CUT AND PASTE INVARIANTS OF MANIFOLDS VIA ALGEBRAIC K-THEORY

RENEE S. HOEKZEMA, MONA MERLING, LAURA MURRAY, CARMEN ROVI, AND JULIA SEMIKINA

Abstract. Recent work of Inna Zakharevich and Jonathan Campbell has focused on building machinery for studying scissors congruence problems via algebraic K-theory, and applying these tools to studying the Grothendieck ring of varieties. In this paper we give a new application of their framework: we construct a K-theory space that recovers the classical SK ("schneiden und kleben," German for "cut and paste") groups for manifolds on π_0, and we construct a derived version of the Euler characteristic.

Contents

2020 *Mathematics Subject Classification.* Primary 19D55, 19D99; Secondary 19A49, 57K99.
Key words and phrases. K-theory, cut and paste, K-theory of manifolds.

1. Introduction

The classical scissors congruence problem asks whether given two polyhedra with the same volume P and Q in $\mathbb{R}^3$, one can cut P into a finite number of smaller polyhedra and reassemble these to form Q. Precisely, P and Q are scissors congruent if $P = \bigcup_{i=1}^m P_i$ and $Q = \bigcup_{i=1}^m Q_i$, where $P_i \cong Q_i$ for all i, and the subpolyhedra in each set only intersect each other at edges or faces. There is an analogous definition of an SK (German "schneiden und kleben," cut and paste) relation for manifolds: Given a closed smooth oriented manifold M, one can cut it along a separating codimension 1 submanifold Σ with trivial normal bundle and paste back the two pieces along an orientation preserving diffeomorphism $\Sigma \to \Sigma$ to obtain a new manifold, which we say is "cut and paste equivalent" or "scissors congruent" to it. We give a pictorial example of this relation:

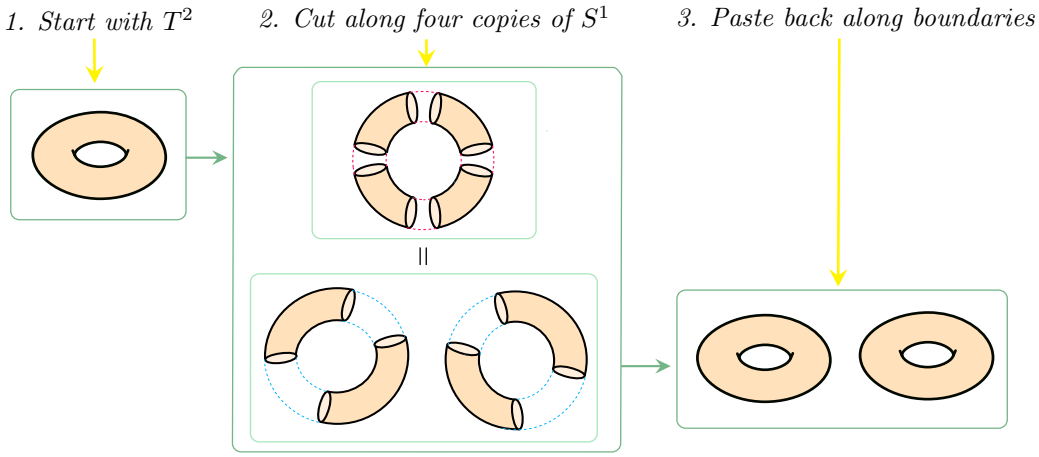


Figure 1. Example of a cut and paste operation

Zakharevich has formalized the notion of scissors congruence via the notion of an *assembler*–this is a Grothendieck site with a few extra properties, whose topology encodes the cut and paste operation. She constructs an associated K-theory spectrum, which on π_0 recovers classical scissors congruence groups [Zak17b]. Specific examples of assemblers recover scissors congruence groups for polytopes and the Grothendieck ring of varieties, as π_0 of their corresponding K-theory spectra. The higher K-groups encode further geometric information. Independently, Campbell has introduced the formalism of *subtractive categories*, a modification of the definition of Waldhausen categories, to define a K-theory spectrum of varieties that recovers the Grothendieck ring of varieties on π_0 [Cam19]. Though the approaches to encoding scissors congruence abstractly are different, the resulting spectra of Zakharevich and Campbell are shown to be equivalent in [CZ19].

The focus of Zakharevich and Campbell has been to construct and study a K-theory spectrum of varieties, and this spectrum level lift of the Grothendieck ring of varieties has led to a fruitful research program to better understand varieties. For example, an analysis of K_1 for the K-theory spectrum of varieties allowed Zakharevich to elucidate structure on the annihilator of the Lefschetz motive [Zak17a], and Campbell, Wolfson and Zakharevich use a lift of the zeta function for varieties to show that π_1 of the K-theory spectrum for varieties contains nontrivial geometric information [CWZ19]. Studying cut and paste relations for manifolds via K-theoretic machinery remains as of yet unexplored. We start this exploration in this paper.

Unfortunately, the framework from [Zak17b, Cam19] does not directly apply to the case of manifolds. The problem is that if one tries to find a common refinement of two different SK-decompositions of a manifold, one might have to cut boundaries and one gets manifolds with corners. This makes some of the axioms in both the assembler approach and the subtractive category approach break down. However, work in progress of Campbell and Zakharevich on "K-theory with squares," $K^{\square}$, a further synthetization of scissors congruence relations as K-theory that generalizes Waldhausen K-theory, does give the right framework to construct the desired scissors congruence K-theory for manifolds. Encompassing the manifold example was also one of the motivations behind Campbell and Zakharevich's development of "K-theory with squares".

The study of SK-invariants and SK-groups in [KKNO73] focuses on closed manifolds. However, in order for the $K^{\square}$-theoretic scissors congruence machinery to apply, we need to work in the category of manifolds with boundary, since the pieces in an SK-decomposition have boundary. This is not well-explored classically, as most of the existing work on SK-groups is for closed manifolds. We generalize the notion of SK-equivalence to the case of manifolds with boundary and denote the corresponding group by SK_n^{∂}. Our definition of SK_n^{∂} is different from the one mentioned in [KKNO73] in that we insist that every boundary along which we cut gets pasted, and this is crucial for the further application of the K-theoretic technology.

We formulate a suitable notion of a category with squares $\mathrm{Mfd}_n^{\partial}$, that fits into the K-theory with squares framework, and whose distinguished squares exactly encode the "cut-and-paste" relations for n-dimensional manifolds with boundary. We show that the K-theory space obtained from the construction of Campbell and Zakharevich, applied to $\mathrm{Mfd}_n^{\partial}$, which we denote by $K^{\square}(\mathrm{Mfd}_n^{\partial})$, recovers the SK_n^{∂} as its zeroth homotopy group:

Theorem A. *There is an isomorphism* $K_0^{\square}(\mathrm{Mfd}_n^{\partial}) \cong \mathrm{SK}_n^{\partial}$, *where* $K_0^{\square}(\mathrm{Mfd}_n^{\partial})$ *is* π_0 *of a scissors congruence* K*-theory space* $K^{\square}(\mathrm{Mfd}_n^{\partial})$.

Scissors congruence invariants for manifolds (SK-invariants) are abelian group valued homomorphisms from the monoid of manifolds under disjoint union, which factor through the SK-group. It is well known classically that for closed manifolds the Euler characteristic, the signature, and linear combinations thereof, are the only SK-invariants, and these are also SK-invariants of manifolds with boundary. In this paper, we show that the Euler characteristic as a map to $\mathbb{Z}$, viewed as the zeroth K-theory group of $\mathbb{Z}$, is the π_0 level of a map of K-theory spaces from the scissors congruence space for manifolds with boundary that we define. In future work, we plan to also investigate the signature map to the zeroth L-theory group of $\mathbb{Z}$.

Theorem B. *There is a map of K-theory spaces*

$$K^{\square}(\mathrm{Mfd}^{\partial}) \to K(\mathbb{Z}),$$

which on π_0 agrees with the Euler characteristic for smooth compact manifolds with boundary.

All the scissors congruence space constructions are in fact infinite loop spaces, and it is not hard to see that all of our maps of K-theory spaces lift to the spectrum level. The spectrum level elborations will appear in future work of Campbell, Zakharevich and collaborators.

The paper is organized as follows. In Section 2 we introduce the definition of SK-groups for smooth compact manifolds with boundary and we prove that they are related to the classical SK groups for smooth closed manifolds via an exact sequence. In Section 3 we review the set-up of categories with squares and their K-theory as defined by Campbell and Zakharevich. In Section 4 we construct the category of squares for smooth compact manifolds with boundary and prove Theorem A, and in Section 5 we prove Theorem B.

Conventions. All manifolds in this paper are smooth, compact and oriented. We will distinguish between closed manifolds and manifolds with boundary. We will use the notation $\bar{M}$ for the manifold M with reversed orientation.

Acknowledgements. We are greatly indebted to Jonathan Campbell and Inna Zakharevich for their generosity in sharing their work in progress on squares K-theory, which our project relies on, and for their extensive patience in explaining it to us and answering our questions. We thank the anonymous referee and Jim Stasheff for close readings and suggestions that greatly improved the paper. We particularly thank George Raptis for catching a mistake in a previous version of the paper regarding our definition of an SKK group for manifolds with boundary—we owe the observation in Scholium 2.11 to him. It is a great pleasure to also acknowledge the contributions to this project arising from discussions with Andrew Blumberg,

Anna Marie Bohmann, Jonathan Block, Jim Davis, Greg Friedman, Søren Galatius, Teena Gerhardt, Herman Gluck, Mike Hill, Fabian Hebestreit, Matthias Kreck, Malte Lackmann, Wolfgang Lück, Mike Mandell, Cary Malkiewich, Kate Ponto, Jan Steinebrunner, Peter Teichner, Andrew Tonks, and Chuck Weibel.

Finally, we thank the organizers of the Women in Topology III program and the Hausdorff Research Institute for Mathematics for their hospitality during the workshop. The WIT III workshop was supported through grants NSF-DSM 1901795, NSF-HRD 1500481 - AWM ADVANCE grant and the Foundation Compositio Mathematica, and we are very grateful for their support. The first named author was supported by the Max Planck Society. The second named author was supported by grants NSF-DMS 1709461, NSF CAREER DMS 1943925 and NSF FRG DMS-2052988. The third named author was supported by grant NSF-DMS 1547292. The fourth named author is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC2181/1-390900948 (the Heidelberg STRUCTURES Excellence Cluster), and also wishes to acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) 281869850 (RTG 2229). The fifth named author was supported by the Max Planck Society and Wolfgang Lück's ERC Advanced Grant "KL2MG-interactions" (no. 662400).

2. Scissors congruence groups for manifolds with boundary

2.1. **SK-groups for closed manifolds.** We start by reviewing the definitions of the classical scissors congruence groups of smooth closed oriented manifolds, namely the SK_n-groups introduced in [KKNO73]. The "scissors congruence" or "cut and paste" relation on smooth closed oriented manifolds is given as follows: cut an n-dimensional manifold M along a codimension 1 smooth submanifold Σ with trivial normal bundle that separates M in the sense that the complement of Σ in M is a disjoint union of two components M_1 and M_2, each with boundary diffeomorphic to Σ. Then paste back the two pieces together along an orientation preserving diffeomorphism $\phi\colon \Sigma \to \Sigma$. We say M and $M_1 \cup_\phi M_2$ are "cut and paste equivalent" or "scissors congruent."

Note that for a codimension 1 submanifold Σ with trivial normal bundle that does not separate M (for example the inclusion of $S^1 \times \{0\}$ into $S^1 \times S^1$) we can take the union with a second copy of Σ embedded close to it, and the disjoint union $\Sigma \sqcup \Sigma$ then separates M.

Definition 2.1. Two smooth closed manifolds M and N are *SK-equivalent* (or *scissors congruent* or *cut and paste equivalent*) if N can be obtained from M by a finite sequence of cut and paste operations.

Example 2.2. In Figure 2 we can see that $T^2 \sharp T^2 \sqcup S^2$ is SK-equivalent to $T^2 \sqcup T^2$.

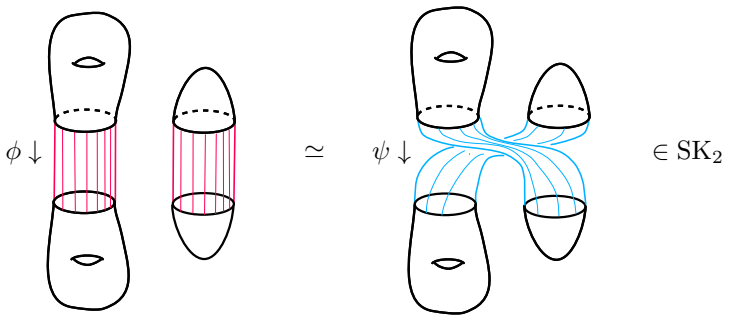


FIGURE 2. Example of an SK-relation

Let $\mathcal{M}_n$ be the monoid of diffeomorphism classes of smooth closed oriented n-dimensional manifolds $[M]$ under disjoint union. The SK_n-group from [KKNO73] is defined to satisfy the universal property that any abelian valued monoid map from $\mathcal{M}_n$ which respects SK-equivalence (also called an SK-invariant) factors through it.

Definition 2.3. The *scissors congruence group* SK_n *for smooth closed oriented* n*-dimensional manifolds* is the quotient of the Grothendieck group $\mathrm{Gr}(\mathcal{M}_n)$ by the SK-equivalence relation.

Explicitly, SK_n is the free abelian group on diffeomorphism classes $[M]$ modulo the following relations:

(1) $[M \sqcup N] = [M] + [N]$;
(2) Given compact oriented manifolds M_1, M_2 and orientation preserving diffeomorphisms $\phi, \psi \colon \partial M_1 \to \partial M_2$,
$$[M_1 \cup_\phi \bar{M}_2] = [M_1 \cup_\psi \bar{M}_2],$$
where $\bar{M}_2$ is M_2 with reversed orientation.

Remark 2.4. It is shown in [KKNO73, Corollary 1.4] that any SK-invariant for smooth oriented manifolds is a linear combination of the Euler characteristic and the signature.

2.2. **SK-groups for manifolds with boundary.** We note that in order to define a scissors congruence K-theory, we need to work in a category of manifolds with boundary since the pieces in the cut and paste relation are manifolds with boundary. Therefore, we introduce a definition of SK-groups for manifolds with boundary; these are the groups which we will recover as π_0 of a scissors congruence K-theory space.

We define the "cut and paste relation" on smooth compact manifolds with boundary analogously to that on closed manifolds: cut an n-dimensional manifold M along a codimension 1 smooth submanifold Σ with trivial normal bundle, which separates M, and for which $\Sigma \cap \partial M = \emptyset$. Then paste back the two pieces together along an orientation preserving diffeomorphism $\phi\colon \Sigma \to \Sigma$. We emphasize that we do not allow boundaries to be cut, and we require that all boundaries which come from cutting to be pasted back together, leaving the existing boundaries of a manifold untouched by the cut and paste operation.

Definition 2.5. Two smooth compact manifolds with boundary will be called SK-*equivalent* if one can be obtained from the other via a finite sequence of cut and paste operations in the sense described above.

Remark 2.6. Our definition of the cut and paste relation for manifolds with boundary is different than the one in [KKNO73, Chapter 5], where $M_1 \cup_\phi M_2 \sim M_1 \sqcup M_2$. Namely, they allow pieces that are cut to not be pasted back together. In order to apply the K-theoretic machinery to obtain the SK_n^∂-group as π_0 of a K-theory space, it is important to use our definition of SK_n^∂.

Definition 2.7. Let $\mathcal{M}_n^\partial$ be the monoid of diffeomorphism classes of smooth compact oriented n-dimensional manifolds with boundary under disjoint union. The *scissors congruence group* SK_n^∂ *for smooth compact oriented manifolds* is the quotient of the Grothendieck group $\mathrm{Gr}(\mathcal{M}_n^\partial)$ by the SK-equivalence relation.

Explicitly, SK_n^∂ is the free abelian group on diffeomorphism classes of smooth compact oriented n-dimensional manifolds (with or without boundary) modulo the following relations:

(1) $[M \sqcup N] \sim [M] + [N]$;
(2) Given compact oriented manifolds M_1, M_2, closed submanifolds $\Sigma \subseteq \partial M_1$ and $\Sigma' \subseteq \partial M_2$, and orientation preserving diffeomorphisms $\phi, \psi\colon \Sigma \to \Sigma'$,

$$[M_1 \cup_\phi \bar{M}_2] = [M_1 \cup_\psi \bar{M}_2].$$

Example 2.8. In Figure 3 we see an example of an SK_n^∂-relation.

Proposition 2.9. *The Euler characteristic and the signature are* SK^∂*- invariants.*

Proof. That the signature of manifolds with boundary is a SK^∂-invariant follows from Novikov additivity [Nov70] and the fact that in our definition of the SK^∂-relation the boundary components of SK^∂-equivalent manifolds remain unchanged by the cut and paste operation.

The argument that shows that the Euler characteristic is a SK^∂-invariant is an exact analog to the argument for closed manifolds.

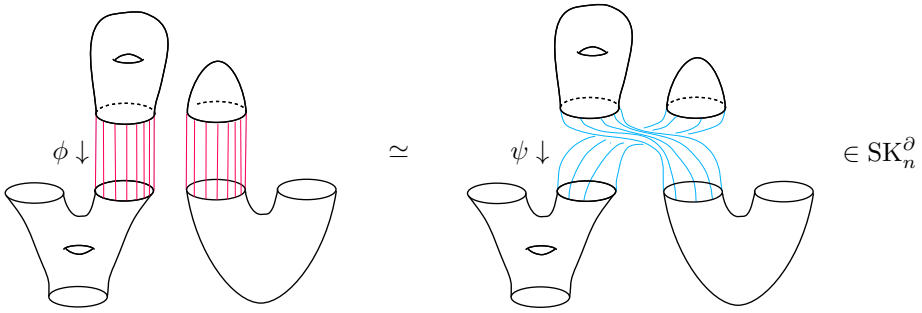


FIGURE 3. Example of an SK^∂-relation

□

2.3. **Exact sequence for SK and SK^∂.** We now relate our definition of SK_n^∂ with the classical SK_n via an exact sequence. Denote by C_n the Grothendieck group of the monoid of diffeomorphism classes of smooth closed oriented n-dimensional nullcobordant manifolds under disjoint union. Note that C_n is a free abelian group for the following reason. The Grothendieck group $\mathrm{Gr}(\mathcal{M}_n)$ is a free abelian group on diffeomorphism classes of connected n-manifolds. Due to the cancellation property, the inclusions of the monoid of diffeomorphism classes of n-dimensional nullcobordant manifolds into its Grothendieck group and into $\mathrm{Gr}(\mathcal{M}_n)$ are injective. Hence the Grothendieck group C_n is a subgroup of $\mathrm{Gr}(\mathcal{M}_n)$ and therefore is free abelian.

Theorem 2.10. *For every $n \geq 1$ the following sequence is exact*

$$0 \longrightarrow \mathrm{SK}_n \xrightarrow[[M]\mapsto[M]]{\alpha} \mathrm{SK}_n^\partial \xrightarrow[[N]\mapsto[\partial N]]{\beta} C_{n-1} \longrightarrow 0.$$

Proof. Note that the map $\alpha\colon \mathrm{SK}_n \to \mathrm{SK}_n^\partial$ taking a class of manifolds in SK_n to a class containing the same manifolds in SK_n^∂ is well-defined, since every relation from the definition of SK_n is also a relation in the definition of SK_n^∂. The map β that takes a class of manifolds to the diffeomorphism class of the boundary is well-defined, since the equivalence relation from the definition of SK_n^∂ preserves the boundary.

We show exactness at the middle term. It is clear from the definition that $\mathrm{Im}\,\alpha \subseteq \ker\beta$. Let us show the reverse inclusion. Let $x \in \ker\beta$. Every element of SK_n^∂ can be written in the form $x = [M] - [N]$, where M, N are compact smooth oriented n-manifolds with boundary (not necessarily connected).

Let $\bar{M}$ be the copy of M with the opposite orientation and let DM be the double of M, i.e. $DM = M \cup_{\mathrm{id}} \bar{M}$. Note that DM is a closed manifold. Since C_{n-1} is a free abelian group and $\beta(x) = [\partial M] - [\partial N] = 0$ we conclude that the ∂M and ∂N

are diffeomorphic. Hence we may glue $\bar{M}$ to N along the boundary. We will call this gluing diffeomorphism ϕ (it does not have to be unique, we just pick one) and denote by L the closed manifold, which is the result of this gluing. Therefore,

$$DM = M \cup_{\mathrm{id}} \bar{M},$$

and

$$L = N \cup_{\phi} \bar{M}.$$

Hence in SK_n^{∂},

$$\begin{aligned} [N] + [DM] &= [N \cup_{\mathrm{id}} (\partial N \times [0,1])] + [M \cup_{\mathrm{id}} \bar{M}] \\ &= [N \cup_{\phi} \bar{M}] + [(\partial N \times [0,1]) \cup_{\phi} M] \\ &= [L] + [M]. \end{aligned}$$

Consequently,

$$x = [M] - [N] = [DM] - [L] \in \mathrm{Im}\alpha.$$

See Figure 4 for an illustration of such an element.

Finally, let us show injectivity of the map α. Let R_n be the subgroup of $\mathrm{Gr}(\mathcal{M}_n)$ generated by the SK-relation $[M_1 \cup_{\phi} \bar{M}_2] - [M_1 \cup_{\psi} \bar{M}_2]$, so that $\mathrm{SK}_n = \mathrm{Gr}(\mathcal{M}_n)/R_n$. Note that the set of elements that generate this relation is closed under summation,

$$\begin{aligned} &\left([M_1 \cup_{\phi} \bar{M}_2] - [M_1 \cup_{\psi} \bar{M}_2]\right) + \left([M_1' \cup_{\phi'} \bar{M}_2'] - [M_1' \cup_{\psi'} \bar{M}_2']\right) \\ &= [(M_1 \sqcup M_1') \cup_{\phi \sqcup \phi'} (\bar{M}_2 \sqcup \bar{M}_2')] - [(M_1 \sqcup M_1') \cup_{\psi \sqcup \psi'} (\bar{M}_2 \sqcup \bar{M}_2')]. \end{aligned}$$

Thus R_n is precisely the set of elements of this form, and similarly for the subgroup R_n^{∂} of $\mathrm{Gr}(\mathcal{M}_n^{\partial})$, which generates the SK-relation for manifolds with boundary. Suppose that $[M] - [N] \in R_n^{\partial} \cap \mathrm{Gr}(\mathcal{M}_n)$. Since $[M] - [N] \in R_n^{\partial}$, by Proposition 2.9 we have that $M \sim_{SK^{\partial}} N$, and thus they have the same Euler characteristic and signature. Since by Remark 2.4 the Euler characteristic and signature are the only SK-invariants, it follows that $M \sim_{SK} N$. Therefore,

$$R_n^{\partial} \cap \mathrm{Gr}(\mathcal{M}_n) = R_n,$$

and injectivity of α follows. □

Scholium 2.11. Classically, there is a more refined relation than that of cutting and pasting called SKK ("scheiden und kleben, kontrollierbar"="controllable cutting and pasting") in which we keep track of the gluing diffeomorphisms. The SKK-equivalence relation is:

$$[M_1 \cup_{\phi} \bar{M}_1'] \; - \; [M_1 \cup_{\psi} \bar{M}_1'] \;\; = \;\; [M_2 \cup_{\phi} \bar{M}_2'] \; - \; [M_2 \cup_{\psi} \bar{M}_2']$$

for compact oriented manifolds M_1, M_1' and M_2, M_2' such that $\partial M_1 = \partial M_2$ and $\partial M_1' = \partial M_2'$, and orientation preserving diffeomorphisms $\phi, \psi \colon \partial M_1 \to \partial M_1'$. The resulting SKK_n-groups obtained by modding out by the SKK-equivalence relation have been interpreted as Reinhardt vector field bordism groups in [KKNO73] and

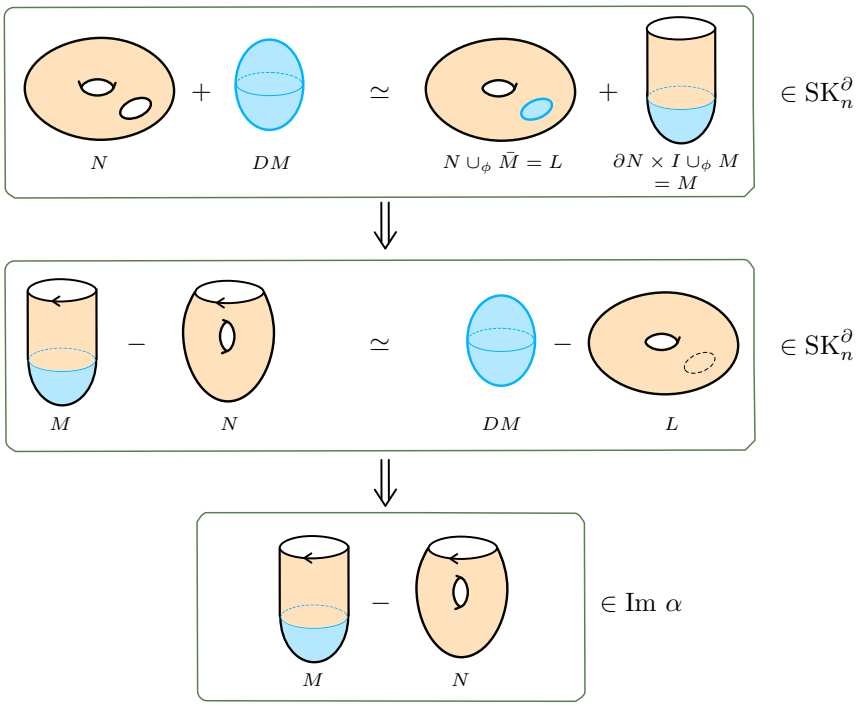


FIGURE 4. Example of an element in $\mathrm{Im}(\alpha\colon \mathrm{SK}_n \to \mathrm{SK}_n^\partial)$

have also been shown to arise as π_0 of the Madsen-Tillman spectra $MTSO(n)$ [Ebe13]. If we define an SKK^∂ relation analogously but allow M_1, M_1', M_2, M_2' to be manifolds with boundary, $\Sigma \subseteq \partial M_i$ and $\Sigma' \subseteq \partial M_i'$ closed submanifolds for $i = 1, 2$, and $\phi, \psi\colon \Sigma \to \Sigma'$ orientation preserving diffeomorphisms, then we would have

$$[M_1 \cup_\phi \bar{M}_1'] \; - \; [M_1 \cup_\psi \bar{M}_1'] \;\; = \;\; [\Sigma \times I \cup_\phi \Sigma' \times I] - [\Sigma \times I \cup_\psi \Sigma' \times I],$$

where the latter is zero. Thus if we defined an SKK_n^∂ group by modding out by this relation, we would have $SKK_n^\partial \cong SK_n^\partial$, in contrast to the classical case where we have a surjective map $SKK_n \to SK_n$, which is not an isomorphism.[1]

3. K-theory of categories with squares

3.1. **Overview of Campbell and Zakharevich's squares K-theory.** This subsection is an exposition of the definitions and results from Campbell and Zakharevich's work in progress on K-theory of categories with squares.

[1]This observation is due to George Raptis.

Definition 3.1. A category with squares is a category $\mathcal{C}$ equipped with a choice of basepoint object O, two subcategories $c\mathcal{C}$ and $f\mathcal{C}$ of morphisms referred to as cofibrations (denoted $\rightarrowtail$) and cofiber maps (denoted $\twoheadrightarrow$), and distinguished squares

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

satisfying the following conditions:

1) $\mathcal{C}$ has coproducts and distinguished squares are closed under coproducts.
2) Distinguished squares are commutative squares in $\mathcal{C}$ and compose horizontally and vertically.
3) Both $c\mathcal{C}$ and $f\mathcal{C}$ contain all isomorphisms of $\mathcal{C}$.
4) If a commutative square satisfies the property that either both horizontal maps or both vertical maps are isomorphisms, then the square is distinguished.

A map of categories of squares is a functor that preserves distinguished basepoint objects and distinguished squares.

Campbell and Zakharevich developed the framework of categories with squares in order to describe a generalized construction of K-theory spaces, inspired by the Waldhausen construction. We review their construction of K-theory for a category with squares. Let $[k]$ denote the category $0 \to 1 \to \cdots \to k$.

Definition 3.2. Let $\mathcal{C}$ be a category with squares. Define $\mathcal{C}^{(k)}$ to be the subcategory of $\mathrm{Fun}([k], \mathcal{C})$ whose objects are sequences of cofibration maps

$$C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_k,$$

and whose morphisms are natural transformations in which every commutative square is distinguished.

Varying over k by composing cofibrations and distinguished squares, we get a simplicial category, denoted $\mathcal{C}^\bullet$. The squares K-theory of $\mathcal{C}$ is defined, analogously to the definition for Waldhausen categories, as follows:

Definition 3.3. Let $\mathcal{C}$ be a category with squares. The *squares K-theory space of* $\mathcal{C}$ is

$$K^\square(\mathcal{C}) \simeq \Omega_O |N_\bullet \mathcal{C}^\bullet|$$

where Ω_O is the based loop space, based at the distinguished object $O \in N_0\mathcal{C}^{(0)}$.

The following computation of K_0 for categories with squares is due to Campbell and Zakharevich; the proof will appear in upcoming work. We record the result here since we will need it later on.

Lemma 3.4 (Campbell-Zakharevich)**.** *Let $\mathcal{C}$ be a category with squares with basepoint O satisfying:*

(1) O is initial or terminal in $c\mathcal{C}$.
(2) O is initial or terminal in $f\mathcal{C}$.
(3) For all objects $A, B \in \mathcal{C}$, there exists some object $X \in \mathcal{C}$ and distinguished squares:

$$\begin{array}{ccc} O & \rightarrowtail & A \\ \downarrow & \square & \downarrow \\ B & \rightarrowtail & X \end{array} \qquad \begin{array}{ccc} O & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ A & \rightarrowtail & X \end{array}$$

Then

$$K_0^{\square}(\mathcal{C}) \cong \mathbb{Z}\{ob\mathcal{C}\}/\sim$$

where $\sim$ is the equivalence relation generated by

(1) $[O] = 0$

(2) $[A] + [D] = [B] + [C]$ for every distinguished square $\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$.

The proof that the K-theory space $K^{\square}(\mathcal{C})$ is an infinite loop space will also appear in upcoming work of Campbell, Zakharevich, and collaborators.

3.2. **Category with squares from a Waldhausen category.** The idea of Campbell's and Zakharevich's squares K-theory is to provide a generalization of both Waldhausen and subtractive categories. In particular, given a Waldhausen category $\mathcal{C}$ one can associate to it a category with squares such that the Waldhausen and squares K-theories agree. Actually, there are two different choices of categories with squares that one can associate to a Waldhausen category, and here we describe the one that will serve our purposes in Section 5. We comment on our choices in Remark 5.3 below.

Definition 3.5. Let $\mathcal{C}$ be a Waldhausen category. Define an associated category with squares $\mathcal{C}^{\square}$ in the following way. The horizontal maps are the cofibrations $\rightarrowtail$

in $\mathcal{C}$, and the vertical maps are all maps. The distinguished squares are the squares

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

with the property that the unique map $C \cup_A B \to D$ is a weak equivalence. The distinguished basepoint object is the zero object.

Proposition 3.6. *The category $\mathcal{C}^{\square}$ satisfies the axioms of a category with squares from Definition 3.1.*

Proof. We check the four axioms. For (1), $\mathcal{C}$ has coproducts because it is a Waldhausen category. Suppose that

$$C \cup_A B \xrightarrow{\simeq} D \text{ and } C' \cup_{A'} B' \xrightarrow{\simeq} D'.$$

Note that since pushouts and coproducts commute with each other, and since

$$C \cup_A B \sqcup C' \cup_{A'} B' \xrightarrow{\simeq} D \sqcup D'$$

by the gluing axiom ([Wal87, p. 326]), distinguished squares are closed under coproducts.

To check axiom (2), suppose we compose two distinguished squares horizontally

$$\begin{array}{ccccc} A & \rightarrowtail & B & \rightarrowtail & E \\ \downarrow & \square & \downarrow & \square & \downarrow \\ C & \rightarrowtail & D & \rightarrowtail & F \end{array}$$

We have a chain of weak equivalences

$$C \cup_A E \cong (C \cup_A B) \cup_B E \xrightarrow{\simeq} D \cup_B E \xrightarrow{\simeq} F,$$

where the first weak equivalence is by the gluing axiom.

Now suppose we compose two distinguished squares vertically

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \\ \downarrow & \square & \downarrow \\ E & \rightarrowtail & F \end{array}$$

Similarly, we have

$$E \cup_A B \cong E \cup_C (C \cup_A B) \xrightarrow{\simeq} E \cup_C D \xrightarrow{\simeq} F,$$

where again the first weak equivalence is by the gluing axiom.

Axiom (3) is immediate since the isomorphisms are contained in the cofibrations in a Waldhausen category, and we don't have any restrictions on the vertical maps.

To check axiom (4), suppose first that the two vertical morphisms in a commuting square in $\mathcal{C}$

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

are isomorphisms. Then $C \cup_A B \cong B \cong D$ and the square is a pushout square. Similarly, if the horizontal maps are isomorphisms, $C \cup_A B \cong C \cong D$, and again the square is a pushout. □

Proposition 3.7. *The Waldhausen K-theory $K^{\mathrm{Wald}}(\mathcal{C})$ agrees with the K-theory $K^{\square}(\mathcal{C}^{\square})$ of the associated category with squares from Definition 3.5.*

Proof. By definition, $K^{\square}(\mathcal{C}^{\square})$ is the realization of the bisimplicial set with (p,q)-simplices given by

$$\begin{array}{ccccccc} A_{00} & \rightarrowtail & A_{01} & \rightarrowtail & \cdots & \rightarrowtail & A_{0p} \\ \downarrow & \square & \downarrow & \square & & \square & \downarrow \\ A_{10} & \rightarrowtail & A_{11} & \rightarrowtail & \cdots & \rightarrowtail & A_{1p} \\ \downarrow & \square & \downarrow & \square & & \square & \downarrow \\ \vdots & & \vdots & & \ddots & & \vdots \\ \downarrow & \square & \downarrow & \square & & \square & \downarrow \\ A_{q0} & \rightarrowtail & A_{q1} & \rightarrowtail & \cdots & \rightarrowtail & A_{qp} \end{array}$$

in which each square is distinguished. Thus it is the nerve of the category whose objects are sequences of cofibrations

$$A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n$$

and morphisms maps of such diagrams that satisfy the condition that for every $i \leq j$ the induced map

$$A'_i \cup_{A_i} A_j \to A'_j$$

is a weak equivalence. Thus the above is precisely the bisimplicial set obtained by applying the nerve to Thomason's simplicial category $wT_\bullet\mathcal{C}$ defined in [Wal87, page 334].

By Thomason-Waldhausen, there is a zig-zag of equivalences via some intermediate construction

$$wT_\bullet\mathcal{C} \xleftarrow{\simeq} wT^+_\bullet\mathcal{C} \xrightarrow{\simeq} wS_\bullet\mathcal{C}.$$

Therefore, via a zig-zag, we have an equivalence of K-theory spaces

$$K^{\square}(\mathcal{C}^{\square}) \simeq K^{\mathrm{Wald}}(\mathcal{C}). \qquad \square$$

Remark 3.8. Campbell and Zakharevich noticed that for a Waldhausen category $\mathcal{C}$, one can also associate to it a category with squares where the vertical maps are cofiber maps, and they prove directly that the square K-theory of this category is equivalent to $K^{\mathrm{Wald}}(\mathcal{C})$. Therefore, it is also equivalent to the square K-theory of the category from Definition 3.5.

4. K-theory of manifolds with boundary

In this section we use the framework described in Section 3 to define a K-theory space for the category of n-dimensional compact smooth manifolds with boundary, which recovers as π_0 the scissors congruence group SK_n^∂.

4.1. The category with squares for manifolds with boundary.

We start by defining a category with squares structure on the category Mfd_n^∂ of smooth compact n-dimensional manifolds with boundary and smooth maps.

Definition 4.1. Let Mfd_n^∂ be the category of smooth compact n-dimensional manifolds with boundary and smooth maps. We define the subcategories $c\mathrm{Mfd}_n^\partial$ of horizontal maps (denoted $\rightarrowtail$) and and $f\mathrm{Mfd}_n^\partial$ of vertical maps (denoted $\hookrightarrow$) to both be given by the morphisms in Mfd_n^∂ which are smooth embeddings of manifolds with boundary $f\colon N \to M$ such that ∂N is mapped to a submanifold with trivial normal bundle, and such that each connected component of the boundary ∂N is either mapped entirely onto a boundary component or entirely into the interior of M. We define distinguished squares to be those commutative squares in Mfd_n^∂

$$\begin{array}{ccc} N & \rightarrowtail & M \\ \downarrow & & \downarrow \\ M' & \rightarrowtail & M \cup_N M'. \end{array}$$

that are pushout squares, i.e. such that $M \cup_N M'$ is a smooth manifold. The chosen basepoint object is the empty manifold.

Example 4.2. Figure 5 gives pictorial examples of distinguished squares in $\mathrm{Mfd}_n^{\partial}$.

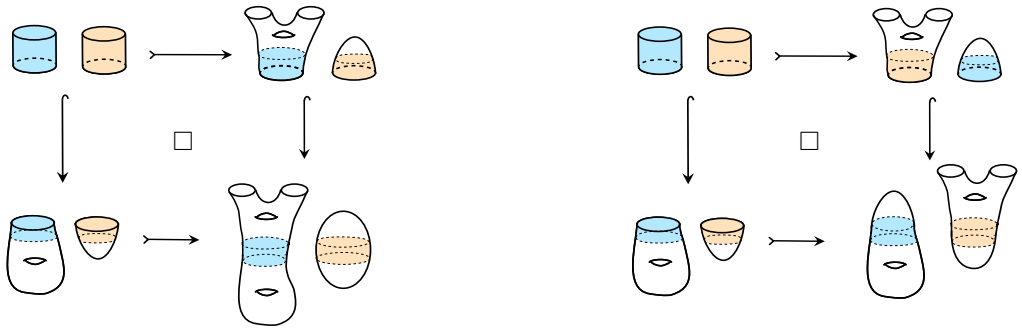
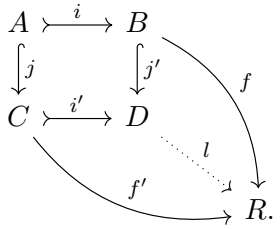


FIGURE 5. Two examples of distinguished squares

Lemma 4.3. *The category* Mfd_n^∂ *with the structure from Definition 4.1 satisfies the axioms of a category with squares from Definition 3.1.*

Proof. The coproduct in Mfd_n^∂ is given by disjoint union of manifolds, and the collection of distinguished squares is closed under disjoint union. Pushout squares are commutative and compose horizontically and vertically. Consider the diagram



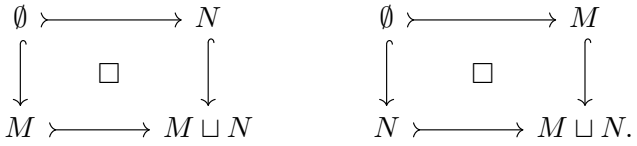
If j' is an isomorphism then we can define the map l uniquely as fj'^{-1}; similarly if i' is an isomorphism. Therefore in both cases this is a pushout diagram. Hence Mfd_n^∂ satisfies the definition of a category with squares. □

4.2. **The computation of $K_0(\mathbf{Mfd}_n^\partial)$.** Using Lemma 3.4 for the category with squares Mfd_n^∂ defined above, we show that the $K_0^\square$-group agrees with the SK_n^∂-group.

Theorem 4.4. *For the manifold category with squares* Mfd_n^∂ *from Definition 4.1,*

$$K_0^\square(\mathrm{Mfd}_n^\partial) \cong \mathrm{SK}_n^\partial.$$

Proof. The empty set is initial in both $c\mathrm{Mfd}_n^\partial$ and $f\mathrm{Mfd}_n^\partial$. Moreover, for all objects M and N in Mfd_n^∂, there exist pushout squares



Therefore Mfd_n^∂ satisfies the conditions of Lemma 3.4, which gives a description of the relations of the left hand side.

First, assume that the relations from $K_0^\square$ hold. To show that these imply the relations in SK_n^∂, we first need to check that the generating objects are compatible (note that SK_n^∂ is generated by diffeomorphism classes of manifolds, whereas K_0 a priori is generated by manifolds). Consider a diffeomorphism $M \xrightarrow{\phi} M'$. Then

$$\begin{array}{ccc} \emptyset & \rightarrowtail & M \\ \downarrow & \square & \downarrow{\scriptstyle \phi} \\ \emptyset & \rightarrowtail & M' \end{array}$$

is a distinguished square; and so the relations in $K_0^{\square}$ give that:

$$[M] + [\emptyset] = [M'] + [\emptyset]$$
$$[M] = [M']$$

Next, consider the square

$$\begin{array}{ccc} \emptyset & \rightarrowtail & M \\ \downarrow & \square & \downarrow \\ M' & \rightarrowtail & M \sqcup M'. \end{array}$$

This is a distinguished square, which means that

$$\begin{aligned} [M] + [M'] &= [M \sqcup M'] + [\emptyset] \\ &= [M \sqcup M']. \end{aligned}$$

For the other relation in SK_n^{∂}, consider compact oriented manifolds M, M', closed submanifolds $\Sigma \subseteq \partial M$ and $\Sigma' \subseteq \partial M'$, and orientation preserving diffeomorphisms $\phi, \psi \colon \Sigma \to \Sigma'$. We want to show that

$$[M \cup_\phi M'] = [M \cup_\psi M'].$$

Consider $(\Sigma \times \epsilon)$ where $\epsilon = [0, \varepsilon]$ for some small $\varepsilon > 0$. We can extend the maps ϕ, ψ by the identity to maps $\tilde{\phi}, \tilde{\psi}$ from $(\Sigma \times \epsilon)$ to $(\Sigma' \times \epsilon)$, which we consider inside M and M' respectively as collars of the boundary components. This is possible as the boundary has trivial normal bundle. We have that $M \cup_\phi M'$ is diffeomorphic to $M \cup_{\tilde{\phi}} M'$. Using the maps ϕ, ψ, consider the squares

$$\begin{array}{ccc} (\Sigma \times \epsilon) & \rightarrowtail & M \\ {\scriptstyle \tilde{\phi}}\downarrow & \square & \downarrow \\ M' & \rightarrowtail & M \cup_\phi M' \end{array} \qquad \begin{array}{ccc} (\Sigma \times \epsilon) & \rightarrowtail & M \\ {\scriptstyle \tilde{\psi}}\downarrow & \square & \downarrow \\ M' & \rightarrowtail & M \cup_\psi M'. \end{array}$$

The relation given by distinguished squares implies:

$$\begin{aligned}[M \cup_\phi M'] + [(\Sigma \times \epsilon)] &= [M] + [M'] \\ &= [M \cup_\psi M'] + [(\Sigma \times \epsilon)]\end{aligned}$$

Thus, $[M \cup_\phi M'] = [M \cup_\psi M']$.

In the other direction, assume that the relations for SK_n^∂ hold. Consider relation (1) in Definition 2.7 applied to the following:

$$\begin{aligned}[\emptyset \sqcup \emptyset] &= [\emptyset] + [\emptyset] \\ [\emptyset] &= [\emptyset]\end{aligned}$$

Thus, for $\emptyset$, the initial object in our category with squares, we have $[\emptyset] = 0$.

Finally, for relation (2) of Definition 2.7, suppose the following is a distinguished square:

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

Define $N := A \cap cl(B - A) \subseteq \partial A$, where $cl(B - A)$ is the closure of the complement of A in B, i.e. N is the part of the boundary of A that is mapped to the interior of B. We define

$$\begin{aligned}M &:= cl(B - A) \sqcup (N \times \epsilon), \\ M' &:= A \sqcup C.\end{aligned}$$

Let $\mathrm{id} : N \sqcup N \to N \sqcup N$ be the identity map; let $\tau : N \sqcup N \to N \sqcup N$ be the twist map. Note that $M \cup_{\mathrm{id}} M' \cong B \sqcup C$ and $M \cup_\tau M' \cong A \sqcup D$. Then the fact that $[M \cup_{\mathrm{id}} M'] = [M \cup_\tau M']$ gives the relations

$$\begin{aligned}[B \sqcup C] &= [A \sqcup D], \\ [B] + [C] &= [A] + [D].\end{aligned}$$

□

5. The derived Euler characteristic for manifolds with boundary

The Euler characteristic map $\chi\colon \mathcal{M}_n^\partial \to \mathbb{Z}$ from the monoid of diffeomorphism classes of smooth compact manifolds is an SK-invariant, since $\chi(M \cup_\Sigma N) = \chi(M) + \chi(N) - \chi(\Sigma)$; thus it factors through SK_n^∂. We show that the Euler characteristic map $\chi\colon \mathrm{SK}_n^\partial \to \mathbb{Z}$ lifts to a map of K-theory spaces. The strategy will be to construct a map of categories with squares from the category of smooth compact oriented manifolds with boundary to the category with squares from Definition 3.5

associated to the Waldhausen category of homologically bounded $\mathbb{Z}$-chain complexes. The main theorem we prove in this section is the following.

Theorem 5.1. *There is a map of K-theory spaces*

$$K^{\square}(\mathrm{Mfd}^{\partial}) \to K(\mathbb{Z}),$$

which on π_0 agrees with the Euler characteristic for smooth compact manifolds with boundary.

We first prove the propositions we need in the next section and give the proof of the theorem at the end of the final section.

5.1. **The lift of the singular chain functor.** Let $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$ be the Waldhausen category of homologically bounded chain complexes, i.e., those complexes that are quasi-isomorphic to bounded finitely generated $\mathbb{Z}$-complexes, with cofibrations given by levelwise injective maps and weak equivalences given by quasi-isomorphisms. Consider the associated category with squares $(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})^{\square}$ as defined in Definition 3.5.

Consider the singular chain functor

$$S\colon \mathrm{Mfd}_n^{\partial} \to \mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$$

which sends a compact manifold with boundary to its singular chain complex. The homology of this complex is finitely generated in each degree and bounded since our manifolds are compact.

Proposition 5.2. *The map S is a map of categories with squares*

$$S\colon \mathrm{Mfd}_n^{\partial} \to (\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})^{\square}$$

Proof. Suppose we have a distinguished square

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \square & \downarrow \\ C & \rightarrowtail & D \end{array}$$

in Mfd^{∂}, and we apply S to it. In the resulting square in $\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$, the horizontal maps are levelwise injective, as required. So in order to show that it is a distinguished square, it remains to show that the map

$$S(B) \cup_{S(A)} S(C) \to S(D)$$

is a quasisomorphism.

Note that by our construction of distinguished squares in Mfd^{∂} the union of the interiors of B and C covers D. Let $S_n(B+C)$ be the subgroup of $S_n(D)$ consisting

of n-chains that are sums of n-chains in B and n-chains in C. By the standard Mayer-Vietoris argument, the following sequence is exact

$$0 \longrightarrow S_n(A) \xrightarrow[x\mapsto(x,-x)]{} S_n(B)\oplus S_n(C) \xrightarrow[(y,z)\mapsto y+z]{} S_n(B+C) \longrightarrow 0.$$

Hence the chain complex $S_*(B+C)$ is a pushout $S_*(B)\cup_{S_*(A)} S_*(C)$. On the other hand by [Hat02, Proposition 2.21], the inclusions $S_n(B+C)\to S_n(D)$ induce isomorphisms on homology groups, which finishes the proof. □

Remark 5.3. The reason for the choices in our Definition 3.5 of a category with squares associated to a Waldhausen category is precisely to make the above proposition work. Note that we allow *all* maps as vertical maps as opposed to only the cofiber maps, which is the other way one could associate a category with squares to a Waldhausen category. This more relaxed definition of the distinguished squares is crucial in allowing us to show that distinguished squares in the category of manifolds map to distinguished squares in the category of chain complexes.

5.2. **Recovering the Euler characteristic on π_0.** Lastly, we claim that $K(\mathrm{Ch}^{\mathrm{hb}}_{\mathbb{Z}})\simeq K(\mathbb{Z})$ via an isomorphism under which $S(M)$ corresponds to $\chi(M)$ on π_0 for a smooth compact oriented manifold M.

Denote by $\mathrm{Ch}^{b}_{\mathbb{Z}}$ the category of bounded complexes of finitely generated $\mathbb{Z}$-modules; the homologically complexes are those that are quasi-isomorphic to complexes in $\mathrm{Ch}^{b}_{\mathbb{Z}}$. Note that $\mathrm{Ch}^{b}_{\mathbb{Z}}\subseteq \mathrm{Ch}^{\mathrm{hb}}_{\mathbb{Z}}$ and moreover by the discussion in [Wei13, V. 2.7.2] (or alternatively directly by the Waldhausen approximation theorem) this inclusion, which we denote by j, induces an isomorphism j_* on K-groups. A similar argument for cohomology appears in [CWZ19, Lemma 2.8].

Consider the following chain of maps of K-theory spaces

$$K(\mathbb{Z})=K(\mathrm{Mod}^{\mathrm{proj}}_{\mathbb{Z}}) \xrightarrow[i_*]{\simeq} G(\mathbb{Z})=K(\mathrm{Mod}^{\mathrm{fg}}_{\mathbb{Z}}) \xrightarrow[t_*]{\simeq} K(\mathrm{Ch}^{b}_{\mathbb{Z}}) \xrightarrow[j_*]{\simeq} K(\mathrm{Ch}^{\mathrm{hb}}_{\mathbb{Z}}),$$

where $\mathrm{Mod}^{\mathrm{proj}}_{\mathbb{Z}}$ is the category of finitely generated projective $\mathbb{Z}$-modules, $\mathrm{Mod}^{\mathrm{fg}}_{\mathbb{Z}}$ is the category of finitely generated $\mathbb{Z}$-modules. Let i_* be the map induced by the inclusion of categories $i\colon \mathrm{Mod}^{\mathrm{proj}}_{\mathbb{Z}}\to \mathrm{Mod}^{\mathrm{fg}}_{\mathbb{Z}}$. Let $t\colon \mathrm{Mod}^{\mathrm{fg}}_{\mathbb{Z}}\to \mathrm{Ch}^{b}_{\mathbb{Z}}$ be the inclusion of categories that sends a module M to the chain complex with M in degree 0 and zeroes in all other degrees. For a regular noetherian ground ring, in particular for $\mathbb{Z}$, the map i_* is an isomorphism on K-groups by the resolution theorem [Wei13, Theorem V.3.3.]. By the Gillet-Waldhausen theorem [Wei13, Theorem V.2.2.] t_* is a homotopy equivalence and hence induces an isomorphism on K-groups. The map j_* is also a homotopy equivalence by the Waldhausen approximation theorem [Wei13, V.2.7.4.].

The next proposition describes an inverse of j_* on the zeroth K-groups.

Proposition 5.4. *The map $q\colon K_0(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}) \to K_0(\mathrm{Ch}_{\mathbb{Z}}^{b})$ sending a chain complex C_* to the class of the corresponding quasi-isomorphic chain complex $H(C_*) \in \mathrm{Ch}_{\mathbb{Z}}^{b}$ is well-defined and is an isomorphism.*

Proof. The map is well-defined since quasi-isomorphic chain complexes have isomorphic homology, and it is surjective because of the inclusion $\mathrm{Ch}_{\mathbb{Z}}^{b} \subseteq \mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}$. On the other hand if $Y = q(X)$ vanishes in $K_0(\mathrm{Ch}_{\mathbb{Z}}^{b})$ then we may identify X with Y in $K_0(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})$ and it will also vanish there, because the set of defining relations (which we quotient out in the presentation for K_0) of $K_0(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})$ contains the defining relations of $K_0(\mathrm{Ch}_{\mathbb{Z}}^{b})$. □

Now, recall that the map $\phi\colon K_0(\mathrm{Ch}_{\mathbb{Z}}^{b}) \to K_0(\mathbb{Z})$ given by $[C_*] \mapsto \chi(C_*) = \sum_i (-1)^i [C_i]$ is an isomorphism [Wei13, Proposition II.6.6.] and coincides with the composition $i_*^{-1} \circ t_*^{-1}$ on the zeroth K-groups [Wei13, Theorem II.9.2.2.]. By an easy exercise using the additivity property, the Euler characteristic of a bounded complex only depends on its homology and $\chi(C_*) = \sum_i (-1)^i [H_i(C_*)]$. Thus the composition $\phi \circ q\colon K_0(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}) \to K_0(\mathbb{Z})$ is also an isomorphism and maps $[C_*]$ to $\chi(C_*) = \sum_i (-1)^i [H_i(C_*)]$. It coincides with the composition $i_*^{-1} \circ t_*^{-1} \circ j_*^{-1}\colon K_0(\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}}) \to K_0(\mathbb{Z})$.

Proof of Theorem 5.1. From Proposition 5.2, the singular chain functor $S\colon \mathrm{Mfd}_n^{\partial} \to (\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})^{\square}$ is a map of categories with squares when the right hand side is given the structure of a category with squares from Definition 3.5. This then induces a map on K-theory spaces

$$K^{\square}(\mathrm{Mfd}^{\partial}) \to K^{\square}\big((\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{hb}})^{\square}\big).$$

By Proposition 3.7 the target is $K(\mathbb{Z})$. By Proposition 5.4 and the discussion following it, the map on π_0 agrees with the Euler characteristic. □

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