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A Risk Extended Version of Merton's Optimal Consumption and Portfolio Selection

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Abstract. The objective of this paper is to study the optimal consumption and portfolio choice problem of risk-controlled investors who strive to maximize total expected discounted utility of both consumption and terminal wealth. Risk is measured by the variance of terminal wealth, which introduces a nonlinear function of the expected value into the control problem. The control problem presented is no longer a standard stochastic control problem but rather, a mean field-type control problem. The optimal portfolio and consumption rules are obtained explicitly. Numerical results shed light on the importance of controlling variance risk. The optimal investment policy is nonmyopic, and consumption is not sacrificed.

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Keywords: consumption and portfolio choice • risk management • mean field-type control • fixed point problem • time inconsistency

1. Introduction

Risk is no doubt essential in portfolio choices. The comprehensive review of portfolio choices with various risk measures can be found in Mitra and Ji (2010), Krokhmal et al. (2011), and Kolm et al. (2014). Also, utility theory is the foundation for the theory of choice under uncertainty. Hence, it is natural to combine the expected utility maximization framework with risk measures (utility-risk management framework hereafter) when considering a risk-controlled individual's optimal investment-consumption decision under uncertainty. The research along this line is scarce and growing (e.g., Basak and Shapiro 2001, Pfeiffer 2016, Wong et al. 2017), and we aim to contribute to literature in this direction.

In this study, we combine the utility maximization framework of Merton (1969, 1971) with the variance minimization idea from the Markowitz (1952) mean-variance analysis to study risk management on optimal portfolio and consumption decisions. Our primary interest is to observe how optimal consumption and portfolio rules are altered because of the introduction of the variance risk measurement. We consider an agent who strives to maximize total expected discounted utility of both consumption and terminal wealth while minimizing the variance of terminal wealth. The inclusion of the variance term introduces a nonlinear function of the expected value into the objective of the control problem. The problem is no longer a standard stochastic control problem but rather, a mean field-type control (MFTC hereafter) problem. We use the Hamilton–Jacobi–Bellman and Fokker–Planck equations (HJB-FP hereafter) framework of Bensoussan et al. (2013) to solve the MFTC problem and obtain a solution depending on the initial condition.

Our work makes four significant contributions. First, despite the growing attention to investigating portfolio selections under a utility-risk management framework, most work does not consider portfolio selections and intermediate consumption simultaneously. Our model fills the gap by integrating intermediate consumption, portfolio selections, and utility-risk management in a unified framework.

Second, our study makes a technical contribution to the literature. It turns out that solving a fixed point equation is the key difficulty of studying the MFTC problem by the HJB-FP framework in our study. We are not only able to rigorously prove the uniqueness and existence of the solution to the fixed point equation but also, able to obtain explicit formulas for the optimal consumption and portfolio choices. Furthermore, the fixed point has a crucial economic interpretation, the average terminal wealth. As a by-product, we show that the optimal terminal wealth is deterministic for an individual whose penalty of the variance risk is infinitely large. Moreover, we rigorously prove that this deterministic optimal terminal wealth is less than the expected optimal terminal wealth in the classical Merton's model (i.e., zero penalty of the variance risk), showing that the conservative portfolio choice because of a zero tolerance of variance risk results in lower expected terminal wealth.

Third, we demonstrate the significance of embedding variance risk management criteria on optimal consumption and portfolio selections. Numerical analysis results show that a consumer-investor's investment in risky assets is inversely related to his perspective on the importance of the variance risk as well as the progress of time. More importantly, numerical results demonstrate the increasing-decreasing shape of optimal consumption rate with respect to a consumer-investor's perspective on the importance of the variance risk. We view this nonlinear relation as a significant finding, which reveals that our model not only can allow a consumer-investor to control the variance risk but also, can allow a consumer-investor to increase his consumption rate. This desirable feature is robustness regardless of values of a consumer-investor's risk-aversion coefficient and the market price of risk.

Fourth, our model theoretically derives the expected terminal wealth depending on the different levels of variance risk. It allows a consumer-investor to choose the proper level of variance risk considering his risk aptitude by inputting a target expected terminal wealth. As a result, the extent of risk management for the investor is measurable and observable.

1.1. Literature Review: General Literature

Much of the current research on portfolio theory emanates from the path-breaking mean-variance portfolio model of Markowitz (1952), who refines the economic logic of diversification and offers a practical way to choose an "optimal portfolio" of assets by explicitly recognizing investment risk as measured by variance of return. Since then, there have been a considerable number of studies devoted to the mean-variance framework, including the extension from the single-period setting to the dynamic continuous-time formulation; see, for example, Li and Ng (2000), Zhou and Li (2000), Li et al. (2002), Zhou and Yin (2003), Cesarone et al. (2013), and Qin (2015), among others. In the extension to the multiperiod and continuous-time framework, before the works of Li and Ng (2000) and Zhou and Li (2000) who study the problem by the embedding technique and the stochastic linear-quadratic control framework, respectively, there is no analytical result. Moreover, to tackle the computational efficiency and to accommodate a broader class of risk measures for various considerations, many researchers have applied the concept of mean-variance analysis to the adoption of different risk measures, such as mean-absolute deviation risk measures, value-at-risk (VaR) risk measures, and coherent risk measures; see, for instance, Konno and Yamazaki (1991); Rockafellar and Uryasev (2000, 2002); Ahmadi-Javid (2012); Campbell et al. (2013); He et al. (2015); and Gao et al. (2016, 2017), among others.

In view that most analyses of portfolio, whether they are of Markowitz's mean variance, maximized over one period, Samuelson (1969) formulates and solves a many-period generalization of portfolio selection as lifetime planning of consumption and investment decisions in a discrete time setup using expected utility maximization. Merton (1969, 1971) extends the work of Samuelson (1969) to a continuous-time setting. From that point on, dynamic portfolio optimization through expected utility maximization has been extensively studied; see, for example, Lehoczky et al. (1983), Karatzas et al. (1986, 1987), Cox and Huang (1989), Shreve and Soner (1994), and Brown and Smith (2011), among others.

1.2. Literature Review: Mean Field-Type Control

Because of time inconsistency, two different optimal strategies of MFTC are both studied. Bensoussan et al. (2013) develop a coupled system of HJB-FP to solve the MFTC problem and obtain a time-inconsistent solution (or precommitment solution), which is dependent on the initial condition. An alternative way is to find time-consistent strategies. Björk et al. (2014) study the mean-variance problem within a game framework. Using the method introduced in Björk et al. (2017), they derive time-consistent equilibrium control by solving the extended HJB equation. Laurière and Pironneau (2014) and Pham and Wei (2017) adopt the dynamic programming for mean field-type control and derive a solution, which is independent of the initial condition. The current study uses the HJB-FP framework to obtain the optimal consumption and portfolio rules. The HJB-FP framework of Bensoussan et al. (2013) has been applied to study a number of MFTC problems: for example, Bensoussan et al. (2013, 2019) apply this framework to study the continuous-time Markowitz portfolio with short-selling prohibition and capital investment problem, respectively, in which closed form solutions are obtained.

1.3. Literature Review: Risk Management and Utility Maximization Unified Framework

Our proposed problem follows the recent trend of embedding risk management criteria into the utility maximization framework. In this section, three studies closely related to our work are discussed. Basak and Shapiro (2001) present the first analytical research to embed the concept of risk management into a utility maximizing problem to analyze optimal dynamic portfolio and wealth/consumption policies. They first consider that a risk-managing investor, constrained to maintain the VaR of horizon wealth at a prespecified level for managing market-risk exposure, attempts to maximize the utility of terminal wealth. Extending the economic setting to a standard pure-exchange equilibrium model, the study then examines the problem that a VaR manager, who must comply with a VaR constraint imposed at some horizon (shorter than the agent's lifetime), strives to maximize the intertemporal utility of consumption over the lifetime. The dynamic optimization problems are solved using the martingale representation approach. Compared with the study of Basak and Shapiro (2001), our work integrates intermediate consumption, portfolio selections, and utility-risk management in a unified framework to study the optimal consumption and portfolio choice problem using the MFTC approach.

Wong et al. (2017) study the utility-risk portfolio selection problem by maximizing an investor's utility of terminal wealth with deviation risk. Although the work of Wong et al. (2017) is closely related to our study, they are different in several aspects. First, unlike our work, the work of Wong et al. (2017) does not consider intermediate consumption over the investment horizon in the utility-risk optimization framework. Consequently, optimal portfolio rules are different, and our study is able to additionally provide the insights with respect to the optimal consumption policy, which is not intuitively guessable. Second, Wong et al. (2017) do not perform numerical analyses. We, on the other hand, quantitatively demonstrate the optimal consumption policy, the optimal investment policy, and the wealth process through numerical studies and obtain several important implications/insights. Third, Wong et al. (2017) convert the utility-risk problem into an equivalent nonlinear moment problem, whereas we study the optimal investment-consumption problem using the HJB-FP framework of Bensoussan et al. (2013).

Recently, Pfeiffer (2016) studies a continuous-time Merton's portfolio choice problem with cost functionals involving the probability distribution of the state variable. The problem takes the form of a mean field-type control problem. In this study, three cost functions, which are a cost involving the semideviation, the conditional value-at-risk, and a cost with a penalization term with a target, are considered to allow for a risk-averse consumer's risk management. Pfeiffer (2016) tackles the mean field-type control problem by solving a coupled system of HJB-FP equations numerically with an iterative method. The main idea of the work of Pfeiffer (2016) is close to the primary purpose of our study, and we both approach the problem by studying the coupled system of HJB-FP equations. However, as in the work of Wong et al. (2017), Pfeiffer (2016) does not consider intermediate consumption in the optimization problem. In addition, we not only rigorously prove the existence and uniqueness of the solution by a fixed point argument but also, obtain analytical results. To the best of our knowledge, our study proposes the first analytical optimal investment-consumption policy for an extended Merton's model incorporating the idea of risk management proposed by Markowitz's portfolio selection problem.

1.4. Literature Review: Literature Related to Our Numerical Study

In our numerical example, we investigate the dynamic portfolio behavior of a consumer-investor who has a constant relative risk-aversion (CRRA) utility function on consumption $(x^{1-\gamma})/(1-\gamma)$ and terminal wealth (log utility) with variance control (importance measured by ϵ). Note that the parameter γ in utility functions represents the consumer-investor's absolute risk aversion. Our MFTC model becomes the traditional Merton's portfolio selection problem when ϵ equals zero, which implies that the consumer-investor is not willing to manage the variation of terminal wealth. In the MFTC model, therefore, the consumer-investor chooses a nonmyopic dynamic portfolio regardless of γ by controlling the variance risk of final wealth.

Dai et al. (2021) set up a mean-variance model for log returns, which is different from the standard mean-variance model for terminal wealth. They study the time-consistent portfolio investment in a complete market and an incomplete market. They show that in a complete market, the mean-variance optimization and the CRRA utility are equivalent. Therefore, the optimal investment strategy is to invest a constant fraction of wealth in the risky asset as Merton's classical result ($\epsilon = 0$ in our case). However, in incomplete markets, they point out that the investment is decreasing as time progresses. We obtain the similar observation in our MFTC model with constant investment opportunity set. In general, the risk-aversion coefficient is too difficult to measure in industry practice as well as academic research. Dai et al. (2021) suggest that the risk aversion can be inferred by inputting a target return because they prove that there exists a one-to-one mapping between γ and annual target return in complete market. By analogy, our model demonstrates the expected terminal wealth ρ_ϵ depending on ϵ. Therefore, our MFTC model allows an investor to choose the proper ϵ considering his risk aptitude by inputting a target terminal wealth.

Sotomayor and Cadenillas (2009) study the optimal consumption-investment problem with regime switching and obtain exact solutions for specific hyperbolic absolute risk aversion (HARA) utility functions. They observe a positive effect of consumption as in Merton

(1969). That is, the investor increases his consumption as wealth increases. Interestingly, for all ϵ, the consumption in our model is increasing, whereas the wealth is decreasing for some time t. In addition, Sotomayor and Cadenillas (2009) observe very high consumption wealth ratios (greater than one) for investors, which are also observed in our model.

Liu (2007) introduces the analytical solutions for the dynamic portfolio selection problem in a continuous-time model with CRRA class utility functions in stochastic environments. His model indicates that the optimal terminal wealth for $\gamma = \infty$ becomes constant if the investment opportunity is constant because an infinite risk-averse investor constructs the optimal portfolio using only the riskless asset. Similarly, we investigate the extreme case for $\epsilon = \infty$ in the MFTC model. In this case, the final wealth (ρ_∞) becomes deterministic because of no variation in the optimal final wealth. The consumer-investor steadily reduces the weight of risk asset as the remaining time horizon goes to zero and eventually, allocates all his wealth to the risk-free asset at the terminal time horizon to attain the best deterministic final wealth.

In the remainder, Section 2 presents the model. Section 3 summarizes the theory and methodology of the mean field-type control approach document in Bensoussan et al. (2013, 2021) and lays out sufficient condition of optimality. Section 4 studies the existence and uniqueness solution of Merton's problem with variance control and presents the optimal feedback and the optimal value of our model. Properties and solutions to two extreme cases are also studied. Section 5 reports the results of our numerical analysis, further highlighting the importance and benefits of our model. Section 6 lays out investment-consumption insights that can benefit investors. Section 7 presents some concluding remarks.

2. Merton's Problem with Variance Control

Merton (1969, 1971) extended the Samuelson (1969) optimal investment-consumption model in a discrete-time setup to a continuous-time setting. A consumer-investor must choose his consumption and asset allocation strategy between risky assets (stocks) and a risk-free asset optimally so as to maximize expected utility. Merton used stochastic optimal control methodology to obtain the optimal portfolio strategy.

A potential risk presented to a consumer-investor in Merton's model is the deviation of his terminal wealth from the expected. In view of this, an extension of the classical Merton's problem that incorporates the variance of a consumer-investor's terminal wealth to measure the risk is proposed. The inclusion of the variance term results in a mean field-type control problem that cannot be solved by classical stochastic control methods. In the following, the financial market where a consumer-investor bases his investment-consumption decision is introduced first, followed by a brief review of the classical Merton's problem. This section is concluded by presenting the model of Merton's problem with variance control.

2.1. Financial Market

A financial market consists of one nonrisky asset with a constant interest rate r and n risky assets. Prices of risky assets $Y_i(t)$, $i = 1, 2, \ldots n$, evolve as

$$\begin{aligned} dY_i(t) &= Y_i(t)\left[\alpha_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j\right], \\ Y_i(0) &= Y_i^0, \end{aligned} \tag{1}$$

where $w_j(t)$ are independent standard Wiener processes, constructed on a probability space $(\Omega, \mathcal{A}, P)$ and a filtration $\mathcal{F}^t$, and the coefficients $\alpha_i(t)$, $\sigma_{ij}(t)$ are deterministic functions.

The volatility matrix, $\boldsymbol{\sigma}(t) = \{\boldsymbol{\sigma}_{ij}(t)\}_{n\times n}$, is invertible. We assume that $\boldsymbol{\alpha}(t) := (\boldsymbol{\alpha}_1(t), \alpha_2(t), \cdots, \boldsymbol{\alpha}_n(t))^T$, $\boldsymbol{\sigma}(t)$, and $\boldsymbol{\sigma}^{-1}(t)$ are bounded. The Sharpe ratio is given by

$$\boldsymbol{\theta}(t) = \boldsymbol{\sigma}^{-1}(t)(\boldsymbol{\alpha}(t) - r\mathbb{1}),$$

where $\mathbb{1} := (1, 1, \ldots)^T$ denotes a vector of R^n. Define the process $Z(t)$ by

$$\begin{aligned} dZ(t) &= -Z(t)\boldsymbol{\theta}(t).d\boldsymbol{w}(t), \\ Z(0) &= 1, \end{aligned} \tag{2}$$

which is called a martingale measure (market indicator). In addition, the process,

$$Z(t)Y_i(t)e^{-rt},$$

is a $(P, \mathcal{F}^t)$ martingale.

2.2. Classical Merton's Problem

Consider a consumer-investor whose unique source of income comes from his portfolio investment on the market. The wealth at time s is

$$X(s) = \pi_0(s)e^{r(s-t)} + \sum_{i=1}^{n} \pi_i(s)Y_i(s), s > t, X(t) = x, \tag{3}$$

where $\pi_0(s)$ and $\pi_i(s)$ are the amount of cash and the number of shares invested in the risky asset i, respectively. The portfolio is self-financed, and the dynamics of controlled wealth process are given by

$$\begin{aligned} dX(s) &= r\pi_0(s)e^{r(s-t)}ds + \sum_{i=1}^{n} \pi_i(s)dY_i(s) - C(s)ds, \\ & s > t, X(t) = x, \end{aligned} \tag{4}$$

where $C(s)$ represents the consumption rate and $\pi_0(s)$ and $\boldsymbol{\pi}(s) := (\pi_1(s),\pi_2(s), \cdots ,\pi_n(s))^T$ are control variables.

Using (1) and introducing

$$\varpi_i(s) = \frac{\pi_i(s)Y_i(s)}{X(s)}, i = 1, 2, \cdots, n \tag{5}$$

(i.e., $\varpi_i(s)$ denotes the proportion of wealth invested in the risky asset i), it follows after rearrangements that

$$dX(s) = rX(s)ds + X(s)\boldsymbol{\sigma}^*(s)\boldsymbol{\varpi}(s).(\boldsymbol{\theta}(s)ds + d\boldsymbol{w}(s)) - C(s)ds, s > t, X(t) = x. \tag{6}$$

Then, $X(s)$ is the state of a dynamic system, with controls $\boldsymbol{\varpi}(.) := (\varpi_1(.),\varpi_2(.), \cdots ,\varpi_n(.))^T$ and $C(.)$.

The consumer-investor considers the intertemporal portfolio choice over a finite horizon T, where consumption and wealth allocation between risky assets and a risk-free asset must be made. The investment-consumption performance is measured by utility functions $U_1(c)$ for consumption and $U_2(x)$ for final wealth defined by

$$J(\boldsymbol{\varpi}(.), C(.)) = E\int_0^T U_1(C(s))e^{-rs}ds + EU_2(X(T))e^{-rT}, \tag{7}$$

with

$$dX(s) = rX(s)ds + X(s)\boldsymbol{\sigma}^*(s)\boldsymbol{\varpi}(s).(\boldsymbol{\theta}(s)ds + d\boldsymbol{w}(s)) - C(s)ds, \quad X(0) = x_0. \tag{8}$$

The consumer-investor's dynamic portfolio optimization problem is to maximize his expected utility over a finite horizon T through his choice of consumption and portfolio investments: that is,

$$\Phi(x_0, 0) = \sup_{\boldsymbol{\varpi}(.), C(.)} J(\boldsymbol{\varpi}(.), C(.)). \tag{9}$$

The optimization problem of (9) can be solved applying dynamic programming, and the value function, $\Phi(x_0, 0)$, is the solution of the Bellman equation.

2.3. Model of Merton's Problem with Variance Control

One potential problem associated with the classical Merton's model (cf. (7) and (9)) is that a consumer-investor's terminal wealth may vary significantly. To control the variation risk, the penalty term, the variance of the terminal wealth, is added to the classical Merton's performance function (7). The consumer-investor's performance function becomes

$$J_\epsilon(\boldsymbol{\varpi}(.), C(.)) = E\int_0^T U_1(C(s))e^{-rs}ds + EU_2(X(T))e^{-rT} - \epsilon e^{-rT}\operatorname{var}(X(T)), \tag{10}$$

subject to (8). In (10), $\epsilon \in [0,\infty)$ is a coefficient that weights the importance of variance.

The dynamic optimization problem is to maximize $J_\epsilon(\boldsymbol{\varpi}(.), C(.))$: that is,

$$u(x_0, 0) = \sup_{\boldsymbol{\varpi}(.), C(.)} J_\epsilon(\boldsymbol{\varpi}(.), C(.)). \tag{11}$$

The consumer-investor's optimization problem now deals not only with maximizing expected utility over a finite horizon T but also, with minimizing the variance of the terminal wealth. Because of the presence of the variance term in (10), standard stochastic control cannot be applied to solve the optimization problem. The mean field-type control theory is the right tool to study such a control problem.

Remark 1. When $\epsilon = 0$, (10) reduces to the classical Merton's problem (cf. (7) and (9)).

Utility functions in (10) satisfy the following assumption.

Assumption 1. *The utility function $U_1(C), U_2(x) : R^+ \to R^+$ is concave and twice differentiable in the interior, $U_i'(0) = +\infty, U_i'(+\infty) = 0, i = 1, 2.$*

3. General Mean Field-Type Control

As stated in Section 2.3, the proposed Merton's problem with variance control can be solved applying the mean field-type control theory. In this section, the mean field-type control problem is briefly presented, followed by the sufficient conditions of optimality. Details can be found in Bensoussan et al. (2013, 2021).

3.1. The Mean Field-Type Control Problem

Let $(\Omega, \mathcal{A}, P)$ be a probability space and a filtration $\mathcal{F}^t$ generated by an n-dimensional standard Wiener process $w(t)$. Consider a diffusion process in R^n given by

$$\begin{aligned} d\boldsymbol{x} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}, s))ds + \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{v}(\boldsymbol{x}, s))dw, \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0, \end{aligned} \tag{12}$$

where $\boldsymbol{x}_0 \in R^n$ represents the initial state of the system, $\boldsymbol{v}(x, s) \in R^m$ is the control obtained by feedback, and $\boldsymbol{\sigma}(x)$ is an $n \times n$ matrix that is invertible.

Define payoff to be maximized as

$$J(\boldsymbol{v}(.)) = \int_0^T e^{-rs}Ef(\boldsymbol{x}(s), \boldsymbol{v}(s))ds + e^{-rT}Eh(\boldsymbol{x}(T)) + e^{-rT}F(E\boldsymbol{x}(T)), \tag{13}$$

where $\boldsymbol{v}(s) = \boldsymbol{v}(x(s), s)$ and $\boldsymbol{x}(s)$ is the solution of (12) after inserting the feedback. In (13), $F(.)$ is a nonlinear function of the expected value of $\boldsymbol{x}(T)$. Because of this term, this is not a standard control problem, but a mean field-type control problem.

Next, transform the stochastic problem (13) into a deterministic control problem for a partial differential equation (PDE) by introducing the Fokker–Planck equation:

$$\frac{\partial m_{v(.)}}{\partial s} - \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(\boldsymbol{x}, v(s))m_{v(.)}) + \operatorname{div}(g(\boldsymbol{x}, v(s))m_{v(.)}) = 0,$$
$$m_{v(.)}(\boldsymbol{x}, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{14}$$

and the solution is denoted by $m_{v(.)}(\boldsymbol{x}, s)$, which is the probability distribution of $\boldsymbol{x}_{v(.)}(s)$. Then, the payoff function (13) can be rewritten as

$$J(v(.)) = \int_0^T e^{-rs} \int_{R^n} f(\boldsymbol{x}, v(s)) m_{v(.)}(\boldsymbol{x}, s) dx ds + e^{-rT} \int_{R^n} h(\boldsymbol{x}) m_{v(.)}(\boldsymbol{x}, T) dx + e^{-rT} F\left(\int_{R^n} \boldsymbol{x} m_{v(.)}(\boldsymbol{x}, T) dx\right) \int_{R^n} m_{v(.)}(\boldsymbol{x}, T) dx. \tag{15}$$

Remark 2. In order to use standard variations of the control, the term $\int_{R^n} m_{v(.)}(x, T)dx$ equal to one is inserted in (15). So, the functional (15) coincides with (13). The problem described by (14) and (15) is now considered with $m_{v(.)} \in L^2(R^n) \cap L^1(R^n)$ and not in the space of probability densities. A linear variation will not respect the normalization because the space of probability densities is not a vector space.

3.2. Sufficient Conditions of Optimality

Following Bensoussan et al. (2021), the sufficient conditions of optimality are briefly presented. First, introduce the Lagrangian function

$$L(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}, v) = f(\boldsymbol{x}, v) + \boldsymbol{q} \cdot \boldsymbol{g}(\boldsymbol{x}, v) + \operatorname{tr}\,(\boldsymbol{a}(\boldsymbol{x}, v)\boldsymbol{M}), \tag{16}$$

where $\boldsymbol{a}(\boldsymbol{x}, v) = \frac{1}{2}\boldsymbol{\sigma}(\boldsymbol{x}, v)\boldsymbol{\sigma}^*(\boldsymbol{x}, v)$, $\boldsymbol{q} \in R^n$, $\boldsymbol{M} \in \mathcal{L}(R^n; R^n)$, and the Hamiltonian function

$$H(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}) = \sup_v L(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}, v). \tag{17}$$

Let $\hat{v}(x, q, M)$ denote a measurable function, which attains the maximum in v in the Lagrangian, and write

$$H(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}) = L(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}, \hat{v}(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M})), \tag{18}$$
$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}) = \boldsymbol{g}(\boldsymbol{x}, \hat{v}(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M})), \tag{19}$$
$$\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M}) = \boldsymbol{a}(\boldsymbol{x}, \hat{v}(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{M})). \tag{20}$$

Next, look for two functions $u(x, t) \in R$, $\Psi(x, t; T) \in R^n$ solutions of the coupled system of PDEs:

$$-\frac{\partial u}{\partial t} + ru = H(\boldsymbol{x}, \boldsymbol{D}u, \boldsymbol{D}^2\boldsymbol{u}),$$
$$u(\boldsymbol{x}, T) = h(\boldsymbol{x}) + \boldsymbol{x}.\boldsymbol{DF}(\boldsymbol{\rho}) + F(\boldsymbol{\rho}), \tag{21}$$

$$-\frac{\partial \boldsymbol{\Psi}}{\partial t} = \operatorname{tr}\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{D}u, \boldsymbol{D}^2u)\boldsymbol{D}^2\boldsymbol{\Psi} + \boldsymbol{D}\boldsymbol{\Psi}.\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{D}u, \boldsymbol{D}^2u),$$
$$\boldsymbol{\Psi}(\boldsymbol{x}, T; T) = \boldsymbol{x}, \tag{22}$$

where

$$\boldsymbol{\Psi}(\boldsymbol{x}, t; T) = E_{x,t}[\hat{\boldsymbol{x}}(T)] \tag{23}$$

and

$$\boldsymbol{\rho} = \boldsymbol{\Psi}(\boldsymbol{x}_0, 0; T) \tag{24}$$

is the expected value of the optimal final state.

Solving this system, one obtains the optimal feedback

$$\hat{v}(\boldsymbol{x}, t) = \hat{v}(\boldsymbol{x}, \boldsymbol{D}u, \boldsymbol{D}^2\boldsymbol{u}) \tag{25}$$

and the optimal value

$$J(\hat{v}(.)) = u(\boldsymbol{x}_0, 0) - e^{-rT}\boldsymbol{\rho}.\boldsymbol{DF}(\boldsymbol{\rho}). \tag{26}$$

Note that $u(\boldsymbol{x}_0, 0)$ is not the optimal value.

4. The Existence and Uniqueness Solutions for Merton's Problem with Variance Control

In this section, solutions for Merton's problem with variance control are obtained. The correspondence of notation is stated first in order to apply the general theory presented in Section 3:

$$v = \begin{pmatrix} \boldsymbol{\varpi} \\ C \end{pmatrix},$$
$$f(x, v) = U_1(C), g(x, v) = rx + \boldsymbol{x\varpi}^*\sigma\theta - C,$$
$$h(x) = U_2(x) - \epsilon x^2,$$
$$a(x, v) = \frac{1}{2}x^2\boldsymbol{\varpi}^*\boldsymbol{\sigma}^*\boldsymbol{\sigma\varpi},$$
$$F(x) = \epsilon x^2,$$
$$\beta_i = (U_i')^{-1},$$
$$L(x, q, M, v) = U_1(C) + q(r\boldsymbol{x} + \boldsymbol{x\varpi}^*\boldsymbol{\sigma\theta} - C) + tr\left(\frac{1}{2}x^2\boldsymbol{\varpi}^*\boldsymbol{\sigma}^*\boldsymbol{\sigma\varpi}M\right),$$
$$U_1'(\hat{C}) = q, \hat{C} = \beta_1(q),$$
$$\hat{\boldsymbol{\varpi}} = -\frac{q}{Mx}(\boldsymbol{\sigma}^*)^{-1}\boldsymbol{\theta}, M < 0,$$
$$H(x, q, M) = U_1(\beta_1(q)) - q\beta_1(q) + qrx - \frac{1}{2}\frac{q^2\,|\,\boldsymbol{\theta}|^2}{M},$$
$$g(x, \hat{v}) = rx - \frac{q\,|\,\boldsymbol{\theta}|^2}{M} - \beta_1(q) = G(x, q, M),$$
$$a(x, \hat{v}) = \frac{1}{2}\frac{q^2\,|\,\boldsymbol{\theta}|^2}{M^2} = P(x, q, M). \tag{27}$$

In order for the Lagrangian, $L(x, q, M, v)$, to admit a maximum, we need to assume that $M < 0$.

Using these notations, from Section 3.2, the system of coupled PDEs for the solutions of Merton's problem with variance control is

$$-\frac{\partial u}{\partial t} + ru = U_1\left(\beta_1\left(\frac{\partial u}{\partial x}\right)\right) - \frac{\partial u}{\partial x}\beta_1\left(\frac{\partial u}{\partial x}\right) + rx\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\left(\frac{\partial u}{\partial x}\right)^2 |\boldsymbol{\theta}|^2}{\frac{\partial^2 u}{\partial x^2}},$$
$$u(x, T) = U_2(x) - \epsilon x^2 + \epsilon\rho_\epsilon^2 + 2\epsilon x\rho_\epsilon, \tag{28}$$

$$-\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial x}\left(rx - \frac{|\boldsymbol{\theta}|^2 \frac{\partial u}{\partial x}}{\frac{\partial^2 u}{\partial x^2}} - \beta_1\left(\frac{\partial u}{\partial x}\right)\right) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}\frac{|\boldsymbol{\theta}|^2\left(\frac{\partial u}{\partial x}\right)^2}{\left(\frac{\partial^2 u}{\partial x^2}\right)^2},$$

$$\Psi(x, T; T) = x, \tag{29}$$

and

$$\rho_\epsilon = \Psi(x_0, 0; T). \tag{30}$$

In (28)–(30), we have written ρ_ϵ instead of ρ to emphasize the dependence in ϵ. Then, the optimal feedback is given by

$$\hat{C}_\epsilon(x,t) = \beta_1\left(\frac{\partial u}{\partial x}\right),$$

$$\hat{\boldsymbol{\varpi}}_\epsilon(x,t) = -\frac{\frac{\partial u}{\partial x}}{x\frac{\partial^2 u}{\partial x^2}}(\boldsymbol{\sigma}^*)^{-1}\boldsymbol{\theta}, \tag{31}$$

and the optimal value is

$$J_\epsilon(\hat{C}_\epsilon, \hat{\boldsymbol{\varpi}}_\epsilon) = u(x_0, 0) - 2\epsilon\rho_\epsilon^2 e^{-rT}. \tag{32}$$

From (28)–(30), ρ_ϵ is the solution of a fixed point problem. As shown in Online Appendix EC.1, by introducing $\lambda(x,t) = \frac{\partial u}{\partial x}(x,t)$ and with some transformation, we can reduce the study to a linear PDE and obtain ρ_ϵ as a solution of a fixed point equation given here:

$$E\beta_{2\epsilon}(\lambda_\epsilon(\rho_\epsilon)\xi_0(T) - 2\epsilon\rho_\epsilon) = \rho_\epsilon, \tag{33}$$

where $\beta_{2\epsilon}(\mu)$ is the solution of $U_2'(x) - 2\epsilon x = \mu$, $\beta_{20}(\mu) = \beta_2(\mu)$ and μ must be greater than zero. The number ρ_ϵ has a crucial economic interpretation. It represents the average wealth at the horizon T. Namely, it is the target to which the final wealth must be close to. Solving this fixed point equation becomes the key difficulty of the mean field-type control problem.

Assumption 2.

$$xU_2'(x) \text{ is concave}, \tag{34}$$

$$xU_2'(x) \le \gamma x + 1 - \gamma,\ 0 \le \gamma \le 1, \tag{35}$$

$$\gamma < \frac{1}{2}\bar{\lambda}, \text{ with } \bar{\lambda} \text{ the solution of}$$

$$E\int_0^T \xi_0(s)\beta_1(\bar{\lambda}\xi_0(s))e^{-rs}ds = x_0, \tag{36}$$

$$U_2'''(x) > 0. \tag{37}$$

Theorem 1. *Define $\mathbb{T}_\epsilon(\rho) = E\beta_{2\epsilon}(\lambda_\epsilon(\rho)\xi_0(T) - 2\epsilon\rho)$. Under Assumption 2, there exists a unique $\rho_\varepsilon \in \mathbb{R}^+$ such that $(\rho_\varepsilon) = \rho_\varepsilon$. Moreover, $\rho_\varepsilon \in [0, \bar{\rho}]$, where $\bar{\rho} = \frac{8\epsilon(1-\gamma)+\gamma^2+\bar{\lambda}^2\exp|\boldsymbol{\theta}|^2T-(\bar{\lambda}-\gamma)^2}{4\epsilon(\bar{\lambda}-2\gamma)}$.*

The proof is given in Online Appendix EC.2.

By Theorem 1, the existence and uniqueness of solutions for Merton's problem with variance control can be stated explicitly in the following theorem. For notational convenience, we omit ρ_ϵ in the subscript.

Theorem 2. *The optimal consumption, investment, and wealth are, respectively,*

$$\hat{C}_\epsilon(s) = \beta_1(\lambda_\epsilon\xi_0(s)), \tag{38}$$

$$\hat{\boldsymbol{\varpi}}_\epsilon(s) = -\frac{\lambda_\epsilon\xi_0(s)\frac{\partial G_\epsilon}{\partial \lambda}(\lambda_\epsilon\xi_0(s), s)}{G_\epsilon(\lambda_\epsilon\xi_0(s), s)}(\boldsymbol{\sigma}^*)^{-1}\boldsymbol{\theta}, \tag{39}$$

and

$$\hat{X}_\epsilon(s) = G_\epsilon(\lambda_\epsilon\xi_0(s), s).$$

The optimal value is

$$\begin{aligned} J_\epsilon(\hat{C}_\epsilon, \hat{\boldsymbol{\varpi}}_\epsilon) = {} & E\int_0^T U_1(\beta_1(\lambda_\epsilon\xi_0(s)))e^{-rs}\,ds \\ & + E[U_2(\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon)) \\ & - \epsilon\beta_{2\epsilon}^2(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon) - \epsilon\rho_\epsilon^2 \\ & + 2\epsilon\rho_\epsilon\beta_{2\epsilon}(\lambda_\epsilon\xi_0(T) - 2\epsilon\rho_\epsilon)]e^{-rT}, \end{aligned} \tag{40}$$

where ρ_ϵ represents the optimal expected wealth at time T and $\lambda_\epsilon = \lambda_\epsilon(x_0, 0)(\lambda_\epsilon > \bar{\lambda})$.

See Online Appendix EC.3 for the proof.

4.1. Extreme Cases

Because the proposed model of Merton's problem with variance control aims at managing the variance of a consumer-investor's wealth at the end of investment horizon T, a natural question to ask is how the expected optimal final wealth at the end of investment horizon is affected by a consumer-investor's perspective on the importance of variance risk measured by $\epsilon \in [0, \infty)$. The following subsections explore two extreme cases: that is, the cases when a consumer-investor displays no concern (i.e., $\epsilon = 0$) and extreme concern (i.e., $\epsilon = \infty$) of the variance risk. Finally, a proven relationship between these two expected optimal final wealth values is presented.

4.1.1. $\epsilon = 0$.

Proposition 1. *In this case, the problem reduces to the classical Merton's problem, and*

$$\rho_0 = E\beta_2(\lambda_0\xi_0(T)). \tag{41}$$

See Online Appendix EC.4 for the proof.

Equation (41) gives an explicit formula for the optimal expected final wealth.

4.1.2. $\epsilon \to +\infty$. Proposition 2. In this case, the enormous penalty imposed on the variance risk leads to a deterministic optimal wealth, and

$$\rho_\infty = \beta_2(\lambda_\infty), \tag{42}$$

with $\lambda_\infty = \lambda(\rho_\infty)$ *solution of the equation*

$$\beta_2(\lambda_\infty)e^{-rT} + E\int_0^T \xi_0(s)\beta_1(\lambda_\infty \xi_0(s))e^{-rs}\,ds = x_0. \tag{43}$$

See Online Appendix EC.5 for the proof.

Equation (42) is the best deterministic final wealth guaranteed when $\epsilon \to +\infty$. It is no surprise because it costs too much for a consumer-investor to pay the penalty arising from the variance risk, leading to the case of zero variance.

4.1.3. Comparison Between ρ_0 ($\epsilon = 0$) and ρ_∞ ($\epsilon \to +\infty$). From Section 4.1.2, the expected terminal wealth ρ_∞ turns out to be deterministic: that is, no variation in the optimal final wealth. It is thus natural to study the relation between ρ_0 when no control is made in the variance of the final wealth and ρ_∞ with zero variance.

Proposition 3. *Assume that* $\beta_1, \beta_2 > 0$ *decrease on* $(0, \infty)$ *and also, that* β_2 *is strictly convex. Then,* $\rho_0 > \rho_\infty$.

See Online Appendix EC.6 for the proof.

Proposition 3 is intuitively explained because ρ_0 is the expected wealth at T without considering the variance risk of the final wealth and ρ_∞ is the best deterministic wealth at T. To be of no surprise at the final wealth (that is, no deviation from the expected), a consumer-investor allocates all his wealth to risk-free assets. Consequently, ρ_∞ is smaller than ρ_0 as suggested by Proposition 3 because risk-free assets yield less return than risky assets.

In addition, the proof of the relationship between ρ_ϵ and ρ_0 for small ϵ and $\gamma_1 = 1$ is given in Online Appendix EC.9.

5. Numerical Analysis

A primary contribution of this research is to solve for optimal consumption-investment policies of Merton's problem with variance control. In this section, numerical studies are performed to illustrate the quantitative results. In the study, only one risky stock is considered, and utility functions take the following CRRA form, which satisfies Assumptions 1 and 2:

$$U_1(x) = \begin{cases} \dfrac{x^{1-\gamma_1}}{1-\gamma_1}, & 0 < \gamma_1 < 1, \\ \ln x, & \gamma_1 = 1, \end{cases} \tag{44}$$

$$U_2(x) = \ln x. \tag{45}$$

The base parameter values used for the numerical studies are

$$x_0 = 1;\ T = 1;\ r = 0.04;\ \sigma = 0.2.$$

Remark 3. The optimal controls for the CRRA utility function with variance control problem are given by

$$\hat{C}_\epsilon(s) = (\lambda_\epsilon \xi_0(s))^{-\frac{1}{\gamma_1}}, \tag{46}$$

$$\hat{\varpi}_\epsilon(s) = -\frac{\lambda_\epsilon \xi_0(s)\dfrac{\partial G_\epsilon}{\partial \lambda}(\lambda_\epsilon \xi_0(s), s)}{G_\epsilon(\lambda_\epsilon \xi_0(s), s)}(\sigma^*)^{-1}\theta, \tag{47}$$

and

$$\hat{X}_\epsilon(s) = G_\epsilon(\lambda_\epsilon \xi_0(s), s).$$

Remark 4. When $\gamma_1 = 1$ and $\epsilon = 0$, the optimal controls are given by

$$\begin{aligned} \hat{C}_0(t) &= \beta_1(\lambda_0 \xi_0(t)) = \frac{r}{1 + (r-1)e^{-r(T-t)}}\hat{X}_0(t), \\ \hat{\varpi}_0(t) &= \frac{\theta}{\sigma}, \end{aligned} \tag{48}$$

which recover the results obtained by Merton (1969).

See Online Appendix EC.7 for the proof.

Remark 5. When $\gamma_1 = 1$ and $\epsilon \to +\infty$, the optimal controls are given by

$$\hat{C}_0(t) = \hat{C}_\infty(t), \tag{49}$$

$$\begin{aligned} \hat{\varpi}_\infty(t) &= \frac{\frac{1}{r\lambda_\infty \xi_0(t)}(1 - e^{-r(T-t)})\frac{\theta}{\sigma}}{\rho_\infty e^{-r(T-t)} + \frac{1}{r\lambda_\infty \xi_0(t)}(1 - e^{-r(T-t)})} \\ &= \frac{\theta}{\sigma} - \left(\frac{\rho_\infty e^{-r(T-t)}}{\hat{X}_\infty(t)}\right)\left(\frac{\theta}{\sigma}\right). \end{aligned} \tag{50}$$

Unlike Merton's classical result, the investment policy, $\hat{\varpi}_\infty(t)$, depends on both the current wealth and the terminal wealth and decreases as time progresses.

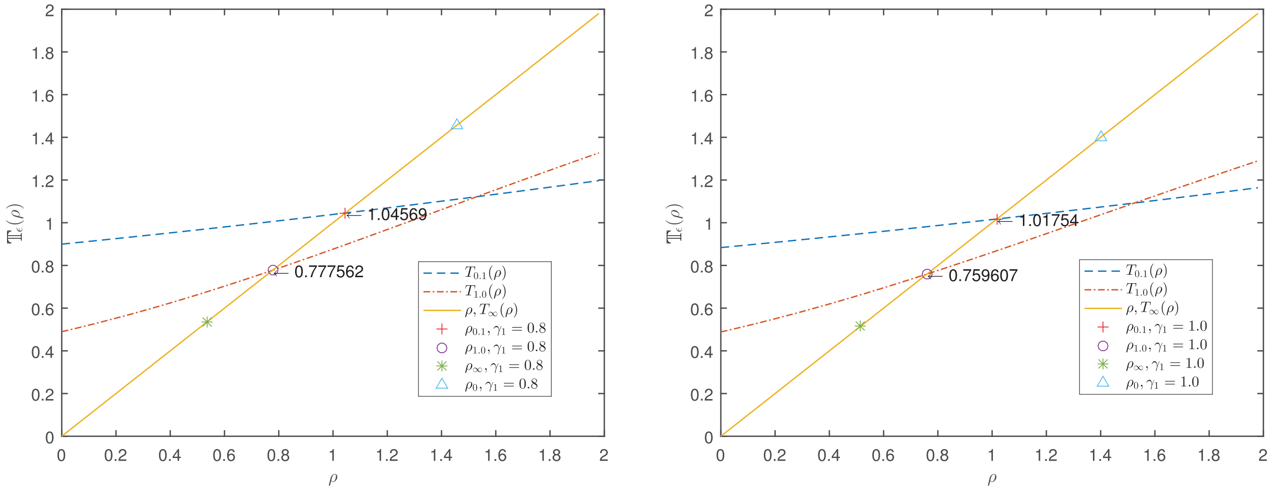
See Online Appendix EC.8 for the proof.

Under log utility functions, Remark 5 shows a "surprising property." A consumer-investor's optimal consumption behavior coincides in two extreme cases: that is, when he completely disregards the variation risk of terminal wealth and when he is excessively concerned about the variation risk of terminal wealth. In addition, when a consumer-investor is extremely concerned about the variation of terminal wealth, the investment policy consists of myopic demand (the first term of Equation (50), Merton's classical result) and hedging demand (the second term of Equation (50)).

5.1. Numerical Verification of Theorem 1

The fixed point Equation (33) is numerically studied to verify Theorem 1. Figure 1, which plots the solutions obtained by the intersection of ρ and $\mathbb{T}_\epsilon$, confirms a uniquely determined fixed point.

Figure 1. (Color online) Fixed Point Problems Depending on $\gamma_1 \in [0.8, 1.0]$ with $\theta = 1.0$



5.2. Impact of ϵ on Expected Value and Variance of Optimal Terminal Wealth, ρ_ϵ and $Var(\hat{X}_\epsilon(T))$

The numerical study exhibits that ρ_ϵ decreases with respect to ϵ (the left panels of Figures 2 and 3). A proof of this relationship for ϵ small and $\gamma_1 = 1$ is given in Online Appendix EC.9. This inverse relation persists regardless of values of γ_1 and θ as shown in the left panels of Figures 4 and 5. In other words, regardless of γ_1 and θ, the more a consumer-investor is concerned about the variance risk of his final wealth, the smaller his expected optimal final wealth will be. The result is expected because a consumer-investor decreases his portfolio holding in risky assets in an effort to reduce the variation of his final wealth as ϵ increases. The immediate effect of investing less wealth in risky assets and more wealth in risk-free assets is a decrease in expected investment returns because risky assets yield higher rates of return. This also explains why ρ_0 and ρ_∞ set the ceiling and the floor of the expected optimal terminal wealth shown in the left panels of Figures 2 and 3.

Figures 6 (upper left panel) and 7 (left panel) reveal the necessity of the MFTC model (Merton's problem with variance risk control) with preferable features. With ϵ small, the MFTC model allows a consumer-investor to enjoy a higher consumption rate with lower variation from his optimal terminal wealth. Furthermore, Figure 7 (left panel) shows that at terminal time T, $\hat{X}_0(T)$ (i.e., optimal terminal wealth from the traditional Merton's model) is extremely volatile.

Figure 2. (Color online) ρ_ϵ and λ_ϵ Depending on $\epsilon \in (0, 50]$ with $\gamma_1 = 0.8$ and $\theta = 1.0$

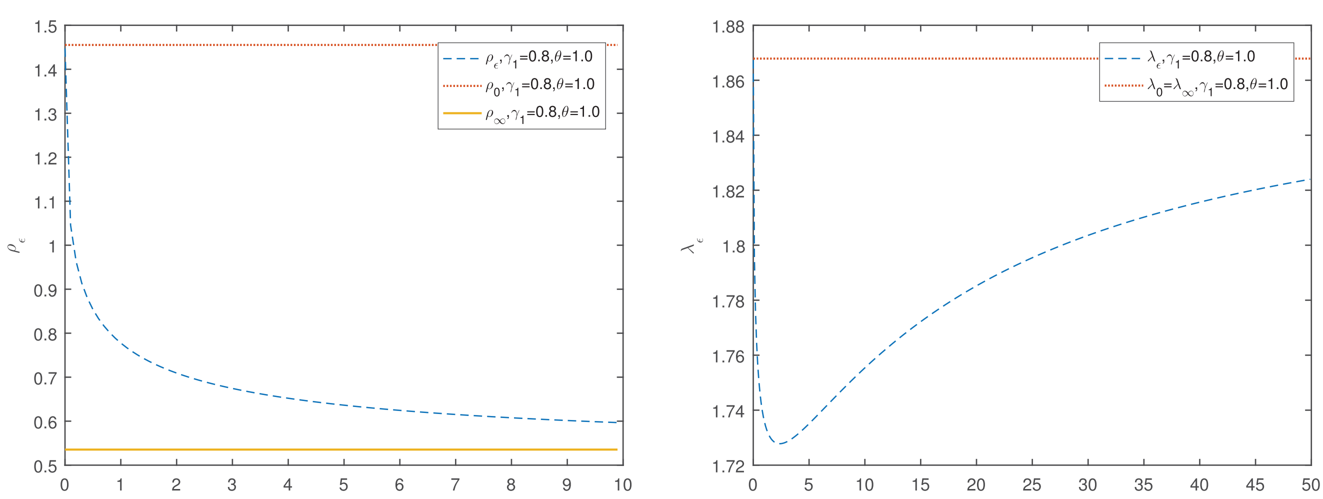
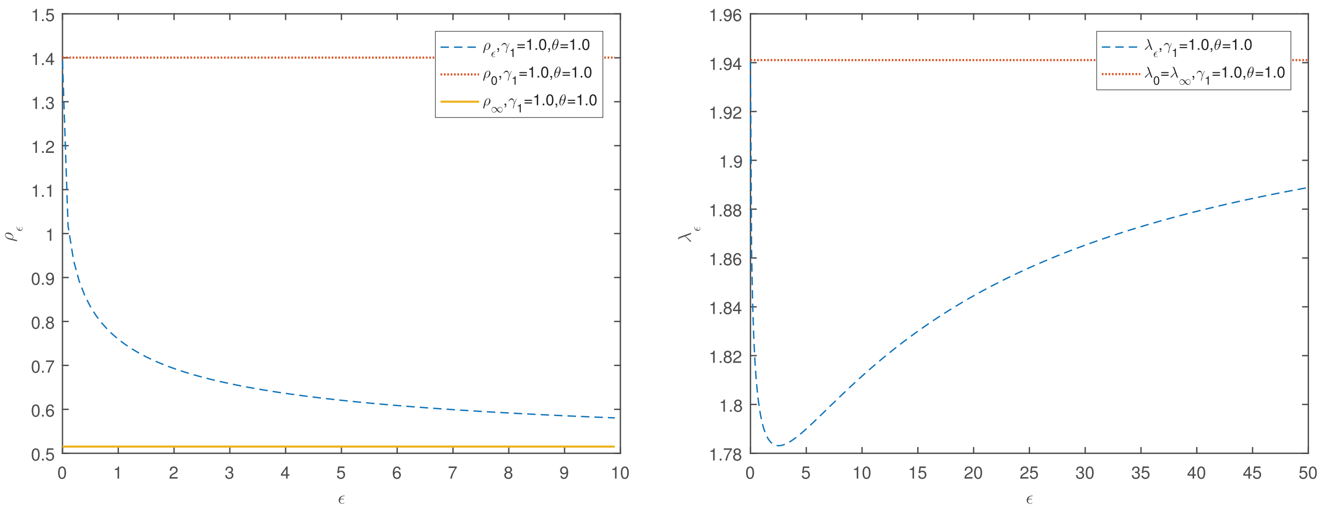


Figure 3. (Color online) ρ_ϵ and λ_ϵ Depending on $\epsilon \in (0,50]$ with $\gamma_1 = 1.0$ and $\theta = 1.0$



Obviously, the variance of terminal wealth in the traditional Merton's model is too substantial for a consumer-investor to ignore.

The optimal terminal wealth from the MFTC model when ϵ goes to $+\infty$, $\hat{X}_\infty(T)$, is deterministic, a property proven in Section 4.1.2. In other words, a consumer-investor with excessive concerns about the variance risk in the terminal wealth can steer clear of variation in his terminal wealth. Given γ_1, ϵ can then be viewed as the risk aversion of a consumer-investor. In this respect, the observation of deterministic terminal wealth is in line with the work of Liu (2007). Liu (2007) studies dynamic portfolio choice with stochastic variation in investment opportunities and predicts that the optimal terminal wealth for an investor with infinite risk aversion is a constant. More importantly, our model exhibits a distinct feature that a consumer-investor with constant terminal wealth can enjoy the same consumption rate as if the traditional Merton's model was implemented as shown in the upper left panel of Figure 6 and Remark 5.

5.3. Impact of ϵ on Optimal Consumption Rate $C_\epsilon(t)$

From Equation (46), optimal consumption rate is inversely related to λ_ϵ; therefore, we will study λ_ϵ for the consumption rate behavior. The right panels of Figures 2 and 3 depict a nonlinear relation between ϵ and $1/\lambda_\epsilon$, and the relation holds regardless values of γ_1 and θ as shown in the right panels of Figures 4 and 5.

As ϵ increases, the optimal consumption rate grows rapidly to a positive maximum and then decreases at a decreasing rate. This nonlinear shape is a consequence of a risk-averse consumer-investor's risk-reward

Figure 4. (Color online) ρ_ϵ and λ_ϵ Depending on $\epsilon \in (0,30]$ with $\gamma_1 \in \{0.8, 0.9, 1.0\}$ and $\theta = 1.0$

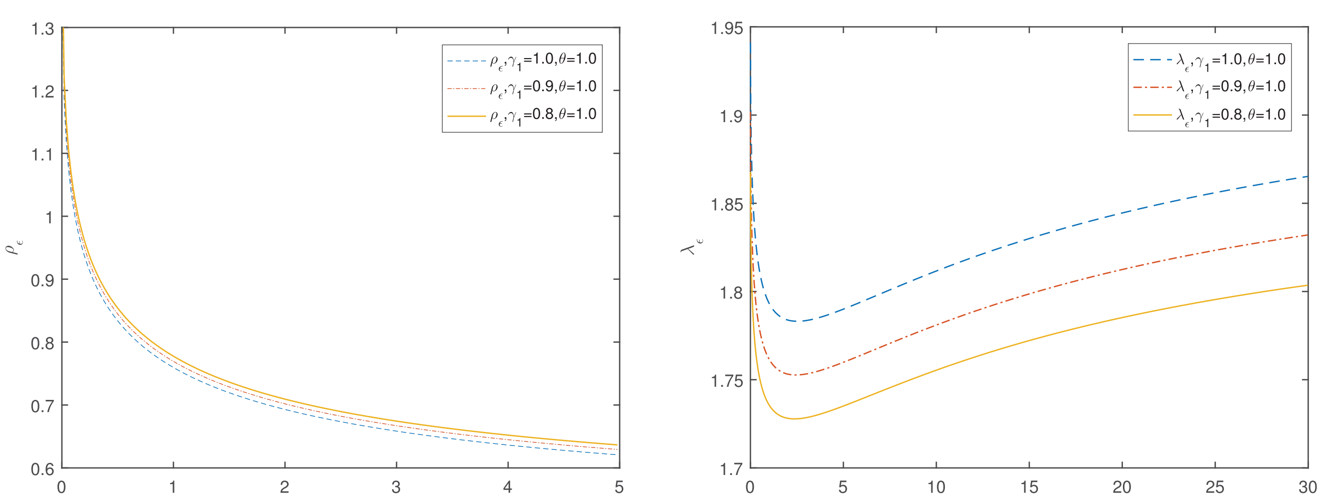
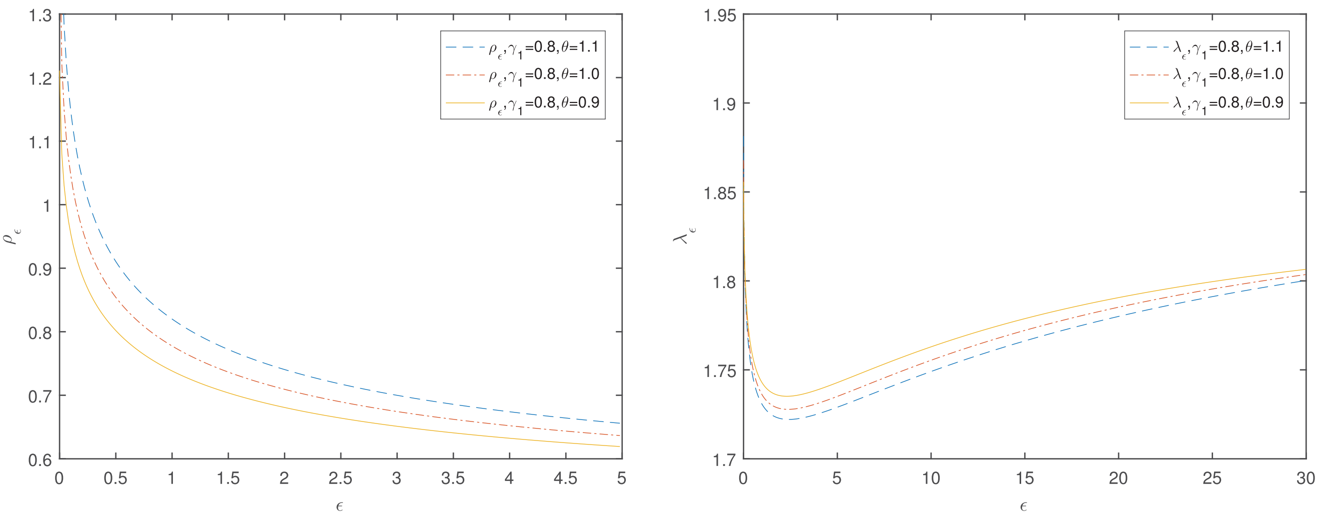


Figure 5. (Color online) ρ_ϵ and λ_ϵ Depending on $\epsilon \in (0, 30]$ with $\gamma_1 = 0.8$ and $\theta \in \{0.9, 1.0, 1.1\}$



trade-off (measured by the ratio between the expected optimal terminal wealth and the standard deviation of the terminal wealth $\rho_\epsilon / SD(\hat{X}_\epsilon(T))$) as well as the decreasing return from increasingly investing in risk-free assets. As shown in the right panel of Figure 7, the reward (ρ_ϵ) per unit of risk ($SD(\hat{X}_\epsilon(T))$) for a consumer-investor is increasing as ϵ increases. Therefore, a consumer-investor is willing to rebalance his optimal investment and consumption policies to reduce the variance of the terminal wealth when ϵ starts kicking in. The consumer-investor can accomplish this goal by buying or selling assets at the market to change the asset allocation of his portfolio from risky assets to risk-free assets and to finance more consumption. Hence, the consumption rate increases as ϵ increases. However, when the variance of the terminal wealth is reduced to extremely small, most of the consumer-investor's asset investment is allocated toward the risk-free assets. The amount of wealth that the consumer-investor can finance his consumption starts decreasing due to low returns from risk-free assets. It explains why the consumption rate starts decreasing after the point where the variance of the terminal wealth is close to the level of zero.

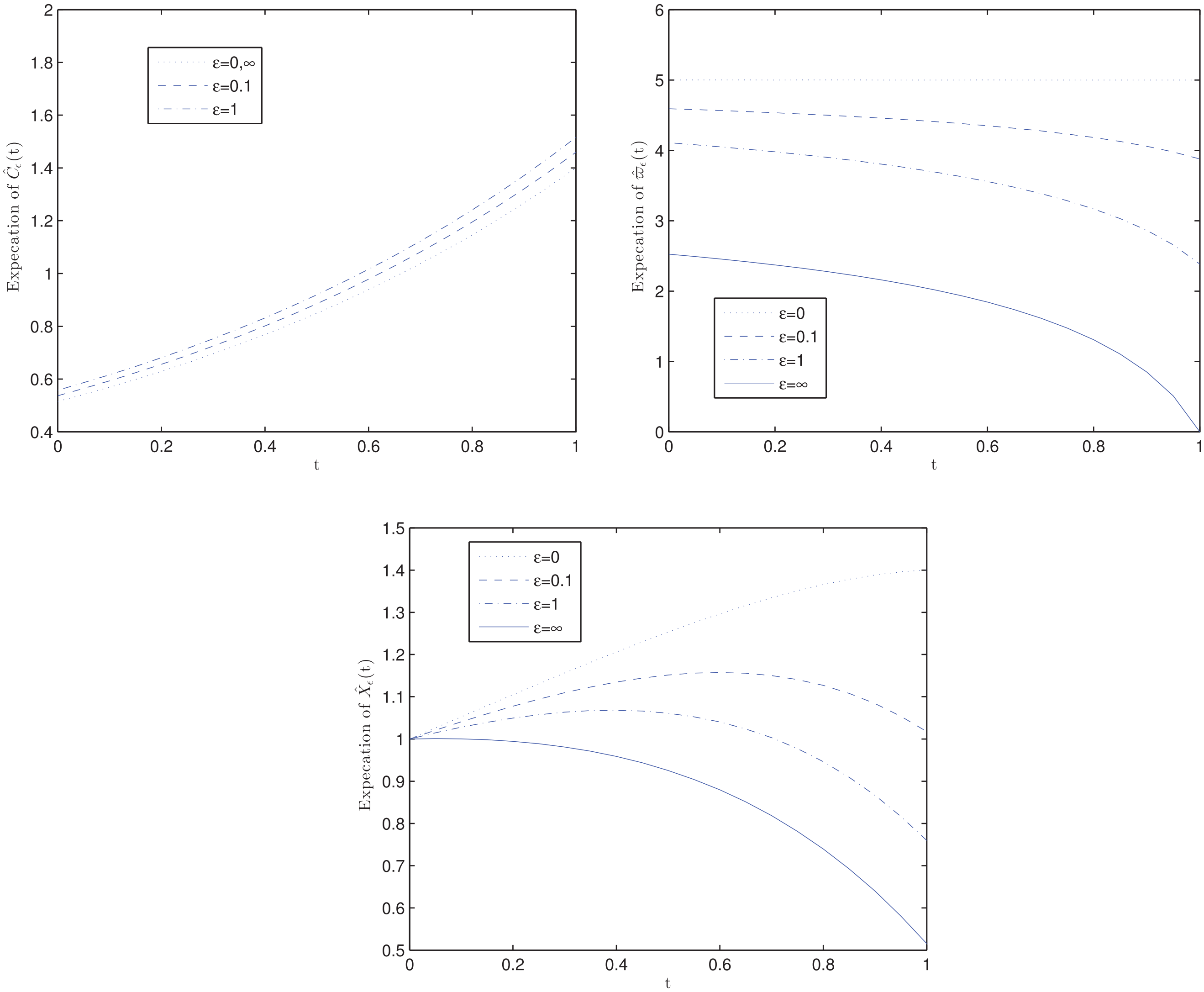
5.4. Expected Optimal Consumption Rate, Expected Optimal Percentage Allocations in Risky Assets, and Expected Optimal Wealth, $E\hat{C}_\epsilon(t)$, $E\hat{\varpi}_\epsilon(t)$, and $E\hat{X}_\epsilon(t)$

Figure 6 graphically studies the expectation of optimal consumption, the expectation of optimal percentage allocations in risky assets, and the expectation of optimal wealth against time. The upper left panel confirms the increasing-decreasing pattern of the optimal consumption process as ϵ increases, discussed in Section 5.3. For a given ϵ, the expected optimal consumption is monotonically increasing against t similar to Merton (1969).

The upper right panel plots the expected optimal proportion of wealth allocated in risky assets, $E[\hat{\varpi}_\epsilon(t)]$, for different values of ϵ against time. It shows that percentage allocations in risky assets continuously decrease as the remaining investment horizon approaches zero and as ϵ increases. For a given γ_1, ϵ can be viewed as a consumer-investor's risk aversion toward terminal wealth given. As such, these observations are comparable with the empirical study by Barberis (2000). Barberis (2000) proposes the optimal portfolio choice for an investor who has a CRRA class utility over terminal wealth. This research shows that the allocation to stocks for the investor, optimally rebalancing the portfolio, steadily decreases as the remaining time horizon goes to zero, and the stock allocation falls as the investor's risk aversion over terminal wealth increases. In addition, Dai et al. (2021), who study a dynamic portfolio choice model with the mean-variance criterion for log returns, also derive that $E[\hat{\varpi}_\epsilon(t)]$ decreases as time proceeds toward the end of the investment horizon under the incomplete market setting. It is noted that, under a complete market setting, Dai et al. (2021) obtain the optimal fraction of the total wealth in risky assets as a constant independent of time and wealth, same as in Merton (1969).

Finally, the lower panel presents the expected optimal wealth process, a result of a consumer-investor's investment-consumption decision, for various ϵ. It shows that the expected optimal wealth process decreases as ϵ increases. Compared with Merton (1969), our model leads to a nonmonotonic expected optimal wealth process against time. As time t increases, the expected optimal wealth process increases first and

Figure 6. (Color online) Expectation of $\hat{C}_\epsilon(t), \hat{\varpi}_\epsilon(t)$, and $\hat{X}_\epsilon(t)$ for $\epsilon \in \{0, 0.1, 1.0, +\infty\}$ with $\gamma_1 = 1$ and $\theta = 1$



then decreases. The nonmonotonic shape holds true even for the extreme case, $\epsilon \to \infty$.

5.5. Expected Consumption-Wealth Ratio

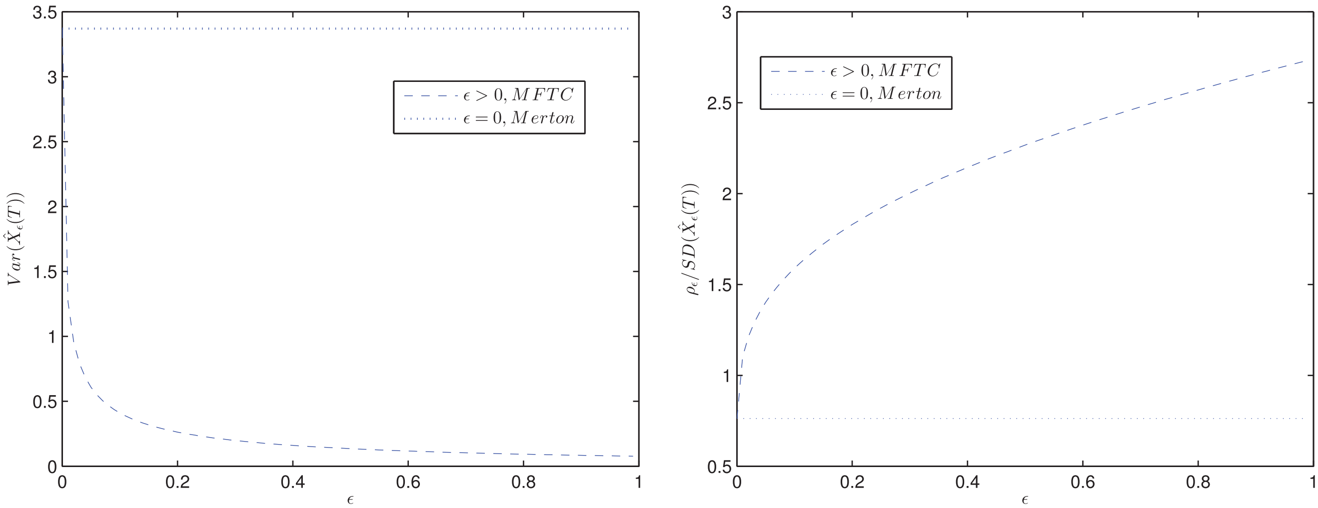
In Figure 8, the expected consumption-wealth ratio is nonmonotonic in ϵ at the beginning investment horizon. However, as t progresses, the expected consumption-wealth ratio becomes monotonically increasing in ϵ. The observation is expected. The differences in the expected wealth among ϵ are not significant at the beginning of investment horizon and then increase at an increasing rate as time progresses; see Figure 6, lower panel. On the contrary, the differences in expected consumption among ϵ stay quite constant over the entire investment horizon. Consequently, the consumption-wealth ratio becomes monotonically increasing in ϵ after the time when decreases in the expected wealth dominate.

It is not surprising that the consumption-wealth ratio in our model is higher than Merton's because our model predicts higher consumption and lower wealth. Moreover, as shown in the figure, the ratios in our model can be greater than one, whereas in Merton's model, the ratios are increasing to one at time T. Sotomayor and Cadenillas (2009), who study investment-consumption problems with regime switching under the utility maximization framework, also observe ratios higher than 1 for the power utility x^α with $0 < \alpha < 1$ in every market regime (bull or bear).

6. Investment-Consumption Insights

Our MFTC model, combining the investment-consumption model of Merton (1969) and the mean-variance framework of Markowitz (1952), investigates the optimal investment and consumption policies when

Figure 7. (Color online) Variance of Terminal Wealth Depending on $\epsilon \in [0,1]$ and the Ratio Between the Expected Terminal Wealth and Standard Deviation of Terminal Wealth Depending on $\epsilon \in [0,1]$ with $\gamma_1 = 1$ and $\theta = 1$



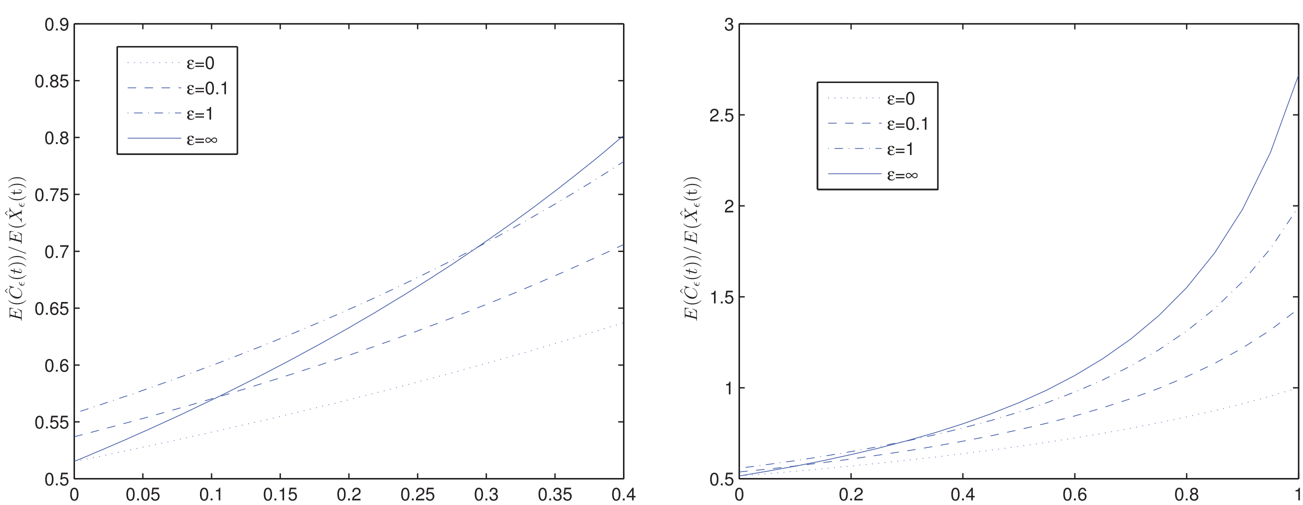
the variance risk is explicitly incorporated into the investor-consumer's portfolio selection framework. We obtain the following investment-consumption insights that can benefit investors.

1. Ignoring variance risk of terminal wealth in a consumer-investor's portfolio selection framework is likely to end up yielding terminal wealth significantly lower than the expected. This message is important to both individual investors and professional investors because neither of them would want to be surprised at much lower accumulated terminal wealth than the expected at the end of the investment horizon.

2. The lower the variance risk of terminal wealth that a consumer-investor achieves, the lower the expected terminal wealth will be. This is consistent with the popular investment quote: "In investing, what is comfortable is rarely profitable" (by Robert Arnott (Arnott et al. 2011)). There is no free lunch; it is a risk-reward trade-off for the investing comfort zone. Consequently, it is important to be able to quantify the variance risk in the portfolio selection instead of risk blindness. As said by Ben Graham (Graham 1976), "The individual investor should act consistently as an investor and not as a speculator."

3. Consumer-investors do not control variance risk at the expense of consumption. In fact, consumer-investors enjoy at least the same consumption rate as if they were not to control the variance risk. This again points out the necessity of incorporating the variance risk in the portfolio selection framework. The framework has the benefit of feeding two birds with one stone; that is,

Figure 8. (Color online) Ratio of the Expectation of $\hat{C}_\epsilon(t)$ and the Expectation of $\hat{X}_\epsilon(t)$ with $\gamma_1 = 1$ and $\theta = 1$



consumer-investors not only achieve the target terminal wealth at lower risk but also, enjoy higher consumption rates.

4. For a consumer-investor to achieve a guaranteed terminal wealth, it does not mean he will only invest in risk-free assets over the entire investment horizon. If the consumer-investor holds purely risk-free assets during his investment horizon, the risk-free return will not be able to finance him the consumption rate as one who does not control the variance risk at all. Our MFTC model can actually help a consumer-investor achieve the goal through the investment policy, which properly balances portfolios between risky assets and risk-free assets over the investment horizon.

7. Conclusion

A consumer-investor's investment-consumption problem is studied through integrating intermediate consumption, portfolio selections, and utility-risk management in a unified framework. Applying the mean field-type control theory and overcoming the key difficulty of solving a fixed point equation, explicit formulas for the optimal consumption and portfolio choices are obtained. When $\epsilon = 0$, closed form solutions for the traditional Merton's problem with logarithmic utilities ($U_1(x) = U_2(x) = \ln x$) are recovered. For comparison purposes, closed form solutions for our MFTC model are derived when $\epsilon \to \infty$. By inspecting the closed form solutions obtained, it reveals that, by implementing our MFTC model, a consumer-investor can obtain guaranteed terminal wealth and meanwhile, enjoy the same consumption rate as the traditional Merton's model, which bears high variation in the terminal wealth.

Numerical analysis results show that our MFTC model not only can effectively control the variance risk but also, can allow a consumer-investor to increase his consumption rate. This desirable feature is illustrated by the increasing-decreasing shape of optimal consumption rate with respect to ϵ. Regardless of values of γ_1 and θ, the optimal consumption rate increases quickly to a positive maximum before starting to decrease at a decreasing rate as ϵ increases. Furthermore, numerical analysis results also show that the allocation to stocks decreases as the remaining time horizon goes to zero, and the stock allocation falls as the investor's risk aversion over terminal wealth increases.

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