

# Dynamic mean-variance problem with frictions

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#### **Abstract**

We study a dynamic mean–variance portfolio selection problem with return predictability and trading frictions from price impact. Applying mean-field type control theory, we provide a characterisation of an equilibrium trading strategy for an investor facing stochastic investment opportunities. An explicit equilibrium strategy is derived in terms of the solution to a generalised matrix Riccati differential equation, and a sufficient condition is also provided to ensure the latter's well-posedness. Our solution indicates that the investor should trade gradually towards a target portfolio which accounts for return predictability, price impact and time-consistency. Moreover, an asymptotic analysis around small liquidity costs shows that the investor's target portfolio is an equilibrium portfolio without price impact in the first-order sense, and that her first-order approximated value function does not deteriorate significantly for sufficiently small liquidity costs. Finally, our numerical results demonstrate that the target portfolio is more conservative than an equilibrium portfolio without price impact.

**Keywords** Dynamic mean–variance problem  $\cdot$  Price impact  $\cdot$  Time-inconsistency  $\cdot$  Asymptotics  $\cdot$  Mean-field type control problems

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#### 1 Introduction

The dynamic mean–variance portfolio analysis in Basak and Chabakauri [3] has inspired many researchers to derive equilibrium or time-consistent solutions to various time-inconsistent problems in finance and economics. In particular, Björk and Murgoci [9] and Björk et al. [8] derived extended Hamilton–Jacobi–Bellman (HJB) equations to obtain discrete- and continuous-time equilibrium strategies for a wide range of time-inconsistent problems with no market frictions. Since then, several extensions have also been proposed. Readers are referred to, for example, Wang and Forsyth [40], Czichowsky [14], Bensoussan et al. [6, 7], Gu et al. [25] and the references therein for representative works in this direction. More recently, He and Jiang [29] and Huang and Zhou [31] proposed various characterisations of strong, weak and regular equilibrium strategies for a class of time-inconsistent problems under Markovian settings.

At the other end of the spectrum, incorporating transaction costs into portfolio selection problems began with the seminal works of Constantinides [13] and Davis and Norman [15], where a no-trade region was introduced and rigorously analysed. The existing literature on transaction costs predominantly considers the time-consistent utility maximisation framework and typically focuses mainly on a single-asset analysis due to the complexity of the no-trade space for multiple assets. As closed-form solutions are typically unavailable in the presence of transaction costs, several asymptotic methods have also been proposed to obtain some tractable almost-optimal policies with transactions costs and constant investment opportunities (see e.g. Guasoni and Muhle-Karbe [26], Gerhold et al. [21], Altarovici et al. [2], Liu et al. [32], Guasoni and Weber [27, 28] and the references therein).

When the trading frictions are modelled by means of a linear transient price impact, Gârleanu and Pedersen [19] showed that the portfolio selection problem with return predictability of an infinitely-lived mean-variance investor admits a closed-form solution under a multi-asset framework. Specifically, to avoid the time-inconsistent nature of the mean-variance utility, Gârleanu and Pedersen [19] simply considered a local mean-variance investor whose utility function is a discounted sum of the single-period mean-variance functions of her portfolio returns over infinite time. The local mean-variance problem was reformulated into a standard linear-quadratic control problem whose solution can be found up to a system of matrix Riccati algebraic equations. Their discrete-time framework has attracted instant, widespread attention (see e.g. Glasserman and Xu [22], DeMiguel et al. [16], Moallemi and Sağlam [35], Mei and Nogales [34], Collin-Dufresne et al. [12]). Later, Gârleanu and Pedersen [20] extended their local mean-variance analysis to a continuous-time framework and demonstrated rigorously that the optimal trading strategy in continuous time can be seen as the limit of that in the discrete-time model for some specific forms of price impact. Representative extensions of Gârleanu and Pedersen [20] include, but are not limited to, Muhle-Karbe et al. [37, 38], Moreau et al. [36], Bouchard et al. [10], Ekren



and Muhle-Karbe [18], Ma et al. [33], Cayé et al. [11], Herdegen et al. [30] and the references therein.

In this paper, we consider a dynamic mean-variance problem with return predictability and frictions. Specifically, we adopt the asset and return-predicting factor dynamics in Gârleanu and Pedersen [20] in which correlations among assets and factors are allowed. Each trade incurs a linear transient price impact, which conveniently leads to quadratic execution costs for the investor. Differently from Gârleanu and Pedersen [20], we consider that the investor optimally trades to maximise a mean-variance utility function of her terminal wealth, which makes the stochastic control problem time-inconsistent. We derive an explicit equilibrium solution of that problem. To the best of our knowledge, such a problem has not been explored in the existing literature.

To obtain an explicit equilibrium solution to our dynamic mean-variance problem, we first derive an extended HJB equation by means of the mean-field type control techniques introduced in Bensoussan et al. [5, Chap. 4] and characterise the solution in terms of a coupled system of partial differential equations (PDEs). In addition, we also establish a verification theorem to demonstrate that the solvability of the original stochastic control problem is equivalent to the well-posedness of the coupled system of PDEs. Due to the wealth-separability of an corresponding equilibrium value function, we show that finding an equilibrium solution can be reduced to solving a coupled system of matrix Riccati differential equations (hereafter referred to as coupled Riccati system). It is important to mention that the inconsistent nature of the mean-variance problem makes the coupled Riccati system non-canonical, even in a block-matrix form, and it has not been well studied in the literature. Indeed, the coupling feature of the Riccati system in our paper makes it structurally different from those in the finite-time analogues of Gârleanu and Pedersen [20] and Ma et al. [33], where the Riccati systems are decoupled in their block matrix forms and the global existence of solutions can be established by means of a standard comparison principle for matrix Riccati differential equations.

Our work makes the following contributions to the existing literature. Since the unique structure of our coupled Riccati system prevents us from using any existing approach, we first derive a sufficient condition to establish the existence and uniqueness of a solution to the coupled Riccati system in this paper. The sufficient condition is easy to check as it is shown to be equivalent to finding a scalar solution of a nonlinear equation. Second, by conducting an asymptotic analysis around small liquidity costs, we show that the coupling feature in our Riccati system appears in the higher order terms since the first-order approximated equations of the coupled Riccati system admit closed-form expressions.

Third, our equilibrium trading strategy indicates that the mean–variance investor should trade gradually towards a dynamic *target portfolio*, which is the discounted value of a future equilibrium portfolio in the absence of price impact, with some adjustment terms. Differently from the aim portfolio in Gârleanu and Pedersen [20], the target portfolio here not only accounts for the persistence of return-predicting signals and price impacts, but also for the investor not deviating from future strategies. More importantly, the volatilities of the return-predicting factors have a direct impact on the dynamics of the target portfolio, whereas they do not appear in the aim portfolio in Gârleanu and Pedersen [20].



Fourth, by means of an asymptotic analysis around small liquidity costs, an investor's frictionless equilibrium portfolio is the target portfolio in the first-order sense when the liquidity cost is sufficiently small. In addition, the equilibrium value function derived from pursuing such a first-order approximated trading strategy is almost optimal in the sense that it does not deteriorate significantly around small liquidity costs. Fifth, our numerical results indicate that the target portfolio appears to be a conservative form of its corresponding equilibrium portfolio without price impact.

The rest of this paper is organised as follows. In Sect. 2, we present a dynamic mean–variance problem with return predictability and price impact, and the definition of an equilibrium strategy. In Sect. 3, we derive an analytical equilibrium solution to the extended HJB equation in terms of a coupled system of matrix Riccati differential equations. The latter's well-posedness is also provided in Sect. 4. Section 5 provides some financial implications of our main results and conducts an asymptotic analysis around small liquidity costs. Section 6 provides numerical illustrations. Section 7 concludes the paper with suggestions for future research.

## 2 Dynamic mean-variance problem with frictions

### 2.1 Dynamics with return predictability

Consider a financial market where one risk-free asset and n risky assets are traded continuously on a finite interval [0,T]. Let  $(\Omega,\mathcal{F},\mathbb{F},\mathbb{F})$  be a complete filtered probability space, where  $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$  denotes the filtration generated by a standard n-dimensional Brownian motion  $(W_t^f)_{t\in[0,T]}$  and a k-dimensional Brownian motion  $(W_t^f)_{t\in[0,T]}$  with instantaneous correlation matrix  $\rho=d\langle W^p,W^f\rangle_t/dt$ .

Denote by  $(p_t^0)_{t \in [0,T]}$  and  $(p_t)_{t \in [0,T]}$  the risk-free asset price and the vector of risky asset prices, respectively. The dynamic of the risk-free asset is given by

$$dp_t^0 = rp_t^0 dt,$$

where r > 0 is the constant risk-free interest rate. The dynamics of the risky asset prices  $(p_t)_{t \in [0,T]}$  and the return-predicting factors  $(f_t)_{t \in [0,T]}$  are modelled as

$$dp_t = (rp_t + Bf_t)dt + \sigma_p dW_t^p, \tag{2.1}$$

$$df_t = -\Phi f_t dt + \sigma_f dW_t^f, \qquad (2.2)$$

where B is an  $n \times k$  matrix of factor loadings of full column rank,  $\Phi$  is a symmetric positive definite matrix which represents the factors' mean-reversion rates, and  $\sigma_p$  and  $\sigma_f$  are  $n \times n$  and  $k \times k$  constant matrices, respectively. Denote the covariance matrices by  $\Sigma_p = \sigma_p^\top \sigma_p = d\langle p, p \rangle_t / dt$ ,  $\Sigma_f = \sigma_f^\top \sigma_f = d\langle f, f \rangle_t / dt$  and

$$\Sigma_{\rho} = \sigma_{p}^{\top} \rho \sigma_{f} = d \langle p, f \rangle_{t} / dt,$$

where  $^{\top}$  denotes the transpose of a matrix or a vector.



### 2.2 Wealth dynamics with price impact

Denote by  $(x_t)_{t \in [0,T]}$  the investor's portfolio position, which contains the number of shares of the risky assets. As pointed out by Gârleanu and Pedersen [20], non-smooth changes in portfolios, such as a discrete jump or quadratic variation (e.g. Brownian motion) in position, would result in infinite transaction costs under the continuous-time framework. Consequently, we adopt their assumption that the investor chooses a *trading strategy*  $(\tau_t)_{t \in [0,T]}$  that determines the instantaneous rate of change of her portfolio position  $(x_t)_{t \in [0,T]}$ , i.e.,

$$dx_t = \tau_t dt. (2.3)$$

To model trading frictions, we follow Gârleanu and Pedersen [20] and assume that when the investor trades  $\tau_t dt$  units of the assets at time  $t \ge 0$ , her trade has a *transient linear price impact* on the asset price  $(p_t)_{t \in [0,T]}$  as

$$p_t^{\mathrm{E}} := p_t + \frac{1}{2}\Lambda \tau_t, \tag{2.4}$$

where  $\Lambda$  is a symmetric positive definite  $n \times n$  matrix that measures the level of transaction costs. The matrix  $\Lambda$  is referred to as the multi-dimensional Kyle's lambda (see e.g. Grossman and Miller [24] and Greenwood [23]). In (2.4),  $(p_t^E)_{t \in [0,T]}$  denotes the vector of *execution prices* and represents the amount the investor pays (receives) for each unit of her buy (sell)  $\tau_t dt$ . We assume that the investor's trade size  $\tau_t dt$  affects only the execution price and that her trade has no permanent impact on the asset price dynamics in (2.1). In terms of the execution price in (2.4), the total cost associated with the trading strategy is expressed as

$$Cost(\tau_t dt) := \tau_t^{\top} p_t dt + TC(\tau_t) dt,$$

where the first term denotes the cost of trading  $\tau_t dt$  units of the assets at the market price  $p_t$ , and the second term,  $TC(\tau_t) := \frac{1}{2}\tau_t^\top \Lambda \tau_t$ , captures the quadratic execution costs of trading  $\tau_t dt$  units.

With the specifications of the asset price dynamics and execution costs in place, we can now derive the wealth dynamics of an investor with trading frictions. Let  $(y_t^{\tau})_{t \in [0,T]}$  be the investor's wealth process under the trading strategy  $(\tau_t)_{t \in [0,T]}$ . Invoking the self-financing condition and Itô's lemma yields

$$dy_t^{\tau} = \left(ry_t^{\tau} + x_t^{\top} B f_t - \frac{1}{2} \tau_t^{\top} \Lambda \tau_t\right) dt + x_t^{\top} \sigma_p dW_t^p.$$

We restrict our attention to admissible feedback trading strategies as defined below.

**Definition 2.1** An admissible feedback trading strategy is a map

$$\tau:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}^n$$



such that for each initial point  $(t, x, y, f) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$ , the stochastic differential equation (SDE) system

$$\begin{cases} dx_t = \tau(t, x_t, y_t^{\tau}, f_t)dt, \\ dy_t^{\tau} = \left(ry_t^{\tau} + x_t^{\top} B f_t - \frac{1}{2} \tau(t, x_t, y_t^{\tau}, f_t)^{\top} \Lambda \tau(t, x_t, y_t^{\tau}, f_t)\right) dt \\ + x_t^{\top} \sigma_p dW_t^p, \end{cases}$$
(2.5)

admits a unique strong solution with  $\mathbb{E}[\sup_{t\in[0,T]}|\tau(t,x_t,y_t^{\tau},f_t)|^2]<\infty$ .

The class of admissible feedback trading strategies is denoted by  $A_{[0,T]}$ , and we use  $\tau$  and  $\tau(t, x, y, f)$  interchangeably throughout the paper.

#### 2.3 Dynamic mean-variance portfolio selection problem

Suppose that the investor seeks to choose a trading strategy  $(\tau_t)_{t \in [0,T]}$  that maximises the expected mean–variance utility of her terminal wealth  $y_T^{\tau}$ , i.e.,

$$\max_{\tau} \left( \mathbb{E}[y_T^{\tau}] - \frac{\eta}{2} \text{Var}[y_T^{\tau}] \right), \tag{2.6}$$

with the return-predicting factors  $(f_t)_{t\in[0,T]}$  in (2.2), her portfolio position  $(x_t)_{t\in[0,T]}$  in (2.3) and her wealth process  $(y_t^{\tau})_{t\in[0,T]}$  in (2.5). The expected mean–variance utility function in (2.6) is different from the local mean–variance utility function in Gârleanu and Pedersen [20] and Collin-Dufresne et al. [12] in that the variance operator in (2.6) violates the tower property of conditional expectations, as stated in Basak and Chabakauri [3]. To circumvent this problem, we follow the existing literature (see e.g. Ekeland and Pirvu [17] and Bensoussan et al. [6]) and provide a game-theoretic characterisation of a so-called equilibrium trading strategy. For a fixed initial point  $(t, x, y, f) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$ , we denote by

$$\mathbb{E}_{t}^{x,y,f}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|x_{t} = x,\,y_{t}^{\tau} = y,\,f_{t} = f\,],$$

$$\text{Var}_{t}^{x,y,f}[\,\cdot\,] := \text{Var}[\,\cdot\,|x_{t} = x,\,y_{t}^{\tau} = y,\,f_{t} = f\,]$$

the conditional expectation and conditional variance, respectively. The criterion in (2.6) can then be rewritten in terms of the objective functional J defined as

$$J(t, x, y, f; \tau) := \mathbb{E}_{t}^{x, y, f}[y_{T}^{\tau}] - \frac{\eta}{2} \operatorname{Var}_{t}^{x, y, f}[y_{T}^{\tau}],$$

$$= \mathbb{E}_{t}^{x, y, f}[F(y_{T}^{\tau})] + G(\mathbb{E}_{t}^{x, y, f}[y_{T}^{\tau}]), \tag{2.7}$$

where  $F(y) := y - \frac{\eta}{2}y^2$  and  $G(y) := \frac{\eta}{2}y^2$ .

Due to the quadratic function  $G(\mathbb{E}_t^{x,y,f}[y_T^{\tau}])$  in the mean-field term, the maximisation of the objective functional in (2.7) is a mean-field type control problem. We follow the guiding philosophy proposed in Bensoussan et al. [5, Chap. 4] to solve our problem more explicitly. In this paper, we consider an equilibrium solution of this problem as defined below.



**Definition 2.2** An *equilibrium solution* (or *equilibrium trading strategy*) is an admissible feedback trading strategy  $\tau^* : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^n$  such that for any  $(t, x, y, f) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$ ,

$$\liminf_{\epsilon \to 0+} \frac{J(t,x,y,f;\tau^*) - J(t,x,y,f;\tau^\epsilon)}{\epsilon} \ge 0.$$

Here  $\tau^{\epsilon}$  is an arbitrary spike variation of  $\tau^* \in \mathcal{A}_{[0,T]}$  defined as

$$\tau^{\epsilon}(s, x, y, f) = \begin{cases} \bar{\tau}(s, x, y, f), & s \in [t, t + \epsilon), x \in \mathbb{R}^n, y \in \mathbb{R}, f \in \mathbb{R}^k, \\ \tau^*(s, x, y, f), & s \in [t + \epsilon, T], x \in \mathbb{R}^n, y \in \mathbb{R}, f \in \mathbb{R}^k, \end{cases}$$
(2.8)

where  $\bar{\tau} \in \mathcal{A}_{[0,T]}$  and  $\epsilon > 0$  are arbitrarily chosen. The corresponding *equilibrium* value function V is defined as  $V(t, x, y, f) := J(t, x, y, f; \tau^*)$ .

**Remark 2.3** According to He and Jiang [29] and Huang and Zhou [31], an equilibrium solution as in Definition 2.2 constitutes a *weak equilibrium solution*, which is defined by means of a first-order condition.

## 3 Equilibrium solution

In this section, we provide an analytical expression for an equilibrium trading strategy  $\tau^*$  in Definition 2.2. Specifically, we first apply mean-field type control techniques to derive an extended HJB equation which is similar to the works of Bensoussan et al. [5, Chap. 4] and Björk et al. [8] and show that the corresponding value function under an equilibrium trading strategy  $\tau^*$  satisfies a system of coupled PDEs. A verification theorem is also established to demonstrate that the solution of the coupled PDE system is indeed the corresponding equilibrium value function V in Definition 2.2. Secondly, we exploit the explicit form of V and show that it can be decomposed into wealth-dependent and wealth-independent components. The modified value function is then shown to be a quadratic function of the investor's current position and the current levels of the return-predicting factors, with the coefficients of the quadratic functions being expressed as solutions to a coupled system of matrix Riccati differential equations.

We first characterise an equilibrium trading strategy  $\tau^*$  and the corresponding equilibrium value function in terms of a system of coupled PDEs. We say that a function  $\ell(t, x, y, f)$  on  $[0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$  belongs to the space  $L^2([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$  if it satisfies the condition

$$\mathbb{E}_t^{x,y,f} \left[ \int_t^T \left( |\partial_y \ell(s,x_s,y_s^\tau,f_s) x_s^\top \sigma_p|^2 + |\partial_f \ell(s,x_s,y_s^\tau,f_s) \sigma_f|^2 \right) ds \right] < \infty$$

for any admissible feedback trading strategy  $\tau$  and every (t, x, y, f).

**Theorem 3.1** Suppose there exists an equilibrium strategy  $\tau^*$  whose corresponding equilibrium value function V(t, x, y, f) and  $\Psi(t, x, y, f) := \mathbb{E}_t^{x, y, f}[y_t^{\tau^*}]$  are both in



 $C^{1,1,2,2}([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k) \cap L^2([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ . Then they satisfy the coupled system of PDEs

$$\begin{cases} \max_{\tau \in \mathbb{R}^n} \mathcal{L}^{\tau} V = \eta \mathcal{N} \Psi, \\ \mathcal{L}^{\tau^*} \Psi = 0 \end{cases}$$
 (3.1)

with  $V(T, x, y, f) = \Psi(T, x, y, f) = y$ , where the operators  $\mathcal{L}^{\tau}$  and  $\mathcal{N}$  are defined, for  $v \in \mathcal{C}^{1,1,2,2}([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ , by

$$\mathcal{L}^{\tau}v := \partial_{t}v + \partial_{x}v^{\top}\tau - \partial_{f}v^{\top}(\Phi f) + \left(ry + x^{\top}Bf - \frac{1}{2}\tau^{\top}\Lambda\tau\right)\partial_{y}v$$
$$+ \frac{1}{2}x^{\top}\Sigma_{\rho}x\partial_{yy}v + x^{\top}\Sigma_{\rho}\partial_{fy}v + \frac{1}{2}\operatorname{tr}(\partial_{ff}v\Sigma_{f}),$$
$$\mathcal{N}v := \frac{1}{2}x^{\top}\Sigma_{\rho}x(\partial_{y}v)^{2} + x^{\top}\Sigma_{\rho}\partial_{f}v\partial_{y}v + \frac{1}{2}\partial_{f}v^{\top}\Sigma_{f}\partial_{f}v,$$

and  $tr(\cdot)$  represents the trace of a matrix.

**Proof** For  $\epsilon > 0$ , consider the spike modification control  $\tau^{\epsilon}$  defined in (2.8) and the corresponding objective functional  $J(t, x, y, f; \tau^{\epsilon})$ . By the Feynman–Kac formula,  $\Psi(t, x, y, f)$  is indeed the solution to the second equation in (3.1). We also have

$$J(t, x, y, f; \tau^{\epsilon}) - J(t, x, y, f; \tau^{*}) = I_1 - I_2, \tag{3.2}$$

where

$$I_{1} := \mathbb{E}_{t}^{x,y,f} [V(t+\epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon})] - V(t, x, y, f),$$

$$I_{2} := G \left( \mathbb{E}_{t}^{x,y,f} [\Psi(t+\epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon})] \right) - \mathbb{E}_{t}^{x,y,f} [G \left( \Psi(t+\epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon}) \right)].$$

In (3.2), the equality follows from the definition of  $J(t, x, y, f; \tau^{\epsilon})$  and V(t, x, y, f) in (2.7) and Definition 2.2, respectively, and the tower property of the conditional expectation.

Applying the Dynkin formula on  $I_1$  and using the definition of the spike modification control in (2.8), we have

$$I_1 = \mathbb{E}_t^{x,y,f} \left[ \int_t^{t+\epsilon} \mathcal{L}^{\bar{\tau}} V(s, x_s, y_s^{\bar{\tau}}, f_s) ds \right], \tag{3.3}$$

where the equality holds due to the assumption that  $V \in L^2([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ . We now turn to the calculation of  $I_2$ . Noting that  $G(y) = \frac{\eta}{2}y^2$ , we apply the Dynkin formula on  $G(\Psi(t+\epsilon,x_{t+\epsilon},y_{t+\epsilon}^{\tau^\epsilon},f_{t+\epsilon}))$  and have

$$\mathbb{E}_{t}^{x,y,f} \Big[ G \big( \Psi(t+\epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon}) \big) \Big]$$

$$= G \big( \Psi(t, x, y, f) \big) + \mathbb{E}_{t}^{x,y,f} \Big[ \int_{t}^{t+\epsilon} \mathcal{L}^{\bar{\tau}} G \big( \Psi(s, x_{s}, y_{s}^{\bar{\tau}}, f_{s}) \big) ds \Big].$$



On the other hand, applying the Taylor expansion on G(y) gives

$$\begin{split} &G\left(\mathbb{E}_{t}^{x,y,f}\left[\Psi(t+\epsilon,x_{t+\epsilon},y_{t+\epsilon}^{\tau^{\epsilon}},f_{t+\epsilon})\right]\right) \\ &= G\left(\Psi(t,x,y,f)\right) + \eta\Psi(t,x,y,f)\mathbb{E}_{t}^{x,y,f}\left[\int_{t}^{t+\epsilon}\mathcal{L}^{\bar{\tau}}\Psi(s,x_{s},y_{s}^{\bar{\tau}},f_{s})ds\right] + o(\epsilon). \end{split}$$

Combining these together leads to

$$I_2 = -\eta \mathbb{E}_t^{x,y,f} \left[ \int_t^{t+\epsilon} \mathcal{N}\Psi(s, x_s, y_s^{\bar{\tau}}, f_s) ds \right] + o(\epsilon). \tag{3.4}$$

Therefore, expressing (3.2) in terms of (3.3) and (3.4) leads to

$$\begin{split} &J(t,x,y,f;\tau^{\epsilon}) - J(t,x,y,f;\tau^{*}) \\ &= \mathbb{E}_{t}^{x,y,f} \left[ \int_{t}^{t+\epsilon} \mathcal{L}^{\bar{\tau}}V(s,x_{s},y_{s}^{\bar{\tau}},f_{s}) - \eta \mathcal{N}\Psi(s,x_{s},y_{s}^{\bar{\tau}},f_{s}) ds \right] + o(\epsilon), \end{split}$$

where we have suppressed the arguments of the functions for simplicity. Dividing by  $\epsilon$  on both sides, taking  $\epsilon \to 0+$  and using Definition 2.2 leads to

$$\mathcal{L}^{\bar{\tau}}V < \eta \mathcal{N}\Psi. \tag{3.5}$$

Consequently, if we choose the maximiser of  $\mathcal{L}^{\bar{\tau}}V$  to be  $\tau^*$ , the inequality in (3.5) becomes an equality, leading to the PDE for V.

Next, we establish a verification theorem to show that a solution to the coupled PDE system in (3.1) is an equilibrium value function in Definition 2.2, thereby proving that a maximiser in (3.1) constitutes an equilibrium trading strategy.

**Theorem 3.2** Suppose that the functions V(t, x, y, f),  $\Psi(t, x, y, f)$  and  $\tau^*$  satisfy the following assumptions:

- 1) V(t, x, y, f) and  $\Psi(t, x, y, f)$  are in  $C^{1,1,2,2}([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$  and are solutions of the coupled PDE system in (3.1).
  - 2)  $\tau^*$  is a maximiser in (3.1) and  $\tau^* \in A_{[0,T]}$ .
  - 3) V(t, x, y, f) and  $\Psi(t, x, y, f) \in L^2([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ .

Then  $\tau^*$  is an equilibrium trading strategy, and V(t, x, y, f) is the corresponding equilibrium value function.

**Proof** We first show that V(t, x, y, f) is the value function associated with the admissible trading strategy  $\tau^*$  which is a maximiser in (3.1). Since V(t, x, y, f) is a solution of the HJB equation (3.1), we have  $\mathcal{L}^{\tau^*}V = \eta \mathcal{N}\Psi$ . Applying the Dynkin formula to V(t, x, y, f) leads to

$$\mathbb{E}_{t}^{x,y,f}[V(T,x_{T},y_{T}^{\tau^{*}},f_{T})] = V(t,x,y,f) + \mathbb{E}_{t}^{x,y,f} \left[ \int_{t}^{T} \mathcal{L}^{\tau^{*}}V(s,x_{s},y_{s}^{\tau^{*}},f_{s})ds \right].$$



where the stochastic integral part becomes zero due to the assumption that V is in  $L^2([0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ . Combining the above leads to

$$\mathbb{E}_{t}^{x,y,f}[V(T,x_{T},y_{T}^{\tau^{*}},f_{T})] - V(t,x,y,f) 
= \eta \mathbb{E}_{t}^{x,y,f} \left[ \int_{t}^{T} \mathcal{N}\Psi(s,x_{s},y_{s}^{\tau^{*}},f_{s})ds \right] 
= \mathbb{E}_{t}^{x,y,f} \left[ G\left(\Psi(T,x_{T},y_{T}^{\tau^{*}},f_{T})\right) \right] - G\left(\Psi(t,x,y,f)\right).$$
(3.6)

By means of the terminal conditions for V and  $\Psi$ , we can rewrite (3.6) as

$$V(t,x,y,f) = \mathbb{E}_t^{x,y,f}[y_T^{\tau^*} - G(y_T^{\tau^*})] + G(\mathbb{E}_t^{x,y,f}[y_T^{\tau^*}]) = J(t,x,y,f;\tau^*).$$

Next, we show that  $\tau^*$  is an equilibrium trading strategy in the sense of Definition 2.2. We first take a spike modification  $\tau^{\epsilon}$  of  $\tau^*$  as in (2.8). Applying the Dynkin formula and the Taylor expansion on the objective function J in (2.7), it follows that

$$J(t, x, y, f; \tau^{\epsilon}) = \mathbb{E}_{t}^{x, y, f} \left[ V(t + \epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon}) - \eta \int_{t}^{t+\epsilon} \mathcal{N}\Psi(s, x_{s}, y_{s}^{\bar{\tau}}, f_{s}) ds \right] + o(\epsilon).$$

On the other hand, from the first equation in (3.1), we have

$$\mathcal{L}^{\tau^{\epsilon}} V \leq \eta \mathcal{N} \Psi.$$

Integrating from t to  $t + \epsilon$  and taking conditional expectations leads to

$$0 \ge \mathbb{E}_{t}^{x,y,f} [V(t+\epsilon, x_{t+\epsilon}, y_{t+\epsilon}^{\tau^{\epsilon}}, f_{t+\epsilon})] - V(t, x, y, f)$$
$$- \eta \mathbb{E}_{t}^{x,y,f} \left[ \int_{t}^{t+\epsilon} \mathcal{N}\Psi(s, x_{s}, y_{s}^{\bar{\tau}}, f_{s}) ds \right]$$
$$= J(t, x, y, f; \tau^{\epsilon}) - V(t, x, y, f) + o(\epsilon).$$

Therefore, it follows that  $J(t,x,y,f;\tau^*) = V(t,x,y,f) \ge J(t,x,y,f;\tau^\epsilon) + o(\epsilon)$ , yielding  $\liminf_{\epsilon \to 0+} \frac{J(t,x,y,f;\tau^*) - J(t,x,y,f;\tau^\epsilon)}{\epsilon} \ge 0$ , which completes the proof.  $\square$ 

The following proposition shows that the equilibrium value function V and the auxiliary function  $\Psi$  in Theorem 3.1 can be decomposed into wealth-dependent and wealth-independent components. This wealth-separable property of V and  $\Psi$  allows one to conveniently focus on finding solutions as functions that are independent of wealth, which greatly enhances the analytical tractability of V in Definition 2.2.



### Proposition 3.3 Let

$$V(t, x, y, f) = h(t)y - \frac{1}{2}x^{\top}A_{xx}(t)x + x^{\top}A_{xf}(t)f + \frac{1}{2}f^{\top}A_{ff}(t)f + A_0(t),$$
(3.7)

$$\Psi(t, x, y, f) = h(t)y - \frac{1}{2}x^{\top}C_{xx}(t)x + x^{\top}C_{xf}(t)f + \frac{1}{2}f^{\top}C_{ff}(t)f + C_{0}(t)$$
(3.8)

and

$$\tau^*(t, x, y, f) = \frac{\Lambda^{-1}}{h(t)} (A_{xf}(t)f - A_{xx}(t)x), \tag{3.9}$$

where  $h(t) := e^{r(T-t)}$  and  $A_{xx}(t)$ ,  $A_{xf}(t)$ ,  $A_{ff}(t)$ ,  $C_{xx}(t)$ ,  $C_{xf}(t)$ ,  $C_{ff}(t)$  solve the coupled Riccati system,

$$\begin{cases}
\dot{A}_{xx} = \frac{1}{h} A_{xx} \Lambda^{-1} A_{xx} - \eta \left( h^{2} \Sigma_{p} + C_{xf} \Sigma_{f} C_{xf}^{\top} + h (\Sigma_{\rho} C_{xf}^{\top} + C_{xf} \Sigma_{\rho}^{\top}) \right), \\
\dot{A}_{xf} = -hB + \frac{1}{h} A_{xx} \Lambda^{-1} A_{xf} + A_{xf} \Phi + \eta (C_{xf} \Sigma_{f} C_{ff} + h \Sigma_{\rho} C_{ff}), \\
\dot{A}_{ff} = -\frac{1}{h} A_{xf}^{\top} \Lambda^{-1} A_{xf} + (A_{ff} \Phi + \Phi A_{ff}) + \eta C_{ff} \Sigma_{f} C_{ff}, \\
\dot{C}_{xx} = -\frac{1}{h} A_{xx} \Lambda^{-1} A_{xx} + \frac{1}{h} (A_{xx} \Lambda^{-1} C_{xx} + C_{xx} \Lambda^{-1} A_{xx}), \\
\dot{C}_{xf} = -hB + C_{xf} \Phi + \frac{1}{h} (A_{xx} \Lambda^{-1} C_{xf} + C_{xx} \Lambda^{-1} A_{xf}) - \frac{1}{h} (A_{xx} \Lambda^{-1} A_{xf}), \\
\dot{C}_{ff} = (C_{ff} \Phi + \Phi C_{ff}) - \frac{1}{h} (A_{xf}^{\top} \Lambda^{-1} C_{xf} + C_{xf}^{\top} \Lambda^{-1} A_{xf}) + \frac{1}{h} A_{xf}^{\top} \Lambda^{-1} A_{xf}
\end{cases}$$

with

$$A_{xx}(T) = C_{xx}(T) = 0_{n \times n},$$

$$A_{xf}(T) = C_{xf}(T) = 0_{n \times k},$$

$$A_{ff}(T) = C_{ff}(T) = 0_{k \times k},$$

$$A_{0}(t) = \frac{1}{2} \int_{t}^{T} \operatorname{tr} \left( A_{ff}(s) \Sigma_{f} \right) ds,$$

$$C_{0}(t) = \frac{1}{2} \int_{t}^{T} \operatorname{tr} \left( C_{ff}(s) \Sigma_{f} \right) ds.$$

If the coupled system in (3.10) is well defined, then  $\tau^*$  in (3.9) is an equilibrium trading strategy, and V(t, x, y, f) is the corresponding equilibrium value function.

**Proof** According to Theorem 3.2, it suffices to show that the functions V(t, x, y, f) in (3.7),  $\Psi(t, x, y, f)$  in (3.8) and the candidate equilibrium trading strategy  $\tau^*$  in (3.9) satisfy the assumptions in Theorem 3.2.



First, substituting the ansatz in (3.7) and (3.8) into the coupled PDE system in (3.1) and matching coefficients for the terms  $y, x^{\top}(\cdot)x, x^{\top}(\cdot)f, f^{\top}(\cdot)f$  directly leads to the coupled system (3.10). In other words, if (3.10) is well defined, then V(t, x, y, f) and  $\Psi(t, x, y, f)$  in (3.7) and (3.8) are in  $\mathcal{C}^{1,1,2,2}([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$  and solve the coupled PDE system in (3.1). In addition, since  $\Lambda > 0_{n \times n}$  and  $\partial_y V > 0$ , it follows that  $\tau^*$  in (3.9) is a maximiser in (3.1).

Next we show that  $\tau^* \in \mathcal{A}_{[0,T]}$ . As the state dynamics (2.2) of  $(f_t)_{t \in [0,T]}$  is linear and independent of the trading strategy, it follows readily that  $\mathbb{E}[\sup_{t \in [0,T]} |f_t|^2] < \infty$ . Since  $\tau^*$  in (3.9) is linear in x and f, the equilibrium portfolio  $(x_t^*)_{t \in [0,T]}$  in (2.3) is linear in  $(f_t)_{t \in [0,T]}$  and so  $\mathbb{E}[\sup_{t \in [0,T]} |x_t^*|^2] < \infty$ . Consequently, we have  $\mathbb{E}[\sup_{t \in [0,T]} |\tau_t^*|^2] < \infty$ . As  $\tau^*$  in (3.9) is independent of y, the equilibrium wealth process  $(y_t^{\tau^*})_{t \geq 0}$  satisfies a linear SDE in (2.5) and admits a unique strong solution. Therefore, it follows that  $\tau^* \in \mathcal{A}_{[0,T]}$ .

From (3.7) and (3.8), the functions V(t,x,y,f) and  $\Psi(t,x,y,f)$  are linear in y and quadratic in f. Consequently, the term  $\partial_y \ell(t,x,y,f)$  is bounded and the term  $\partial_f \ell(t,x,y,f)$  is linear in x and f for  $\ell \in \{V,\Psi\}$ . For any admissible trading strategy  $\tau$ , we have  $\mathbb{E}[\sup_{t \in [0,T]} |\tau_t|^2] < \infty$ , which implies that  $\mathbb{E}[\int_0^T |x_s|^2 ds] < \infty$ . Combining this with  $\mathbb{E}[\int_0^T |f_s|^2 ds] < \infty$ , we obtain

$$\mathbb{E}_t^{x,y,f} \left[ \int_t^T \left( |\partial_y \ell(s,x_s,y_s^{\tau},f_s) x_s^{\tau} \sigma_p|^2 + |\partial_f \ell(s,x_s,y_s^{\tau},f_s) \sigma_f|^2 \right) ds \right] < \infty,$$

proving that 
$$V(t, x, y, f)$$
,  $\Psi(t, x, y, f)$  are in  $L^2([0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$ .

Up to now, we have shown that V and  $\Psi$  in Proposition 3.3 can be represented as linear functions of current wealth y and quadratic functions of the portfolio position  $x_t$  and the return-predicting factors  $f_t$  at the time  $t \ge 0$ . The corresponding coefficient functions can be obtained by solving the system of matrix Riccati differential equations in (3.10). We provide a rigorous analysis on the well-posedness of (3.10) in Sect. 4.

We have also derived an explicit form of an equilibrium trading strategy  $\tau^*$  in Proposition 3.3. The equilibrium strategy  $\tau^*$  in (3.9) is of feedback form on the current gap between the return-predicting factors and the position. More importantly, the equilibrium strategy  $\tau^*$  in (3.9) indicates that the more illiquid the stocks are (larger  $\Lambda$ ), the smaller the magnitude of the investor's equilibrium trading strategy becomes (smaller  $\tau^*$ ). We provide more financial interpretation on  $\tau^*$  in Sect. 5 after proposing in Sect. 4 a sufficient condition to ensure the well-posedness of the coupled system (3.10).

**Remark 3.4** In Proposition 3.3, since  $\Lambda > 0_{n \times n}$  and  $\partial_y V > 0$ , there exists a unique interior solution  $\tau^*$  in (3.9). In view of Remark 2.3,  $\tau^*$  in (3.9) is a weak equilibrium solution in the sense of He and Jiang [29] and Huang and Zhou [31]. In addition, by Theorem 5.2 (iii) in He and Jiang [29],  $\tau^*$  in (3.9) is also a *regular equilibrium solution*.



## 4 The well-posedness of the coupled Riccati differential system

In the previous section, we have demonstrated that an equilibrium value function V as in Definition 2.2 can be characterised by the extended HJB equation in (3.1). Moreover, existence and uniqueness of the latter's solution is equivalent to well-posedness of the coupled system of matrix Riccati differential equations in (3.10). In this section, we provide a time-dependent sufficient condition to address the latter.

We reformulate (3.10) in block matrix form as

$$\begin{cases} \dot{A} = D_1 + AK_1 + K_1A - AN_1A + \eta(CK_2 + K_2^{\top}C + CN_2C), \\ \dot{C} = D_2 + CK_1 + K_1C + AN_1A - AN_1C - CN_1A \end{cases}$$
(4.1)

with terminal conditions  $A(T) = C(T) = 0_{(n+k)\times(n+k)}$  and

$$A(t) = \begin{pmatrix} -A_{xx}(t) & A_{xf}(t) \\ A_{xf}^{\top}(t) & A_{ff}(t) \end{pmatrix}, \qquad C(t) = \begin{pmatrix} -C_{xx}(t) & C_{xf}(t) \\ C_{xf}^{\top}(t) & C_{ff}(t) \end{pmatrix},$$

$$D_1(t) = \begin{pmatrix} \eta h^2(t) \Sigma_p & -h(t)B \\ -h(t)B^{\top} & 0_{k \times k} \end{pmatrix}, \qquad D_2(t) = \begin{pmatrix} 0_{n \times n} & -h(t)B \\ -h(t)B^{\top} & 0_{k \times k} \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0_{n \times n} & 0_{n \times k} \\ 0_{k \times n} & \Phi \end{pmatrix}, \qquad K_2(t) = \begin{pmatrix} 0_{n \times n} & 0_{n \times k} \\ h(t) \Sigma_{\rho}^{\top} & 0_{k \times k} \end{pmatrix},$$

$$N_1(t) = \begin{pmatrix} \frac{\Lambda^{-1}}{h(t)} & 0_{n \times k} \\ 0_{k \times n} & 0_{k \times k} \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0_{n \times n} & 0_{n \times k} \\ 0_{k \times n} & \Sigma_f \end{pmatrix}.$$

To demonstrate the difficulty in solving the coupled Riccati system in (4.1), we reexpress (A, C) in the form of a  $2(n + k) \times 2(n + k)$  block matrix M as

$$M(t) := \begin{pmatrix} A(t) & 0_{(n+k)\times(n+k)} \\ 0_{(n+k)\times(n+k)} & C(t) \end{pmatrix}.$$

In terms of M, the coupled system in (4.1) can be reformulated into the matrix Riccati differential equation

$$\dot{M} = D_3 + MK_3 + K_2^{\top} M - MN_3 M + p(M), \tag{4.2}$$

where  $p(M) = MK_4MP + PMK_4^{\top}M + P(MK_5 + K_5^{\top}M + MN_4M)P$  and

$$D_3 = \operatorname{diag}(D_1, D_2), \quad K_3 = \operatorname{diag}(K_1, K_1), \quad N_3 = \operatorname{diag}(N_1, 0_{(n+k)\times(n+k)}),$$

 $N_4 = \text{diag}(N_1, \eta N_2), \quad K_5 = \text{diag}(0_{(n+k)\times(n+k)}, \eta K_2),$ 

$$P = \begin{pmatrix} 0_{(n+k)\times(n+k)} & \operatorname{Id}_{(n+k)} \\ \operatorname{Id}_{(n+k)} & 0_{(n+k)\times(n+k)} \end{pmatrix}, \qquad K_4(t) = \begin{pmatrix} 0_{(n+k)\times(n+k)} & 0_{(n+k)\times(n+k)} \\ -N_1(t) & 0_{(n+k)\times(n+k)} \end{pmatrix},$$

with diag(A, B) representing a block diagonal matrix with matrices A and B on its main diagonal. The presence of p(M) makes the matrix Riccati differential equation



(4.2) non-canonical. In fact, (4.2) is documented in Abou-Kandil et al. [1, Chap. 6] and called a *generalised* or *perturbed* matrix Riccati differential equation. As discussed there, only a few algorithms have been developed to solve (4.2) numerically, and no proof of existence of a solution to (4.2) is available in the existing literature. This motivates us to derive a sufficient condition to establish well-posedness of (4.1) (or, equivalently, (4.2)) in the remainder of this section.

### 4.1 A decoupled case

When the return-predicting factors are deterministic, the following result shows that well-posedness of the coupled Riccati system in (4.1) can be easily established by means of a comparison principle.

**Theorem 4.1** If  $\sigma_f = 0_{k \times k}$ , the coupled Riccati system (4.1) becomes decoupled as

$$\begin{cases} \dot{A} = D_1 + AK_1 + K_1A - AN_1A, \\ \dot{C} = D_2 + CK_1 + K_1C + AN_1A - AN_1C - CN_1A. \end{cases}$$

In addition, it admits a unique global solution on [0, T].

**Proof** When  $\sigma_f = 0_{k \times k}$ , both  $K_2$  and  $N_2$  become zero matrices, which makes the last bracket term in (4.1) disappear. This essentially makes the first equation decoupled from the second equation in (4.1), which allows us to solve for A independently. In this case, existence and uniqueness of the first equation in (4.1) can be established globally by invoking the comparison principles for matrix Riccati equations by applying [1, Theorem 4.1.4]. Once the solution A is obtained, we can proceed to solve the second equation for C in (4.1), which is a linear matrix differential equation of C, in terms of A.

#### 4.2 A sufficient condition

For the general case where  $\sigma_f$  is not degenerate, there is no existing theory to ensure existence of a solution to (4.1). We now provide a sufficient condition to establish well-posedness of the coupled system (4.1). First, we introduce some notations for convenience. Denote by  $\mathbb{S}_n$  the space of real symmetric  $n \times n$  matrices and define a partial order  $< (\leq)$  in  $\mathbb{S}_n$  as

$$A_1 < (\leq) A_2 \iff A_2 - A_1 \text{ is positive (semi-)definite.}$$

We begin with the following four lemmas that will be useful later.

**Lemma 4.2** *The instantaneous correlation matrix*  $\rho$  *satisfies* 

$$0_{n \times n} \le \rho \rho^{\top} \le \mathrm{Id}_n$$

where  $Id_n$  denotes the  $n \times n$  identity matrix.



**Proof** For any vector  $v \in \mathbb{R}^n$ , we have  $v^{\top} \rho \rho^{\top} v = ||v\rho||_2^2 \ge 0$ , which implies that  $\rho \rho^{\top} \ge 0_{n \times n}$ . On the other hand, for  $t \ge 0$ , consider the covariance matrix  $\Sigma_t$  of the vector  $(W_t^{P,\top}, W_t^{f,\top})^{\top}$ , i.e.,

$$\Sigma_t = \begin{pmatrix} \mathrm{Id}_n & \rho \\ \rho^\top & \mathrm{Id}_k \end{pmatrix} t.$$

We have  $\Sigma_t \ge 0_{(n+k)\times(n+k)}$  as a covariance matrix is always positive semi-definite. Note that both  $\Sigma_t$  and its second entry  $\mathrm{Id}_k$  in the diagonal are positive definite. Consequently, it follows that the *Schur complement* of the block matrix  $\mathrm{Id}_k$  of the block  $\Sigma_t$  takes the form

$$0_{n \times n} \le \Sigma_t / \mathrm{Id}_k := (\mathrm{Id}_n - \rho \mathrm{Id}_k^{-1} \rho^\top) t = (\mathrm{Id}_n - \rho \rho^\top) t,$$

which completes the proof.

Let  $\sigma_i(A)$  be the *i*th eigenvalue of a real matrix A. The spectral norm of A is the largest singular value of A, i.e.,  $\sigma_{\max}(A) := \sqrt{\max_i \sigma_i(A^\top A)}$ . We also define  $\sigma_{\min}(A) := \sqrt{\min_i \sigma_i(A^\top A)}$ . Obviously, we have  $\sigma_{\max}(A) = \max_i |\sigma_i(A)|$  and  $\sigma_{\min}(A) = \min_i |\sigma_i(A)|$  for any  $A \in \mathbb{S}_n$ . Denote by  $\mathcal{Z} := C([0,T]; \mathbb{R}^{(n+k)\times(n+k)})$  the set of  $\mathbb{R}^{(n+k)\times(n+k)}$ -valued continuous functions with the norm

$$\rho_{\infty}(A) := \max_{0 \le t \le T} \sigma_{\max} \left( A(t) \right) = \max_{0 \le t \le T} \sqrt{\max_{i} \sigma_{i} \left( A^{\top}(t) A(t) \right)} \qquad \text{for } A(\cdot) \in \mathcal{Z}.$$

**Lemma 4.3** For  $A_1, A_2 \in C([0, T]; \mathbb{S}_{n+k})$ , the norm  $\rho_{\infty}$  has the following properties:

- 1)  $\rho_{\infty}(A_1A_2) \le \rho_{\infty}(A_1)\rho_{\infty}(A_2)$ .
- 2)  $\rho_{\infty}(A_1 + A_2) \le \rho_{\infty}(A_1) + \rho_{\infty}(A_2)$ .
- 3)  $\rho_{\infty}(A) \le \kappa \iff -\kappa \operatorname{Id}_{n+k} \le A(t) \le \kappa \operatorname{Id}_{n+k} \text{ for all } t \in [0, T].$

**Proof** The first two properties can be obtained directly from the definition of the norm. Hence it remains to prove the last one. Since A(t) is a real symmetric matrix for any  $t \in [0, T]$ , it can be diagonalised as  $A(t) = \bar{P}(t) \operatorname{diag}(\sigma_i(A(t))) \bar{P}^{\top}(t)$ , where  $\bar{P}(t)$  is a real orthogonal matrix. It is obvious that

$$\rho_{\infty}(A) \le \kappa \iff |\sigma_i(A(t))| \le \kappa \text{ for } t \in [0, T], i = 1, \dots, n + k.$$

For any  $t \in [0, T]$  and i = 1, ..., n + k, we have

$$\left|\sigma_{i}(A(t))\right| \leq \kappa \iff -\kappa \operatorname{Id}_{n+k} = -\kappa \,\bar{P}(t) \,\bar{P}^{\top}(t) \leq A(t) \leq \kappa \,\bar{P}(t) \,\bar{P}^{\top}(t) = \kappa \operatorname{Id}_{n+k}.$$

Combining things completes the proof for property 3).

**Lemma 4.4** For any positive definite (respectively, semi-definite)  $H \in \mathbb{S}_n$ , we have  $\sigma_{\max}(e^{-H}) < 1$  (respectively,  $\leq 1$ ).



**Proof** Since H is symmetric positive definite (respectively, semi-definite), there exists an orthogonal matrix  $\hat{P}$  such that  $H = \hat{P} \operatorname{diag}(H) \hat{P}^{-1}$ , where  $\operatorname{diag}(H)$  is a real diagonal matrix whose main diagonal contains positive (respectively, nonnegative) eigenvalues. Since  $e^{-H} = e^{-\hat{P}\operatorname{diag}(H)\hat{P}^{-1}} = \hat{P}e^{-\operatorname{diag}(H)}\hat{P}^{-1}$ , we have

$$\sigma_i(e^{-H}) = \sigma_i(e^{-\operatorname{diag}(H)}) = e^{-\sigma_i(H)} < 1$$

(respectively,  $\leq 1$ ), which completes the proof.

Consider now the auxiliary system

$$\dot{A}_1 = D_1 - \eta N_4 + A_1 K_1 + K_1 A_1 - A_1 N_1 A_1 \tag{4.3}$$

with terminal condition  $A_1(T) = 0_{(n+k)\times(n+k)}$  and

$$N_4 := \begin{pmatrix} h^2 \sigma_p^\top \rho \rho^\top \sigma_p & 0_{n \times k} \\ 0_{k \times n} & 0_{k \times k} \end{pmatrix}. \tag{4.4}$$

The following result shows that (4.3) admits a unique solution.

**Lemma 4.5** For any T > 0, (4.3) admits a unique solution  $A_1$ .

**Proof** We rewrite  $A_1$  componentwise as

$$A_1(t) := \begin{pmatrix} -A_{1,xx}(t) & A_{1,xf}(t) \\ A_{1,xf}^{\top}(t) & A_{1,ff}(t) \end{pmatrix}$$

and perform a number of direct calculations to show that the components satisfy

$$\begin{cases} \dot{A}_{1,xx} = \frac{1}{h} A_{1,xx} \Lambda^{-1} A_{1,xx} - \eta h^2 \sigma_p^{\top} (\mathrm{Id}_n - \rho \rho^{\top}) \sigma_p, \\ \dot{A}_{1,xf} = -hB + \frac{1}{h} A_{1,xx} \Lambda^{-1} A_{1,xf} + A_{1,xf} \Phi, \\ \dot{A}_{1,ff} = -\frac{1}{h} (A_{1,xf})^{\top} \Lambda^{-1} A_{1,xf} + (A_{1,ff} \Phi + \Phi A_{1,ff}). \end{cases}$$
(4.5)

From (4.5), it suffices to show that the first equation admits a unique solution, as the other equations for  $A_{1,xf}$  and  $A_{1,ff}$  are both linear. By Lemma 4.2, we have  $\eta h^2 \sigma_p^\top (\mathrm{Id}_n - \rho \rho^\top) \sigma_p \geq 0_{n \times n}$ . Since  $\Lambda^{-1}/h(t) \geq 0_{n \times n}$ , it follows readily that  $A_{1,xx}$  admits a unique solution for any T > 0 by the comparison principle; see [1, Theorem 4.1.4]. Consequently, the existence and uniqueness of  $A_{1,xf}$  and  $A_{1,ff}$  are also established. This gives the result.

We now provide a sufficient condition to ensure well-posedness of the system (4.1).

**Theorem 4.6** Given T > 0, suppose there exists an R > 0 satisfying the inequality

$$T\left(\rho_{\infty}(D_2) + \rho_{\infty}(N_1)\left(L^2(R) + 2RL(R)\right)\right) \le R,\tag{4.6}$$



where

$$L(R) := \max \{ \rho_{\infty}(A_1), T(\rho_{\infty}(D_1) + b(R)) \}, \tag{4.7}$$

$$b(R) := \eta (2\rho_{\infty}(K_2)R + \rho_{\infty}(N_2)R^2). \tag{4.8}$$

Then the coupled Riccati system (4.1) admits a unique solution. In addition, the solution satisfies the estimates

$$\rho_{\infty}(A) \le L(R), \qquad \rho_{\infty}(C) \le R.$$

**Remark 4.7** The sufficient condition in (4.6) implies that (4.1) is well posed for small T. Such a feature is not unique to our system, as sufficient conditions for local existence of solutions to other non-canonical, coupled Riccati systems can also be found in Bensoussan et al. [4] and Herdegen et al. [30]. We provide some numerical demonstrations on finding a positive R satisfying the inequality in (4.6).

**Proof** of Theorem 4.6 We first define in the space  $C([0, T]; \mathbb{S}_{n+k})$  a ball with radius R > 0 as

$$\mathbb{B}_R := \{ C \in C([0, T]; \mathbb{S}_{n+k}) : \rho_{\infty}(C) \le R \},$$

and then define a map  $\mathcal T$  on the ball  $\mathbb B_R$  as  $\mathcal T(\bar C):=C$ , where

$$\begin{cases} \dot{A} = D_1 + AK_1 + K_1A - AN_1A + \eta(\bar{C}K_2 + K_2^{\top}\bar{C}^{\top} + \bar{C}N_2\bar{C}), \\ \dot{C} = D_2 + CK_1 + K_1C + AN_1A - AN_1\bar{C} - \bar{C}N_1A \end{cases}$$
(4.9)

with terminal conditions  $A(T) = C(T) = 0_{(n+k)\times(n+k)}$ . We prove that the map  $\mathcal{T}$  admits a fixed point via the Schauder fixed-point theorem in the following two steps.

**Step 1.** We show that  $\mathcal{T}$  maps  $\mathbb{B}_R$  to itself. We first derive an estimate for A in (4.9). For  $\bar{C} \in \mathbb{B}_R$ , applying Lemma 4.3 and invoking the definition of b in (4.8) yields

$$\rho_{\infty} \left( \eta(\bar{C}K_2 + K_2^{\top}\bar{C}^{\top} + \bar{C}N_2\bar{C}) \right) \le b(R), \tag{4.10}$$

which implies  $\eta(\bar{C}K_2 + K_2^{\top}\bar{C} + \bar{C}N_2\bar{C}) \leq b(R)\mathrm{Id}_{n+k}$ . On the other hand, we have

$$\bar{C}K_{2} + K_{2}^{\top}\bar{C} + \bar{C}N_{2}\bar{C}$$

$$= \begin{pmatrix} \bar{C}_{xf}\Sigma_{f}\bar{C}_{xf}^{\top} + h(\bar{C}_{xf}\Sigma_{\rho}^{\top} + \Sigma_{\rho}\bar{C}_{xf}^{\top}), & \bar{C}_{xf}\Sigma_{f}\bar{C}_{ff} + h\Sigma_{\rho}\bar{C}_{ff} \\ \bar{C}_{ff}\Sigma_{f}\bar{C}_{xf}^{\top} + h\bar{C}_{ff}\Sigma_{\rho}^{\top}, & \bar{C}_{ff}\Sigma_{f}\bar{C}_{ff} \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^{\top} & b^{\top} \end{pmatrix} - N_{4} \ge -N_{4}, \qquad (4.11)$$

where  $a := \bar{C}_{xf}\sigma_f + h\sigma_p^\top \rho$ ,  $b := \bar{C}_{ff}\sigma_f$  and  $N_4$  is defined in (4.4). The last inequality results from the fact that

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^\top & b^\top \end{pmatrix} \ge 0_{(n+k)\times(n+k)}.$$



Combining (4.10) and (4.11) yields

$$-\eta N_4 \le \eta (\bar{C} K_2 + K_2^{\top} \bar{C} + \bar{C} N_2 \bar{C}) \le b(R) \mathrm{Id}_{n+k}.$$

According to the comparison principle of matrix Riccati differential equations (see Abou-Kandi et al. [1, Theorem 4.1.4]), we have the estimate

$$A_2(t) \le A(t) \le A_1(t),$$

where  $A_1(t)$  is defined in (4.3) and  $A_2(t)$  is

$$A_2(t) = -\int_t^T e^{-\int_t^s K_1(u)du} (D_1(s) + b(R)\mathrm{Id}_{n+k}) e^{-\int_t^s K_1(u)du} ds.$$

As  $K_1$  is positive semi-definite, applying Lemma 4.4 leads to  $\rho_{\infty}(e^{-K_1(s-t)}) \le 1$  for all  $s \ge t$ . Consequently, it follows that  $\rho_{\infty}(A_2) \le T(\rho_{\infty}(D_1) + b(R))$ . By Lemmas 4.3 and 4.4, we have

$$-T(\rho_{\infty}(D_1) + b(R)) \operatorname{Id}_{n+k} \le A_2(t) \le A(t) \le A_1(t) \le \rho_{\infty}(A_1) \operatorname{Id}_{n+k},$$

which implies that  $\rho_{\infty}(A) \leq L(R)$ , where L(R) is defined in (4.7).

Next, we derive an estimate for C in (4.9). In terms of  $\bar{C}$  and A, C reads

$$C(t) = -\int_{t}^{T} e^{-K_{1}(s-t)} (D_{2}(s) + A(s)N_{1}(s)A(s) - A(s)N_{1}(s)\bar{C}(s)$$
$$-\bar{C}(s)N_{1}(s)A(s))e^{-K_{1}(s-t)}ds,$$

yielding  $\rho_{\infty}(C) \leq T(\rho_{\infty}(D_2) + \rho_{\infty}(N_1)(L^2(R) + 2RL(R)))$ . Therefore, if there exists an R > 0 such that the latter term is  $\leq R$ , we have  $\rho_{\infty}(C) \leq R$ , which verifies that  $\mathcal{T}$  maps the ball  $\mathbb{B}_R$  to itself.

Step 2. It is clear that  $\mathbb{B}_R$  is a closed convex subset and that the mapping  $\mathcal{T}$  is continuous. Hence it remains to show that the image  $\mathcal{T}(\mathbb{B}_R)$  is a relatively compact subset of  $\mathbb{B}_R$ . Consider any sequence  $(\mathcal{T}C_n)_{n\in\mathbb{N}}$  in  $\mathcal{T}(\mathbb{B}_R)$ . Since  $\mathcal{T}$  maps  $\mathbb{B}_R$  to itself, the sequence is uniformly bounded, i.e., we have  $\rho_\infty(\mathcal{T}C_n) \leq R$  for any n. Since each element in the sequence is defined in (4.9), it is obvious that their first derivatives are also uniformly bounded, which implies that the sequence is equicontinuous. By the Arzelà–Ascoli theorem, the sequence  $(\mathcal{T}C_n)_{n\in\mathbb{N}}$  thus has a uniformly convergent subsequence, proving that  $\mathcal{T}(\mathbb{B}_R)$  is a relatively compact subset of  $\mathbb{B}_R$ . Consequently, by the Schauder fixed-point theorem (see Zeidler [41, Theorem 1.C]),  $\mathcal{T}$  admits a fixed point  $C^\infty$ . Accordingly, we can define  $A^\infty$  in terms of  $C^\infty$ .

In addition, the uniqueness of the solution  $(A^{\infty}, C^{\infty})$  follows because the right-hand side of the equations in (4.1) is locally Lipschitz-continuous with respect to t.

A direct consequence of Theorem 4.6 is that  $A_{xx}$  in (4.1) is positive definite.

**Corollary 4.8** *If the condition in Theorem* **4.6** *holds, we have* 

$$A_{rr}(t) > 0_{n \times n}, \qquad t \in [0, T).$$



**Proof** We rewrite the first equation in (3.10) by completing the square with respect to the term  $\sigma_f C_{xf}^\top + h \rho^\top \sigma_p$  and come to the expression

$$\dot{A}_{xx} = \frac{1}{h} A_{xx} \Lambda^{-1} A_{xx} - \eta (\sigma_f C_{xf}^\top + h \rho^\top \sigma_p)^\top (\sigma_f C_{xf}^\top + h \rho^\top \sigma_p)$$
$$- \eta h^2 \sigma_p^\top (\mathrm{Id}_n - \rho \rho^\top) \sigma_p \tag{4.12}$$

with  $A_{xx}(T) = 0_{n \times n}$ . By Lemma 4.2, we know that  $\mathrm{Id}_n - \rho \rho^\top \ge 0_{n \times n}$ . The result follows by applying the comparison principle on (4.12) (see [1, Theorem 4.1.4]).  $\square$ 

## 5 Financial implications

In this section, we first provide some financial implications behind the equilibrium trading strategy  $\tau^*$  in (3.9) in Proposition 3.3, whose uniqueness is a direct consequence of Theorem 4.6 under the sufficient condition in (4.6). We then compare our results with the dynamic mean–variance model without price impact. To explore more financial implications of this equilibrium trading strategy, we also perform an asymptotic expansion on the coupled system in (4.1) around small liquidity costs and derive the corresponding *almost* equilibrium trading strategy.

### 5.1 Equilibrium portfolio and target portfolio

To provide more financial interpretations on the equilibrium trading strategy  $\tau^*$ , we begin with the following corollary which is a direct consequence of Proposition 3.3 and Corollary 4.8.

**Corollary 5.1** When the coupled Riccati system in (4.1) is well posed, the equilibrium trading strategy  $\tau^*$  admits the form

$$\tau^*(t, x, y, f) = M^{\text{rate}}(t) \left( M^{\text{targ}}(t, f) - x \right), \tag{5.1}$$

where the tracking rate matrix is  $M^{\text{rate}}(t) = \Lambda^{-1} A_{xx}(t)/h(t)$  and the target portfolio is

$$M^{\text{targ}}(t, f) = A_{xx}^{-1}(t)A_{xf}(t)f.$$
 (5.2)

The target portfolio is obviously well defined since  $A_{xx}$  in (3.10) is positive definite by Corollary 4.8 when (4.1) is well posed. Financially, the target portfolio plays a critical role to guide the mean–variance investor in rebalancing her portfolio. The presence of price impact forbids the investor from rebalancing her portfolio simultaneously whenever the return-predicting factors change. Instead, she should trade gradually towards the target portfolio. It is observed from  $M^{\text{rate}}$  that a larger price impact coefficient  $\Lambda$  results in a slower trading rate. In addition, the target portfolio  $M^{\text{targ}}$  in (5.2) is directly proportional to the return-predicting factors, with the coefficient matrix  $A_{xx}^{-1}A_{xf}$  capturing the persistence of the return-predicting signals and the price impact.



Next, we show that the target portfolio  $M^{\text{targ}}$  degenerates into the Markowitz portfolio in (5.3) as the time to invest goes to 0. We follow the notation in Gârleanu and Pedersen [20] to denote the *Markowitz portfolio* by

Markowitz 
$$(t, f) := \frac{1}{\eta h(t)} \Sigma_p^{-1} Bf.$$
 (5.3)

**Corollary 5.2** When  $t \to T$ , the target portfolio  $M^{\text{targ}}$  converges to Markowitz, i.e.,

$$\lim_{t \to T} M^{\text{targ}}(t, f_t) = \text{Markowitz}(T, f_t).$$

**Proof** Applying the Taylor expansion to  $A_{xx}(t)$  and  $A_{xf}(t)$  around T, we have

$$A_{xx}(t) = A_{xx}(T) + (t - T)\dot{A}_{xx}(T) + o(t - T) = -\eta h^{2}(T)\Sigma_{p}(t - T) + o(t - T),$$
  

$$A_{xf}(t) = A_{xf}(T) + (t - T)\dot{A}_{xf}(T) + o(t - T) = -h(T)B(t - T) + o(t - T).$$

Combining these and applying L'Hôpital's rule, we achieve the desired results as

$$\lim_{t \to T} M^{\text{targ}}(t, f_t) = \lim_{t \to T} A_{xx}^{-1}(t) A_{xf}(t) f_t = \frac{1}{\eta h(T)} \sum_{p=0}^{\infty} B f_T = \text{Markowitz}(T, f_T).$$

## 5.2 Comparisons with an equilibrium portfolio without price impact

To explore the effect of Kyle's lambda  $\Lambda$  on the target portfolio  $M^{\text{targ}}$ , let us revisit the dynamic mean–variance problem without frictions, i.e.,  $\Lambda = 0_{n \times n}$ . As in Definition 2.1, we first provide a definition of an *admissible feedback frictionless portfolio*.

**Definition 5.3** An admissible feedback frictionless portfolio is a map

$$x_{\rm np}:[0,T]\times\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}^n$$

such that for any initial point  $(t, y, f) \in [0, T] \times \mathbb{R} \times \mathbb{R}^k$ , the SDE

$$dy_t^{x_{\text{np}}} = (ry_t^{x_{\text{np}}} + x_{\text{np}}(t, y_t^{x_{\text{np}}}, f_t)^{\top} Bf_t) dt + x_{\text{np}}(t, y_t^{x_{\text{np}}}, f_t)^{\top} \sigma_p dW_t^p$$

admits a unique strong solution denoted by  $y^{x_{np}}$ . The class of admissible portfolios is denoted by  $\mathcal{A}_{10,T1}^{np}$ , and we use  $x_{np}$  and  $x_{np}(t, y, f)$  interchangeably.

The mean–variance objective functional without frictions is given by

$$J_{\text{np}}(t, y, f; x_{\text{np}}) = \mathbb{E}_{t}^{y, f}[y_{T}^{x_{\text{np}}}] - \frac{\eta}{2} \text{Var}_{t}^{y, f}[y_{T}^{x_{\text{np}}}] = \mathbb{E}_{t}^{y, f}[F(y_{T}^{x_{\text{np}}})] + G(\mathbb{E}_{t}^{y, f}[y_{T}^{x_{\text{np}}}]),$$

where

$$\mathbb{E}_{t}^{y,f}[\cdot] := \mathbb{E}[\cdot|y_{t}^{x_{\text{np}}} = y, f_{t} = f],$$

$$\text{Var}_{t}^{y,f}[\cdot] := \text{Var}[\cdot|y_{t}^{x_{\text{np}}} = y, f_{t} = f]$$



denote the conditional expectation and conditional variance, respectively. Correspondingly, we define an equilibrium frictionless portfolio  $x_{np}^*$  as follows.

**Definition 5.4** An admissible feedback frictionless portfolio  $x_{np}^*$  is called an *equilib-rium frictionless portfolio* if for any  $(t, y, f) \in [0, T] \times \mathbb{R} \times \mathbb{R}^k$ ,

$$\liminf_{\epsilon \to 0+} \frac{J_{\rm np}(t,\,y,\,f;\,x_{\rm np}^*) - J_{\rm np}(t,\,y,\,f;\,x_{\rm np}^\epsilon)}{\epsilon} \ge 0.$$

Here  $x_{np}^{\epsilon}$  is a spike modification of  $x^{np,*} \in \mathcal{A}_{[0,T]}^{np}$  defined as

$$x_{\rm np}^{\epsilon}(s, y, f) = \begin{cases} \bar{x}_{\rm np}(s, y, f), & s \in [t, t + \epsilon), y \in \mathbb{R}, f \in \mathbb{R}^k, \\ x_{\rm np}^*(s, y, f), & s \in [t + \epsilon, T], y \in \mathbb{R}, f \in \mathbb{R}^k, \end{cases}$$

where  $\bar{x}_{np} \in \mathcal{A}_{[0,T]}^{np}$  and  $\epsilon > 0$  are arbitrarily chosen. The corresponding *equilibrium* frictionless value function is defined as  $V_{np}(t, y, f) := J_{np}(t, y, f; x_{np}^*)$ .

Following Basak and Chabakauri [3], we show that an equilibrium frictionless value function  $V_{np}(t, y, f)$  satisfies an extended HJB equation. For  $\Psi_{np}(t, y, f)$  in  $C^{1,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^k)$ , we have

$$\begin{cases} 0 = \max_{x_{np} \in \mathbb{R}^n} (\mathcal{L}^{x_{np}} V_{np} - \eta \mathcal{N}^{x_{np}} \Psi_{np}), \\ 0 = \mathcal{L}^{x_{np}^*} \Psi_{np} \end{cases}$$

with  $V_{\rm np}(T,y,f) = \Psi_{\rm np}(T,y,f) = y$ . Here, the operators  $\mathcal{L}^x$  and  $\mathcal{N}^x$  are defined, for any  $u \in \mathcal{C}^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}^k)$ , by

$$\mathcal{L}^{x}u := \partial_{t}u + (ry + x^{\top}Bf)\partial_{y}u - \partial_{f}u^{\top}\Phi f$$

$$+ \frac{1}{2}x^{\top}\Sigma_{p}x\partial_{yy}u + x^{\top}\Sigma_{\rho}\partial_{yf}u + \frac{1}{2}\operatorname{tr}(\partial_{ffu}\Sigma_{f}),$$

$$\mathcal{N}^{x}u := \frac{1}{2}x^{\top}\Sigma_{p}x(\partial_{y}u)^{2} + x^{\top}\Sigma_{\rho}\partial_{y}u\partial_{f}u + \frac{1}{2}(\partial_{f}u)^{\top}\Sigma_{f}(\partial_{f}u).$$

We now provide the main result of this section in the form of the following proposition. As the proof is analogous to that of Proposition 3.3, we omit it for brevity.

#### Proposition 5.5 *Let*

$$V_{\rm np}(t, y, f) = h(t)y + \hat{V}_{\rm np}(t, f), \qquad \Psi_{\rm np}(t, y, f) = h(t)y + \hat{g}_{\rm np}(t, f),$$

where

$$\hat{V}_{\rm np}(t,f) = \frac{1}{2} f^{\top} A_{ff}^{(0)}(t) f + A_0^{(0)}(t), \quad \hat{g}_{\rm np}(t,f) = \frac{1}{2} f^{\top} C_{ff}^{(0)}(t) f + C_0^{(0)}(t),$$



with the coefficients

$$\begin{split} C_{ff}^{(0)}(t) &= \frac{2}{\eta} \int_{t}^{T} e^{(\Phi + \Sigma_{p}^{-1} \Sigma_{\rho} B)^{\top}(s - t)} B^{\top} \Sigma_{p} B e^{(\Phi + \Sigma_{p}^{-1} \Sigma_{\rho} B)(s - t)} ds, \\ A_{0}^{(0)}(t) &= \frac{1}{2} \int_{t}^{T} \text{tr} \left( A_{ff}^{(0)}(s) \Sigma_{ff} \right) ds, \quad C_{0}^{(0)}(t) = \frac{1}{2} \int_{t}^{T} \text{tr} \left( C_{ff}^{(0)}(s) \Sigma_{ff} \right) ds, \\ \begin{cases} \dot{A}_{ff}^{(0)} &= (C_{ff}^{(0)} \Sigma_{p}^{\top} \Sigma_{p}^{-1} B + B^{\top} \Sigma_{p}^{-1} \Sigma_{\rho} C_{ff}^{(0)}) + (A_{ff}^{(0)} \Phi + \Phi^{\top} A_{ff}^{(0)}) \\ &+ \eta C_{ff}^{(0)} \sigma_{f}^{\top} (\text{Id} - \rho \rho^{\top}) \sigma_{f} C_{ff}^{(0)} - \frac{B^{\top} \Sigma_{p}^{-1} B}{\eta}, \\ A_{ff}^{(0)}(T) &= 0_{n \times n}, \end{split}$$

and

$$x_{\text{np}}^{*}(t, y, f) = \frac{\Sigma_{p}^{-1}}{\eta h(t)} \Big( \Big( B - \eta \Sigma_{\rho} C_{ff}^{(0)}(t) \Big) f \Big)$$

$$= \text{Markowitz}(t, f) - \underbrace{\frac{1}{h(t)} \Sigma_{p}^{-1} \Sigma_{\rho} C_{ff}^{(0)}(t) f}_{\text{hedging}}. \tag{5.4}$$

Then  $x_{np}^*$  in (5.4) is an equilibrium frictionless portfolio and  $V_{np}(t,y,f)$  is the corresponding equilibrium frictionless value function in Definition 5.4.

Hence, in the absence of price impact, the equilibrium frictionless portfolio  $x_{np}^*$  in (5.4) is independent of  $\hat{g}_{np}$ , which is consistent with Basak and Chabakauri [3].

Proposition 5.5 reveals that the equilibrium frictionless portfolio  $x_{np}^*$  in (5.4) is structurally different from the equilibrium portfolio  $x^*$  defined in (2.3) with respect to  $\tau^*$  in Proposition 3.3 in three aspects. First, the equilibrium frictionless portfolio  $x_{np}^*$  is linear in the return-predicting factor, which implies that the path of  $x_{np}^*$  follows the Brownian path of the return-predicting factors. In contrast, the path of the equilibrium portfolio  $x^*$  is smooth. Secondly, when  $\Lambda \neq 0_{n \times n}$ , Proposition 3.3 shows that the equilibrium trading strategy  $\tau^*$  is expressed in terms of (A, C) in (4.1), and that C cannot be solved for independently. This coupling feature escalates the difficulty of our problem significantly and also demonstrates the essential difference when price impact is taken into account in the dynamic portfolio selection problem. Finally, from (5.4), the hedging demand disappears when  $\Sigma_{\rho} = 0_{n \times k}$ , i.e., when the risky asset prices and the return-predicting factors are independent. The investor invests myopically and follows the Markowitz portfolio. On the other hand, the hedging demand is still present when trading with price impact, as  $\tau^*$  in Proposition 3.3 would not degenerate into the Markowitz portfolio when  $\Sigma_{\rho} = 0_{n \times k}$ . This can be explained by the fact that trading with price impact makes the investor far-sighted, instead of myopic, as she anticipates the expected path of the return-predicting factors and rebalances her portfolio smoothly towards the target portfolio to avoid staggering execution costs.

We conclude this section with the following corollary, which shows that the target portfolio in (5.2) can be expressed in terms of the frictionless equilibrium portfolio without price impact in (5.4).



 $M^{\text{targ}}(t, f)$ 

**Corollary 5.6** *The target portfolio in* (5.2) *admits the representation* 

$$= \eta A_{xx}^{-1}(t) \mathbb{E}_{t}^{y,f} \left[ \int_{t}^{T} e^{-\int_{t}^{s} \frac{A_{xx}(u)\Lambda^{-1}}{h(u)} du} \left( h^{2}(s) \Sigma_{p} x_{np}^{*}(s, y_{s}^{x_{np}^{*}}, f_{s}) + \underbrace{\left( h(s) \Sigma_{\rho} C_{ff}^{(0)}(s) - C_{xf}(s) \Sigma_{f} C_{ff}(s) - h(s) \Sigma_{\rho} C_{ff}(s) \right) f_{s}}_{\rho} \right) ds \right].$$

**Proof** This readily follows from applying Itô's lemma to  $e^{\int_t^T \frac{1}{h(s)} A_{xx}(s) \Lambda^{-1} ds} A_{xf}(t) f_t$ , integrating from t to T and taking conditional expectations.

The representation above comprises an adjusted discounted integral of the future equilibrium frictionless portfolio  $x_{\rm np}^*$ , netting the adjustment term due to the mean-field term in the objective functional in (2.7). In addition, the adjustment term in Corollary 5.6 hinges primarily on the stochastic nature of the return-predicting factors and the coefficient functions  $C_{xf}$  and  $C_{ff}$  that account for the persistence of the return-predicting factors as well as the trading friction  $\Lambda$ . Indeed, when  $\Sigma_f = 0_{k \times k}$ , the adjustment term disappears as there would be no hedging demand needed for the deterministic return-predicting factors.

#### 5.3 Small-liquidity-costs asymptotics

In Sect. 4, we have derived a sufficient condition for the well-posedness of the coupled Riccati system in (4.1). Also, in Sect. 5.2, the frictionless value function and trading strategy are derived. In light of Corollary 5.6, we now conduct a small-liquidity-costs perturbation around the frictionless case to shed more light on the relationship between the investor's equilibrium portfolio without price impact and her target portfolio. Following Gârleanu and Pedersen [20], we assume in this subsection that the liquidity costs are proportional to the amount of stock risk for a scalar  $\lambda > 0$ , i.e.,

$$\Lambda = \lambda \Sigma_p. \tag{5.5}$$

In terms of (5.5), we denote by  $(A^{\lambda}, C^{\lambda})$  the solution of the coupled system in (4.1). We first heuristically provide an explicit  $\sqrt{\lambda}$ -order (hereafter, referred to as the *first-order*) solution to the coupled Riccati system as

$$\tilde{A}^{\lambda}(t) = \begin{pmatrix} -\tilde{A}_{xx}(t) & \tilde{A}_{xf}(t) \\ \tilde{A}_{xf}^{\top}(t) & \tilde{A}_{ff}(t) \end{pmatrix} 
:= \begin{pmatrix} 0_{n \times n} & 0_{n \times k} \\ 0_{k \times n} & A_{ff}^{(0)}(t) \end{pmatrix} + \sqrt{\lambda} \begin{pmatrix} -A_{xx}^{(1)}(t) & A_{xf}^{(1)}(t) \\ (A_{xf}^{(1)}(t))^{\top} & 0_{k \times k} \end{pmatrix},$$

$$\tilde{C}^{\lambda}(t) = \begin{pmatrix} -\tilde{C}_{xx}(t) & \tilde{C}_{xf}(t) \\ \tilde{C}_{xf}^{\top}(t) & \tilde{C}_{ff}(t) \end{pmatrix} 
:= \begin{pmatrix} 0_{n \times n} & 0_{n \times k} \\ 0_{k \times n} & C_{ff}^{(0)}(t) \end{pmatrix} + \sqrt{\lambda} \begin{pmatrix} -C_{xx}^{(1)}(t) & C_{xf}^{(1)}(t) \\ (C_{xf}^{(1)}(t))^{\top} & 0_{k \times k} \end{pmatrix},$$
(5.7)



where the zero-order part  $(A_{ff}^{(0)},C_{ff}^{(0)})$  is defined in Proposition 5.5 and the  $\sqrt{\lambda}$ -order part is given by

$$\begin{split} A_{xx}^{(1)}(t) &:= h^{\frac{3}{2}}(t) \eta^{\frac{1}{2}} \Sigma_p, \qquad A_{xf}^{(1)}(t) := \sqrt{h(t)/\eta} \big( B - \eta \Sigma_\rho C_{ff}^{(0)}(t) \big), \\ C_{xx}^{(1)}(t) &:= h^{\frac{3}{2}}(t) \eta^{\frac{1}{2}} \Sigma_p/2, \qquad C_{xf}^{(1)}(t) := \sqrt{h(t)/\eta} \big( 3B - \eta \Sigma_\rho C_{ff}^{(0)}(t) \big)/2. \end{split}$$

Based on the above explicit first-order solution to the coupled Riccati system, we construct an approximate value function as

$$\tilde{V}(t,x,y,f) := h(t)y - \frac{1}{2}x^{\top}\tilde{A}_{xx}(t)x + x^{\top}\tilde{A}_{xf}(t)f + \frac{1}{2}f^{\top}A_{ff}^{(0)}(t)f + A_{0}^{(0)},$$
(5.8)

$$\tilde{\Psi}(t,x,y,f) := h(t)y - \frac{1}{2}x^{\top}\tilde{C}_{xx}(t)x + x^{\top}\tilde{C}_{xf}(t)f + \frac{1}{2}f^{\top}C_{ff}^{(0)}(t)f + C_{0}^{(0)},$$
(5.9)

with a sub-optimal trading strategy

$$\tilde{\tau}(t, x, y, f) := \frac{\sum_{p}^{-1}}{\lambda h(t)} \partial_x \tilde{V}(t, x, y, f)$$

$$= \sqrt{\frac{\eta h(t)}{\lambda}} \left( \underbrace{\frac{\sum_{p}^{-1} (B - \eta \sum_{p} C_{ff}^{(0)}(t)) f}{\eta h(t)}}_{=x_{nn}^*} - x \right). \tag{5.10}$$

Substituting  $\tilde{V}$  and  $\tilde{\Psi}$  in (5.8) and (5.9) into the coupled PDE in (3.1) yields

$$\begin{cases}
\max_{\tau \in \mathbb{R}^n} \mathcal{L}^{\tau} \tilde{V} - \eta \mathcal{N}^x \tilde{\Psi} = \mathcal{O}(\sqrt{\lambda}), \\
\mathcal{L}^{\tilde{\tau}} \tilde{\Psi} = \mathcal{O}(\sqrt{\lambda}).
\end{cases} (5.11)$$

For more discussions and derivations on small-liquidity-costs asymptotics in a general Markovian setting, readers are referred to Moreau et al. [36], Ekren and Muhle-Karbe [18], Muhle-Karbe et al. [38], and the references therein.

**Remark 5.7** One can also derive an alternative sub-optimal trading strategy by imposing a different assumption on  $\Lambda$  from (5.5). Specifically, by adopting the form of  $\Lambda$  as in Moreau et al. [36] and Ekren and Muhle-Karbe [18], i.e.,

$$\Lambda = \lambda \bar{\Lambda}, \qquad \bar{\Lambda} > 0_{n \times n}, \lambda > 0,$$

the corresponding sub-optimal trading strategy  $\tilde{\tau}$  becomes

$$\tilde{\tau}^{\lambda}(t, x, y, f) := \frac{\eta^{\frac{1}{2}} \bar{\Lambda}^{-\frac{1}{2}} (\bar{\Lambda}^{-\frac{1}{2}} \Sigma_{p} \bar{\Lambda}^{-\frac{1}{2}})^{\frac{1}{2}} \bar{\Lambda}^{\frac{1}{2}}}{(\lambda h_{(t)}^{-1})^{\frac{1}{2}}} \left( \underbrace{\frac{\Sigma_{p}^{-1} (B - \eta \Sigma_{\rho} C_{ff}^{(0)}(t)) f}{\eta h(t)}}_{x_{\text{np}}^{*}} - x \right).$$
(5.12)



Comparing (5.12) with (5.10), the coefficient matrix of the vector  $x_{np}^* - x$  in (5.12) is an identity matrix under the assumption in (5.5), implying that the investor trades towards the frictionless portfolio. This assumption also helps us achieve the desired symmetric structure of the coupled Riccati system (A, C) in (4.1) when proving Proposition 5.9 below.

Important implications can be drawn from the sub-optimal strategy  $\tilde{\tau}$  in (5.10) and the first-order solution pair  $(\tilde{V}, \tilde{\Psi})$  in (5.11). First, the feedback nature of  $\tilde{\tau}$  in (5.10) indicates that the investor is trading towards  $x_{np}^*$  for sufficiently small liquidity cost  $\lambda$ . In other words, when  $\lambda$  is small,  $x_{np}^*$  is the target portfolio for the investor in a first-order sense, and  $\sqrt{\eta h(t)/\lambda}$  is the trading rate. The smaller the value of  $\lambda$ , the larger the trading rate becomes as it becomes less costly for the investor to rebalance her portfolio. When  $\lambda = 0$ , the tracking rate becomes infinite, indicating that the investor should immediately rebalance her portfolio to match the target portfolio in the absence of any trading friction. Second, the approximated PDE for V in (5.11) indicates that the investor's expected utility would not deteriorate significantly when the liquidity cost  $\lambda$  is sufficiently small. Third, from the view of the asymptotic analysis, the first-order ( $\sqrt{\lambda}$ -order) terms  $A_{xx}^{(1)}$ ,  $A_{xf}^{(1)}$ ,  $C_{xx}^{(1)}$  and  $C_{xf}^{(1)}$  all admit closedform expressions, which is in a stark contrast to the coupled Riccati system in (4.1). This observation indicates that the coupling feature in (4.1) appears in the *higher*order terms and one must therefore consider to expand into higher orders to derive a similar sufficient condition in Theorem 4.6 by means of an asymptotic analysis.

Compared to the equilibrium value function in (3.7) with the equilibrium trading strategy in (3.9), the first-order approximations in (5.8) and (5.10) are explicit. To demonstrate that the sub-optimal control is an *almost* optimal control in the sense of Moreau et al. [36], we proceed to conduct an error analysis on (5.8) with respect to (3.7) in the sense that

$$V - \tilde{V} = -\frac{1}{2}x^{\top} (A_{xx}^{\lambda} - \tilde{A}_{xx})x + x^{\top} (A_{xf}^{\lambda} - \tilde{A}_{xf})f$$
$$+ \frac{1}{2}f^{\top} (A_{ff}^{\lambda} - A_{ff}^{(0)})f + (A_{0}^{\lambda} - A_{0}^{(0)}).$$

Obviously, it suffices to show that the coefficients defined in (5.6) and (5.7) are indeed the first-order approximation to  $(A^{\lambda}, C^{\lambda})$  as  $\lambda \to 0$ . We conclude this section by showing that when  $\sigma_f = 0_{k \times k}$ , we have  $(A^{\lambda} - \tilde{A}^{\lambda}, C^{\lambda} - \tilde{C}^{\lambda}) = \mathcal{O}(\sqrt{\lambda})$  as  $\lambda \to 0$  for the spectral norm  $\sigma_{\text{max}}$ . The error analysis for the general case  $(\sigma_f \neq 0_{k \times k})$  is more delicate and will be studied in the future. We begin with the following lemma.

**Lemma 5.8** Suppose that  $H_1$  is symmetric positive definite and  $H_2$  is symmetric positive semi-definite. If they commute, i.e.,  $H_1H_2 = H_2H_1$ , then  $H_1H_2$  is symmetric positive semi-definite.

**Proof** Since  $H_1$  commutes with  $H_2$ , we have  $(H_1H_2)^{\top} = H_2^{\top}H_1^{\top} = H_2H_1 = H_1H_2$ , proving that  $H_1H_2$  is symmetric. Moreover, there exists an invertible matrix  $P_1$  such that  $H_1 = P_1^{\top}P_1$ . Now the set of eigenvalues of  $P_1^{\top}P_1H_2$  (or  $H_1H_2$ ) equals the set of



eigenvalues of  $P_1H_2P_1^{\top}$ . Since  $P_1H_2P_1^{\top}$  is positive semi-definite, all its eigenvalues are nonnegative, i.e.,  $H_1H_2$  is symmetric positive semi-definite.

**Proposition 5.9** When  $\sigma_f = 0_{k \times k}$ , we have the estimate, for  $t \in [0, T]$  and as  $\lambda \to 0$ ,

$$\begin{split} \sigma_{\max} \big( A_{\chi\chi}^{\lambda}(t) - \tilde{A}_{\chi\chi}(t) \big) &= \mathcal{O}(\sqrt{\lambda}), \\ \sigma_{\max} \big( C_{\chi\chi}^{\lambda}(t) - \tilde{C}_{\chi\chi}(t) \big) &= \mathcal{O}(\sqrt{\lambda}), \\ \sigma_{\max} \big( C_{\chi\chi}^{\lambda}(t) - \tilde{C}_{\chi\chi}(t) \big) &= \mathcal{O}(\sqrt{\lambda}), \\ \sigma_{\max} \big( A_{ff}^{\lambda} - A_{ff}^{0} \big) &= \mathcal{O}(\sqrt{\lambda}), \end{split} \qquad \sigma_{\max} \big( C_{\chi f}^{\lambda}(t) - \tilde{A}_{\chi f}(t) \big) &= \mathcal{O}(\sqrt{\lambda}), \\ \sigma_{\max} \big( C_{\chi f}^{\lambda}(t) - \tilde{C}_{\chi f}(t) \big) &= \mathcal{O}(\sqrt{\lambda}), \end{split}$$

**Proof** It is enough to show  $\sigma_{\max}(A_{xx}^{\lambda}(t) - \tilde{A}_{xx}(t)) = o(\sqrt{\lambda})$ ; the other estimates follow similarly. Define the remainder  $A_{xx}^{R}(t) := A_{xx}^{\lambda}(t) - \tilde{A}_{xx}(t)$ . Differentiating the remainder  $A_{xx}^{R}$  leads to

$$\begin{split} \dot{A}_{xx}^{R} &= \dot{A}_{xx}^{\lambda} - \sqrt{\lambda} \dot{A}_{xx}^{(1)} = \frac{(\sqrt{\lambda} A_{xx}^{(1)} + A_{xx}^{R}) \sum_{p}^{-1} (\sqrt{\lambda} A_{xx}^{(1)} + A_{xx}^{R})}{h \lambda} - \eta h^{2} \sum_{p} - \sqrt{\lambda} \dot{A}_{xx}^{(1)} \\ &= A_{xx}^{R} \underbrace{\sqrt{\frac{\eta h}{\lambda}}}_{=:a} + \underbrace{\frac{A_{xx}^{\lambda} \sum_{p}^{-1}}{h \lambda}}_{=:b} A_{xx}^{R} - \sqrt{\lambda} \dot{A}_{xx}^{(1)}. \end{split}$$

According to [1, Theorem 1.1.5], its solution can be obtained as

$$A_{xx}^{R}(t) = \Pi_{b}(t, T) A_{xx}^{R}(T) \Pi_{a^{\top}}^{\top}(t, T) + \int_{t}^{T} \Pi_{b_{1}}(t, s) \sqrt{\lambda} \dot{A}_{xx}^{(1)} \Pi_{a^{\top}}^{\top}(t, s) ds$$

$$= -\sqrt{\eta \lambda} \Pi_{b}(t, T) \Sigma_{p} \Pi_{a^{\top}}^{\top}(t, T)$$

$$- \frac{3r\sqrt{\eta \lambda}}{2} \int_{t}^{T} h^{\frac{3}{2}}(s) \Pi_{b}(t, s) \Sigma_{p} \Pi_{a^{\top}}^{\top}(t, s) ds, \qquad (5.13)$$

where  $\Pi(t, s)_H$  is the fundamental matrix of the matrix differential equation

$$\dot{\Pi}_H(t,s) := \frac{\partial}{\partial t} \Pi_H(t,s) = H(t) \Pi_H(t,s), \quad 0 \le t < s \le T,$$
 (5.14)

with  $\Pi_H(s,s) = \mathrm{Id}_n$ . Next, applying [1, Theorem 3.6.1] to (5.14), we have the estimate

$$\sigma_{\max}(\Pi_H(t,s)) \le 1, \qquad 0 \le t < s \le T, \tag{5.15}$$

if H(t) is positive semi-definite for  $t \in [0, T]$ . Since it is easy to see that  $a^{\top}$  is positive semi-definite, it remains to show that b is positive semi-definite on [0, T]. When  $\sigma_f = 0_{k \times k}$ , the assumption in (5.5) readily implies that  $A_{xx}^{\lambda}$  and  $\Sigma_p$  (and also  $\Sigma_p^{-1}$ ) commute. Moreover, we know that  $A_{xx}^{\lambda}$  is positive semi-definite on [0, T]. According to Lemma 5.8, the matrix b is positive semi-definite. Thus following (5.15), we have for the fundamental matrices the estimates

$$\sigma_{\max}(\Pi_b(t,s)) \le 1, \quad \sigma_{\max}(\Pi_{a^{\top}}^{\top}(t,s)) \le 1, \quad 0 \le t < s \le T.$$



Applying the spectral norm  $\sigma_{\text{max}}$  on both sides of (5.13) gives

$$\frac{\sigma_{\max}(A_{xx}^R(t))}{\sqrt{\lambda}} \le \sqrt{\eta} \,\sigma_{\max}(\Sigma_p) \left(1 + \frac{3r}{2} \int_t^T h^{\frac{3}{2}}(s) ds\right)$$

$$= \sqrt{\eta} \,\sigma_{\max}(\Sigma_p) e^{\frac{3r(T-t)}{2}}.$$
(5.16)

It follows from (5.16) that

$$\lim_{\lambda \to 0} \frac{\sigma_{\max}(A_{xx}^R(t))}{\sqrt{\lambda}} < \infty,$$

i.e., 
$$\sigma_{\max}(A_{xx}^R(t)) = \mathcal{O}(\sqrt{\lambda}).$$

**Remark 5.10** Establishing Proposition 5.9 requires the remainders, such as  $A_{xx}^R$ , to be well defined; this holds when the original coupled Riccati system  $(A^{\lambda}, C^{\lambda})$  in (4.1) with respect to  $\lambda$  is well defined. When  $\sigma_f = 0_{k \times k}$ , as in Proposition 5.9, the original coupled system  $(A^{\lambda}, C^{\lambda})$  is essentially decoupled and therefore its global solution uniquely exists. Hence the error analysis can then be conducted in a similar vein as in Muhle-Karbe et al. [38], since the system of matrix Riccati differential equations admits a global solution that exists unconditionally. However, when  $\sigma_f \neq 0_{k \times k}$ , a more delicate analysis (postponed to later work) is needed to ensure that the remainder terms are well defined in our framework.

## 6 Numerical examples

In Theorem 4.6, we have provided a sufficient condition in (4.6) to ensure the well-posedness of the coupled Riccati system (A, C) in (4.1). This section provides a detailed numerical investigation on (4.6) under various sets of model parameters. To facilitate discussion, we first rephrase (4.6) in terms of a *discriminant function* defined as

$$F(R, T, r, B, \Phi, \rho, \Sigma_p, \Sigma_f, \Lambda, \eta)$$

$$:= \frac{T(\rho_{\infty}(D_2) + \rho_{\infty}(N_1)(L^2(R) + 2RL(R)))}{R} - 1.$$
(6.1)

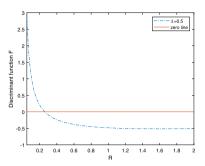
In view of (6.1), establishing (4.6) amounts to finding an R > 0 such that

$$F(R, T, r, B, \Phi, \rho, \Sigma_p, \Sigma_f, \Lambda, \eta) \le 0.$$
(6.2)

We therefore provide several numerical illustrations on F by varying one to two model parameters below.



**Fig. 1** The discriminant function F as a function of R



## 6.1 Single risky asset with single return-predicting factor

In this subsection, we consider a financial market with one risky asset (i.e., index fund) and one return-predicting factor. Unless stated otherwise, the parameters in this subsection are taken from the set

$$\Delta := \{ T = 0.5, r = 0.05, B = 0.3, \Phi = 1, \rho = 0.2,$$

$$\Sigma_{p} = 0.01, \Sigma_{f} = 0.04, \Lambda = 0.5, \eta = 0.5 \}.$$
(6.3)

Figure 1 shows the discriminant function F in (6.1) as a function of R, while the remaining parameters are given in (6.3). From Fig. 1, F crosses the zero line at around  $R \approx 0.30$ . In view of Theorem 4.6, Fig. 1 provides a numerical confirmation that the coupled Riccati system (A, C) in (4.1) under the model parameters in (6.3) is well posed.

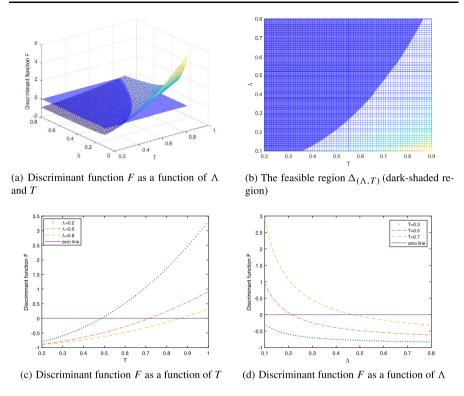
Figure 1 has shown that there exists an R > 0 to satisfy the condition in (6.2) given the model parameters in (6.3). We now provide some visualisations on the sets of feasible regions consisting of two model parameters such that the sufficient condition in (6.2) is satisfied under a given R > 0. Without any loss of generality, we set R = 1 and consider the two feasible regions defined as

$$\begin{split} \Delta_{(\Lambda,T)} &:= \{ (\Lambda,T) \in \mathbb{R}^2_+ : F(R,T,r,B,\Phi,\rho,\Sigma_p,\Sigma_f,\Lambda,\eta) \leq 0, \\ &R = 1; r,B,\Phi,\rho,\Sigma_p,\Sigma_f,\eta \in \Delta \}, \\ \Delta_{(\eta,B)} &:= \{ (\eta,B) \in \mathbb{R}^2_+ : F(R,T,r,B,\Phi,\rho,\Sigma_p,\Sigma_f,\Lambda,\eta) \leq 0, \\ &R = 1; T,r,\Phi,\rho,\Sigma_p,\Sigma_f,\Lambda \in \Delta \}. \end{split}$$

In words, for R=1, the feasible region  $\Delta_{(\Lambda,T)}$  (respectively,  $\Delta_{(\eta,B)}$ ) is the set of  $(\Lambda,T)\in\mathbb{R}^2_+$  (respectively,  $(\eta,B)\in\mathbb{R}^2_+$ ) such that the coupled Riccati system (A,C) in (4.1) under the remaining model parameters in (6.3) is well posed.

Figure 2(a) (respectively, 3(a)) displays a three-dimensional plot of the discriminant function F as a function of  $\Lambda$  and T (respectively,  $\eta$  and B), with the feasible region  $\Delta_{(\Lambda,T)}$  (respectively,  $\Delta_{(\eta,B)}$ ) shown as the dark-shaded region. In addition, the cross-sectional plots are given in Figs. 2(c) and 2(d) (respectively, Figs. 3(c) and 3(d)) to illustrate the behaviour of F with respect to each model parameter.





**Fig. 2** The discriminant function F as a function of  $\Lambda$  and T

We now provide some numerical illustrations on the optimal portfolio  $x^*$ . According to (5.1), the investor trades towards the target portfolio  $M^{\text{targ}}$  in (5.2), and hence it suffices for us to compare the target portfolio  $M^{\text{targ}}$  with the equilibrium frictionless portfolio  $x_{\text{np}}^*$  in (5.4). Since (5.2) and (5.4) are linear in f, we obtain

$$\frac{M^{\text{targ}}(t,f)}{x_{\text{np}}^*(t,y,f)} = \frac{\eta h(t) \Sigma_p}{(B - \eta \Sigma_p C_{ff}^0(t))} \frac{A_{xf}(t)}{A_{xx}(t)}.$$

Figure 4 displays the evolution of the ratio  $M^{\text{targ}}(t,f)/x_{\text{np}}^*(t,y,f)$  and of  $M^{\text{rate}}(t)$ , which is the tracking rate defined in Corollary 5.1, with respect to time. Specifically, in Fig. 4(a),  $M^{\text{targ}}/x_{\text{np}}^*$  is shown to be less than 1 and upward-sloping as time approaches the maturity T, with  $M^{\text{targ}}(T,f_T)/x_{\text{np}}^*(T,y_T^{x_{\text{np}}^*},f_T)=1$ . This observation indicates that the target portfolio is more conservative than the equilibrium frictionless portfolio and the magnitude of that conservatism decreases as the investment horizon shrinks. In other words, the market frictions make the investor trade more prudently in response to the changes of the return-predicting factor. On the other hand, in Fig. 4(b),  $M^{\text{rate}}$  is downward-sloping and converges to 0 as time approaches the maturity T. As the maturity draws nearer, the chance of the investor narrowing the gap between her current portfolio and the target portfolio diminishes, and hence she should slow down the rebalancing speed to avoid staggering trading costs.



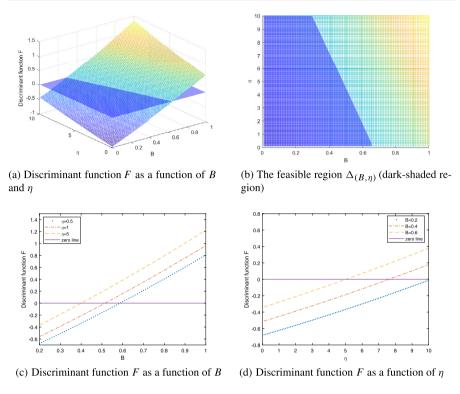


Fig. 3 The discriminant function F as a function of  $\eta$  and B

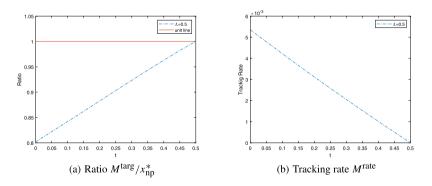


Fig. 4 The ratio (a) and the tracking rate (b) with respect to time

## 6.2 Two assets with idiosyncratic return-predicting signals

In this section, we consider the case of two risky assets each of which only depends on its own level, i.e.,  $\Phi$ , B,  $\Lambda$ ,  $\Sigma_p$  and  $\Sigma_f$  are all diagonal matrices. The parameters



and  $\overline{x}_{nn}^*$ 

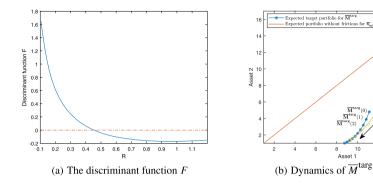


Fig. 5 Two assets with idiosyncratic return-predicting signals

in this example are

$$T = 0.5, \quad r = 0.05, \quad \Phi = \text{diag}(1, 5),$$

$$\Sigma_p = \text{diag}(0.04, 0.04), \quad \Sigma_f = \text{diag}(0.16, 0.16),$$

$$\eta = 0.5, \quad B = \text{diag}(1, 1), \quad \rho = 0_{2 \times 2}, \quad \Lambda = \text{diag}(0.5, 0.5),$$

$$x_0 = (0, 0)^\top, \quad f_0 = (1, 1)^\top. \tag{6.4}$$

Figure 5(a) shows that the discriminant function F crosses the zero line at  $R \approx 0.45$ , indicating that the coupled Riccati system (A,C) in (4.1) under the model parameters in (6.4) is well posed. On the other hand, the study of the price impact in this two-asset example can be done by comparing  $M^{\rm targ}$  and  $x_{\rm np}^*$ . To this end, we define the *expected* target portfolio  $\overline{M}^{\rm targ}$  and the *expected* equilibrium portfolio without price impact  $\overline{x}_{\rm np}^*$  as

$$\overline{M}^{\text{targ}}(t, f) := \mathbb{E}[M^{\text{targ}}(t, f_t) | f_0 = f] = A_{xx}^{-1} A_{xf} e^{-\Phi t} f_0,$$

$$\overline{x}_{\text{np}}^*(t, y, f) := \mathbb{E}[x_{\text{np}}^*(t, y_t^{x_{\text{np}}^*}, f_t) | y_0^{x_{\text{np}}^*} = y, f_0 = f],$$

where  $x_{np}^*$  in (5.4) is seen to be independent of  $y_{np}^{x_{np}^*}$ .

Figure 5(b) displays the allocations in assets 1 and 2 in  $\overline{M}^{\text{targ}}$  and  $\overline{x}_{\text{np}}^*$ , with the arrows indicating the direction of the evolution with respect to time. The straight line denotes a 45-degree line which represents equal allocations in assets 1 and 2. When  $t=0, \overline{x}_{\text{np}}^*$  shows an equal allocation in assets 1 and 2, but more allocations are placed in asset 1 than in asset 2 as time progresses. On the other hand,  $\overline{M}^{\text{targ}}$  follows  $\overline{x}_{\text{np}}^*$  from the left as time progresses, and converges to  $\overline{x}_{\text{np}}^*$  as time approaches the maturity T.

The observation in Fig. 5(b) can be explained by inspecting the elements of the mean-reversion matrix  $\Phi$ , as the other parameters in assets 1 and 2 are identical. Since  $\Phi_2$  is larger than  $\Phi_1$ , the return-predicting signal for asset 2 is less persistent than that for asset 1. Consequently, the investor would invest more in asset 1 over time,



resulting in convex paths with respect to asset 1 for both  $\overline{M}^{\text{targ}}$  and  $\overline{x}_{\text{np}}^*$ . However, trading frictions forbid the investor to react aggressively, and this explains why  $\overline{M}^{\text{targ}}$  is to the left of  $\overline{x}_{\text{np}}^*$  over time. Finally, the close proximity between  $\overline{M}^{\text{targ}}$  and  $\overline{x}_{\text{np}}^*$  in Fig. 4(b) provides a visual confirmation of the small-liquidity-costs asymptotics in Sect. 5.3 in that  $\overline{x}_{\text{np}}^*$  serves as a good approximation of  $\overline{M}^{\text{targ}}$  for sufficiently small liquidity costs.

#### 7 Conclusions

In this paper, we derive an explicit equilibrium strategy of a dynamic mean–variance investor in which asset returns are driven by stochastic factors, which effectively captures return predictability, and where each trade incurs a linear transient price impact. We employ mean-field type control techniques to characterise an equilibrium trading strategy in terms of an extended HJB equation, which can be solved up to a coupled system of matrix Riccati differential equations. After establishing the well-posedness of the coupled Riccati system, we derive a sufficient condition under which a local solution exists uniquely.

Our equilibrium solution admits several important financial implications when trading with price impact. In addition, we conduct an asymptotic analysis by perturbing our matrix Riccati differential equations around small liquidity costs, and show that the equilibrium trading strategy without price impact can serve as a target portfolio for the investor in a first-order sense for sufficiently small liquidity costs.

This paper also initiates many new research directions. First, equilibrium solutions under different asset price dynamics with price impact can also be reformulated as mean-field type control problems and therefore a general, extended dynamic programming principle can be derived accordingly. Second, a precommitment solution of the dynamic mean-variance problem with price impacts can also be characterised via solutions of fully coupled forward-backward PDE equations by means of the Hamilton-Jacobi-Bellman-Fokker-Planck (HJB-FP) approach. Finally, further explorations of the portfolio selection problem with time-inconsistency and price impacts may not conveniently yield any explicit formulations and further numerical analysis in a similar vein as in Ekren and Muhle-Karbe [18] and Van Staden et al. [39] become necessary. We leave these ideas for future research.

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father during the drafting of the present article at their home. Although he just lost his father at the final stage of the review of this work, his father's devotion to him and his family will be long remembered in his heart; and he used this work in memory of his father's brave battle against liver cancer.

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