltonian is a step towards understanding CIM's overall Hamiltonian reduction properties based on the c-number equations alone.

When $\Delta_i \neq 0$ (see (4)), the linear term $-\sum_i \Delta_i c_i$ in (22) modifies this Lyapunov-Hamiltonian correspondence. Mathematically, this term is analogous to the one arising from frequency variations in OIM's oscillators [14, Section 3.4, Equation 19]. If the values of Δ_i are small, key features of the Lyapunov function, such as the locations of local minima, do not change much and the Lyapunov-Hamiltonian correspondence is largely retained. Large values of Δ_i will upset this correspondence, likely affecting CIM's ability to find good Ising solutions.

3. Significance of the Result

The stability of the bistable solutions (17) and (18) perturbed by weak coupling (as established in Sec. 2.5) immediately implies that every one of the 2^n possible states in a weakly coupled CIM is a stable equilibrium point, *i.e.*, a local minimum of the Lyapunov function (7). The only mechanism for breaking out of a local minimum is disturbances from the uncorrelated noise terms in (1), *i.e.*, W_{i1} and W_{i2}. Through their random influence, the system can probabilistically break out of the basin of attraction of a local minimum and find its way into that of another. This mechanism, a purely probabilistic one, is similar to Gibbs sampling, *i.e.*, the core technique used in simulated annealing for updating spins [15]. This argument also implies that at the level of the c-number equations, CIM is not a "genuinely quantum" scheme, but is more akin to classical annealing.

Although CIM preceded OIM and BLIM by several years, our result shows that its core operating mechanism is governed by similar principles, *i.e.*, local minimization of a global Lyapunov function. Indeed, the fact that Lyapunov functions have emerged as a common thread in these three Ising machine schemes suggests that other schemes may fall in this class, too—we may term these **Lyapunov Ising Machines (LIM)**. This common mathematical framework will, we hope, lead to a more unified view of Ising machine schemes, and a more precise understanding of their similarities and differences.

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