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The average-distance problem with an Euler elastica penalization

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Abstract. We consider the minimization of an average-distance functional defined on a two-dimensional domain Ω with an Euler elastica penalization associated with $\partial\Omega$, the boundary of Ω. The average distance is given by

$$\int_{\Omega} \operatorname{dist}^p(x, \partial\Omega)\, \mathrm{d}x,$$

where $p \geq 1$ is a given parameter and $\operatorname{dist}(x, \partial\Omega)$ is the Hausdorff distance between $\{x\}$ and $\partial\Omega$. The penalty term is a multiple of the Euler elastica (i.e., the Helfrich bending energy or the Willmore energy) of the boundary curve $\partial\Omega$, which is proportional to the integrated squared curvature defined on $\partial\Omega$, as given by

$$\lambda \int_{\partial\Omega} \kappa_{\partial\Omega}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial\Omega},$$

where $\kappa_{\partial\Omega}$ denotes the (signed) curvature of $\partial\Omega$ and $\lambda > 0$ denotes a penalty constant. The domain Ω is allowed to vary among compact, convex sets of $\mathbb{R}^2$ with Hausdorff dimension equal to two. Under no a priori assumptions on the regularity of the boundary $\partial\Omega$, we prove the existence of minimizers of $E_{p,\lambda}$. Moreover, we establish the $C^{1,1}$-regularity of its minimizers. An original construction of a suitable family of competitors plays a decisive role in proving the regularity.

1. Introduction

The curvature of boundaries plays an important role in many physical and biological models. For instance, the elasticity of cell membranes is strongly correlated to its bending, and thus to its curvature. One way to quantify the bending energy per unit area of closed lipid bilayers was proposed by Helfrich in [11], and is now commonly referred to as "Helfrich bending energy". A related notion, from differential geometry, is the "Willmore energy", which measures how much a surface differs from the sphere [10]. In 2D, the Willmore energy simplifies to be a multiple of the integrated squared curvature, which is also commonly referred as the Euler elastica.

Easy access to the boundary is also relevant in nature: many processes such as heat dissipation, waste disposal, and nutrient absorption are more efficient when the whole

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body has “easy access” to its boundary. One way to quantify the “average accessibility” for points of a set $\Omega \subset \mathbb{R}^2$ to the boundary $\partial\Omega$ is an energy functional of the form

$$\Omega \mapsto \int_{\Omega} \mathrm{dist}^p(x, \partial\Omega)\,\mathrm{d}x \tag{1.1}$$

for a given parameter $p \geq 1$.

There are other energy functionals sharing similar geometric features with (1.1). For example, (1.1) is formally similar to the average-distance functional associated with a *given* domain $\Omega \subset \mathbb{R}^2$,

$$\Sigma \mapsto \int_{\Omega} \mathrm{dist}^p(x, \Sigma)\,\mathrm{d}x, \tag{1.2}$$

where the unknown Σ varies among compact subsets of $\overline{\Omega}$. In many existing studies, Σ is assumed to be a connected set with its Hausdorff dimension equal to one, and its one-dimensional Hausdorff measure is to be bounded from above by a specified constant. Problems of this type are used in many modeling applications, such as urban planning and optimal pricing. For a (non-exhaustive) list of references on the average-distance problem we refer to the works by Buttazzo et al. [2, 3, 5, 6] and [4, Chapter 3.3]. Also related are the papers by Paolini and Stepanov [14], Santambrogio and Tilli [15], Tilli [18], Lemenant and Mainini [13], Slepčev [16], and the review paper by Lemenant [12]. Alternatively, if Σ is assumed to vary among sets Ω consisting of discrete points with a fixed cardinality, say k, then the minimization of the functional in (1.2), often named the quantization error in this case, is related to the centroidal Voronoi tessellations [8] and k-means, which are widely studied in subjects such as vector quantization, signal compression, sensor and resource placement, geometric meshing, and so on [9].

In this work, we consider the average-distance energy functional as a functional of the domain Ω with $\Sigma = \partial\Omega$ penalized by the Euler elastica of $\partial\Omega$, as given by

$$E_{p,\lambda}(\Omega) = \int_{\Omega} \mathrm{dist}^p(x, \partial\Omega)\, dx + \lambda \int_{\partial\Omega} \kappa_{\partial\Omega}^2 \, d\mathcal{H}^1_{\llcorner\partial\Omega},$$

where $p \geq 1, \lambda > 0$ are given parameters, with λ proportional to a bending constant, and $\mathcal{H}^1_{\llcorner\partial\Omega}$ denotes the Hausdorff measure restricted on $\partial\Omega$. For further properties of the Hausdorff measure, we refer to [1]. We consider a free boundary problem associated with the minimization of $E_{p,\lambda}$ among domains Ω in the admissible set

$$\mathcal{A} := \left\{\Omega : \Omega \subset \mathbb{R}^2 \text{ is compact, convex, and Hausdorff two-dimensional}\right\}.$$

For any $\Omega_1, \Omega_2 \in \mathcal{A}$, define the metric in $\mathcal{A}$ as

$$d(\Omega_1, \Omega_2) := \mathcal{H}^2(\Omega_1 \triangle \Omega_2), \tag{1.3}$$

where $\triangle$ denotes the symmetric difference of the two sets and $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure.

The term

$$\int_{\partial\Omega} \kappa_{\partial\Omega}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega}$$

is the integrated squared curvature [7]. Since we do not make any a priori assumptions on the regularity of the boundary $\partial\Omega$, we need to make sense of the integrand $\kappa_{\partial\Omega}$. For future reference, we define it as follows: let γ be an arc-length parameterization of $\partial\Omega$ and define

$$\int_{\partial\Omega} \kappa_{\partial\Omega}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} := \begin{cases} \int_0^{\mathcal{H}^1(\partial\Omega)} |\gamma''|^2 \, \mathrm{d}s & \text{if } \gamma \in H^2((0, \mathcal{H}^1(\partial\Omega)); \mathbb{R}^2), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.4}$$

Here, $\mathcal{H}^1(\partial\Omega)$ denotes the total length of $\partial\Omega$. That is, we are reducing our minimization problem to quite regular sets, i.e., domains Ω whose boundaries admit an H^2 regular arc-length parameterization. Therefore, we are considering the minimization problem

$$\inf \left\{ \int_\Omega \mathrm{dist}^p(x, \partial\Omega) \, \mathrm{d}x + \lambda \int_0^{\mathcal{H}^1(\partial\Omega)} |\gamma''|^2 \, \mathrm{d}s : \mathcal{H}^1(\partial\Omega) > 0, \right.$$
$$\left. |\gamma'| = 1, \ \gamma([0, \mathcal{H}^1(\partial\Omega)]) \text{ is the boundary of a compact convex set} \right\}.$$

We note first a simple rescaling analysis where the domain Ω is stretched by a factor $\varepsilon > 0$. Given the two-dimensional nature, if $\varepsilon > 1$, then the average-distance functional is scaled by no more than ε^{2+p} but no less than ε^2. Meanwhile, the Euler elastica gets scaled by $1/\varepsilon$. This shows that the optimal Ω, if it exists, must have a suitable and finite size for any prescribed $\lambda > 0$. Indeed, the energy considered might be viewed as a competition between access to the boundary and the elastic stiffness of the boundary.

The main result of this paper is the following:

Theorem 1.1. *Given $p \geq 1$ and $\lambda > 0$, any minimizer Ω of $E_{p,\lambda}$ is $C^{1,1}$-regular with a Lipschitz constant of at most*

$$C = C(p, \lambda) := \sqrt{\lambda^{-1} p (C_1 + 1)^{p-1} \pi C_1^2 + 2C_2},$$

where

$$C_1 = C_1(p, \lambda) := (p+1)(p+2)\left(\frac{24}{\lambda}\right)^{p+1} (2(1 + \pi\lambda))^{p+2}, \tag{1.5}$$

$$C_2 = C_2(p, \lambda) := 32(\lambda^{-1} + \pi)^2 + 32\sqrt{2C_1\pi}(\lambda^{-1} + \pi)^{5/2} \tag{1.6}$$

are constants independent of Ω. That is, the boundary $\partial\Omega$ admits a $C^{1,1}$-regular arc-length parameterization $\gamma : [0, \mathcal{H}^1(\partial\Omega)] \to \mathbb{R}^2$ such that

$$|\gamma'(t_1) - \gamma'(t_2)| \leq C|t_1 - t_2|$$

for any t_1, t_2.

The rest of the paper is organized as follows: Section 2 is dedicated to proving some auxiliary estimates on elements of minimizing sequences. Existence of minimizers is shown in Section 3, while $C^{1,1}$-regularity is established in Section 4. Finally, in Section 5, we explore several future directions to further our understanding of the penalized average-distance problem. Technical results concerning properties of convex sets used in this paper will be presented in Appendix A.

2. Estimates

This section is dedicated to establishing quantitative bounds on the diameter and the area of any domain associated with the minimizing sequences of $E_{p,\lambda}$. In particular, the main result is Lemma 2.3, which provides a uniform upper bound on the diameter, crucial to the proof of the existence of minimizers.

Remark. It is worth noting that, due to (1.4), any set Ω whose boundary is not C^1-regular will have infinite energy, since a corner on $\partial\Omega$ with a discontinuous tangent corresponds to the Dirac measure in the curvature measure $\kappa_{\partial\Omega}$. Thus, we can restrict ourselves to C^1-regular sets.

Lemma 2.1. *Given $p \geq 1$ and $\lambda > 0$, for any $\Omega \in \mathcal{A}$ it holds that*

$$\operatorname{diam}(\Omega) \geq \frac{4\pi\lambda}{E_{p,\lambda}(\Omega)}. \tag{2.7}$$

Then, for any minimizing sequence $\Omega_n \subseteq \mathcal{A}$ (that is, $E_{p,\lambda}(\Omega_n) \to \inf_{\mathcal{A}} E_{p,\lambda}$), it holds that

$$\operatorname{diam}(\Omega_n) \geq \frac{2\pi\lambda}{1+\pi\lambda} \tag{2.8}$$

for all sufficiently large n.

Proof. Consider an arbitrary $\Omega \in \mathcal{A}$. Choose $x, y \in \partial\Omega$ such that $|x-y| = \operatorname{diam}(\Omega)$. Note that $\Omega \subseteq B(x, \operatorname{diam}(\Omega))$ and hence, due to the convexity of Ω (see Lemma A.1), it follows that

$$\mathcal{H}^1(\partial\Omega) \leq \pi \operatorname{diam}(\Omega).$$

As $\partial\Omega$ is a closed convex curve with winding number equal to one and our restriction on the curvature term ensures that the boundary is H^2 regular, it follows that

$$\int_{\partial\Omega} |\kappa_{\partial\Omega}| \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} = 2\pi,$$

and by Hölder's inequality we have

$$\begin{aligned} E_{p,\lambda}(\Omega) &\geq \lambda \int_{\partial\Omega} \kappa^2_{\partial\Omega} \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \\ &\geq \frac{4\pi^2\lambda}{\mathcal{H}^1(\partial\Omega)} \\ &\geq \frac{4\pi\lambda}{\operatorname{diam}(\Omega)} \end{aligned}$$

and hence, equation (2.7).

To prove (2.8), we show first that $\inf_{\mathcal{A}} E_{p,\lambda} < +\infty$. Consider the unit ball $B := B((0,0),1)$, and note that

$$\inf_{\mathcal{A}} E_{p,\lambda} \leq E_{p,\lambda}(B) = \int_B \operatorname{dist}^p(x, \partial B) \, \mathrm{d}x + \lambda \int_{\partial B} \kappa^2_{\partial B} \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial B} \leq \frac{\pi}{3} + 2\pi\lambda. \tag{2.9}$$

Let $\Omega_n \subseteq \mathcal{A}$ be an arbitrary minimizing sequence. Clearly, since $E_{p,\lambda}(\Omega_n) \to \inf_{\mathcal{A}} E_{p,\lambda}$, for all sufficiently large n it holds that

$$E_{p,\lambda}(\Omega_n) \leq \inf_{\mathcal{A}} E_{p,\lambda} + 2 - \frac{\pi}{3} \overset{(2.9)}{\leq} 2 + 2\pi\lambda, \tag{2.10}$$

and (2.7) gives

$$\operatorname{diam}(\Omega_n) \geq \frac{2\pi\lambda}{1+\pi\lambda}$$

and hence, (2.8). ■

In the following, we will use the *total variation* of a function u, which we now define. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^1(\Omega)$. Then,

$$\|u\|_{TV} := \sup\left\{\int_\Omega u \operatorname{div}\phi \, dx : \phi \in C_c^1(\Omega; \mathbb{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1\right\},$$

and

$$\|u\|_{BV} = \|u\|_{L^1} + \|u\|_{TV}.$$

Lemma 2.2. *Given $p \geq 1$, $\lambda > 0$, and $\Omega \in \mathcal{A}$, it holds that*

$$\mathcal{H}^2(\Omega) \geq \frac{\pi\lambda^2}{2E_{p,\lambda}(\Omega)^2}. \tag{2.11}$$

Moreover, given a minimizing sequence $\Omega_n \subseteq \mathcal{A}$ (that is, $E_{p,\lambda}(\Omega_n) \to \inf_{\mathcal{A}} E_{p,\lambda}$), we have

$$\mathcal{H}^2(\Omega_n) \geq \frac{\pi\lambda^2}{8(1+\pi\lambda)^2} \tag{2.12}$$

for all sufficiently large n.

To simplify notations, given a point $z \in \mathbb{R}^2$, we let z_x (resp. z_y) denote the x (resp. y) coordinate of z, and given points $x, y \in \mathbb{R}^2$, we denote by

$$[\![x, y]\!] := \{(1-s)x + sy : s \in [0,1]\}$$

the line segment between x and y.

Proof. Consider an arbitrary $\Omega \in \mathcal{A}$. Choose arbitrary points $\bar{x}, \bar{y} \in \partial\Omega$ such that $|\bar{x} - \bar{y}| = \operatorname{diam}(\Omega)$. Endow $\mathbb{R}^2$ with a Cartesian coordinate system, with the origin at the midpoint $(\bar{x} + \bar{y})/2$ (see Figure 1), such that

$$\bar{x} = (-\operatorname{diam}(\Omega)/2, 0), \qquad \bar{y} = (\operatorname{diam}(\Omega)/2, 0).$$

Let $\gamma : [0, \mathcal{H}^1(\partial\Omega)] \to \partial\Omega$ be an arc-length parameterization, and without loss of generality, we impose $\gamma(0) = \bar{x}$. We make and prove the following claim:

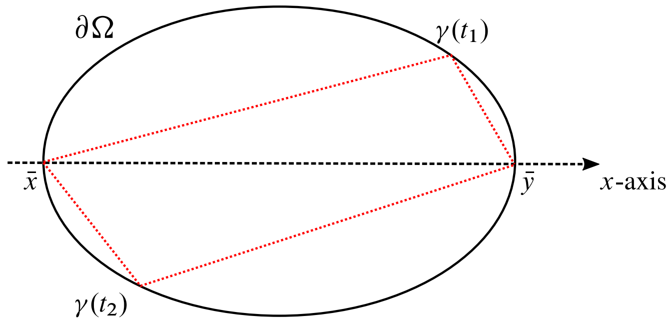

Figure 1. Schematic representation of the construction.

$\boldsymbol{\gamma'(0)_x = 0}$. Assume the opposite, i.e., $\gamma'(0)_x \neq 0$. For $|\varepsilon| \ll 1$, since γ is C^1-regular, it holds that $\gamma(\varepsilon) = \bar{x} + \varepsilon\gamma'(0) + v_\varepsilon$ for some vector v_ε with $|v_\varepsilon| = o(\varepsilon)$ as $\varepsilon \to 0$. Since $\bar{y} - \bar{x}$ is parallel to the x-axis, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{y} - \gamma(t)|\Big|_{t=0} = \lim_{\varepsilon\to 0} \frac{|\bar{y} - (\bar{x} + \varepsilon\gamma'(0))| - |\bar{y} - \bar{x}| + o(\varepsilon)}{\varepsilon} = \gamma'(0)_x \neq 0,$$

hence, $t = 0$ is not a maximum for $t \mapsto |\bar{y} - \gamma(t)|$. This contradicts

$$|\bar{y} - \bar{x}| = \mathrm{diam}(\Omega) = \max_{x\in\partial\Omega} |\bar{y} - x|,$$

and the claim is proven.

Without loss of generality, we can further impose $\gamma'(0) = (0, 1)$. Consider the region $\Omega \cap \{y \geq 0\}$. Set

$$t_1 := \inf\{t : \gamma'(t)_y = 1/2\},$$

where $\gamma(t)_y$ denotes the y-coordinate of $\gamma(t)$. By Hölder's inequality, it follows that

$$\frac{1}{4t_1} \leq \frac{\|\gamma_y'\|^2_{TV(0,t_1)}}{t_1} \leq \int_{\partial\Omega} \kappa^2_{\partial\Omega}\, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \leq \frac{E_{p,\lambda}(\Omega)}{\lambda} \Longrightarrow t_1 \geq \frac{\lambda}{4E_{p,\lambda}(\Omega)}. \tag{2.13}$$

Since $1/2 \leq \gamma_y'(t) \leq 1$ for any $t \in [0, t_1]$, it holds that $\gamma(t_1)_y \geq t_1/2$. Due to the convexity of $\Omega \cap \{y \geq 0\}$, both line segments $[\![\gamma(t_1), \bar{x}]\!]$ and $[\![\gamma(t_1), \bar{y}]\!]$ are contained in Ω, hence, $\triangle\bar{x}\gamma(t_1)\bar{y} \subseteq \Omega$. By construction, the triangle $\triangle\bar{x}\gamma(t_1)\bar{y}$ has base $[\![\bar{x}, \bar{y}]\!]$ and height $[\![\gamma(t_1), (\gamma(t_1)_x, 0)]\!]$. Therefore,

$$\mathcal{H}^2(\triangle\bar{x}\gamma(t_1)\bar{y}) = \frac{1}{2}|\bar{x} - \bar{y}| \cdot |\gamma(t_1)_y| \geq \frac{\mathrm{diam}(\Omega)t_1}{4}. \tag{2.14}$$

By repeating the same construction for $\Omega \cap \{y \leq 0\}$, we get the existence of

$$t_2 \geq \frac{\lambda}{4E_{p,\lambda}(\Omega)}$$

such that the triangle $\triangle \bar{x}\gamma(t_2)\bar{y}$ satisfies

$$\mathcal{H}^2(\triangle \bar{x}\gamma(t_2)\bar{y}) = \frac{1}{2}|\bar{x}-\bar{y}|\cdot|\gamma(t_2)_y| \geq \frac{\operatorname{diam}(\Omega)t_2}{4}. \tag{2.15}$$

Combining (2.14) and (2.15) gives

$$\mathcal{H}^2(\Omega) \geq \frac{\operatorname{diam}(\Omega)t_1}{4} + \frac{\operatorname{diam}(\Omega)t_2}{4} \overset{(2.7),(2.13)}{\geq} \frac{\pi\lambda^2}{2E_{p,\lambda}(\Omega)^2},$$

and hence, (2.11).

We conclude the proof that (2.12) holds by noting that the above arguments give

$$\mathcal{H}^2(\Omega_n) \overset{(2.11)}{\geq} \frac{\pi\lambda^2}{2E_{p,\lambda}(\Omega_n)^2} \overset{(2.10)}{\geq} \frac{\pi\lambda^2}{8(1+\pi\lambda)^2}$$

for any sufficiently large n. ∎

Lemma 2.3. *Given $p \geq 1$ and $\lambda > 0$, for any $\Omega \in \mathcal{A}$ it holds that*

$$\operatorname{diam}(\Omega) \leq (p+1)(p+2)\left(\frac{24}{\lambda}\right)^{p+1} E_{p,\lambda}(\Omega)^{p+2}. \tag{2.16}$$

Moreover, for any minimizing sequence $\Omega_n \subseteq \mathcal{A}$ (that is, $E_{p,\lambda}(\Omega_n) \to \inf_{\mathcal{A}} E_{p,\lambda}$) it holds that

$$\operatorname{diam}(\Omega_n) \leq C_1 \tag{2.17}$$

for all sufficiently large n, with C_1 defined in (1.5).

Proof. Similar to the proof of Lemma 2.2, consider an arbitrary $\Omega \in \mathcal{A}$ and choose arbitrary points $\bar{x}, \bar{y} \in \partial\Omega$ such that $|\bar{x}-\bar{y}| = \operatorname{diam}(\Omega)$. Endow $\mathbb{R}^2$ again with a Cartesian coordinate system, with the origin at the midpoint $(\bar{x}+\bar{y})/2$ (see Figure 2), such that

$$\bar{x} = (-\operatorname{diam}(\Omega)/2, 0), \qquad \bar{y} = (\operatorname{diam}(\Omega)/2, 0).$$

In the proof of Lemma 2.2 we have shown the existence of a point $q \in \partial\Omega$ (e.g., the point $\gamma(t_1)$) such that

$$\triangle \bar{x}q\bar{y} \subseteq \Omega, \qquad |q_y| \geq \frac{\lambda}{8E_{p,\lambda}(\Omega)}. \tag{2.18}$$

Let q_c be the incenter of $\triangle \bar{x}q\bar{y}$, and note that for any $z \in \triangle \bar{x}q_c\bar{y}$ we have

$$\operatorname{dist}(z, \partial(\triangle \bar{x}q\bar{y})) = \operatorname{dist}(z, [\![\bar{x},\bar{y}]\!]).$$

Denote by $q_c^\perp \in [\![\bar{x},\bar{y}]\!]$ the projection of q_c on $[\![\bar{x},\bar{y}]\!]$ and set

$$D_1 := |\bar{x}-q_c^\perp|, \qquad D_2 := |\bar{y}-q_c^\perp|, \qquad r := |q_c - q_c^\perp|.$$

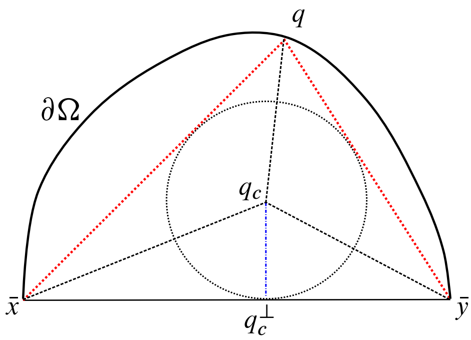

Figure 2. Schematic representation of the construction. Only the region $\Omega \cap \{y \geq 0\}$ is represented. Notice that the sides of $\triangle \bar{x} q \bar{y}$ are tangents of the incircle.

Clearly, $D_1 + D_2 = \operatorname{diam}(\Omega)$, and direct computation gives

$$\begin{aligned}\int_\Omega \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z &\geq \int_{\triangle \bar{x} q_c \bar{y}} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z \\ &\geq \int_{\triangle \bar{x} q_c \bar{y}} \operatorname{dist}^p(z, [\![\bar{x}, \bar{y}]\!])\,\mathrm{d}z \\ &= \int_{\triangle \bar{x} q_c \bar{y}} z_y^p\,\mathrm{d}z \\ &= \int_0^{D_1} \int_0^{\frac{r}{D_1}x} y^p\,\mathrm{d}y\,\mathrm{d}x + \int_0^{D_2} \int_0^{\frac{r}{D_2}x} y^p\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{r^{p+1}(D_1 + D_2)}{(p+1)(p+2)} \\ &= \frac{r^{p+1}\operatorname{diam}(\Omega)}{(p+1)(p+2)}. \end{aligned} \tag{2.19}$$

To estimate r, note that the sides $[\![\bar{x}, q_c]\!]$ and $[\![\bar{y}, q_c]\!]$ satisfy

$$|\bar{x} - \bar{y}| = \operatorname{diam}(\Omega) \geq \max\{|\bar{x} - q_c|, |\bar{y} - q_c|\}.$$

Since q_c is the incenter of $\triangle \bar{x} q \bar{y}$ and the sides of $\triangle \bar{x} q \bar{y}$ are tangents of the incircle, we have

$$\mathcal{H}^2(\triangle \bar{x} q \bar{y}) = \frac{1}{2}\operatorname{diam}(\Omega)|q_y| = \frac{1}{2}(\operatorname{diam}(\Omega) + |\bar{x} - q_c| + |\bar{y} - q_c|)r.$$

Thus, we infer

$$r \geq \frac{|q_y|}{3} \overset{(2.18)}{\geq} \frac{\lambda}{24 E_{p,\lambda}(\Omega)}.$$

Substituting into (2.19) gives

$$\Big(\frac{\lambda}{24E_{p,\lambda}(\Omega)}\Big)^{p+1}\cdot\frac{\mathrm{diam}(\Omega)}{(p+1)(p+2)}\le\int_\Omega \mathrm{dist}^p(z,\partial\Omega)\,\mathrm{d}z\le E_{p,\lambda}(\Omega),$$

and hence, (2.16).

To prove (2.17), note that for any minimizing sequence it holds that

$$\begin{aligned} &E_{p,\lambda}(\Omega_n)\overset{(2.10)}{\le}2(1+\pi\lambda)\\ &\qquad\Longrightarrow \mathrm{diam}(\Omega_n)\overset{(2.16)}{\le}(p+1)(p+2)\Big(\frac{24}{\lambda}\Big)^{p+1}(2(1+\pi\lambda))^{p+2}=C_1\end{aligned}$$

for all sufficiently large n. ∎

3. Proof of existence

Set

$$\overline{\mathcal{A}}:=\text{completion of }\mathcal{A}\text{ with respect to the metric }d,$$

where d is defined in (1.3).

Lemma 3.1. *Given a compact set $\Sigma\subseteq\mathbb{R}^2$ and a sequence of curves $\{\gamma_k\}:[0,1]\to\Sigma$ satisfying*

$$\sup_k\|\gamma_k'\|_{BV}<+\infty,\qquad \sup_k\mathcal{H}^1(\gamma_k([0,1]))<+\infty,$$

where $\|\cdot\|_{BV}$ denotes the BV norm, there exists a curve $\gamma:[0,1]\to\Sigma$ such that (upon subsequence) it holds that:

(1) *$\gamma_k\to\gamma$ in C^α for any $\alpha\in[0,1)$,*

(2) *$\gamma_k'\to\gamma'$ in L^p for any $p\in[1,\infty)$, and*

(3) *$\gamma_k''\overset{*}{\rightharpoonup}\gamma''$ in the space of signed Borel measures.*

This is a classical result (see for instance [16], to which we refer for the proof).

Lemma 3.2. *If a minimizing sequence $\Omega_n\subseteq\mathcal{A}$ converges to some $\Omega\in\overline{\mathcal{A}}\backslash\mathcal{A}$, then Ω must be either be a point or a line segment.*

Proof. The compactness of Ω can be guaranteed by Lemma 2.3. To see that $\Omega\in\overline{\mathcal{A}}\backslash\mathcal{A}$ is convex, consider an arbitrary pair of points $P,Q\in\Omega$ and $t\in(0,1)$. We now show that $(1-t)P+tQ\in\Omega$. Consider sequences $P_n,Q_n\in\Omega_n$ such that $P_n\to P$, $Q_n\to Q$. Since each Ω_n is convex, $(1-t)P_n+tQ_n\in\Omega_n$. By Lemma 3.1, we know

$$\|\gamma_n-\gamma\|_{C^0([0,1];\mathbb{R}^2)}\to 0.$$

As a consequence,

$$d_{\mathcal{H}}(\partial\Omega_n,\partial\Omega)\to 0.$$

This allows us to choose, for each n, another point $z_n \in \Omega$ such that $|z_n - ((1-t)P_n + tQ_n)| \le d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega)$. By construction, now the sequences $(1-t)P_n + tQ_n$ and z_n have the same limit. As $(1-t)P_n + tQ_n \to (1-t)P + tQ$ and $z_n \to z$, hence $z = (1-t)P + tQ$, using the compactness of Ω finally gives $z \in \Omega$. Then, if Ω contains non-collinear points x, y, z, by convexity we have $\triangle xyz \subseteq \Omega$, which would give the contradiction $\Omega \in \mathcal{A}$. ■

Lemma 3.3. *Given a sequence $\Omega_n \subseteq \mathcal{A}$ such that $E_{p,\lambda}(\Omega_n)$ is bounded, there exists $\Omega \in \mathcal{A}$ such that a subsequence of Ω_n (still denoted by Ω_n) converges to Ω with respect to the metric d defined in* (1.3).

Proof. By (2.7) and (2.16), we have

$$\begin{aligned}(p+1)(p+2)\Big(\frac{24}{\lambda}\Big)^{p+1} E_{p,\lambda}(\Omega_n)^{p+2} &\ge \operatorname{diam}(\Omega_n)\\ &\ge \frac{4\pi\lambda}{E_{p,\lambda}(\Omega_n)},\end{aligned}$$

hence, $\sup_n \operatorname{diam}(\Omega_n) < +\infty$, that is, Ω_n has uniform bounded diameters. Note also that $\sup_n E_{p,\lambda}(\Omega_n) < +\infty$ implies

$$\sup_n \int_{\partial\Omega_n} \kappa^2_{\partial\Omega_n}\, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_n} < +\infty,$$

i.e., the curvatures of Ω_n are uniformly square integrable. Then, by letting $\gamma_n : [0, 2\pi] \to \mathbb{R}^2$ be constant speed parameterizations of $\partial\Omega_n$, we have that a subsequence γ_n satisfies all the hypotheses of Lemma 3.1, thus concluding the proof. ■

Based on Lemmas 2.1, 2.2, 2.3, and 3.3 we get the following corollary:

Corollary 3.4. *Any minimizer (if they exist at all) satisfies estimates* (2.8), (2.12), *and* (2.17).

Lemma 3.5. *For any $p \ge 1$ and $\lambda > 0$, the functional $E_{p,\lambda}$ admits a minimizer in $\mathcal{A}$.*

Proof. Consider a minimizing sequence $\Omega_n \subseteq \mathcal{A}$. Since $E_{p,\lambda}$ is invariant under rigid movements, we can assume that $(0,0) \in \Omega_n$ for any n. In view of (2.10), without loss of generality, we can also impose

$$\sup_n E_{p,\lambda}(\Omega_n) \le 2(1 + \pi\lambda).$$

Then, by Lemma 2.3 we get $\sup_n \operatorname{diam}(\Omega_n) \le C_1$, and hence

$$\Omega_n \subseteq B((0,0), C_1) \quad \text{for any } n.$$

Thus, Ω_n is a sequence of uniformly bounded, compact sets and there exists (upon subsequence, which we do not relabel) a limit set $\Omega \in \overline{\mathcal{A}}$ such that $\Omega_n \to \Omega$ in the metric d (defined in (1.3)).

We claim

$$\int_{\Omega} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z = \lim_{n\to+\infty} \int_{\Omega_n} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z, \tag{3.20}$$

$$\int_{\partial\Omega} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \le \liminf_{n\to+\infty} \int_{\partial\Omega_n} \kappa_{\partial\Omega_n}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_n}. \tag{3.21}$$

Inequality (3.21) follows rather straightforwardly from the lower semicontinuity of the H^2 norm.

We need to prove (3.20). In the following, it is useful to recall Lemma A.1, which states that the diameter is continuous with respect to the convergence in $\mathcal{A}$. We split the sums

$$\int_{\Omega_n} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z = \int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z + \int_{\Omega_n\cap\Omega} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z,$$

$$\int_{\Omega} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z = \int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z + \int_{\Omega_n\cap\Omega} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z,$$

and note that

$$\begin{aligned} &\left|\int_{\Omega_n} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z - \int_{\Omega} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z\right| \\ &\qquad \le \int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z + \int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z \qquad (3.22)\\ &\qquad\quad + \int_{\Omega_n\cap\Omega} |\operatorname{dist}^p(z, \partial\Omega_n) - \operatorname{dist}^p(z, \partial\Omega)|\,\mathrm{d}z. \qquad (3.23) \end{aligned}$$

Moreover,

$$\int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z \le \mathcal{H}^2(\Omega_n\setminus\Omega)\operatorname{diam}(\Omega_n)^p \le \mathcal{H}^2(\Omega_n\setminus\Omega)C_1^p \to 0,$$

$$\int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z \le \mathcal{H}^2(\Omega\setminus\Omega_n)\operatorname{diam}(\Omega)^p \le \mathcal{H}^2(\Omega\setminus\Omega_n)C_1^p \to 0,$$

and hence,

$$\lim_{n\to+\infty} \int_{\Omega_n\setminus\Omega} \operatorname{dist}^p(z, \partial\Omega_n)\,\mathrm{d}z = \lim_{n\to+\infty} \int_{\Omega\setminus\Omega_n} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z = 0.$$

To prove

$$\lim_{n\to+\infty} \int_{\Omega_n\cap\Omega} |\operatorname{dist}^p(z, \partial\Omega_n) - \operatorname{dist}^p(z, \partial\Omega)|\,\mathrm{d}z = 0,$$

denote by $d_{\mathcal{H}}$ the Hausdorff distance, and by the mean value theorem, it holds that

$$\int_{\Omega_n\cap\Omega} |\operatorname{dist}^p(z, \partial\Omega_n) - \operatorname{dist}^p(z, \partial\Omega)|\,\mathrm{d}z$$

$$\leq \int_{\Omega_n \cap \Omega} |\operatorname{dist}(z, \partial\Omega_n) - \operatorname{dist}(z, \partial\Omega)|$$
$$\cdot\, p \sup_{z \in \Omega_n \cap \Omega} \big(\max\{\operatorname{dist}(z, \partial\Omega_n), \operatorname{dist}(z, \partial\Omega)\}\big)^{p-1} \, \mathrm{d}z$$
$$\leq \mathcal{H}^2(\Omega_n \cap \Omega) d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega) \cdot pC_1^{p-1}$$
$$\leq \pi C_1^2 d_{\mathcal{H}}(\partial\Omega_n, \partial\Omega) \cdot pC_1^{p-1} \to 0.$$

Thus, both terms (3.22) and (3.23) converge to zero, and (3.20) is proven.

Combining with (3.20) gives

$$E_{p,\lambda}(\Omega) \leq \liminf_{n\to+\infty} E_{p,\lambda}(\Omega_n) = \inf_{\bar{\mathcal{A}}} E_{p,\lambda},$$

hence, Ω is effectively a minimizer of $E_{p,\lambda}$. Since Ω is the limit of $\Omega_n \subseteq \mathcal{A}$, based on Lemma 3.2 and Corollary 3.4, we have $\Omega \in \mathcal{A}$. ■

4. Proof of regularity

This section completes the proof of Theorem 1.1 by establishing the desired regularity of the minimizers. A few technical estimates used in the proof are left as separate lemmas proved at the end of the section.

Proof of Theorem 1.1. Let Ω be a minimizer of $E_{p,\lambda}$ and let γ be an arc-length parameterization of $\partial\Omega$. Assume there exist $M, \varepsilon, t_1 < t_2$ such that

$$|\gamma'(t_2) - \gamma'(t_1)| = M\varepsilon, \qquad t_2 - t_1 = \varepsilon. \tag{4.24}$$

The quantity ε is assumed to be vanishingly small, and estimates involving ε will be in general valid for sufficiently small ε, rather than all ε. The goal is to find an upper bound for M.

Without loss of generality, upon rigid movements, we can assume $t_1 = 0$, $t_2 = \varepsilon$. Endow $\mathbb{R}^2$ with a Cartesian coordinate system with

$$\gamma(0) \in \{x \geq 0, y = 0\}, \qquad \gamma(\varepsilon) \in \{y \geq 0, x = 0\}, \qquad \gamma'(0) = (0, 1). \tag{4.25}$$

We first give an estimate of $\gamma(\varepsilon)_y$. Using Hölder's inequality and recalling the fact that

$$\kappa_{\partial\Omega} \ll \mathcal{H}^1_{\llcorner\partial\Omega}, \qquad \frac{\mathrm{d}\kappa_{\partial\Omega}}{\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega}} \in L^2(0, \mathcal{H}^1(\partial\Omega); \mathbb{R}),$$

for any $t \in [0, \varepsilon]$ it holds that

$$\frac{E_{p,\lambda}(\Omega)}{\lambda} \geq \int_{\gamma([0,t])} \kappa^2_{\partial\Omega} \, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} = \int_0^t |\gamma''|^2 \, \mathrm{d}s \geq \frac{|\gamma'(t) - \gamma'(0)|^2}{\varepsilon} \geq \frac{|\gamma'(t)_y - 1|^2}{\varepsilon}$$
$$\Longrightarrow \; |\gamma'(t)_y - 1| = 1 - \gamma'(t)_y \leq \sqrt{\varepsilon E_{p,\lambda}(\Omega)/\lambda}$$

$$\implies \gamma'(t)_y \geq 1 - \sqrt{\varepsilon E_{p,\lambda}(\Omega)/\lambda},$$

and hence,

$$\gamma(\varepsilon)_y = \int_0^\varepsilon \gamma'(t)_y \,\mathrm{d}t \geq \int_0^\varepsilon \big[1 - \sqrt{\varepsilon E_{p,\lambda}(\Omega)/\lambda}\big] \,\mathrm{d}t = \varepsilon\big[1 - \sqrt{\varepsilon E_{p,\lambda}(\Omega)/\lambda}\big].$$

On the other hand, as we imposed $\gamma(\varepsilon)_y \geq 0$, we have

$$\gamma(\varepsilon)_y = |\gamma(\varepsilon)_y| = \Big|\int_0^\varepsilon \gamma'(t)_y \,\mathrm{d}t\Big| \leq \int_0^\varepsilon |\gamma'(t)_y| \,\mathrm{d}t \leq \varepsilon.$$

In particular,

$$\varepsilon - \sqrt{E_{p,\lambda}(\Omega)/\lambda}\,\varepsilon^{3/2} \leq \gamma(\varepsilon)_y \leq \varepsilon,$$

and therefore

$$\gamma(\varepsilon)_y = \varepsilon + O(\varepsilon^{3/2}). \tag{4.26}$$

Construct the competitor Ω_ε in the following way:

(1) Denote by $t_\pm$ the two times such that $\gamma'(t_\pm) = (\pm 1, 0)$ and by $t_\perp$ the time such that $\gamma'(t_\perp) = (0, -1)$. Since we imposed $\gamma'(0) = (0, 1)$, without loss of generality we can assume that the tangent direction turns *counterclockwise*, i.e.,

$$\varepsilon < t_- < t_\perp < t_+ < \mathcal{H}^1(\partial\Omega).$$

Note that

$$\begin{aligned}\frac{2}{t_\perp - t_-} = \frac{|\gamma'(t_\perp) - \gamma'(t_-)|^2}{t_\perp - t_-} &\leq \int_{\gamma([t_-, t_\perp])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \leq \frac{E_{p,\lambda}(\Omega)}{\lambda} \\ &\overset{(2.9)}{\leq} 2\lambda^{-1} + 2\pi \\ \implies t_\perp - t_- \geq \frac{1}{\lambda^{-1} + \pi}.\end{aligned}$$

Similarly, we get

$$\min\big\{\mathcal{H}^1(\partial\Omega) - t_+,\ t_+ - t_\perp,\ t_-\big\} \geq \frac{1}{\lambda^{-1} + \pi}. \tag{4.27}$$

(2) Define the vector field $v : [t_-, t_+] \to \mathbb{R}^2$ as

$$v(s) := \begin{cases} \Big(\cos\big(\frac{\pi}{2}\big(1 + \frac{s - t_-}{t_\perp - t_-}\big)\big), \sin\big(\frac{\pi}{2}\big(1 + \frac{s - t_-}{t_\perp - t_-}\big)\big)\Big), & \text{if } s \in [t_-, t_\perp], \\ \Big(\cos\big(\frac{\pi}{2}\big(1 + \frac{t_+ - s}{t_+ - t_\perp}\big)\big), -\big(\frac{t_+ - s}{t_+ - t_\perp}\big)^2 \sin\big(\frac{\pi}{2}\big(1 + \frac{t_+ - s}{t_+ - t_\perp}\big)\big)\Big), & \text{if } s \in [t_\perp, t_+]. \end{cases} \tag{4.28}$$

Note first that v is continuous (smooth outside $t_\perp$), and direct computation gives

$$v'(s) = \begin{cases} \frac{\pi/2}{t_\perp - t_-}\Big(-\sin\big(\frac{\pi}{2}\big(1+\frac{s-t_-}{t_\perp-t_-}\big)\big), \cos\big(\frac{\pi}{2}\big(1+\frac{s-t_-}{t_\perp-t_-}\big)\big)\Big), & \text{if } s \in [t_-, t_\perp), \\ \Big(\frac{\pi/2}{t_+-t_\perp}\sin\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big), & \\ \quad \frac{\pi/2}{t_+-t_\perp}\big(\frac{t_+-s}{t_+-t_\perp}\big)^2\cos\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big) & \\ \quad +2\frac{t_+-s}{(t_+-t_\perp)^2}\sin\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big)\Big), & \text{if } s \in (t_\perp, t_+]. \end{cases}$$

In particular,

$$\lim_{t\to t_\perp^-} v'(t) = \frac{\pi/2}{t_\perp - t_-}(0,-1), \qquad \lim_{t\to t_\perp^+} v'(t) = \frac{\pi/2}{t_+ - t_\perp}(0,-1),$$

that is, the left and right limit differ just by a multiplicative constant. This observation is crucial, since it implies that the tangent derivative of the arc-length reparameterization of v does *not* jump at $t = t_\perp$ (recall also that γ' does not jump at $t = t_\perp$, hence, the tangent derivative of the arc-length reparameterization of $\gamma + cv$ does not jump at $t = t_\perp$, for any $c > 0$). We claim:

$$\|v'\|_{L^\infty} \le \max\left\{\frac{\pi/2}{t_\perp - t_-}, \frac{4}{t_+ - t_\perp}\right\} \le 4(\lambda^{-1} + \pi) < +\infty, \tag{4.29}$$

$$\|v''\|_{L^\infty} \le 16(\lambda^{-1} + \pi)^2 < +\infty. \tag{4.30}$$

The proofs of both claims are presented in Lemma 4.1 below.

(3) Let γ_ε be the curve such that

$$\gamma_\varepsilon(t) := \begin{cases} (2\gamma(t)_x, 2\gamma(t)_y) & \text{if } t \in [0,\varepsilon], \\ \gamma(t) + (0, \gamma(\varepsilon)_y) & \text{if } t \in [\varepsilon, t_-], \\ \gamma(t) + \gamma(\varepsilon)_y v(t) & \text{if } t \in [t_-, t_+], \\ \big(\gamma(t)_x\big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\big) - \frac{\gamma(t_+)_x\gamma(0)_x}{\gamma(0)_x-\gamma(t_+)_x}\big) & \text{if } t \in [t_+, \mathcal{H}^1(\partial\Omega)]. \end{cases} \tag{4.31}$$

Note that γ_ε defined in (4.31) is injective. Let $\partial\Omega_\varepsilon$ be the image of γ_ε, and Ω_ε be the bounded region of the plane delimited by $\partial\Omega_\varepsilon$. This will be our competitor. Observe first that, as $\gamma'(t_+) = (1,0)$,

$$\lim_{t\to t_+^-} \gamma_\varepsilon'(t) = \Big(1 + \gamma(\varepsilon)_y \frac{\pi/2}{t_+ - t_\perp}\Big)(1,0),$$

$$\lim_{t\to t_+^+} \gamma_\varepsilon'(t) = \Big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\Big)(1,0),$$

that is, the left and right limit differ just by a multiplicative constant. This observation is again crucial, since it implies that the tangent derivative of the arc-length reparameterization of $\gamma + \gamma(\varepsilon)_y v$ does *not* jump at $t = t_+$.

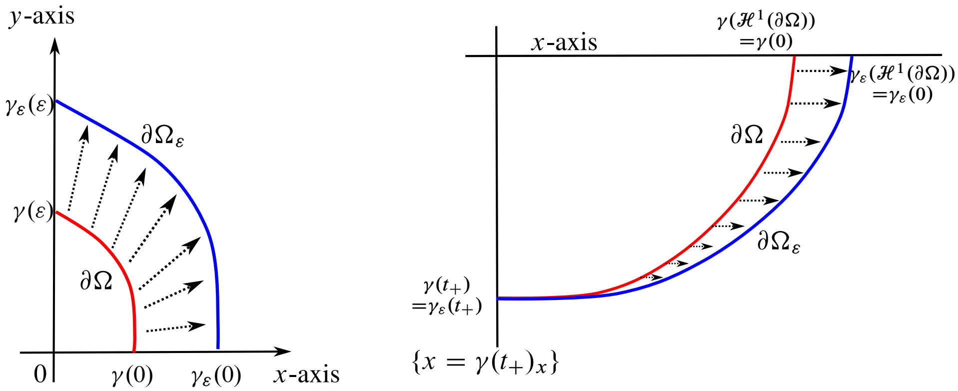

Figure 3. Representation of the construction of the competitor γ_ε, for $t \in [0, \varepsilon]$ (left) and $t \in [t_+, \mathcal{H}^1(\partial\Omega)]$ (right).

Intuitively, for $t \in [0, \mathcal{H}^1(\partial\Omega)]$ the competitor γ_ε is constructed from γ (see Figure 3) by:

(1) a homothety of center $(0, 0)$ and ratio 2 for $t \in [0, \varepsilon]$,

(2) a translation of the vector $(0, \gamma(\varepsilon)_y)$ for $t \in [\varepsilon, t_-]$,

(3) adding the smooth vector field $\gamma(\varepsilon)_y v(t)$ for $t \in [t_-, t_+]$,

(4) a scaling of factor

$$1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}$$

and then translation to the right by

$$\frac{\gamma(t_+)_x \gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}$$

in the x direction for $t \in [t_+, \mathcal{H}^1(\partial\Omega)]$.

It is straightforward to check compactness and convexity for Ω_ε. Moreover, denoting by $\tilde{\gamma}_\varepsilon$ the arc-length reparameterization of γ_ε, the curvature of $\tilde{\gamma}_\varepsilon$ is still a *function* (instead of a more generic measure), as γ_ε is always constructed from γ via translation, scaling, or summing with smooth vector fields, and the tangent derivative $\tilde{\gamma}'_\varepsilon$ never jumps at “junction points” (i.e., for $t = \varepsilon, t_-, t_\perp, t_+$, and $\mathcal{H}^1(\partial\Omega)$, respectively).

Next, to estimate $E_{p,\lambda}(\Omega_\varepsilon) - E_{p,\lambda}(\Omega)$, we claim

$$\int_{\Omega_\varepsilon} \operatorname{dist}^p(z, \partial\Omega_\varepsilon)\,\mathrm{d}z - \int_{\Omega} \operatorname{dist}^p(z, \partial\Omega)\,\mathrm{d}z \\ \leq \varepsilon \cdot p(C_1 + 1)^{p-1}\pi C_1^2/2 + (2\varepsilon)^{p+1}\pi(C_1 + 1) \tag{4.32}$$

and

$$\int_{\partial\Omega_\varepsilon} \kappa^2_{\partial\Omega_\varepsilon}\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} - \int_{\partial\Omega} \kappa^2_{\partial\Omega}\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \leq \varepsilon\Big(C_2 - \frac{M^2}{2}\Big) + O(\varepsilon^{3/2}), \tag{4.33}$$

with

$$C_2 = 32(\lambda^{-1} + \pi)^2 + 32\sqrt{2C_1\pi}(\lambda^{-1} + \pi)^{5/2},$$

as defined in (1.6).

Step 1. Proof of (4.33). Using the notation from (4.31), we make the following claims:

$$\int_{\gamma([\varepsilon,t_-])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} = \int_{\gamma_\varepsilon([\varepsilon,t_-])} \kappa_{\partial\Omega_\varepsilon}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} \tag{4.34}$$

and

$$\int_{\gamma_\varepsilon([t_+,\mathcal{H}^1(\partial\Omega)])} \kappa_{\partial\Omega_\varepsilon}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} - \int_{\gamma([t_+,\mathcal{H}^1(\partial\Omega)])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \le O(\varepsilon^{3/2}), \tag{4.35}$$

$$\int_{\gamma_\varepsilon([t_-,t_+])} \kappa_{\partial\Omega_\varepsilon}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} - \int_{\gamma([t_-,t_+])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} \le \varepsilon C_2 + O(\varepsilon^{3/2}), \tag{4.36}$$

$$\int_{\gamma([0,\varepsilon])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} - \int_{\gamma_\varepsilon([0,\varepsilon])} \kappa_{\partial\Omega_\varepsilon}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} \ge \frac{M^2\varepsilon}{2}. \tag{4.37}$$

The proof of all four assertions are quite technical, and for the reader's convenience, will be done in Lemmas 4.2 and 4.3 below. Combining (4.34), (4.35), (4.36), and (4.37) gives

$$\int_{\Omega_\varepsilon} \kappa_{\partial\Omega_\varepsilon}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\Omega_\varepsilon} \le \int_{\Omega} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\Omega} + \varepsilon\Big(C_2 - \frac{M^2}{2}\Big) + O(\varepsilon^{3/2}),$$

and hence, (4.33).

Step 2. Proof of (4.32). Recall that the construction of the competitor Ω_ε in (4.31) also gives

(1) $\gamma(\varepsilon)_y > \gamma(0)_x > 0$.

(2) For $t \in [\varepsilon, \mathcal{H}^1(\partial\Omega)]$ the competitor $\gamma_\varepsilon(t)$ is obtained by translating $\gamma(t)$ by a vector of length at most 2ε. Moreover, it holds that

$$\frac{\gamma(t)_x - \gamma(t_+)_x}{\gamma(0)_x - \gamma(t_+)_x}\gamma(0)_x \le 2\varepsilon, \quad \forall t \in [\varepsilon, \mathcal{H}^1(\partial\Omega)],$$

since by construction we have $\gamma(0)_x \le 2\varepsilon$, and $\gamma(0)$ is the point with the most positive x coordinate, which ensures that

$$\frac{\gamma(t)_x - \gamma(t_+)_x}{\gamma(0)_x - \gamma(t_+)_x} \le 1.$$

(3) For $t \in [0, \varepsilon]$, the competitor $\gamma_\varepsilon(t)$ is obtained by scaling $\gamma(t)$ by a factor of two.

One readily checks for all t that $|\gamma_\varepsilon(t) - \gamma(t)| \le 2\varepsilon$ holds, and

$$d_{\mathcal{H}}(\partial\Omega_\varepsilon, \partial\Omega) \le 2\varepsilon.$$

Thus, by the mean value theorem, for each point $x \in \Omega_\varepsilon \cap \Omega$ we have

$$\begin{aligned}\operatorname{dist}^p(x,\partial\Omega_\varepsilon) - \operatorname{dist}^p(x,\partial\Omega) &\le (\operatorname{dist}(x,\partial\Omega_\varepsilon) - \operatorname{dist}(x,\partial\Omega))\\ &\qquad \cdot p\Big(\sup_{z\in\Omega_\varepsilon\cap\Omega} \max\{\operatorname{dist}(x,\partial\Omega_\varepsilon), \operatorname{dist}(x,\partial\Omega)\}\Big)^{p-1}\\ &\le 2\varepsilon\cdot p(\operatorname{diam}(\Omega)+2\varepsilon)^{p-1}\\ &\overset{(2.17)}{\le} \varepsilon\cdot 2p(C_1+1)^{p-1},\end{aligned}$$

with C_1 defined in (1.5). Thus, by convexity of Ω,

$$\mathcal{H}^2(\Omega_\varepsilon\cap\Omega) \le \mathcal{H}^2(\Omega) \le \pi(\operatorname{diam}(\Omega)/2)^2.$$

It follows that

$$\begin{aligned}\int_{\Omega_\varepsilon\cap\Omega} \operatorname{dist}^p(x,\partial\Omega_\varepsilon)\,\mathrm{d}x - \int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x &\le \varepsilon\cdot 2p(C_1+1)^{p-1}\mathcal{H}^2(\Omega_\varepsilon\cap\Omega)\\ &\le \varepsilon\cdot p(C_1+1)^{p-1}\pi C_1^2/2. \end{aligned}\tag{4.38}$$

Then note that, since by construction we have $d_{\mathcal{H}}(\partial\Omega_\varepsilon,\partial\Omega)\le 2\varepsilon$, it follows that

$$\begin{aligned}\int_{\Omega_\varepsilon\setminus\Omega} \operatorname{dist}^p(x,\partial\Omega_\varepsilon)\,\mathrm{d}x \le (2\varepsilon)^p\mathcal{H}^2(\Omega_\varepsilon\setminus\Omega) &\le (2\varepsilon)^p\cdot 2\varepsilon\mathcal{H}^1(\partial\Omega_\varepsilon)\\ &\le (2\varepsilon)^{p+1}\pi(\operatorname{diam}(\Omega)+4\varepsilon)\\ &\le (2\varepsilon)^{p+1}\pi(C_1+1). \end{aligned}\tag{4.39}$$

The inequality $\mathcal{H}^2(\Omega_\varepsilon\setminus\Omega)\le 2\varepsilon\mathcal{H}^1(\partial\Omega_\varepsilon)$ is due to the convexity of Ω_ε, the fact that $d_{\mathcal{H}}(\partial\Omega_\varepsilon,\partial\Omega)\le 2\varepsilon$, and Lemma A.3. Combining (4.38) and (4.39) gives

$$\begin{aligned}&\int_{\Omega_\varepsilon} \operatorname{dist}^p(x,\partial\Omega_\varepsilon)\,\mathrm{d}x - \int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x\\ &\qquad\le \varepsilon\cdot p(C_1+1)^{p-1}\pi C_1^2/2 + (2\varepsilon)^{p+1}\pi(C_1+1). \end{aligned}\tag{4.40}$$

Thus, (4.32) is proven.

Combining (4.32) and (4.33) we finally infer

$$\begin{aligned}E_{p,\lambda}(\Omega_\varepsilon) - E_{p,\lambda}(\Omega) &= \int_{\Omega_\varepsilon} \operatorname{dist}^p(x,\partial\Omega_\varepsilon)\,\mathrm{d}x - \int_\Omega \operatorname{dist}^p(x,\partial\Omega)\,\mathrm{d}x\\ &\qquad + \lambda\bigg(\int_{\partial\Omega_\varepsilon} \kappa^2_{\partial\Omega_\varepsilon}\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} - \int_{\partial\Omega} \kappa^2_{\partial\Omega}\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega}\bigg)\\ &\le \varepsilon\cdot p(C_1+1)^{p-1}\pi C_1^2/2 + (2\varepsilon)^{p+1}\pi(C_1+1)\\ &\qquad + \lambda\Big(\varepsilon\Big(C_2 - \frac{M^2}{2}\Big) + O(\varepsilon^{3/2})\Big).\end{aligned}$$

Note also that the term $(2\varepsilon)^{p+1}\pi(C_1+1)$ can be absorbed into $O(\varepsilon^{3/2})$ due to the condition $p\ge 1$, therefore,

$$E_{p,\lambda}(\Omega_\varepsilon) - E_{p,\lambda}(\Omega) \le \lambda\Big(\varepsilon\Big(\lambda^{-1}p(C_1+1)^{p-1}\pi C_1^2/2 + C_2 - \frac{M^2}{2}\Big) + O(\varepsilon^{3/2})\Big).$$

The minimality assumption on Ω and the arbitrariness of $\varepsilon > 0$ then imply

$$\lambda^{-1} p(C_1+1)^{p-1}\pi C_1^2/2 + C_2 - \frac{M^2}{2} \geq 0$$
$$\Longrightarrow \; M \leq \sqrt{\lambda^{-1} p(C_1+1)^{p-1}\pi C_1^2 + 2C_2} = C,$$

and the proof is complete. ∎

Lemma 4.1. *Under the hypotheses of Theorem 1.1, assertions (4.29) and (4.30) hold.*

Proof. We use the same notations from the proof of Theorem 1.1. Since

$$v'(s) = \begin{cases} \frac{\pi/2}{t_\perp - t_-}\Big(-\sin\big(\frac{\pi}{2}\big(1+\frac{s-t_-}{t_\perp - t_-}\big)\big), \\ \qquad \cos\big(\frac{\pi}{2}\big(1+\frac{s-t_-}{t_\perp-t_-}\big)\big)\Big), & \text{if } s \in [t_-, t_\perp), \\ \Big(\frac{\pi/2}{t_+ - t_\perp}\sin\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big), \\ \qquad \frac{\pi/2}{t_+-t_\perp}\big(\frac{t_+-s}{t_+-t_\perp}\big)^2\cos\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big) \\ \qquad +2\frac{t_+-s}{(t_+-t_\perp)^2}\sin\big(\frac{\pi}{2}\big(1+\frac{t_+-s}{t_+-t_\perp}\big)\big)\Big) & \text{if } s \in (t_\perp, t_+], \end{cases}$$

it follows that

$$|v'(s)| \leq \frac{\pi/2}{t_\perp - t_-} \quad \text{for any } s \in [t_-, t_\perp),$$

and

$$\begin{aligned} |v'(s)| &= \Big[\Big(\frac{\pi/2}{t_+ - t_\perp}\Big)^2 \sin^2\Big(\frac{\pi}{2}\Big(1+\frac{t_+-s}{t_+-t_\perp}\Big)\Big) \\ &\quad + \Big(\frac{\pi/2}{t_+-t_\perp}\Big)^2\Big(\frac{t_+-s}{t_+-t_\perp}\Big)^4 \cos^2\Big(\frac{\pi}{2}\Big(1+\frac{t_+-s}{t_+-t_\perp}\Big)\Big) \\ &\quad + 4\frac{(t_+-s)^2}{(t_+-t_\perp)^4}\sin^2\Big(\frac{\pi}{2}\Big(1+\frac{t_+-s}{t_+-t_\perp}\Big)\Big) \\ &\quad + 4\frac{\pi/2}{t_+-t_\perp}\frac{(t_+-s)^3}{(t_+-t_\perp)^4}\cos\Big(\frac{\pi}{2}\Big(1+\frac{t_+-s}{t_+-t_\perp}\Big)\Big)\sin\Big(\frac{\pi}{2}\Big(1+\frac{t_+-s}{t_+-t_\perp}\Big)\Big)\Big]^{1/2} \\ &\leq \Big[\Big(\frac{\pi/2}{t_+-t_\perp}\Big)^2 + \frac{4+\pi}{(t_+-t_\perp)^2}\Big]^{1/2} \\ &\leq \frac{4}{t_+-t_\perp} \end{aligned}$$

for any $s \in (t_\perp, t_+]$. Thus, (4.29) is proven.

To prove (4.30), note that for $s \in [t_-, t_\perp)$ we have

$$v''(s) = -\Big(\frac{\pi/2}{t_\perp - t_-}\Big)^2\Big(\cos\Big(\frac{\pi}{2}\Big(1+\frac{s-t_-}{t_\perp-t_-}\Big)\Big), \sin\Big(\frac{\pi}{2}\Big(1+\frac{s-t_-}{t_\perp-t_-}\Big)\Big)\Big),$$

and hence,

$$|v''(s)| \leq \Big(\frac{\pi/2}{t_\perp - t_-}\Big)^2.$$

Similarly, for $s \in (t_\perp, t_+]$ it holds that

$$
\begin{aligned}
v''(s) = \Bigg(&- \Big(\frac{\pi/2}{t_+ - t_\perp}\Big)^2 \cos\Big(\frac{\pi}{2}\Big(1 + \frac{t_+ - s}{t_+ - t_\perp}\Big)\Big), \\
&\frac{\pi/2}{t_+ - t_\perp}\Big[\frac{-2(t_+ - s)}{(t_+ - t_\perp)^2} \cos\Big(\frac{\pi}{2}\Big(1 + \frac{t_+ - s}{t_+ - t_\perp}\Big)\Big) \\
&+ \Big(\frac{t_+ - s}{t_+ - t_\perp}\Big)^2 \frac{\pi/2}{t_+ - t_\perp} \sin\Big(\frac{\pi}{2}\Big(1 + \frac{t_+ - s}{t_+ - t_\perp}\Big)\Big)\Big] \\
&- 2\frac{t_+ - s}{(t_+ - t_\perp)^2} \frac{\pi/2}{t_+ - t_\perp} \cos\Big(\frac{\pi}{2}\Big(1 + \frac{t_+ - s}{t_+ - t_\perp}\Big)\Big) \\
&- \frac{2}{(t_+ - t_\perp)^2} \sin\Big(\frac{\pi}{2}\Big(1 + \frac{t_+ - s}{t_+ - t_\perp}\Big)\Big)\Bigg).
\end{aligned}
$$

Therefore, using (4.27) we have

$$
|v''(s)| \leq \Big(\frac{4}{t_+ - t_\perp}\Big)^2 \leq 16(\lambda^{-1} + \pi)^2,
$$

hence, (4.30) is proven. ∎

The rest of the section contains the proofs of the technical estimates needed in the above proof.

Lemma 4.2. *Under the hypotheses of Theorem 1.1, assertions (4.34), (4.35), and (4.37) hold.*

Proof. We use the same notations from the proof of Theorem 1.1.

Proof of (4.34). By construction, for any $t \in [\varepsilon, t_-]$, $\gamma_\varepsilon(t)$ differ from $\gamma(t)$ by a translation. Thus, the curvature of these two segments are always equal, hence (4.34).

Proof of (4.35). For $t \in [t_+, \mathcal{H}^1(\partial\Omega)]$ we have

$$
\begin{aligned}
\gamma_\varepsilon(t) &= \Big(\gamma(t)_x\Big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\Big) - \frac{\gamma(t_+)_x \gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}, \gamma(t)_y\Big), \\
\gamma_\varepsilon'(t) &= \Big(\gamma'(t)_x\Big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\Big), \gamma'(t)_y\Big), \\
\gamma_\varepsilon''(t) &= \Big(\gamma''(t)_x\Big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\Big), \gamma''(t)_y\Big).
\end{aligned}
$$

We claim

$$
\gamma(0)_x \leq \int_0^\varepsilon |\gamma'(t)_x| \,\mathrm{d}t \leq \varepsilon^{3/2}\sqrt{E_{p,\lambda}(\Omega)/\lambda}, \qquad \gamma(0)_x - \gamma(t_+)_x \geq \frac{\lambda}{2E_{p,\lambda}(\Omega)}. \tag{4.41}
$$

In view of (4.25), and noting that for any $t \in [0, \varepsilon]$ it holds that

$$
\frac{E_{p,\lambda}(\Omega)}{\lambda} \geq \int_{\gamma([0,t])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} = \int_0^t |\gamma''|^2 \,\mathrm{d}s \geq \frac{|\gamma'(t) - \gamma'(0)|^2}{\varepsilon} \geq \frac{|\gamma'(t)_x|^2}{\varepsilon}
$$

$$\implies |\gamma'(t)_x| \le \sqrt{\varepsilon E_{p,\lambda}(\Omega)/\lambda},$$

it follows that

$$|\gamma(0)_x - \gamma(\varepsilon)_x| = |\gamma(0)_x| \le \int_0^\varepsilon |\gamma'(t)_x|\,\mathrm{d}t \le \varepsilon^{3/2}\sqrt{E_{p,\lambda}(\Omega)/\lambda}.$$

Now, recall that by construction $\gamma'(t_+) = (1,0)$, $\gamma'(\mathcal{H}^1(\partial\Omega)) = \gamma'(0) = (0,1)$, $|\gamma'| \equiv 1$ for a.e. t, and let $\tau \in (t_+, \mathcal{H}^1(\partial\Omega))$ be the time for which $\gamma'(\tau) = (1/2, \sqrt{3}/2)$. Then,

$$\begin{aligned}\frac{E_{p,\lambda}(\Omega)}{\lambda} &\ge \int_{\gamma([t_+,\tau])} \kappa_{\partial\Omega}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} = \int_{t_+}^{\tau} |\gamma''|^2\,\mathrm{d}s \ge \frac{|\gamma'(\tau)-\gamma'(t_+)|^2}{\tau - t_+} = \frac{1}{\tau - t_+} \\ &\implies \tau - t_+ \ge \lambda/E_{p,\lambda}(\Omega),\end{aligned}$$

and since $\gamma_x' \ge 0$ on $[t_+, \mathcal{H}^1(\partial\Omega)]$ and $\gamma_x' \ge 1/2$ for all $t \in [t_+, \tau]$, it follows that $\gamma(0)_x - \gamma(t_+)_x \ge \gamma(\tau)_x - \gamma(t_+)_x \ge \frac{\lambda}{2E_{p,\lambda}(\Omega)}$, hence (4.41) is proven. Consequently,

$$\left|\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right| \le 2(\varepsilon E_{p,\lambda}(\Omega)/\lambda)^{3/2} = O(\varepsilon^{3/2}).$$

Therefore,

$$\begin{aligned}|\gamma_\varepsilon'|^{-4} &= \left(|\gamma_x'|^2\left(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right)^2 + |\gamma_y'|^2\right)^{-2} \\ &= \left(1 + |\gamma_x'|^2\frac{2\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x} + |\gamma_x'|^2\left(\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right)^2\right)^{-2} \\ &= 1 + O(\varepsilon^{3/2}).\end{aligned}$$

Observe that for $t \in [t_+, \mathcal{H}^1(\partial\Omega)]$ we have

$$\begin{aligned}\kappa_{\partial\Omega_\varepsilon} &= \frac{\left|\frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|} - \gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^3}\right|}{|\gamma_\varepsilon'|}, \\ \langle\gamma_\varepsilon'', \gamma_\varepsilon'\rangle &= \gamma_x'\gamma_x''\left(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right)^2 + \gamma_y'\gamma_y'' = \langle\gamma', \gamma''\rangle + O(\varepsilon^{3/2}) = O(\varepsilon^{3/2}).\end{aligned}$$

Here, we use the fact that

$$\langle\gamma'', \gamma'\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\gamma'|^2 = 0, \tag{4.42}$$

since γ is parameterized by arc-length. We also have

$$\begin{aligned}\int_{t_+}^{\mathcal{H}^1(\partial\Omega)} \frac{|\gamma_\varepsilon''|^2}{|\gamma_\varepsilon'|^4}\,\mathrm{d}t = \int_{t_+}^{\mathcal{H}^1(\partial\Omega)} &\Bigg(|\gamma''|^2 + |\gamma_x''|^2\frac{2\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x} \\ &+ |\gamma_x''|^2\left(\frac{2\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right)^2\Bigg)(1 + O(\varepsilon^{3/2}))\,\mathrm{d}t\end{aligned}$$

$$= \int_{t_+}^{\mathcal{H}^1(\partial\Omega)} |\gamma''|^2 \,\mathrm{d}t + O(\varepsilon^{3/2}),$$

hence (4.35).

Proof of (4.37). In the time interval $[0, \varepsilon]$, the competitor is obtained by scaling by a factor of two, and direct computations give that the integrated squared curvature scales by a factor of $1/2$. Thus, based on (4.24) we get

$$\begin{aligned}
\int_0^{\varepsilon} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma'}{|\gamma'|}\right)\right|^2 dt - \frac{1}{2}\int_0^{\varepsilon} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right)\right|^2 dt \\
= \frac{1}{2}\int_0^{\varepsilon} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma'}{|\gamma'|}\right)\right|^2 dt = \frac{1}{2}\int_0^{\varepsilon} |\gamma''|^2 \\
\geq \frac{M^2}{2}\varepsilon, \qquad (4.43)
\end{aligned}$$

hence (4.37). ∎

Lemma 4.3. *Under the hypotheses of Theorem* 1.1, *assertion* (4.36) *holds.*

Proof. We use the same notations from the proof of Theorem 1.1. In the time interval $[t_-, t_+]$, γ_ε is given by

$$\gamma_\varepsilon(t) = \gamma(t) + \gamma(\varepsilon)_y v(t), \quad t \in [t_-, t_+].$$

Note first that since Ω is a minimizer of $E_{p,\lambda}$, it must be that

$$\int_{\gamma([t_-,t_+])} \kappa_{\partial\Omega}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} < +\infty,$$

and recalling our definition of integrated squared curvature in (1.4), it follows that the Radon–Nikodym derivative

$$\frac{\mathrm{d}\kappa_{\partial\Omega}}{\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega}}$$

is square integrable. In terms of the parameterization γ_ε, this gives

$$\frac{1}{|\gamma_\varepsilon'|}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right) = \frac{1}{|\gamma_\varepsilon'|}\left(\frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|} - \gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'', \gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^3}\right) \in L^2(0, \mathcal{H}^1(\partial\Omega); \mathbb{R}).$$

Recall (4.26), that is, $\gamma(\varepsilon)_y = \varepsilon + O(\varepsilon^{3/2})$. This together with the facts that γ is parameterized by arc-length (i.e., $|\gamma'| = 1$ for a.e. t), and v was defined in (4.28) (in particular, $|v'|$ was uniformly bounded from above), it follows that

$$|\gamma_\varepsilon'| = \sqrt{1 + 2\varepsilon\langle\gamma', v'\rangle + O(\varepsilon^{3/2})}.$$

Then, for any $\alpha \in \mathbb{R}$ and sufficiently small ε, we have

$$|\gamma_\varepsilon'|^\alpha = 1 + \alpha\varepsilon\langle\gamma', v'\rangle + O(\varepsilon^{3/2}). \qquad (4.44)$$

Routine calculation shows that

$$\begin{aligned}\frac{1}{|\gamma_\varepsilon'|}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right) &= \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|^2} - \gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^4} \\ &= \frac{\gamma''+\varepsilon v''}{|\gamma_\varepsilon'|^2} - \frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|^4}\big(\langle\gamma'',\gamma'\rangle + \varepsilon^2\langle v'',v'\rangle + \varepsilon\langle\gamma'',v'\rangle + \varepsilon\langle\gamma',v''\rangle\big) \\ &\quad + \text{ higher order terms.}\end{aligned}$$

Based on (4.42), we observe:

(1) As both $|v'|$ and $|v''|$ are uniformly bounded from above, the term $\varepsilon^2\langle v'',v'\rangle$ is of order $O(\varepsilon^2)$.

(2) The norm of $\varepsilon\gamma_\varepsilon'\langle\gamma',v''\rangle/|\gamma_\varepsilon'|^4$ is estimated by

$$\varepsilon\left|\frac{\gamma_\varepsilon'\langle\gamma',v''\rangle}{|\gamma_\varepsilon'|^4}\right| \le \varepsilon\frac{|\gamma'|\cdot|v''|}{|\gamma_\varepsilon'|^3} \le \varepsilon\|v''\|_{L^\infty} + O(\varepsilon^2). \tag{4.45}$$

(3) The norm of $\varepsilon\gamma_\varepsilon'\langle\gamma'',v'\rangle/|\gamma_\varepsilon'|^4$ is estimated by

$$\varepsilon\left|\frac{\gamma_\varepsilon'\langle\gamma'',v'\rangle}{|\gamma_\varepsilon'|^4}\right| \le \varepsilon\frac{|\gamma''|\cdot|v'|}{|\gamma_\varepsilon'|^3}. \tag{4.46}$$

Thus, combining (4.42), (4.45), and (4.46) gives

$$\begin{aligned}\int_{t_-}^{t_+}\left|\gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^4}\right|^2\mathrm{d}t &= \int_{t_-}^{t_+}\left|\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|^4}\big(\langle\gamma'',\gamma'\rangle + \varepsilon^2\langle v'',v'\rangle + \varepsilon\langle\gamma'',v'\rangle + \varepsilon\langle\gamma',v''\rangle\big)\right|^2\mathrm{d}t \\ &\quad + \text{ higher order terms} \\ &\le \int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}\big|\varepsilon\langle\gamma'',v'\rangle + \varepsilon\langle\gamma',v''\rangle\big|^2\,\mathrm{d}t + O(\varepsilon^3) \\ &\le 2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}(|\varepsilon\langle\gamma'',v'\rangle|^2 + |\varepsilon\langle\gamma',v''\rangle|^2)\,\mathrm{d}t + O(\varepsilon^3) \\ &= 2\varepsilon^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}\left(|\langle\gamma'',v'\rangle|^2 + |\langle\gamma',v''\rangle|^2\right)\,\mathrm{d}t + O(\varepsilon^3) \\ &\le 2\varepsilon^2\|v'\|_{L^\infty}^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}|\gamma''|^2\,\mathrm{d}t + O(\varepsilon^2).\end{aligned}$$

In view of (4.44), we get

$$\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}|\gamma''|^2\,\mathrm{d}t \le 2\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t \le 2\int_{\partial\Omega}\kappa_{\partial\Omega}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega} < +\infty,$$

hence

$$2\varepsilon^2\|v'\|_{L^\infty}^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-6}|\gamma''|^2\,\mathrm{d}t \le O(\varepsilon^2).$$

Thus,

$$\int_{t_-}^{t_+}\left|\gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^4}\right|^2\mathrm{d}t \le O(\varepsilon^2)$$

and

$$\int_{\gamma([t_-,t_+])} \kappa^2_{\partial\Omega_\varepsilon}\, \mathrm{d}\mathcal{H}^1_{\llcorner\partial\Omega_\varepsilon} = \int_{t_-}^{t_+} \left| \frac{1}{|\gamma_\varepsilon'|} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right)\right|^2 \mathrm{d}t = \int_{t_-}^{t_+} \left| \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|^2} - \gamma_\varepsilon' \frac{\langle \gamma_\varepsilon'', \gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^4}\right|^2 \mathrm{d}t$$
$$= \int_{t_-}^{t_+} \left| \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|^2}\right|^2 \mathrm{d}t + O(\varepsilon^2).$$

Again, in view of (4.44), it follows that

$$\begin{aligned}
\int_{t_-}^{t_+} \left| \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|^2}\right|^2 \mathrm{d}t &= \int_{t_-}^{t_+} \left| \frac{\gamma'' + \varepsilon v''}{|\gamma_\varepsilon'|^2}\right|^2 \mathrm{d}t \\
&= \int_{t_-}^{t_+} \big(\langle \gamma'' + \varepsilon v'', \gamma'' + \varepsilon v''\rangle\big)\big(1 - 4\varepsilon\langle \gamma', v'\rangle + O(\varepsilon^{3/2})\big)\, \mathrm{d}t \\
&\quad + \text{ higher order terms} \\
&= \int_{t_-}^{t_+} \big(|\gamma''|^2 + 2\varepsilon\langle \gamma'', v''\rangle + \varepsilon^2 |v''|\big)\big(1 - 4\varepsilon\langle \gamma', v'\rangle + O(\varepsilon^{3/2})\big)\, \mathrm{d}t \\
&\quad + \text{ higher order terms} \\
&\le (1 + 4\varepsilon\|v'\|_{L^\infty}) \int_{t_-}^{t_+} |\gamma''|^2\, \mathrm{d}t + 2\varepsilon\|v''\|_{L^\infty} \int_{t_-}^{t_+} |\gamma''|\, \mathrm{d}t + O(\varepsilon^{3/2}) \\
&\le (1 + 4\varepsilon\|v'\|_{L^\infty}) \int_{t_-}^{t_+} |\gamma''|^2\, \mathrm{d}t \\
&\quad + 2\varepsilon\|v''\|_{L^\infty} \left(\mathcal{H}^1(\partial\Omega) \int_{t_-}^{t_+} |\gamma''|^2\, \mathrm{d}t\right)^{1/2} + O(\varepsilon^{3/2}) \\
&\le \int_{t_-}^{t_+} |\gamma''|^2\, \mathrm{d}t + 4\varepsilon\|v'\|_{L^\infty} E_{p,\lambda}(\Omega)/\lambda \\
&\quad + 2\varepsilon\|v''\|_{L^\infty} \sqrt{\mathcal{H}^1(\partial\Omega) E_{p,\lambda}(\Omega)/\lambda} + O(\varepsilon^{3/2}).
\end{aligned} \tag{4.47}$$

Note that

$$\begin{aligned}
&4\|v'\|_{L^\infty} E_{p,\lambda}(\Omega)/\lambda + 2\|v''\|_{L^\infty} \sqrt{\mathcal{H}^1(\partial\Omega) E_{p,\lambda}(\Omega)/\lambda} \\
&\qquad \le 32(\lambda^{-1} + \pi)^2 + 32\sqrt{2C_1\pi}(\lambda^{-1} + \pi)^{5/2} = C_2
\end{aligned}$$

in view of (1.6), (2.17), (4.29), (4.30), and Lemma A.1. Hence, inequality (4.36) follows from (4.47). ∎

5. Conclusion

In this paper we investigated the minimization problem for the average-distance functional defined for a two-dimensional domain with respect to its boundary, subject to a penalty proportional to the Euler elastica of the boundary. We proved the existence and $C^{1,1}$-regularity of minimizers, mainly relying on the method of contradictions by constructing suitable competitors. Echoing the large amount of existing studies that have exclusively focused on either the 1D average-distance problem or the 2D Willmore energy question, by considering variational problems associated with combined energy functional, this study enriches and deepens our understanding of the penalized average-distance problem. Questions on the exact shape of a minimizer are still open and worth investigating in future. Limiting behaviors of the minimizers, properly scaled with respect to λ, as $\lambda \to 0$ and $\lambda \to \infty$ may also shed light on this class of free boundary problems.

A. Technical results concerning properties of convex sets

In this appendix, we collect some results about convex sets, convergence in $\mathcal{A}$, and their effect on geometric quantities such as perimeter and diameter. One elementary yet crucial observation is that, given a two-dimensional convex set $\Omega \in \mathcal{A}$, every point $x \in \mathcal{A}$ has a set $U \subseteq \Omega$ of *positive* area containing x.

Lemma A.1 ([17, p. 1]). *Let $n \geq 2$ and let $A, B \subset \mathbb{R}^n$ be two convex bodies (i.e., compact convex sets with non-empty interior). If $A \subset B$, then the monotonicity of perimeters holds, i.e.,*

$$\mathcal{H}^{n-1}(\partial A) \leq \mathcal{H}^{n-1}(\partial B). \tag{1.48}$$

As a consequence, since any set with diameter d is contained in a ball of diameter d, we have

$$\mathcal{H}^1(\partial\Omega) \leq \pi \operatorname{diam}(\Omega), \quad \textit{for all } \Omega \in \mathcal{A}.$$

Lemma A.2. *Consider a sequence $\Omega_n \subseteq \mathcal{A}$ converging to $\Omega \in \mathcal{A}$ in the topology of $\mathcal{A}$ such that $\bigcup_n \Omega_n \subseteq K$ for some compact set K. Then,* $\operatorname{diam}(\Omega_n) \to \operatorname{diam}(\Omega)$.

Proof. Consider a sequence $\Omega_n \subseteq \mathcal{A}$ converging to Ω in the topology of $\mathcal{A}$. As each Ω_n is compact, there exist $x_n, y_n \in \Omega_n$ such that $|x_n - y_n| = \operatorname{diam}(\Omega_n)$. By our assumption that $\Omega_n \subseteq \mathcal{A}$ are all contained in a given compact set K, we have, up to a subsequence, $x_n \to x$, $y_n \to y$ for some $x, y \in \Omega$. Thus, it is clear that

$$\operatorname{diam}(\Omega_n) = |x_n - y_n| \to |x - y| \leq \operatorname{diam}(\Omega).$$

We need to exclude the strict inequality case. This is achieved by a contradiction argument. Assume the opposite, i.e., there exist $v, w \in \Omega$ such that $|v - w| > |x - y|$. Then, we claim that there exist sequences v_n, w_n of points in Ω_n such that, up to a subsequence, $v_n \to v$, $w_n \to w$. This, because of our assumption, would leave the existence of some set $U \subseteq \Omega$ of positive area, containing either v or w, such that there are no sequence of points in Ω_n that enter into U. This contradicts the fact that Ω_n is converging to Ω in the topology of $\mathcal{A}$. Thus, the proof is complete. ∎

Lemma A.3. *Given convex sets $\Omega_\varepsilon, \Omega$ such that $d_{\mathcal{H}}(\partial\Omega_\varepsilon, \partial\Omega) \le \delta$, it holds that*

$$\mathcal{H}^2(\Omega_\varepsilon \backslash \Omega) \le 2\delta \mathcal{H}^1(\partial\Omega_\varepsilon). \tag{1.49}$$

Proof. Clearly, $\Omega_\varepsilon \backslash \Omega$ is entirely contained in

$$\{x \in \Omega_\varepsilon : \operatorname{dist}(x, \Omega_\varepsilon) \le \delta\},$$

that is, the part of the tubular neighborhood of $\partial\Omega_\varepsilon$ with thickness δ that lies inside Ω_ε.

We now use an approximation argument. We approximate $\partial\Omega_\varepsilon$ with convex polygons $P_n \subseteq \Omega_\varepsilon$, e.g., by choosing n points $x_{1,n}, \dots, x_{n,n} \in \partial\Omega_\varepsilon$ such that $\sup_i |x_{i+1,n} - x_{i,n}| \le 2\mathcal{H}^1(\partial\Omega_\varepsilon)/n$, and then connecting $x_{i+1,n}$ to $x_{i,n}$ with line segments.

Note that the area of the difference is continuous with respect to such an approximation, i.e., $\mathcal{H}^2(P_n \backslash \Omega) \nearrow \mathcal{H}^2(\Omega_\varepsilon \backslash \Omega)$. It is then a straightforward computation to check that

$$\mathcal{H}^2(\{x \in P_n : \operatorname{dist}(x, P_n) \le \delta\}) \le 2\delta \mathcal{H}^1(\partial P_n),$$

for all sufficiently large n. ∎

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