SIAM J. MATH. ANAL.
Vol. 54, No. 3, pp. 3122–3138

THE AVERAGE DISTANCE PROBLEM WITH PERIMETER-TO-AREA RATIO PENALIZATION*

QIANG DU†, XIN YANG LU‡, AND CHONG WANG§

Abstract. In this paper we consider the functional $E_{p,\lambda}(\Omega) := \int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x + \lambda \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}$. Here $p \geq 1$, $\lambda > 0$ are given parameters, the unknown Ω varies among compact, convex, Hausdorff two-dimensional sets of $\mathbb{R}^2$, $\partial\Omega$ denotes the boundary of Ω, and $\operatorname{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y|$. The integral term $\int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x$ quantifies the "easiness" for points in Ω to reach the boundary, while $\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}$ is the perimeter-to-area ratio. The main aim is to prove existence and $C^{1,1}$-regularity of minimizers of $E_{p,\lambda}$.

Key words. perimeter-to-area ratio, regularity

AMS subject classifications. 49Q20, 49K10, 35B65

DOI. 10.1137/21M1422987

1. Introduction.

The perimeter-to-area ratio (in 2D), or surface area-to-volume ratio (in 3D), plays a crucial role in many processes. In biology, for instance, the size of prokaryote cells is limited by the efficiency of diffusion processes, fundamental to transport nutrients across the cell, which is strongly correlated with the surface area-to-volume ratio. A larger surface area-to-volume ratio also gives prokaryote cells a high metabolic rate, fast growth, and short lifespan compared to eukaryote cells (see, for instance, [10]).

In chemistry, higher surface area-to-volume ratio increases the typical speed of chemical reactions. This phenomenon can be observed in many instances, sometimes quite dramatically, such as dust explosions, when dust particles of seemingly nonflammable materials (e.g., aluminum, sugar, flour, etc.) can be ignited due to their very large surface area-to-volume ratio [13, 11].

In this paper we will focus on the two-dimensional case. In the above examples, there are essentially two often competing quantities: one is the "easiness" to access the boundary, and the other is the perimeter-to-area ratio.

A very thin, rod-like, rectangular body would have very good access to the boundary (desirable) but large perimeter-to-area ratio. A disc would have the lowest perimeter-to-area ratio (desirable) among shapes of the same total area, but access to the boundary would be limited. It is also possible to have both a large perimeter-to-area ratio and limited access to the boundary.

*Received by the editors May 27, 2021; accepted for publication (in revised form) January 24, 2022; published electronically May 24, 2022.
https://doi.org/10.1137/21M1422987
Funding: The work of the first author was partially supported by the National Science Foundation grant DMS-2012562 and the ARO MURI grant W911NF-15-1-0562. The work of the second author was supported by the NSERC Discovery grant "Regularity of minimizers and pattern formation in geometric minimization problems."
†Department of Applied Physics and Applied Mathematics, and Data Science Institute, Columbia University, New York, NY 10027 USA (qd2125@columbia.edu).
‡Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON P7B 5E1, Canada (xlu8@lakeheadu.ca).
§Department of Mathematics, Washington and Lee University, Lexington, VA 24450 USA (cwang@wlu.edu).

Until now, we have discussed the "easiness" of accessing the boundary only at a qualitative level. In order to quantify it, we introduce the "average distance" term

$$F_p(\Omega) := \int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x,$$

where $\operatorname{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x-y|$; $p \geq 1$ is a given parameter and $|\cdot|$ denotes the Euclidean distance.

Consider the energy functional

$$E_{p,\lambda}(\Omega) = \int_\Omega \operatorname{dist}^p(x, \partial\Omega)\ dx + \lambda \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}, \tag{1.1}$$

where $p \geq 1, \lambda > 0$ are given parameters. Define the admissible set

$$\mathcal{A} := \{\Omega : \Omega \subset \mathbb{R}^2 \text{ is compact, convex, and Hausdorff two-dimensional}\}.$$

The term $\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}$ is the perimeter-to-area ratio. Note that neither the perimeter $\mathcal{H}^1(\partial\Omega)$ nor the area $\mathcal{H}^2(\Omega)$ is penalized; only their ratio is. This makes compactness results quite challenging to prove, and several estimates (in section 2) will be required. Another issue is that it is not very clear if the average distance term is just a lower order perturbation of $\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}$. The role of convexity is to ensure crucial compactness estimates (Lemmas 2.2 and 3.5). Note that $E_{p,\lambda}$ is invariant under rigid movements. Further details about the space of convex sets, and its topology, will be discussed in section 2. The main result of this paper is the following.

THEOREM 1.1. *Given $p \geq 1, \lambda > 0$, the following assertions hold:*

(1) *$E_{p,\lambda}$ admits a minimizer in $\mathcal{A}$.*
(2) *All minimizers are compact, convex, $C^{1,1}$-regular sets, with Hausdorff dimension equal to 2.*
(3) *The perimeter-to-area ratio of any minimizer Ω satisfies*

$$\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} = \frac{p+2}{\lambda(p+3)} \min_{\mathcal{A}} E_{p,\lambda}.$$

Here, and for future reference, the expression "Ω is C^k-regular" means that its boundary $\partial\Omega$ is C^k-regular; i.e., $\partial\Omega$ admits a C^k-regular parameterization.

Note that the functional F_p is formally similar to the average distance functional

$$\Sigma \mapsto \int_\Gamma \operatorname{dist}^p(x, \Sigma)\,\mathrm{d}\mu,$$

where Γ is a given domain, μ is a given measure on Γ, and Σ varies among compact, pathwise connected sets with Hausdorff dimension equal to 1. The average distance functional has been widely studied and used in several modeling problems. For a (nonexhaustive) list of references, we cite the papers (and books) by Buttazzo and collaborators [2, 3, 5, 4, 8, 9, 6, 7]. Also related are the papers by Paolini and Stepanov [20], Santambrogio and Tilli [21], Tilli [23], Lemenant and Mainini [17], Slepčev [22], and the review paper by Lemenant [16]. Similar variational problems entailing a competition between classical perimeter and nonlocal repulsive interaction were studied by Muratov and Knüpfer [19], Goldman, Novaga, and Ruffini [15], and

Goldman, Novaga, and Röger [14]. Figalli et al. studied a competition between a nonlocal s-perimeter and a nonlocal repulsive interaction term [12].

The rest of the paper is structured as follows: section 2 is dedicated to proving some auxiliary estimates on the area ((2.1) and Corollary 2.1) and perimeter (Lemma 2.2) of elements of minimizing sequences. Existence of minimizers will be shown in section 3, while $C^{1,1}$-regularity will be proven in section 4. Finally, we explore several future directions to further our understanding of the penalized average distance problem.

2. Preliminary estimates. In this section we collect some preliminary estimates that will be used later. First, we remark that given $p \geq 1$ and $\lambda > 0$, for any $\Omega \in \mathcal{A}$ it holds

$$\mathcal{H}^2(\Omega) \geq \frac{4\pi\lambda^2}{E_{p,\lambda}(\Omega)^2}. \tag{2.1}$$

Indeed, consider an arbitrary $\Omega \in \mathcal{A}$. By the isoperimetric inequality, among all convex sets with area $\mathcal{H}^2(\Omega)$, the perimeter-to-area ratio is minimum for a disc, where it attains the value $2\sqrt{\pi}/\sqrt{\mathcal{H}^2(\Omega)}$. Hence

$$\frac{2\lambda\sqrt{\pi}}{\sqrt{\mathcal{H}^2(\Omega)}} \leq \lambda\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} \leq E_{p,\lambda}(\Omega),$$

and (2.1) is proven.

Corollary 2.1. *Given $p \geq 1$, $\lambda > 0$, any minimizing sequence $\Omega_n \subseteq \mathcal{A}$ satisfies*

$$\mathcal{H}^2(\Omega_n) \geq 4\pi\lambda^2\left(\frac{2\pi}{p^2+3p+2} + 2\lambda + 1\right)^{-2} =: C_1, \tag{2.2}$$

$$\frac{\mathcal{H}^1(\partial\Omega_n)}{\mathcal{H}^2(\Omega_n)} \leq \frac{1}{\lambda}\left(\frac{2\pi}{p^2+3p+2} + 2\lambda + 1\right) =: C_2 \tag{2.3}$$

for any sufficiently large n.

Proof. First we prove $\inf_{\mathcal{A}} E_{p,\lambda} < +\infty$. Let $B_1 \in \mathcal{A}$ be a disc of radius 1. Direct computation gives

$$\begin{aligned} \inf_{\mathcal{A}} E_{p,\lambda} \leq E_{p,\lambda}(B_1) &= \int_{B_1} \operatorname{dist}^p(x, \partial B_1)\,\mathrm{d}x + \lambda\frac{\mathcal{H}^1(\partial B_1)}{\mathcal{H}^2(B_1)} \\ &= 2\pi\int_0^1 (1-r)^p r\,\mathrm{d}r + 2\lambda = \frac{2\pi}{p^2+3p+2} + 2\lambda < +\infty. \end{aligned} \tag{2.4}$$

Thus, given a minimizing sequence $\Omega_n \subseteq \mathcal{A}$, there exists N such that for any $n \geq N$ it holds

$$E_{p,\lambda}(\Omega_n) \leq \frac{2\pi}{p^2+3p+2} + 2\lambda + 1, \tag{2.5}$$

and (2.1) gives

$$\mathcal{H}^2(\Omega_n) \geq 4\pi\lambda^2\left(\frac{2\pi}{p^2+3p+2} + 2\lambda + 1\right)^{-2}$$

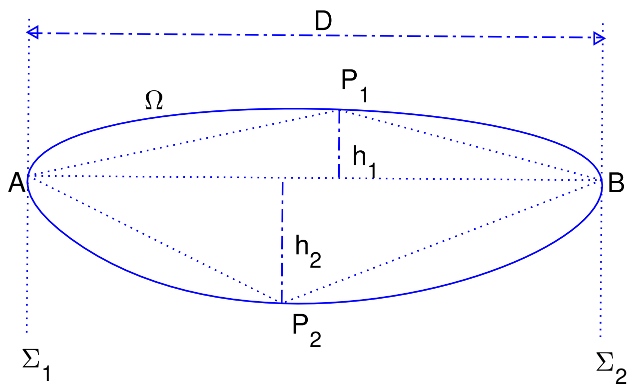

FIG. 1. *A schematic representation of the construction. The points* P_1 *and (resp.,* P_2*) are the points on* $\partial\Omega$ *above (resp., below) the segment* $[\![A, B]\!]$ *furthest away from* $[\![A, B]\!]$.

for any $n \geq N$, hence (2.2). To prove (2.3), note that (2.5) forces

$$\frac{2\pi}{p^2+3p+2} + 2\lambda + 1 \geq E_{p,\lambda}(\Omega_n) \geq \lambda \frac{\mathcal{H}^1(\partial\Omega_n)}{\mathcal{H}^2(\Omega_n)},$$

concluding the proof. ∎

LEMMA 2.2. *Given* $p \geq 1$ *and* $\lambda > 0$*, for any minimizing sequence* $\Omega_n \subseteq \mathcal{A}$*, it holds, for all sufficiently large* n*,*

$$\mathcal{H}^1(\partial\Omega_n) \leq C_3 = C_3(p,\lambda) \tag{2.6}$$

with C_3 *being some computable (but uninfluential) constant.*

Proof. We first claim that for any $\Omega \in \mathcal{A}$ it holds

$$\int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x \geq C \frac{\mathcal{H}^2(\Omega)^{p+1}}{\mathcal{H}^1(\partial\Omega)^p}, \qquad C = 3^{-p}2^{-p-4}. \tag{2.7}$$

Consider an arbitrary $\Omega \in \mathcal{A}$. Let $A, B \in \partial\Omega$ be two points realizing $D := |A - B| = \operatorname{diam}\Omega$. Let Σ_i, $i = 1, 2$ be the lines (see Figure 1) orthogonal to the line segment between A and B (which we denote by $[\![A, B]\!]$). Since Ω is convex and $|A - B| = \operatorname{diam}\Omega$, Ω is entirely contained in the region between Σ_1 and Σ_2. Then let P_i, $i = 1, 2$ be the points on $\partial\Omega$ such that the triangles $\triangle AP_iB$ have maximal areas. As Ω is convex, we have

$$\mathcal{H}^2(\Omega) \leq D(h_1 + h_2), \qquad h_i := \operatorname{dist}(P_i, [\![A, B]\!]).$$

On the other hand, $\mathcal{H}^2(\triangle AP_iB) = Dh_i/2$; hence

$$\frac{\mathcal{H}^2(\triangle AP_1B \cup \triangle AP_2B)}{\mathcal{H}^2(\Omega)} \geq \frac{1}{2}.$$

Now we do the following construction: let O_i (resp., r_i) be the incenter (resp., inradius) of $\triangle AP_iB$, $i = 1, 2$. Denote by $\tilde{A}$ (resp., $\tilde{P}_i$, $\tilde{B}$) the midpoints of the line segments between O_i and A (resp., P_i, B)—see Figure 2.

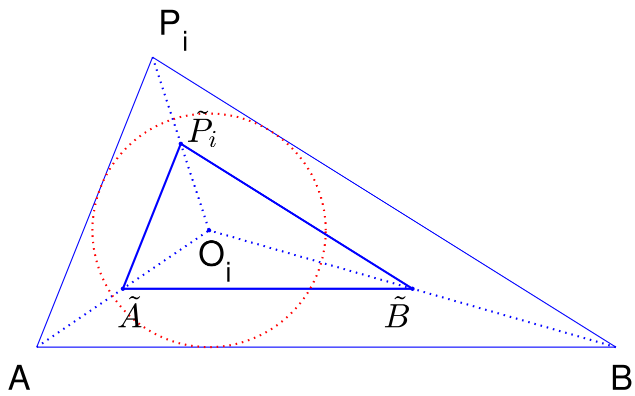

FIG. 2. *A schematic representation of the construction. The points $\tilde{A}$, $\tilde{P}_i$, and $\tilde{B}$ are the midpoints of the segments $[\![O_i, A]\!]$, $[\![O_i, P_i]\!]$, and $[\![O_i, B]\!]$, respectively. The red dotted circle is the incircle of the triangle $\triangle AP_iB$.*

Clearly, $\triangle\tilde{A}\tilde{P}_i\tilde{B}$ is a rescaled copy of $\triangle AP_iB$, with area $\mathcal{H}^2(\triangle AP_iB)/4$. As

$$\text{inradius} = \frac{2\text{Area}}{\text{Perimeter}},$$

we can estimate r_i as follows:

$$r_i = \frac{Dh_i}{D + |A - P_i| + |B - P_i|} \geq \frac{Dh_i}{3D} = \frac{h_i}{3}, \tag{2.8}$$

since by definition we have $D = \operatorname{diam}\Omega \geq |A - P_i|, |B - P_i|$. Then, noting that

$$\operatorname{dist}(x, \partial\Omega) \geq \operatorname{dist}(x, \partial AP_iB) \geq \frac{1}{2}\operatorname{dist}(O_i, \partial AP_iB) \geq \frac{1}{2}r_i$$

for all $x \in \triangle\tilde{A}\tilde{P}_i\tilde{B},\ i = 1, 2$, we have

$$\begin{aligned}
\int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x &\geq \sum_{i=1}^{2}\int_{\triangle\tilde{A}\tilde{P}_i\tilde{B}} \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x \\
&\geq \sum_{i=1}^{2} 2^{-p} r_i^p \mathcal{H}^2(\triangle\tilde{A}\tilde{P}_i\tilde{B}) = \sum_{i=1}^{2} 2^{-p-2} r_i^p \mathcal{H}^2(\triangle AP_iB) \\
&\geq \sum_{i=1}^{2} 3^{-p}2^{-p-3} h_i^{p+1} D \geq 3^{-p}2^{-p-3} D \cdot \max_{i=1,2} h_i^{p+1}. \qquad (2.9)
\end{aligned}$$

Recalling that $\mathcal{H}^2(\Omega) \leq D(h_1 + h_2)$, $\mathcal{H}^1(\partial\Omega) \geq 2D$, we get

$$\frac{\mathcal{H}^2(\Omega)^{p+1}}{\mathcal{H}^1(\partial\Omega)^p} \leq \frac{D^{p+1}(h_1 + h_2)^{p+1}}{(2D)^p} = 2^{-p}D(h_1 + h_2)^{p+1} \leq 2D \cdot \max_{i=1,2} h_i^{p+1}.$$

Hence (2.9) gives

$$\int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x \geq C\frac{\mathcal{H}^2(\Omega)^{p+1}}{\mathcal{H}^1(\partial\Omega)^p}, \qquad C = 3^{-p}2^{-p-4},$$

and (2.7) is proven. From (2.3) we know that

$$\frac{\mathcal{H}^1(\partial\Omega_n)}{\mathcal{H}^2(\Omega_n)} \le C_2 \Longrightarrow \frac{\mathcal{H}^2(\Omega_n)^{p+1}}{\mathcal{H}^1(\partial\Omega_n)^{p+1}} \ge C_2^{-p-1},$$

so the above inequality gives

$$\int_\Omega \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x \ge C\frac{\mathcal{H}^2(\Omega_n)^{p+1}}{\mathcal{H}^1(\partial\Omega_n)^{p+1}}\mathcal{H}^1(\partial\Omega_n) \ge CC_2^{-p-1}\mathcal{H}^1(\partial\Omega_n).$$

Now, any minimizing sequence $\{\Omega_n\}$ is such that, for all sufficiently large n,

$$E_{p,\lambda}(\Omega_n) \le \inf E_{p,\lambda} + 1;$$

thus

$$\inf E_{p,\lambda} + 1 \ge E_{p,\lambda}(\Omega_n) \ge \int_{\Omega_n} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x \ge CC_2^{-p-1}\mathcal{H}^1(\partial\Omega_n),$$

and (2.4) shows that $\inf E_{p,\lambda} < +\infty$, completing the proof. ∎

REMARK 2.3. *We note that it is an interesting geometric question by itself to study what the optimal constant C for the inequality* (2.7) *is. Furthermore, one may ask if the form of the inequality is optimal. That is, one may ask, given $\mathcal{H}^2(\Omega)$ and $\mathcal{H}^1(\partial\Omega)$, what is the minimum of $\int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x$, which is a constrained optimization problem related to the one considered in this work.*

3. Existence.

In this section we will prove that the $E_{p,\lambda}$ admits a minimizer in $\mathcal{A}$. As our arguments rely on a lower semicontinuity result, namely, Lemma 3.4 below, we need first to introduce a metric on $\mathcal{A}$.

For any $\Omega_1, \Omega_2 \in \mathcal{A}$, define

$$d(\Omega_1,\Omega_2) := \mathcal{H}^2(\Omega_1 \triangle \Omega_2), \tag{3.1}$$

where $\triangle$ denotes the symmetric difference. Set

$$\bar{\mathcal{A}} := \text{completion of } \mathcal{A} \text{ with respect to } d.$$

Before we can proceed, we need to characterize the elements of $\bar{\mathcal{A}} \setminus \mathcal{A}$: we cannot exclude a priori that an element $\Omega \in \bar{\mathcal{A}}$ can be quite irregular:

1. $\Omega \in \bar{\mathcal{A}}$ needs not to be closed: indeed it is very possible for a sequence of compact sets to converge to an open set in the metric d. For instance, let Ω_n be the closed ball of radius $1 - 1/n$ centered around the origin; then it converges to the open ball, centered around the origin, of radius 1.
2. As we do not have any a priori bounds on the diameter of elements of $\mathcal{A}$, a set $\Omega \in \bar{\mathcal{A}}$ needs not to be bounded.
3. The distance d is insensitive to perturbations on $\mathcal{H}^2$-negligible sets. Therefore, we cannot exclude that $\bar{\mathcal{A}}$ might contain compact convex sets *up to $\mathcal{H}^2$-negligible sets*. Thus whether a generic element in $\bar{\mathcal{A}}$ is convex or not is unclear.

In view of the above mentioned issues, we cannot assume compactness, or convexity, for elements of $\bar{\mathcal{A}}$. Our goal is to show (see Lemma 3.3 below) that minimizing sequences must converge to some element in $\mathcal{A}$.

The next result, from [22], will be crucial for our convergence arguments.

LEMMA 3.1. *Consider a sequence of constant speed parameterized curves $\gamma_n : [0,1] \longrightarrow K$, where $K \subseteq \mathbb{R}^d$ is some compact set. Assume moreover that*

$$\sup_n L(\gamma_n) < +\infty, \qquad \sup_n \|\gamma_n\|_{BV([0,1];\mathbb{R}^d)} < +\infty, \tag{3.2}$$

where $\|\cdot\|_{BV([0,1];\mathbb{R}^d)}$ denotes the bounded variation norm. Then there exists a curve $\gamma : [0,1] \longrightarrow K$ such that

1. *$\gamma_n \to \gamma$ in $C^\alpha([0,1];\mathbb{R}^d)$ for all $\alpha \in [0,1)$,*
2. *$\gamma_n' \to \gamma'$ in $L^p(0,1;\mathbb{R}^d)$ for all $p < +\infty$,*
3. *$\gamma_n'' \overset{*}{\rightharpoonup} \gamma''$ weakly as measures.*

REMARK 3.2. *We remark that this convergence result is quite strong: consider a sequence $\{\Omega_n\} \subseteq \mathcal{A}$, and let γ_n be constant speed parameterizations of $\partial\Omega_n$. Note that γ_n are all closed curves. Assume that we are under the hypotheses of Lemma 3.1; hence there exists $\gamma : [0,1] \longrightarrow K$ such that $\gamma_n \to \gamma$ in $C^\alpha([0,1];\mathbb{R}^d)$ for all $\alpha \in [0,1)$. In particular, we can define Ω to be the bounded region delimited by the graph of γ, and we have the uniform convergence of the boundaries, which in turn gives $d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega) \to 0$. Here $d_{\mathcal{H}}$ denotes the Hausdorff distance*

$$d_{\mathcal{H}}(X,Y) := \max\left\{ \sup_{x\in X} \operatorname{dist}(x,Y), \sup_{y\in Y} \operatorname{dist}(y,X) \right\}.$$

Such strong convergence also implies that the characteristic functions χ_{Ω_n} converge to χ_Ω in L^p, $p \in [1,+\infty)$, since

$$\|\chi_{\Omega_n} - \chi_\Omega\|^p_{L^p(\mathbb{R}^d)} \le \mathcal{H}^2(\Omega_n \triangle \Omega) \le \max\left\{ \sup_n L(\gamma_n), L(\gamma) \right\} \cdot d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega) \to 0.$$

LEMMA 3.3. *Consider a minimizing sequence $\Omega_n \subseteq \mathcal{A}$; then there exists $\Omega \in \mathcal{A}$ and a sequence $x_n \subseteq \mathbb{R}^n$ such that $\Omega_n + x_n \to \Omega$ in the metric d.*

Note that since our energy is translation invariant, the above convergence result is sufficient for our purposes.

Proof. In this proof it is more convenient to work with constant speed, instead of arc-length, parameterizations.

Consider minimizing sequence $\{\Omega_n\} \subseteq \mathcal{A}$, and let $\varphi_n : [0,1] \longrightarrow \partial\Omega_n$ be constant speed parameterizations. Note all $\partial\Omega_n$ are closed curves, and as $E_{p,\lambda}$ is translation invariant, we can replace Ω_n with translated copies (which, for brevity, we still denote by Ω_n and by φ_n the parameterization of $\partial\Omega_n$) such that $\varphi_n(0) = \varphi_n(1) = 0$. We show that we are under the conditions (3.2): first, the upper bound on the perimeter (2.6) and $\Omega_n \subseteq \mathbb{R}^2$ ensures all Ω_n are contained in some compact set K. As the curves φ_n are parameterized by constant speed, we have $\|\varphi_n'\| = L(\varphi_n) = \mathcal{H}^1(\partial\Omega_n)$ a.e. Then, in view of Lemma 2.2, we infer (3.2). Therefore there exists a limit curve $\varphi : [0,1] \longrightarrow K$ such that the convergences in Lemma 3.1 hold. Since $\varphi_n(0) = \varphi_n(1) = 0$ for all n, we get $\varphi(0) = \varphi(1) = 0$ too. We define Ω to be the *bounded* area delimited by φ, and the graph of φ turn out to be $\partial\Omega$. By construction, Ω is compact.

We need to check it is convex: consider arbitrary $P, Q \in \Omega$, $t \in (0,1)$, and we show that $(1-t)P + tQ \in \Omega$. Consider sequences $P_n, Q_n \in \Omega_n$ such that $P_n \to P$, $Q_n \to Q$: since each Ω_n is convex, $(1-t)P_n + tQ_n \in \Omega_n$. By Lemma 3.1, we know $\|\varphi_n - \varphi\|_{C^0([0,1];\mathbb{R}^2)} \to 0$. As a consequence,

$$d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega) \to 0$$

too. This allows us to choose, for each n, another point $z_n \in \Omega$ such that $|z_n - ((1-t)P_n + tQ_n)| \leq d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega)$. By construction, now the sequences $(1-t)P_n + tQ_n$ and z_n have the same limit. As $(1-t)P_n + tQ_n \to (1-t)P + tQ$ and $z_n \to z$, hence $z = (1-t)P + tQ$, using the compactness of Ω finally gives $z \in \Omega$.

Finally, we check that $\dim_{\mathcal{H}} \Omega = 2$. Since the ambient space $\mathbb{R}^2$ already has Hausdorff dimension two, it suffices to show that Ω contains a set of Hausdorff dimension two. For each n, we can use the construction from the proof of Lemma 2.2 on each Ω_n: we showed the existence of triangles $T_i := \triangle\tilde{A}\tilde{P}_i\tilde{B}$ (see Figure 1) whose distance to the boundary is at least $r_i/2$, with r_i being the incenter which satisfied $r_i \geq h_i/3$. Now, since we showed in the proof of Lemma 2.2 that

$$\sum_{i=1}^{2} h_{i,n} \operatorname{diam}\Omega_n \geq \mathcal{H}^2(\Omega_n)$$

and

$$\mathcal{H}^2(\Omega_n) \geq C_1, \qquad \operatorname{diam}\Omega_n \leq \mathcal{H}^1(\partial\Omega_n) \leq C_3$$

due to Corrollary 2.1, we get

$$\sum_{i=1}^{2} h_{i,n} \geq \frac{\mathcal{H}^2(\Omega_n)}{\operatorname{diam}\Omega_n} \geq \frac{C_1}{C_3} > 0.$$

This shows that at least one of the triangles $T_{i,n}$, $i = 1, 2$, must be nondegenerate since its inradius is bounded from below by

$$\max_{i=1,2} r_{i,n} \geq \max_{i=1,2} \frac{h_{i,n}}{3} \geq \frac{C_1}{6C_3},$$

and the proof is complete. ∎

Lemma 3.3 is of crucial importance: since we are interested in the minimizers of $E_{p,\lambda}$, this allows us to reduce the minimization problem to $\mathcal{A}$ and neglect the highly irregular elements of $\bar{\mathcal{A}} \setminus \mathcal{A}$.

Lemma 3.4. *Given $p \geq 1$, $\lambda > 0$, and a minimizing sequence $\Omega_n \subseteq \mathcal{A}$ converging to $\Omega \in \mathcal{A}$ with respect to d, then it holds*

$$\mathcal{H}^2(\Omega) = \lim_{n\to+\infty} \mathcal{H}^2(\Omega_n), \tag{3.3}$$

$$\mathcal{H}^1(\partial\Omega) \leq \liminf_{n\to+\infty} \mathcal{H}^1(\partial\Omega_n), \tag{3.4}$$

$$\int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{\Omega_n} \operatorname{dist}^p(x, \partial\Omega_n)\,\mathrm{d}x. \tag{3.5}$$

Proof. Estimate (3.3) follows from the definition of the metric d and Remark 3.2.

To prove (3.4), recall that the perimeter $\mathcal{H}^1(\partial\Omega_n)$ is the total variation of the characteristic function of Ω_n. Convergence $\Omega_n \to \Omega$ with respect to d implies (see Remark 3.2)

$$\chi_{\Omega_n} \to \chi_\Omega \text{ strongly in } L^1(\mathbb{R}^2)$$

with "χ" denoting the characteristic function of the subscribed set. Thus (3.4) follows from the lower semicontinuity of the total variation seminorm.

To prove (3.5), note that

$$\int_{\Omega_n} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x = \int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x + \int_{\Omega_n\cap\Omega} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x,$$

$$\int_{\Omega} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x = \int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x + \int_{\Omega_n\cap\Omega} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x;$$

hence

$$\left|\int_{\Omega_n} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x - \int_{\Omega} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x\right|$$

$$\le \int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x \tag{3.6}$$

$$+ \int_{\Omega_n\cap\Omega} |\operatorname{dist}^p(x,\partial\Omega_n) - \operatorname{dist}^p(x,\partial\Omega)|\,\mathrm{d}x. \tag{3.7}$$

By Lemma 2.2,

$$\operatorname{diam}(\Omega_n) \le \mathcal{H}^1(\partial\Omega_n) \le C_3.$$

According to (3.4),

$$\operatorname{diam}(\Omega) \le \mathcal{H}^1(\partial\Omega) \le \liminf_{n\to+\infty} \mathcal{H}^1(\partial\Omega_n) \le C_3.$$

Therefore,

$$\int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x \le \mathcal{H}^2(\Omega_n\setminus\Omega)(\operatorname{diam}(\Omega_n))^p \le \mathcal{H}^2(\Omega_n\setminus\Omega)C_3^p \to 0,$$

$$\int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x \le \mathcal{H}^2(\Omega\setminus\Omega_n)(\operatorname{diam}(\Omega))^p \le \mathcal{H}^2(\Omega\setminus\Omega_n)C_3^p \to 0;$$

hence the sum in (3.6) goes to zero. To estimate (3.7), denote by $d_{\mathcal{H}}$ the Hausdorff distance, and note that, by the mean value theorem, it holds

$$\begin{aligned}
&\int_{\Omega_n\cap\Omega} |\operatorname{dist}^p(x,\partial\Omega_n) - \operatorname{dist}^p(x,\partial\Omega)|\,\mathrm{d}x \\
&\quad\le \int_{\Omega_n\cap\Omega} |\operatorname{dist}(x,\partial\Omega_n) - \operatorname{dist}(x,\partial\Omega)| \\
&\quad\cdot p \sup_{x\in\Omega_n\cap\Omega} \Big(\max\{\operatorname{dist}(x,\partial\Omega_n), \operatorname{dist}(x,\partial\Omega)\}\Big)^{p-1}\,\mathrm{d}x \\
&\quad\le \mathcal{H}^2(\Omega_n\cap\Omega) d_{\mathcal{H}}(\partial\Omega_n,\partial\Omega)\cdot p\Big(\max\{\operatorname{diam}\Omega_n, \operatorname{diam}\Omega\}\Big)^{p-1} \\
&\quad\le \mathcal{H}^2(\Omega_n\cap\Omega)\ d_{\mathcal{H}}(\partial\Omega_n,\partial\Omega)\cdot p\ C_3^{p-1} \to 0.
\end{aligned}$$

Thus the term in (3.7) goes to zero too, and (3.5) is proven. ∎

Now we prove part (1) of Theorem 1.1, i.e., the existence of minimizers in $\mathcal{A}$.

LEMMA 3.5. *For any $p \ge 1$, $\lambda > 0$, the functional $E_{p,\lambda}$ admits a minimizer $\Omega \in \mathcal{A}$, which satisfies*

$$\mathcal{H}^2(\Omega) \ge C_1, \qquad \mathcal{H}^1(\partial\Omega) \le C_3,$$

with C_1 (resp., C_3) defined in (2.2) *(resp.,* (2.6)*).*

Proof. Corollary 2.1 gives $\mathcal{H}^2(\Omega_n) \geq C_1$ for any sufficiently large n, and Lemma 3.4 gives

$$\mathcal{H}^2(\Omega) = \lim_{n\to+\infty} \mathcal{H}^2(\Omega_n) \geq C_1. \tag{3.8}$$

Based on Lemma 3.4 and (2.6),

$$\mathcal{H}^1(\partial\Omega) \leq C_3. \tag{3.9}$$

Lemma 3.4 gives

$$\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} \leq \liminf_{n\to+\infty} \frac{\mathcal{H}^1(\partial\Omega_n)}{\mathcal{H}^2(\Omega_n)}, \tag{3.10}$$

and

$$\int_\Omega \mathrm{dist}^p(x,\partial\Omega)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{\Omega_n} \mathrm{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x. \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\begin{aligned} E_{p,\lambda}(\Omega) &= \int_\Omega \mathrm{dist}^p(x,\partial\Omega)\,\mathrm{d}x + \lambda \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} \\ &\leq \lim_{n\to+\infty} \int_{\Omega_n} \mathrm{dist}^p(x,\partial\Omega_n)\,\mathrm{d}x + \lambda \liminf_{n\to+\infty} \frac{\mathcal{H}^1(\partial\Omega_n)}{\mathcal{H}^2(\Omega_n)} \leq \liminf_{n\to+\infty} E_{p,\lambda}(\Omega_n) = \inf_{\bar{\mathcal{A}}} E_{p,\lambda}; \end{aligned} \qquad \square$$

hence Ω is effectively a minimizer of $E_{p,\lambda}$ in $\bar{\mathcal{A}}$. Lemma 3.3 shows $\Omega \in \mathcal{A}$.

4. Regularity.

Now we prove part (2) of Theorem 1.1. The proof will be split over Lemmas 4.2 and 4.3.

LEMMA 4.1. *Let S be a compact, convex set, with Hausdorff dimension equal to 2. Let w_1, $w_2 \in \partial S$ be arbitrary distinct points, and let σ be the segment with endpoints w_1 and w_2. Denoting by S_1 and S_2 the two connected components of $S\backslash\sigma$, then both S_1, S_2 are convex.*

Proof. Endow $\mathbb{R}^2$ with a Cartesian coordinate system. Upon rotation and reflection, assume that σ lies in the y-axis and $S_1 \subseteq \{x > 0\}$, $S_2 \subseteq \{x < 0\}$. Clearly, given points $u, v \in S_1$, the segment ξ between u and v lies entirely in $S \cap \{x > 0\} = S_1$; hence S_1 is convex. The proof for S_2 is analogous. $\square$

LEMMA 4.2. *(C^1-regularity) For any $p \geq 1$, $\lambda > 0$, any minimizer of $E_{p,\lambda}$ is C^1-regular.*

Proof. Consider an arbitrary minimizer $\Omega \in \mathcal{A}$. Endow $\mathbb{R}^2$ with a polar coordinate system. We parameterize $\partial\Omega$ by a closed Lipschitz curve

$$\gamma : [0, 2\pi] \longrightarrow \partial\Omega.$$

The proof is achieved by a contradiction argument. Assume that Ω is not C^1-regular. That is, γ is not C^1-regular at some point t_0. Upon rotating the coordinates, we can also assume $t_0 \in (0, 2\pi)$. Since Ω is convex, both one-sided derivatives

$$l^- := \lim_{t\to t_0^-} \gamma'(t), \qquad l^+ := \lim_{t\to t_0^+} \gamma'(t)$$

are well defined [1, 18]. Denote by α the angle between l^- and l^+. Clearly, $\alpha \neq \pi$.

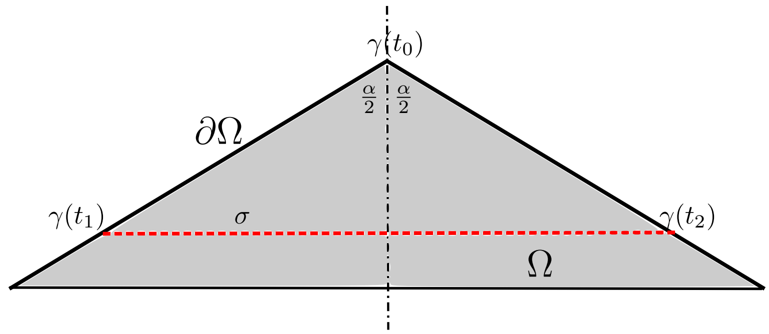

FIG. 3. *A schematic representation (near $\gamma(t_0)$, in first order approximation in ε) of the construction of Ω_ε.*

Figure 3 is a representation (in first order approximation) of $\partial\Omega$ near $\gamma(t_0)$. For small parameters $0 < \varepsilon \ll 1$, construct the competitor Ω_ε as follows:

1. Choose $t_1 < t_0 < t_2$ such that (in first order approximation in ε)

$$\mathcal{H}^1(\gamma([t_1, t_0])) = \mathcal{H}^1(\gamma([t_0, t_2])) = \varepsilon + O(\varepsilon^2).$$

2. Denote by

$$\sigma := \{(1-s)\gamma(t_1) + s\gamma(t_2) : s \in [0, 1]\}$$

the line segment between $\gamma(t_1)$ and $\gamma(t_2)$, and set

$$L := \big(\partial\Omega \backslash \gamma([t_1, t_2])\big) \cup \sigma. \tag{4.1}$$

Note that such L is a convex Jordan curve, and denote by Ω_ε the bounded region delimited by L.

By construction, in first order approximation in ε, it holds

$$\mathcal{H}^1(\partial\Omega_\varepsilon) = \mathcal{H}^1(\partial\Omega) - 2\varepsilon(1 - \sin(\alpha/2)) + O(\varepsilon^2), \tag{4.2}$$

$$\mathcal{H}^2(\Omega_\varepsilon) = \mathcal{H}^2(\Omega) - \frac{\varepsilon^2 \sin\alpha}{2} + o(\varepsilon^2) = \mathcal{H}^2(\Omega) + O(\varepsilon^2). \tag{4.3}$$

Moreover, it is straightforward to show that

$$\int_{\Omega_\varepsilon} \operatorname{dist}^p(x, \partial\Omega_\varepsilon)\,\mathrm{d}x \le \int_\Omega \operatorname{dist}^p(x, \partial\Omega)\,\mathrm{d}x. \tag{4.4}$$

Recalling that $\mathcal{H}^2(\Omega) > 0$ (since Ω is a minimizer), combining (4.2), (4.3), and (4.4) gives (in first order approximation in ε)

$$\begin{aligned} E_{p,\lambda}(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} \operatorname{dist}^p(x, \partial\Omega_\varepsilon)\; dx + \lambda \frac{\mathcal{H}^1(\partial\Omega_\varepsilon)}{\mathcal{H}^2(\Omega_\varepsilon)} \\ &\le \int_\Omega \operatorname{dist}^p(x, \partial\Omega)\; dx + \lambda \frac{\mathcal{H}^1(\partial\Omega) - 2\varepsilon(1 - \sin(\alpha/2)) + O(\varepsilon^2)}{\mathcal{H}^2(\Omega) + O(\varepsilon^2)} \end{aligned}$$

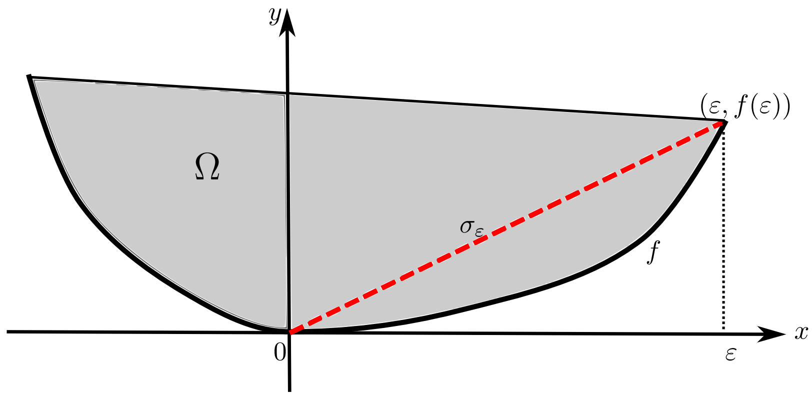

FIG. 4. *A schematic representation of the construction near $p_0 = (0,0)$.*

$$
\begin{aligned}
&= \int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x + \lambda\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} - \frac{2\lambda\varepsilon(1-\sin(\alpha/2))}{\mathcal{H}^2(\Omega)} + O(\varepsilon^2)\\
&= E_{p,\lambda}(\Omega) - \frac{2\lambda\varepsilon(1-\sin(\alpha/2))}{\mathcal{H}^2(\Omega)} + O(\varepsilon^2)\\
&= \min_{\mathcal{A}} E_{p,\lambda} - \frac{2\lambda\varepsilon(1-\sin(\alpha/2))}{\mathcal{H}^2(\Omega)} + O(\varepsilon^2),
\end{aligned}
$$

which is a contradiction for sufficiently small ε. Thus Ω must be C^1-regular. ∎

LEMMA 4.3. *Given $p \geq 1$, $\lambda > 0$, a minimizer Ω of $E_{p,\lambda}$, let $\gamma : [0, \mathcal{H}^1(\partial\Omega)] \longrightarrow \partial\Omega$ be an arc-length parameterization. Then it holds*

$$
\limsup_{h\to 0} \frac{|\gamma(t+2h) - 2\gamma(t) + \gamma(t-2h)|}{h^2} \leq 4C \tag{4.5}
$$

for any t, where C is some constant depending only on λ and p (and independent of Ω).

We remark that (4.5) implies $C^{1,1}$-regularity of $\partial\Omega$.

Proof. Consider an arbitrary point $p_0 \in \partial\Omega$. Since we proved that Ω is C^1-regular, consider a (local) orthogonal coordinate system with origin in p_0 and x-axis oriented along the tangent derivative (at p_0), such that Ω is entirely contained in the half-plane $\{y \geq 0\}$. The boundary $\partial\Omega$ is thus (locally) the graph of some nonnegative function f. Clearly, such f satisfies $f(0) = 0$. See Figure 4 for an illustration.

Choose an arbitrary $0 < \varepsilon \ll 1$. Denote by

$$
\sigma_\varepsilon := \{(x,y) : 0 \leq x \leq \varepsilon,\ y = x \cdot f(\varepsilon)/\varepsilon\}
$$

the segment between the origin and $(\varepsilon, f(\varepsilon))$. Let L_ε be the curve obtained by replacing $f([0,\varepsilon])$ with σ_ε that is,

$$
L_\varepsilon := (\partial\Omega \backslash f([0,\varepsilon])) \cup \sigma_\varepsilon.
$$

By construction (see Lemma 4.1) L_ε is a convex Jordan curve, and let Ω_ε be the bounded region delimited by L_ε. Note the following:

1. Clearly we can infer

$$\int_{\Omega_\varepsilon} \mathrm{dist}^p(x, \partial\Omega_\varepsilon)\,\mathrm{d}x \leq \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x. \tag{4.6}$$

2. For areas, since by construction it holds $\Omega_\varepsilon \subseteq \Omega$, we have

$$\begin{aligned}\mathcal{H}^2(\Omega) - \mathcal{H}^2(\Omega_\varepsilon) &= \mathcal{H}^2(\Omega\backslash\Omega_\varepsilon)\\ &= \mathcal{H}^2(\{(x,y) : 0 \leq x \leq \varepsilon,\ f(x) \leq y \leq x \cdot f(\varepsilon)/\varepsilon\})\\ &= \int_0^\varepsilon [x f(\varepsilon)/\varepsilon - f(x)]\,\mathrm{d}x = \frac{f(\varepsilon)\varepsilon}{2} - \int_0^\varepsilon f(x)\,\mathrm{d}x. \end{aligned}\tag{4.7}$$

3. For perimeters, note that $f'(0) = 0$, so $|f'|$ is small near 0. In particular, by choosing sufficiently small ε, we can ensure that

$$\sqrt{1 + |f'(x)|^2} \leq 2 \qquad \text{for all } x \in (0, \varepsilon),$$

and $|f(\varepsilon)|/\varepsilon$ can also be made as small as we need to satisfy

$$\sqrt{1 + \frac{f(\varepsilon)^2}{\varepsilon^2}} = 1 + \frac{f(\varepsilon)^2}{2\varepsilon^2} - \frac{1}{8}\Big(\frac{f(\varepsilon)^2}{\varepsilon^2}\Big)^2 + O\Big(\Big(\frac{f(\varepsilon)^2}{\varepsilon^2}\Big)^3\Big) \leq 1 + \frac{f(\varepsilon)^2}{2\varepsilon^2}.$$

Therefore,

$$\begin{aligned}\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon) &= \int_0^\varepsilon \Big(\sqrt{1 + |f'(x)|^2} - \sqrt{1 + \frac{f(\varepsilon)^2}{\varepsilon^2}}\Big)\,\mathrm{d}x\\ &\geq \int_0^\varepsilon \Big(\sqrt{1 + |f'(x)|^2} - 1 - \frac{f(\varepsilon)^2}{2\varepsilon^2}\Big)\,\mathrm{d}x,\end{aligned}$$

where, since for sufficiently small $\varepsilon \ll 1$ the quantity $\frac{f(\varepsilon)^2}{2\varepsilon^2}$ can be made arbitrarily small, we have

$$\int_0^\varepsilon \frac{f(\varepsilon)^2}{2\varepsilon^2}\,\mathrm{d}x = o(\varepsilon), \qquad \varepsilon \ll 1.$$

Thus, for all sufficiently small ε,

$$\begin{aligned}\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon) &= \int_0^\varepsilon \Big(\sqrt{1 + |f'(x)|^2} - \sqrt{1 + \frac{f(\varepsilon)^2}{\varepsilon^2}}\Big)\,\mathrm{d}x\\ &\geq \int_0^\varepsilon \Big(\sqrt{1 + |f'(x)|^2} - 1\Big)\,\mathrm{d}x + o(\varepsilon)\\ &= \int_0^\varepsilon \frac{|f'(x)|^2}{\sqrt{1 + |f'(x)|^2} + 1}\,\mathrm{d}x + o(\varepsilon) \geq \frac{1}{3}\int_0^\varepsilon |f'(x)|^2\,\mathrm{d}x.\end{aligned}\tag{4.8}$$

Combining (4.6), (4.7), and (4.8) gives

$$\begin{aligned}E_{p,\lambda}(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} \mathrm{dist}^p(x, \partial\Omega_\varepsilon)\,\mathrm{d}x + \lambda\frac{\mathcal{H}^1(\partial\Omega_\varepsilon)}{\mathcal{H}^2(\Omega_\varepsilon)}\\ &\leq \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x + \lambda\frac{\mathcal{H}^1(\partial\Omega) - \frac{1}{3}\int_0^\varepsilon |f'(x)|^2\,\mathrm{d}x}{\mathcal{H}^2(\Omega) - (\frac{f(\varepsilon)\varepsilon}{2} - \int_0^\varepsilon f(x)\,\mathrm{d}x)}.\end{aligned}\tag{4.9}$$

Since

$$\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}=\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}\cdot\frac{\mathcal{H}^2(\Omega)}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}$$
$$=\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}\cdot\left(1+\frac{\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}\right),$$

estimate (4.9) reads

$$E_{p,\lambda}(\Omega_\varepsilon)\le\int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x-\lambda\frac{\frac{1}{3}\int_0^\varepsilon|f'(x)|^2\,\mathrm{d}x}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}$$
$$+\lambda\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}\cdot\left(1+\frac{\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}\right)$$
$$=E_{p,\lambda}(\Omega)+\lambda\frac{\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)-\frac{1}{3}\int_0^\varepsilon|f'(x)|^2\,\mathrm{d}x}{\mathcal{H}^2(\Omega)-(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x)}. \tag{4.10}$$

Since Ω is a minimizer, Lemma 3.5 gives $\mathcal{H}^2(\Omega)>0$, and note that

$$\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x\le\frac{\mathcal{H}^2(\Omega)}{2}$$

for all sufficiently small ε; hence the denominator in (4.10) is positive. Thus the minimality of Ω forces the numerator in (4.10) to be nonnegative, i.e.,

$$3\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}\left(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x\right)-\int_0^\varepsilon|f'(x)|^2\,\mathrm{d}x\ge 0. \tag{4.11}$$

Equation (4.7) shows

$$\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x=\mathcal{H}^2(\Omega)-\mathcal{H}^2(\Omega_\varepsilon)\ge 0$$

since by construction $\Omega_\varepsilon\subseteq\Omega$. Lemma 3.5 gives

$$\mathcal{H}^2(\Omega)\ge C_1,\qquad \mathcal{H}^1(\partial\Omega)\le C_3;$$

hence

$$3\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}\le\frac{3C_3}{C_1}=:C,$$

and (4.11) forces

$$C\left(\frac{f(\varepsilon)\varepsilon}{2}-\int_0^\varepsilon f(x)\,\mathrm{d}x\right)\ge\int_0^\varepsilon|f'(x)|^2\,\mathrm{d}x. \tag{4.12}$$

Since Ω is convex and we assumed (at the beginning of this proof) that $\Omega\subseteq\{y\ge 0\}$, f is nonnegative; hence (4.12) forces

$$\frac{C}{2}f(\varepsilon)\varepsilon\ge\int_0^\varepsilon|f'(x)|^2\,\mathrm{d}x. \tag{4.13}$$

Note that since $f(0) = 0$, it follows

$$f(\varepsilon) = \int_0^\varepsilon f'(x)\,\mathrm{d}x \le \int_0^\varepsilon |f'(x)|\,\mathrm{d}x. \tag{4.14}$$

By Hölder's inequality,

$$\int_0^\varepsilon |f'(x)|^2\,\mathrm{d}x \ge \frac{1}{\varepsilon}\left(\int_0^\varepsilon |f'(x)|\,\mathrm{d}x\right)^2; \tag{4.15}$$

hence

$$\frac{C}{2}\varepsilon \int_0^\varepsilon |f'(x)|\,\mathrm{d}x \overset{(4.14)}{\ge} \frac{C}{2} f(\varepsilon)\varepsilon \overset{(4.13)}{\ge} \int_0^\varepsilon |f'(x)|^2\,\mathrm{d}x \overset{(4.15)}{\ge} \frac{1}{\varepsilon}\left(\int_0^\varepsilon |f'(x)|\,\mathrm{d}x\right)^2$$
$$\Longrightarrow \frac{C}{2}\varepsilon^2 \ge \int_0^\varepsilon |f'(x)|\,\mathrm{d}x \ge \int_0^\varepsilon f'(x)\,\mathrm{d}x = f(\varepsilon).$$

The above arguments can be repeated for $\varepsilon < 0$, $|\varepsilon| \ll 1$ (or equivalently, when the orientation of x-axis is inverted). The arbitrariness of ε then gives

$$\limsup_{\varepsilon\to 0} \frac{|f(\varepsilon) - 2f(0) + f(-\varepsilon)|}{(\varepsilon/2)^2} \le 4C,$$

concluding the proof. ∎

Now we prove part (3) of Theorem 1.1.

LEMMA 4.4. *Given $p \ge 1$, $\lambda > 0$, any minimizer Ω of $E_{p,\lambda}$ satisfies*

$$\frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} = \frac{p+2}{\lambda(p+3)} \min_{\mathcal{A}} E_{p,\lambda}.$$

Proof. Let Ω be an arbitrary minimizer. Endow $\mathbb{R}^2$ with a Cartesian coordinate system, and assume without loss of generality that $(0,0)$ is in the interior part of Ω. For any $r > 0$, denote by

$$T_r : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad T_r(x) := rx$$

the homothety of center $(0,0)$ and ratio r. Note that $T_r(\Omega) \in \mathcal{A}$ for any $r > 0$, and the scalings are

$$\int_{T_r(\Omega)} \mathrm{dist}^p(x, \partial T_r(\Omega))\,\mathrm{d}x = r^{p+2} \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x,$$
$$\frac{\mathcal{H}^1(\partial T_r(\Omega))}{\mathcal{H}^2(T_r(\Omega))} = \frac{1}{r} \cdot \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}.$$

Define the function

$$f : (0, +\infty) \longrightarrow (0, +\infty),$$
$$f(r) := E_{p,\lambda}(T_r(\Omega)) = r^{p+2} \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x + \frac{\lambda}{r} \cdot \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)}.$$

Since f is smooth and attains a global minimum at $r = 1$, it follows

$$f'(1) = (p+2)\int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x - \lambda \cdot \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} = 0$$
$$\Longrightarrow \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x = \frac{\lambda}{p+2} \cdot \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)};$$

hence

$$E_{p,\lambda}(\Omega) = \frac{\lambda(p+3)}{p+2} \cdot \frac{\mathcal{H}^1(\partial\Omega)}{\mathcal{H}^2(\Omega)} = \min_{\mathcal{A}} E_{p,\lambda},$$

and the proof is complete. □

Let us conclude the paper with some final remarks. In this paper we investigated the minimization problem for the average distance functional, with perimeter-to-area ratio penalization, in the plane. We proved the existence and $C^{1,1}$-regularity of minimizers, mainly relying on constructing suitable competitors. Echoing and developing former studies that exclusively focused on either the one-dimensional average distance problem or purely surface area-to-volume ratio question, by considering optimal sets of combined energy from broader and more eclectic perspectives, this study enriches and deepens our understanding of penalized average distance problem.

We remark that all the main results of this paper, i.e., bounds on the perimeter and area and $C^{1,1}$-regular of minimizers, can also be proven if we replace the perimeter-to-area term with a generalized ratio of the form $\lambda\frac{\mathcal{H}^1(\partial\Omega)^\alpha}{\mathcal{H}^2(\Omega)^\beta}$, symbolizing a perimeter term normalized (by area) with different scaling exponents α and β. That is, we consider an energy of the form

$$E^{\alpha,\beta}_{p,\lambda}(\Omega) := \int_\Omega \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x + \lambda\frac{\mathcal{H}^1(\partial\Omega)^\alpha}{\mathcal{H}^2(\Omega)^\beta}, \tag{4.16}$$

where α, β are given powers satisfying $2\beta > \alpha > \frac{p}{p+1}\beta > 0$. This last bound, combined with Young's inequality, allows us to easily bound the perimeter, and the subsequent results. It can also be quickly checked that if $\alpha > 2\beta$, then minimizers are just single points. One more remark is that, according to (2.7), if in (4.16) we pick $\alpha = p, \beta = p+1$ and $\lambda = C$ as in (2.7), we get

$$E^{p,p+1}_{p,\lambda}(\Omega) \geq C\frac{\mathcal{H}^2(\Omega)^{p+1}}{\mathcal{H}^1(\partial\Omega)^p} + C\frac{\mathcal{H}^1(\partial\Omega)^p}{\mathcal{H}^2(\Omega)^{p+1}} \geq 2C.$$

So in this case if the optimal constant in (2.7) is obtained by a circle, the optimal shape for (1.1) is a circle. An interesting question worthy of further consideration is if the circle would be the minimizer for other parameters, as in similar discussions given in [19, 15, 14, 12]. Another natural question is to ask if in general one may improve the $C^{1,1}$-regularity by combining the established results with elliptic regularity theory, given that the variation of the perimeter-to-area ratio leads to a system of second order differential equations of the boundary parametrization.

In addition, it is interesting to improve the results of this paper to higher dimensions, again with a generalized ratio penalization. However, the geometric complexity of higher dimensional objects can increase significantly, and more work is required to exclude more complicated sets (e.g., "tentacles"), which were not an issue in the planar case; thus we expected to rely on rather different tools and arguments.

REFERENCES

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York, 2000.

[2] G. Buttazzo, E. Mainini, and E. Stepanov, *Stationary configurations for the average distance functional and related problems*, Control Cybernet., 38 (2009), pp. 1107–1130.

[3] G. Buttazzo, E. Oudet, and E. Stepanov, *Optimal transportation problems with free Dirichlet regions*, in Variational methods for discontinuous structures, Birkhäuser, Basel, 2002, pp. 41–65.

[4] G. Buttazzo, A. Pratelli, S. Solimini, and E. Stepanov, *Optimal urban networks via mass transportation*, Lecture Notes in Mathematics 1961, Springer-Verlag, Berlin, 2009.

[5] G. Buttazzo, A. Pratelli, and E. Stepanov, *Optimal pricing policies for public transportation networks*, SIAM J. Optim., 16 (2006), pp. 826–853.

[6] G. Buttazzo and F. Santambrogio, *A model for the optimal planning of an urban area*, SIAM J. Math. Anal., 37 (2005), pp. 514–530.

[7] G. Buttazzo and F. Santambrogio, *A mass transportation model for the optimal planning of an urban region*, SIAM Rev., 51 (2009), pp. 593–610.

[8] G. Buttazzo and E. Stepanov, *Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003), pp. 631–678.

[9] G. Buttazzo and E. Stepanov, *Minimization problems for average distance functionals*, in Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, vol. 14, Aracne, Rome, 2004, pp. 48–83.

[10] N. Campbell, J. B. Reece, L. G. Mitchell, and M. R. Taylor, *Biology: Concepts & Connections*, Benjamin-Cummings, San Francisco, CA, 2003.

[11] R. Eckhoff, *Dust Explosions in the Process Industries*, 3rd ed., Gulf Professional Publishing, Houston, TX, 2003.

[12] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Commun. Math. Phys., 336 (2015), pp. 441–507.

[13] W. Gao, T. Mogi, J. Yu, X. Yan, J. Sun, and R. Dobashi, *Flame propagation mechanisms in dust explosions*, J. Loss Prev. Process Ind., 36 (2015), pp. 186–194.

[14] M. Goldman, M. Novaga, and M. Röger, *Quantitative estimates for bending energies and applications to non-local variational problems*, Proc. Roy. Soc. Edinburgh Sect. A, 150 (2020), pp. 131–169.

[15] M. Goldman, M. Novaga, and B. Ruffini, *On minimizers of an isoperimetric problem with long-range interactions under a convexity constraint*, Anal. PDE, 11 (2018), pp. 1113–1142.

[16] A. Lemenant, *A presentation of the average distance minimizing problem*, J. Math. Sci., 181 (2012), pp. 820–836.

[17] A. Lemenant and E. Mainini, *On convex sets that minimize the average distance*, ESAIM Control Optim. Calc. Var., 18 (2012), pp. 1049–1072.

[18] X. Lu and D. Slepčev, *Properties of minimizers of average-distance problem via discrete approximation of measures*, SIAM J. Math. Anal., 45 (2013), pp. 3114–3131.

[19] C. Muratov and H. Knüpfer, *On an isoperimetric problem with a competing nonlocal term* II: *The general case*, Commun. Pure Appl. Math., 67 (2014), pp. 1974–1994.

[20] E. Paolini and E. Stepanov, *Qualitative properties of maximum distance minimizers and average distance minimizers in* $\mathbb{R}^n$, J. Math. Sci. (N.Y.), 122 (2004), pp. 3290–3309.

[21] F. Santambrogio and P. Tilli, *Blow-up of optimal sets in the irrigation problem*, J. Geom. Anal., 15 (2005), pp. 343–362.

[22] D. Slepčev, *Counterexample to regularity in average-distance problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), pp. 169–184.

[23] P. Tilli, *Some explicit examples of minimizers for the irrigation problem*, J. Convex Anal., 17 (2010), pp. 583–595.