

K -THEORETIC TORSION AND THE ZETA FUNCTION

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ABSTRACT. We generalize to higher algebraic K -theory an identity (originally due to Milnor) that relates the Reidemeister torsion of an infinite cyclic cover to its Lefschetz zeta function. Our identity involves a higher torsion invariant, the *endomorphism torsion*, of a parametrized family of endomorphisms as well as a *higher zeta function* of such a family. We also exhibit several examples of families of endomorphisms having non-trivial endomorphism torsion.

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1. INTRODUCTION

Algebraic K -theory is a natural setting for algebraic torsion invariants such as Reidemeister and Whitehead torsion. In this paper we consider a closely related invariant that we call *endomorphism torsion*. Endomorphism torsion provides a means of conceptually relating Reidemeister torsion to dynamical systems. Specifically, it is related to the “counting” of periodic orbits in a dynamical system by work of Milnor

[25], and subsequently partially developed in the papers of Fried [8], [9] and Geoghegan and Nicas [10].

Milnor considers infinite cyclic coverings $\tilde{X} \rightarrow X$ with generating deck transformation $T: \tilde{X} \rightarrow \tilde{X}$. Assume that $\tilde{X}$ has finite rational homology. One can then consider two kinds of invariants associated with $(\tilde{X}, T)$.

The first invariant associated with $(\tilde{X}, T)$ is the algebraic torsion $\tau(t) \in \mathbb{Q}(t)^\times$ of the contractible chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Q}(t)$. This is the Reidemeister torsion of X with respect to the homomorphism $\pi_1(X) \rightarrow \mathbb{Z} \rightarrow \mathbb{Q}(t)^\times$ coming from the cyclic cover $\tilde{X} \rightarrow X$ and the map $n \mapsto t^n$. Normally $\tau(t)$ would only be defined up to an element of $\mathbb{Z}[t, t^{-1}]^\times = \{\pm t^n\}_{n \in \mathbb{Z}}$, but in this paper we re-interpret $\tau(t)$ as the endomorphism torsion of the complex $C_*(\tilde{X})$. As a result, it is an element of $\mathbb{Q}(t)^\times$ with no indeterminacy.

The other invariant associated with $(\tilde{X}, T)$ is the Lefschetz zeta function

$$\zeta(t) := \exp \left(\sum_{k \geq 1} L(T^{\circ k}) \frac{t^k}{k} \right) \in \mathbb{Q}(t)^\times,$$

which encodes counting the Lefschetz numbers of the iterates of T taken collectively. Here $L(T^{\circ k})$ is the Lefschetz number of the k -fold composite $T^{\circ k}$, which is in some sense a homological count of the k -fold periodic points of T .

Milnor's remarkable formula is the functional equation

$$(1) \quad \zeta(t^{-1})\tau(t) = t^{\chi(\tilde{X})},$$

where $\chi(\tilde{X})$ is the Euler characteristic. In this paper we show that this equation (1) is a consequence of another equation that holds in the higher algebraic K -theory of endomorphisms.

Let End_A denote the endomorphism category of an associative unital ring A . The objects of End_A are pairs (P, f) such that P is a finitely projective right A -module, and $f: P \rightarrow P$ is an endomorphism. A morphism $(P_1, f_1) \rightarrow (P_2, f_2)$ is a homomorphism of A -modules $g: P_1 \rightarrow P_2$ such that $f_2 g = g f_1$.

Grayson [15] considers a variant of the endomorphism category that is based on a localizing parameter. Recall that a polynomial $p \in A[t]$ is *centric* or *central* if it lies in $Z(A[t]) = Z(A)[t]$, where $Z(A)$ is the center of A . Let $S \subset A[t]$ be any multiplicative subset of monic centric polynomials. Define the full subcategory of S -torsion endomorphisms

$$\text{End}_A^S \subset \text{End}_A$$

whose objects are those (P, f) such that $p(f) = 0$ for some $p \in S$. Let

$$A[t]_S := S^{-1}A[t]$$

denote the localization of $A[t]$ with respect to S .

Example 1.1. Suppose A is commutative and $S \subset A[t]$ is the set of all monic polynomials. Then $\text{End}_A^S = \text{End}_A$. If A is an integral domain, then $A[t]_S = A(t)$ is the ring of rational functions.

Example 1.2. Let $S = \{t^n \mid n \geq 0\}$. Then $\text{End}_A^S = \text{Nil}_A$ is the category of nilpotent endomorphisms, and $A[t]_S = A[t, t^{-1}]$ is the ring of Laurent polynomials.

In §3 we construct a map of K -theory spaces

$$(2) \quad \tau(t): K(\text{End}_A^S) \rightarrow \Omega K(A[t]_S)$$

called the *higher endomorphism torsion map*. Note that in contrast to more familiar kinds of torsion, this induces a map on all homotopy groups, not just π_0 , and has no indeterminacy.

Remark 1.3. Throughout this paper, we regard higher K -theory as a space rather than a spectrum. This is only for simplicity – the arguments also work for spectra, we just have to include the iterations of the S -construction and the compatibility between them.

For $p \in S$ of degree d , set $\tilde{p}(t) := p(1/t)t^d$. Define $T \subset A[t]$ to be the set $\{\tilde{p} \mid p \in S\}$. Then T is a multiplicative subset of centric polynomials with leading term 1. In §4 we define the *higher K -theory zeta function*

$$\zeta(t): K(\text{End}_A^S) \rightarrow \Omega K(A[t]_T),$$

following [21]. This zeta function is related to the more classical ones via the trace map to the topological de Rham-Witt space $\text{TR}(A)$, see Remark 4.3 and the recent preprints [4, 6].

Assume that $t \in S$. Then there is a canonical ring homomorphism

$$A[t^{-1}]_T \rightarrow A[t]_S,$$

that is induced by $t^{-1} \mapsto 1/t$. Therefore, if we substitute t by t^{-1} , the zeta function $\zeta(t^{-1})$ can also be regarded as a map into $\Omega K(A[t]_S)$:

$$(3) \quad \zeta(t^{-1}): K(\text{End}_A^S) \rightarrow \Omega K(A[t]_S).$$

Under this assumption we also get that the zero endomorphism $(P, 0)$ is S -torsion, so composing τ with the operation $(P, f) \mapsto (P, 0)$ gives a zero-torsion map

$$(4) \quad \chi(t): K(\text{End}_A^S) \rightarrow \Omega K(A[t]_S).$$

Our main result is the following statement.

Theorem A. *The sum (loop product) of $\tau(t)$ from (2) and $\zeta(t^{-1})$ from (3) is homotopic to $\chi(t)$ from (4).*

This follows from a related statement about free loop spaces that we prove in Theorem 5.1. Taking $S \subset A[t]$ to be all monic centric polynomials and passing to π_0 recovers Milnor's formula (1).

The forgetful functor $(P, f) \mapsto P$ induces a map $K(\text{End}_A^S) \rightarrow K(A)$. Let $\tilde{K}(\text{End}_A^S)$ be its homotopy fiber, so that

$$K(\text{End}_A^S) \simeq \tilde{K}(\text{End}_A^S) \times K(A).$$

Let

$$\tilde{\tau}(t), \tilde{\zeta}(t^{-1}): \tilde{K}(\text{End}_A^S) \rightarrow \Omega K(A[t]_S)$$

denote the respective restrictions of $\tau(t)$ and $\zeta(t^{-1})$ to $\tilde{K}(\text{End}_A^S)$. Since $\chi(t)$ factors through the projection to $K(A)$, its restriction $\tilde{\chi}(t)$ is homotopically trivial.

Corollary B. *The sum of the maps $\tilde{\tau}(t)$ and $\tilde{\zeta}(t^{-1})$ is nullhomotopic.*

Therefore $\tilde{\tau}(t)$ admits a canonical factorization through $\Omega K(A[t^{-1}]_T)$, since it is just the negative of $\tilde{\zeta}(t^{-1})$.

In §8 we construct higher endomorphism torsion and zeta function invariants for families of endomorphisms of spaces, i.e., data of the form

$$(p: E \rightarrow B, f: E \rightarrow E),$$

in which p is fibration with finitely dominated fibers, and f is a self-map covering the identity map of B . Hence, we may regard E as fiberwise space over B equipped with fiberwise $\mathbb{N}$ -action. Both the higher endomorphism torsion $\tau(p, f)$ and zeta function $\zeta(p, f)$ are invariant with respect to $\mathbb{N}$ -equivariant fiberwise weak equivalence.

For any preselected ring homomorphism $\mathbb{Z}[\pi_1(E)] \rightarrow A$ and subset $S \subset A[t]$ as above, the torsion invariant $\tau(p, f)$ is a composition of the form

$$B \xrightarrow{c} K(\text{End}_A^S) \xrightarrow{\tau(t)} \Omega K(A[t]_S),$$

where the map c is induced by the classifying space data for the family of endomorphisms.

In particular, taking $B = S^n$ to be the n -sphere, the torsion produces an invariant in $K_{n+1}(A[t]_S)$. In Theorem 8.5 we prove the following:

Theorem C. *There is a pair*

$$(p: E \rightarrow S^1, f: E \rightarrow E)$$

for which the component of $\tau(p, f)$ in $K_2(\mathbb{Q}(t))$ is non-trivial.

In the above, the fibration p has fiber $S^n \vee S^n$ and the monodromy $\theta: S^n \vee S^n \rightarrow S^n \vee S^n$ has degree given by the matrix

$$Q := \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

As $I - Q$ is invertible over the integers, E is a non-trivial homology circle. The self-map $f: E \rightarrow E$ is the map of mapping tori induced by θ .

Remark 1.4. The example in Theorem C is explicit. In §8.2 we provide several other explicit examples having non-trivial torsion in $K_2(\mathbb{Q}(t))$.

In §8.3 we employ a kind of reverse engineering to exhibit higher dimensional examples with non-trivial torsion in $K_{2n+2}(\mathbb{Q}(t))$ for $n > 1$. The higher dimensional examples are fibrations over homology $(2n+1)$ -spheres equipped with a fiberwise self-map. However, these examples are not explicit and their existence relies on recent work by Andrew Salch [28].

1.1. Further results. The paper also contains three appendices.

Appendix A recalls two “fundamental theorems” of endomorphism K -theory. The first relates $K(\text{End}_A^S)$ to $\Omega K(A[t]_S)$ and can be derived without much effort from Waldhausen’s generic fibration theorem. We sketch a proof since we could not find it stated in the literature. The other result, which is due to Grayson and is better known, relates the reduced K -theory space $\tilde{K}(\text{End}_A^S)$ to $\Omega K(A[t^{-1}]_T)$. We make little claim to originality except to note that the zeta function provides Grayson’s equivalence and the endomorphism torsion provides the other one.

In Appendix B we give an explicit description in terms of Waldhausen’s S . construction of the boundary map ∂ of the K -theory localization sequence associated with the localization $A[t] \rightarrow A[t]_S$. In this case there is a short exact sequence

$$0 \rightarrow K_{n+1}(A[t]) \rightarrow K_{n+1}(A[t]_S) \xrightarrow{\partial} K_n(\text{End}_A^S) \rightarrow 0$$

that is split by the endomorphism torsion map. This result is used in the proof of Theorem C.

In Appendix C we verify the hypotheses of Waldhausen’s “sphere theorem” [29, thm. 1.71] for the categories of endomorphisms End_A and End_A^S . This is somewhat non-trivial due to the discrepancy between the categories of perfect complexes over the rings $A[t]$ and A .

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2. PRELIMINARIES

2.1. K -theory of Waldhausen categories. Let C be a Waldhausen category, i.e., a category with cofibrations $\text{co}C$ and weak equivalences wC . It is customary to denote the cofibrations by $\rightarrowtail$ and the weak equivalences by $\xrightarrow{\sim}$. When the isomorphisms of C are used as the weak equivalences, we denote wC by iC .

The K -theory of $(C, \text{co}C, wC)$ is the based loop space

$$\Omega|wS.C|,$$

where $|wS.C|$ denotes the realization of a simplicial pointed category $wS.C$. The objects of $wS_k C$ are given by filtered objects of C of the form

$$A_\bullet := A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_k$$

together with explicit choices of quotient data

$$A_{ij} := A_j/A_i \in C \quad i \leq j.$$

See [29] or [31] for more details.

In particular, $wS_0 C$ is the trivial category with object $*$, and $wS_1 C = wC$. As a result there is a canonical map

$$(5) \quad |wC| \rightarrow \Omega|wS.C|$$

which is adjoint to the inclusion of the one-skeleton in the simplicial direction.

Example 2.1 (Projective modules). Let A be an associative unital ring. Let $P(A)$ be the category of finitely generated projective (right) A -modules. A cofibration of $P(A)$ is a monomorphism $P \rightarrow Q$ whose cokernel is projective. A weak equivalence of $P(A)$ is an isomorphism. The category of weak equivalences is denoted $iP(A)$. Then the K -theory space of A is

$$K(A) := \Omega|iS.P(A)|.$$

This coincides up to equivalence with the Quillen K -theory of $P(A)$ considered as an exact category [29, §1.9]

Example 2.2 (Endomorphisms of modules). Let $\text{End}_A = \text{End}(P(A))$ be the category in which an object is a pair (P, f) , where

- P a finitely generated projective A -module, and
- $f: P \rightarrow P$ is an A -linear endomorphism.

A morphism $(P, f) \rightarrow (Q, g)$ of End_A is a homomorphism $h: P \rightarrow Q$ such that $gh = hf$.

We equip End_A with a Waldhausen category structure induced by the forgetful functor $\text{End}_A \rightarrow P(A)$. That is, a morphism $h: (P, f) \rightarrow (Q, g)$ is a cofibration/weak equivalence if and only if $h: P \rightarrow Q$ is cofibration/weak equivalence in $P(A)$. The *K-theory of endomorphisms* is

$$K(\text{End}_A) := \Omega |iS.\text{End}_A|.$$

Example 2.3 (Endomorphisms with S -torsion). Each endomorphism (P, f) is an $A[t]$ -module with t acting by f . Here $A[t]$ is the polynomial ring, the free A -algebra on one generator t subject to the relation that t commutes with A . Given a polynomial $p(t)$, we denote its action on P by $p(f)$. Note that if $p(t)$ is in the center $Z(A[t]) = Z(A)[t]$, then $p(f)$ is an $A[t]$ -linear map.

Let $S \subseteq A[t]$ be any multiplicatively closed set of monic centric polynomials. Then we define $\text{End}_A^S = \text{End}(P(A))^S$ be the full subcategory of End_A on the objects (P, f) such that $p(f) = 0$ for some $p(t) \in S$. Equivalently, the localized module $S^{-1}P$ is zero.

When A is commutative and S consists of all monic polynomials, then $\text{End}_A^S = \text{End}_A$. In other words, every endomorphism in End_A is monic-torsion. This is because we could take $p(t)$ to be the characteristic polynomial of any extension of f to a finite free module. This argument does not apply outside the commutative case because we cannot define determinants in the usual way.

Example 2.4 (Automorphisms). The full subcategory $\text{Aut}_A \subset \text{End}_A$ consisting of those objects (P, f) such that f is an automorphism is called the *automorphism category* of A . The *K-theory of automorphisms* is

$$K(\text{Aut}_A) := \Omega |iS.\text{Aut}_A|.$$

Example 2.5 (Extension of scalars). Let $\phi: A \rightarrow B$ be a ring homomorphism. Then B has the structure of an A - A -bimodule. The operation

$$(P, f) \mapsto (P \otimes_A B, f \otimes 1)$$

is an exact functor. Consequently, it induces a map

$$K(\text{End}_A) \rightarrow K(\text{End}_B).$$

For the first segment of the paper these examples will be sufficient, but see §6 for refinements of these examples to chain complexes.

2.2. Categorical loops. For a category C let $\text{End}(C)$ be the category whose objects are self-maps $\phi_a: a \rightarrow a$ of C and whose morphisms $(a, \phi_a) \rightarrow (b, \phi_b)$ are maps $u: a \rightarrow b$ which fit into a commutative diagram

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ \phi_a \downarrow & & \downarrow \phi_b \\ a & \xrightarrow{u} & b. \end{array}$$

There is a “coassembly” map of spaces

$$(6) \quad e: |\text{End}(C)| \rightarrow L|C|,$$

where $L|C|$ is the unbased loop space of $|C|$. This map commutes with the projection of each space to $|C|$, by sending (a, ϕ_a) to a or by evaluating a loop at the basepoint.

The coassembly map may be described in the following way: each k -simplex in the realization $|\text{End}(C)|$ is represented by a functor

$$f: [1] \times [k] \rightarrow C.$$

Here, $[k]$ is the category with objects i for $0 \leq i \leq k$ and a unique morphism from $i \rightarrow j$ whenever $i \leq j$. The functor f also has the property that its restrictions to the two objects $0, 1 \in [1]$ give the same functor $[k] \rightarrow C$. We take realization of f to get a map

$$(\Delta^1 / \partial \Delta^1) \times \Delta^k \rightarrow |C|$$

whose adjoint is a map $\Delta^k \rightarrow L|C|$. It is easy to check this rule respects faces and degeneracies, giving a well-defined map $|\text{End}(C)| \rightarrow L|C|$. Furthermore, restricting to the basepoint of S^1 gives precisely the map $|\text{End}(C)| \rightarrow |C|$ that forgets the endomorphism ϕ_a .

For each $k \geq 0$, we can apply the coassembly construction to the category $iS_k P(A)$. Note that this category consists entirely of isomorphisms, so its endomorphism category is actually an automorphism category, and can be rewritten as $iS_k \text{Aut}_A$. The coassembly map is then a map of the form

$$|iS_k \text{Aut}_A| \rightarrow L|iS_k P(A)|.$$

By the naturality of this construction, as we vary k this gives a map of simplicial spaces. Taking the realization of the adjoint, we obtain a map

$$S^1 \times |iS \cdot \text{Aut}_A| \rightarrow |iS \cdot P(A)|,$$

which after looping once defines a map

$$S^1 \times K(\text{Aut}_A) \rightarrow K(A).$$

Taking the adjoint again defines a coassembly map on K -theory

$$(7) \quad c: K(\mathrm{Aut}_A) \rightarrow LK(A).$$

Remark 2.6. Note that this K -theory coassembly map does not extend to $K(\mathrm{End}_A)$. It is necessary to use isomorphisms to actually get a loop in the target space. More generally, we can define such a coassembly map for any Waldhausen category C , and it takes the form

$$K(\mathrm{Aut}_C) \rightarrow LK(C),$$

where Aut_C is the category of objects $c \in C$ with weak equivalence endomorphisms $c \xrightarrow{\sim} c$.

2.3. Composition. Composition of endomorphisms defines a functor

$$\mathrm{End}(P(A)) \times_{P(A)} \mathrm{End}(P(A)) \xrightarrow{\circ} \mathrm{End}(P(A)).$$

It is straightforward to check that it is strictly associative and has unit given by the functor $P(A) \rightarrow \mathrm{End}_A$ taking each module to its identity morphism. Since K -theory commutes with pullbacks, this gives maps

$$K(\mathrm{End}_A) \times_{K(A)} K(\mathrm{End}_A) \xrightarrow{\circ} K(\mathrm{End}_A)$$

making the space $K(\mathrm{End}_A)$ into a fiberwise topological monoid over the space $K(A)$. Restricting further to automorphisms makes $K(\mathrm{Aut}_A)$ into a fiberwise topological group over $K(A)$.

Remark 2.7. This composition operation does not induce a composition on the homotopy fiber $\tilde{K}(\mathrm{End}_A)$ because $K(\mathrm{End}_A) \rightarrow K(A)$ is not a fibration. This would lead to a contradiction anyway: using the characteristic polynomial map $\tilde{K}_0(\mathrm{End}(A)) \rightarrow (1 + tA[[t]])^\times$, we would get an incorrect identity $p_{f \circ g} = p_f \cdot p_g$, where p_f is the characteristic polynomial of an endomorphism f .

Similarly, the free loop space $LK(A)$ is a fiberwise grouplike A_∞ -space over $K(A)$, with composition $*$ given by composing loops (using the little intervals operad). Note that $LK(A) \rightarrow K(A)$ is a fibration and so this does give a composition on the homotopy fiber, namely the usual composition on the based loop space $\Omega K(A)$.

Lemma 2.8. *The following square commutes up to homotopy:*

$$(8) \quad \begin{array}{ccc} K(\mathrm{Aut}_A) \times_{K(A)} K(\mathrm{Aut}_A) & \xrightarrow{c \times c} & LK(A) \times_{K(A)} LK(A) \\ \circ \downarrow & & \downarrow * \\ K(\mathrm{Aut}_A) & \xrightarrow{c} & LK(A). \end{array}$$

Here, the horizontal maps are induced by the coassembly map (7).

Proof. We first describe a natural homotopy making the following square commute.

$$\begin{array}{ccc}
 |\mathrm{End}(C)| \times_{|C|} |\mathrm{End}(C)| & \xrightarrow{e \times e} & L|C| \times_{|C|} L|C| \\
 \circ \downarrow & & \downarrow * \\
 |\mathrm{End}(C)| & \xrightarrow{e} & L|C|,
 \end{array}$$

where the horizontal maps are induced by the coassembly map (6). Since realization commutes with pullbacks, each k -simplex in the top-left is described as a pair of functors $[1] \times [k] \rightarrow C$ that restrict to the same functor $[k] \rightarrow C$ on the endpoints of $[1]$. These together specify a functor $[2] \times [k] \rightarrow C$, which realizes to a map

$$(9) \quad \Delta^2 \times \Delta^k \rightarrow |C|.$$

The left-bottom route of the commuting diagram is the restriction of (9) to the face $[0 < 2]$ of Δ^2 , while the top-right route is the restriction to the two faces $[0 < 1]$ and $[1 < 2]$, composed together using some fixed rule for composing two loops. The Δ^2 provides a homotopy between them. So if we fix a choice for this homotopy without reference to Δ^k , we get a homotopy that commutes with face and degeneracy maps and therefore gives a well-defined homotopy on the entire realization.

Now we apply this to $C = iS_k P(A)$ for each $k \geq 0$ to get a homotopy that commutes the following maps of simplicial spaces:

$$\begin{array}{ccc}
 |iS_k \mathrm{Aut}_A| \times_{|iS_k P(A)|} |iS_k \mathrm{Aut}_A| & \longrightarrow & L|iS_k P(A)| \times_{|iS_k P(A)|} L|iS_k P(A)| \\
 \circ \downarrow & & \downarrow * \\
 |iS_k \mathrm{Aut}_A| & \longrightarrow & L|iS_k P(A)|,
 \end{array}$$

where the horizontal maps arise from (6). Taking realizations and then based loops on the left-hand side gives the left-hand column of (8). On the right-hand side, this does not give the desired right-hand column, but this can be fixed by another coassembly trick.

Observe that for any simplicial space X , there is a coassembly map

$$|LX| \rightarrow L|X|$$

defined as in (6), whose adjoint is

$$S^1 \times |LX| \cong |S^1 \times LX| \xrightarrow{\mathrm{ev}} |X|.$$

We form a square

$$\begin{array}{ccc} |LX. \times_X LX.| & \longrightarrow & L|X.| \times_{|X.|} L|X.| \\ \downarrow * & & \downarrow * \\ |LX.| & \longrightarrow & L|X.|, \end{array}$$

in which the vertical maps arise from loop composition and the horizontal maps from this new coassembly map. A straightforward verification shows that this square strictly commutes. Therefore we may take $X_k = iS_k P(A)$ and append this square to the earlier one to get a new homotopy-commuting square

$$\begin{array}{ccc} K(\mathrm{Aut}_A) \times_{K(A)} K(\mathrm{Aut}_A) & \longrightarrow & \Omega(L|iS.P(A)|) \times_{|iS.P(A)|} L|iS.P(A)| \\ \downarrow \circ & & \downarrow * \\ K(\mathrm{Aut}_A) & \longrightarrow & \Omega(L|iS.P(A)|). \end{array}$$

Commuting the Ω past the loop product gives the desired square in the statement of the lemma. $\square$

3. HIGHER ENDOMORPHISM TORSION

3.1. Some extensions of the polynomial ring. Let A be an associative unital ring, and let $A[t]$ be the ring of polynomials with coefficients in A . By convention, this means that the indeterminate t commutes with A . A polynomial is centric if it lies in the center $Z(A[t]) = Z(A)[t]$.

Following Grayson [15], fix a multiplicative subset $S \subset A[t]$ of monic centric polynomials in $A[t]$, containing at least the powers of t . Let

$$A[t]_S := S^{-1}A[t]$$

denote the localization of $A[t]$ given by the left fractions with respect to S .

For each polynomial

$$p(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_{n-1} t + a_n \in S,$$

let $\tilde{p}(t)$ denote the re-normalized polynomial

$$\tilde{p}(t) = 1 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + a_n t^n = p(1/t) t^n.$$

This defines a second multiplicative subset

$$T = \{\tilde{p}(t) | p(t) \in S\} \subseteq A[t]$$

of centric polynomials having leading coefficient 1. Let

$$A[t]_T := T^{-1}A[t].$$

The homomorphisms $A[t] \rightarrow A[t]_S$ and $A[t] \rightarrow A[t]_T$ are both flat localizations [30].

We will sometimes write $A[t^{-1}]_T$, substituting t^{-1} for the role of t . When we do this, T is instead the set

$$T = \{\tilde{p}(t^{-1}) \mid p(t) \in S\} \subseteq A[t^{-1}].$$

The universal property of localization gives canonical, injective ring homomorphisms

$$(10) \quad A[t]_T \rightarrow A[[t]] \quad \text{and} \quad A[t^{-1}]_T \rightarrow A[t]_S \rightarrow A(t),$$

where $A[[t]]$ is the ring of formal power series over A and $A(t)$ is the localization of $A[t]$ at the set of all central non-zero-divisors.

The first of these homomorphisms is induced by the inclusion $A[t] \rightarrow A[[t]]$. The second is induced by the canonical inclusion $A[t^{-1}] \rightarrow A[t]_S$ as subsets of $A(t)$. Formally, it rewrites each polynomial $q(t^{-1})$ as the quotient $(t^n q(t^{-1}))/t^n$, where n is the degree of q , since then the numerator is in $A[t]$ and the denominator is in S . Each of the elements $\tilde{p}(t^{-1})$ in the subset T is sent to $p(t)/t^n$, which is invertible in $A[t]_S$ since both $p(t)$ and t^n are in S .

If A is a field, then $A(t)$ is the field of rational functions, and in this case if S consists of all monic polynomials, then the homomorphism $A[t]_S \rightarrow A(t)$ is an isomorphism.

For a finitely generated projective (right) A -module P we define

$$P[t] := P \otimes_A A[t] \cong \bigoplus_{i=0}^{\infty} P, \quad P[[t]] := P \otimes_A A[[t]] \cong \prod_{i=0}^{\infty} P.$$

Note that the second isomorphism holds because P is a summand of a finite free module.

3.2. Higher endomorphism torsion. Recall from Example 2.3 that End_A^S is the category of all endomorphisms of finitely generated projective A -modules that are S -torsion. If (P, f) is an object of End_A^S then there is a short exact sequence of $A[t]$ -modules

$$(11) \quad 0 \rightarrow P[t] \xrightarrow{t-f} P[t] \rightarrow P \rightarrow 0,$$

called the *characteristic sequence* of P . Here $t - f$ is shorthand for $1 \otimes t - f \otimes 1$. The action of t on the first two terms is by $\cdot t$, and on the third term is by f . Let

$$\tau_f(t) : P[t]_S \xrightarrow[\cong]{t-f} P[t]_S$$

denote the localization of the first map. Note that $\tau_f(t)$ is an isomorphism, by the assumption that $P_S = S^{-1}P$ vanishes.

The rule $(P, f) \mapsto (P[t]_S, \tau_f(t))$ defines an exact functor

$$\mathrm{End}_A^S \rightarrow \mathrm{Aut}_{A[t]_S}$$

and therefore a map of K -theory spaces $K(\mathrm{End}_A^S) \rightarrow K(\mathrm{Aut}_{A[t]_S})$. We compose with the coassembly map (7) to obtain an *extended endomorphism torsion* map

$$(12) \quad \underline{\tau}^{\mathrm{End}}(t): K(\mathrm{End}_A^S) \rightarrow LK(A[t]_S).$$

Since $K(A[t]_S)$ is an H-space, its free loop space canonically splits as

$$LK(A[t]_S) \simeq \Omega K(A[t]_S) \times K(A[t]_S).$$

Composing $\underline{\tau}^{\mathrm{End}}(t)$ with the projection to $\Omega K(A[t]_S)$, we obtain the *endomorphism torsion* map

$$(13) \quad \tau^{\mathrm{End}}(t): K(\mathrm{End}_A^S) \rightarrow \Omega K(A[t]_S).$$

This is the map (2) from the introduction.

Notation 3.1. When the indeterminate t or the decoration ‘End’ are understood, we drop them from that notation, writing

$$\tau^{\mathrm{End}} = \tau^{\mathrm{End}}(t) = \tau(t) = \tau.$$

On path components, τ defines a homomorphism

$$\tau_*: K_0(\mathrm{End}_A) \rightarrow K_1(A[t]_S).$$

It sends the object (P, f) to the class in $K_1(A[t]_S)$ of the automorphism $t - f$. This can be viewed as the Reidemeister torsion of the short chain complex $P[t] \xrightarrow{t-f} P[t]$ with respect to extension $P[t] \rightarrow P[t]_S$. This chain complex is a resolution of the $A[t]$ -module P by free modules, so we can think of this as the Reidemeister torsion of the pair (P, f) .

On higher homotopy groups, τ defines a homomorphism

$$\tau_*: K_n(\mathrm{End}_A) \rightarrow K_{n+1}(A[t]_S).$$

In Appendices A and B, we recall how localization theorems in K -theory provide a short exact sequence

$$0 \rightarrow K_{n+1}(A[t]) \rightarrow K_{n+1}(A[t]_S) \xrightarrow{\partial} K_n(\mathrm{End}_A) \rightarrow 0.$$

We then prove in Theorem B.1 that τ_* gives a section of ∂ . As a result,

Proposition 3.2. *τ_* is always split injective.*

In fact, more is true: the space $\Omega K(A[t]_S)$ splits into a product, and τ is the inclusion of one of the two factors. See Theorems A.1 and B.1.

3.3. Examples. Let A be a field and let $S \subset A[t]$ consist of all monic polynomials. Then $A[t]_S = A(t)$ is the field of rational functions. There is an isomorphism

$$(14) \quad \bigoplus_p K_*(A[t]/(p)) \rightarrow K_*(\text{End}_A^S) = K_*(\text{End}_A),$$

where $p := p(t)$ ranges over the irreducible monic polynomials [27], [14, p. 233], [11, th. 1.4], [24, th. 2.3]. The restriction to the p -th summand is the map $K_*(A[t]/(p)) \rightarrow K_*(\text{End}_A)$ induced by the exact functor which sends a finite dimensional vector space V over $A[t]/(p)$ to the pair $(V, t \cdot)$.

Fix an irreducible monic polynomial p , and a nontrivial unit

$$u \in (A[t]/(p))^\times$$

in the field $A[t]/(p)$. Since K_1 of a field is the group of units, we can think of u as a (nonzero) class in $K_1(A[t]/(p))$. It is represented by the vector space $V = A[t]/(p)$ and the automorphism of V that multiplies by u . Applying the isomorphism (14), we get a (nonzero) class

$$\hat{u} \in K_1(\text{End}_A).$$

As a result of Proposition 3.2,

Proposition 3.3. *The element $\tau_*(\hat{u}) \in K_2(A(t))$ is non-trivial.*

Note that a class in K_1 of endomorphisms can sometimes be represented by an object, an endomorphism, and an automorphism that commutes with it; in this case, the object is the A -vector space $V = A[t]/(p)$, the endomorphism is $t \cdot$, and the automorphism is $u \cdot$.

For an example in higher dimensions, let $A = \mathbb{Q}$, and $p(t) \in \mathbb{Q}[t]$ be an irreducible monic polynomial. Then $\mathbb{Q}[t]/(p)$ is a number field. Let r_1 denote its number of distinct embeddings into $\mathbb{R}$ and let r_2 its number of distinct conjugate pairs of embeddings into $\mathbb{C}$. By a theorem of Borel [3], for $n \geq 2$ there is an isomorphism of vector spaces

$$K_n(\mathbb{Q}[t]/(p)) \otimes \mathbb{Q} \cong \begin{cases} 0 & n \text{ even;} \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \pmod{4}; \\ \mathbb{Q}^{r_2} & n \equiv 3 \pmod{4}. \end{cases}$$

For example, let $p(t) = t^2 + bt + c$ have negative discriminant $D := b^2 - 4c$. Then $\mathbb{Q}[t]/(p) = \mathbb{Q}(\sqrt{D})$ is a quadratic number field with $r_1 = 0$ and $r_2 = 1$. Hence, the abelian group $K_{2n+1}(\mathbb{Q}[t]/(p))$ has rank one for $n \geq 1$. Let $x \in K_{2n+1}(\mathbb{Q}[t]/(p))$ be an element of infinite order and let

$$\hat{x} \in K_{2n+1}(\text{End}_{\mathbb{Q}})$$

be its image with respect to the homomorphism $K_{2n+1}(\mathbb{Q}[t]/(p)) \rightarrow K_{2n+1}(\text{End}_{\mathbb{Q}})$. Then $\hat{x}$ also has infinite order (cf. (14)). By Proposition 3.2, the endomorphism torsion $\tau_*(\hat{x}) \in K_{2n+2}(\mathbb{Q}(t))$ has infinite order as well. This produces examples of nonvanishing torsion in arbitrarily high dimensions.

4. THE HIGHER ZETA FUNCTION

If (P, f) is an object of End_A^S then we can define a different endomorphism of finitely generated projective $A[t]$ -modules

$$(15) \quad P[t] \xrightarrow{1-ft} P[t].$$

Recall the multiplicative subset $T \subset A[t]$ of §3.1.

Lemma 4.1. *After localizing at T , the map $1 - ft$ is an isomorphism.*

Proof. Before localization, the map is injective. To see this just write $P[t]$ as $\oplus_{i=0}^{\infty} P$, and observe that in terms of this decomposition the map is given by the triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ -f & 1 & 0 & \cdots \\ 0 & -f & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Its cokernel then rearranges into the colimit of the maps

$$P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

which is the localization $f^{-1}P$. Under this identification, the $A[t]$ -module structure has t acting by f^{-1} .

Therefore after localizing at T , the map $1 - ft$ is still injective and its cokernel is $T^{-1}f^{-1}P$. So it suffices to show that some polynomial in T annihilates $f^{-1}P$. By assumption some $p(t) \in S$ annihilates P , so we take $\tilde{p}(t) = t^n p(t^{-1}) \in T$. The action of this polynomial on $f^{-1}P$ is by $\tilde{p}(f^{-1}) = f^{-n}p(f)$, which vanishes because $p(f)$ vanishes on P . $\square$

Let

$$\zeta_f(t) = (1 - ft)^{-1}: P[t]_T \xrightarrow{\cong} P[t]_T$$

denote the inverse map. Again the rule $(P, f) \mapsto (P[t]_T, \zeta_f(t))$ defines an exact functor

$$\text{End}_A^S \rightarrow \text{Aut}_{A[t]_T}$$

and therefore a map of K -theory spaces $K(\text{End}_A^S) \rightarrow K(\text{Aut}_{A[t]_T})$. We compose with the coassembly map (7) to obtain the *extended K -theoretic zeta function*

$$\underline{\zeta}(t): K(\text{End}_A^S) \rightarrow LK(A[t]_T).$$

Projecting from the free loop space to the based loop space gives the *K -theoretic zeta function*

$$(16) \quad \zeta(t): K(\text{End}_A^S) \rightarrow \Omega K(A[t]_T).$$

The operation $t \mapsto t^{-1}$ defines an involution $A(t) \rightarrow A(t)$ which restricts to a ring isomorphism $A[t]_T \rightarrow A[t^{-1}]_T$. As $t \in S$, we also have the canonical inclusion $A[t^{-1}]_T \rightarrow A[t]_S$ from §3.1. Assembling these, we obtain two more variants of the zeta function that we call $\underline{\zeta}(t^{-1})$ and $\zeta(t^{-1})$:

$$(17) \quad \begin{array}{ccccccc} & & \xrightarrow{\underline{\zeta}(t^{-1})} & & & & \\ & & \text{---} & & \text{---} & & \\ K(\text{End}_A^S) & \xrightarrow{\underline{\zeta}(t)} & LK(A[t]_T) & \xrightarrow{t \mapsto t^{-1}} & LK(A[t^{-1}]_T) & \longrightarrow & LK(A[t]_S) \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ K(\text{End}_A^S) & \xrightarrow{\zeta(t)} & \Omega K(A[t]_T) & \xrightarrow{t \mapsto t^{-1}} & \Omega K(A[t^{-1}]_T) & \longrightarrow & \Omega K(A[t]_S) \\ & & \xrightarrow{\zeta(t^{-1})} & & & & \end{array}$$

Remark 4.2. The term “zeta function” is motivated by the observation that if we extend scalars along the embedding $A[t]_T \rightarrow A[[t]]$ we get the expression

$$\zeta_f(t) = \sum_{k=0}^{\infty} f^k t^k : P[[t]] \xrightarrow{\cong} P[[t]],$$

where $f^k t^k$ is shorthand for $f^{\circ k} \otimes t^k$. Note that this further variant requires no torsion assumptions, defining maps

$$\begin{aligned} \underline{\zeta}(t) &: K(\text{End}_A) \rightarrow LK(A[[t]]), \\ \zeta(t) &: K(\text{End}_A) \rightarrow \Omega K(A[[t]]). \end{aligned}$$

The zeta function is therefore more general than endomorphism torsion.

Remark 4.3. The zeta function is related to the topological de Rham-Witt complex, studied by Hesselholt [17], Betley and Schlichtkrull [1] and Lindenstrauss and McCarthy [21]. In particular, the latter authors examine the map

$$\tilde{\zeta}(t): \tilde{K}(\text{End}_A) \rightarrow \Omega K(A[[t]]),$$

where $\tilde{K}(\text{End}_A)$ is the homotopy fiber of the forgetful map $K(\text{End}_A) \rightarrow K(A)$. Generalizing this to coefficients, this induces an equivalence of Taylor towers, whose homotopy limits are $W(A) = \text{TR}(A)$. We therefore get a commutative diagram up to homotopy

$$\begin{array}{ccc} \tilde{K}(\text{End}_A) & \xrightarrow{\zeta} & \Omega K(A[[t]]) \\ & \searrow \text{trc} & \swarrow \text{"det"} \\ & \text{TR}(A) & \end{array}$$

where the lower-left map is the trace to TR as in [4, 21]. However there doesn't appear to be an interesting lift of Milnor's identity to this setting, because we don't have an obvious ring map from $A[t]_S$ or $A[t^{-1}]_S$ to $A[[t]]$.

5. A GENERALIZATION OF MILNOR'S IDENTITY

We are now in a position to generalize Milnor's identity on the level of K -theory. Our formulation will use the extended torsion $\underline{\tau}(t)$ from (12), the extended zeta function $\underline{\zeta}(t^{-1})$ from (17), and an extended zero-torsion map $\underline{\chi}(t)$, defined by composing $\underline{\tau}(t)$ with the maps

$$K(\text{End}_A^S) \longrightarrow K(A) \longrightarrow K(\text{End}_A^S)$$

sending (P, f) to P , and then sending P to $(P, 0)$. We therefore get a commuting diagram

$$(18) \quad \begin{array}{ccccccc} & & \underline{\chi}(t) & & & & \\ & \nearrow & & \searrow & & & \\ K(\text{End}_A^S) & \xrightarrow{(P,f) \mapsto P} & K(A) & \xrightarrow{P \mapsto (P,0)} & K(\text{End}_A^S) & \xrightarrow{\underline{\tau}(t)} & LK(A[t]_S) \\ & \searrow & & \nearrow & \searrow & \downarrow & \\ & & \underline{\chi}(t) & & \tau(t) & & \Omega K(A[t]_S) \end{array}$$

For the following theorem, recall our assumptions that A is an associative unital ring, $S \subset A[t]$ is a multiplicative subset of central monic polynomials, and $t \in S$. Recall that $*$ denotes the operation $LX \times_X LX \rightarrow X$ that composes loops with the same basepoint.

Theorem 5.1. *The three maps*

$$\underline{\zeta}(t^{-1}), \underline{\tau}(t), \underline{\chi}(t): K(\text{End}_A^S) \rightarrow LK(A[t]_S)$$

all agree (up to canonical homotopy) when projected down to $K(A[t]_S)$. Therefore their loop composites $$ can be defined. Furthermore, there is*

a homotopy

$$\underline{\zeta}(t^{-1}) * \underline{\tau}(t) \sim \underline{\chi}(t).$$

Proof. The following diagram commutes by the naturality of the coassembly map.

$$(19) \quad \begin{array}{ccc} K(\mathrm{End}_A^S) & & \\ \downarrow (P,f) \mapsto (P[t]_T, 1-ft) & & \\ K(\mathrm{Aut}_{A[t]_T}) & \xrightarrow{c} & LK(A[t]_T) \\ \downarrow t \mapsto t^{-1} & & \downarrow t \mapsto t^{-1} \\ K(\mathrm{Aut}_{A[t^{-1}]_T}) & \xrightarrow{c} & LK(A[t^{-1}]_T) \\ \downarrow & & \downarrow \\ K(\mathrm{Aut}_{A[t]_S}) & \xrightarrow{c} & LK(A[t]_S) \end{array}$$

The map $\underline{\zeta}(t^{-1})$ is described in (17) as the right-hand route through this diagram. Therefore, it is equal to the left-hand route, in which the coassembly map is applied last. The composite along the left-hand column is induced by the exact functor that sends (P, f) to

$$\begin{aligned} P[t^{-1}]_T \otimes_{A[t^{-1}]_T} A[t]_S &\cong (P \otimes_A A[t^{-1}]_T) \otimes_{A[t^{-1}]_T} A[t]_S \\ &\cong P \otimes_A A[t]_S \\ &= P[t]_S. \end{aligned}$$

It is equipped with an endomorphism as well, the inverse of:

$$P[t]_T \otimes_{A[t^{-1}]_T} A[t]_S \xrightarrow{(1-ft^{-1}) \otimes 1} P[t]_T \otimes_{A[t^{-1}]_T} A[t]_S.$$

As an endomorphism of $P[t]_S$, this map is identified with

$$P[t]_S \xrightarrow{(t-f) \cdot t^{-1}} P[t]_S,$$

and we know that $(t-f)$ is invertible in this localization, so this inverse endomorphism can be expressed as $(t-f)^{-1} \cdot t$. In summary, the exact functor along the left-hand column of (19) can be re-expressed up to canonical isomorphism as

$$(P, f) \mapsto (P[t]_S, (t-f)^{-1}t).$$

This modifies the corresponding map on the K -theory spaces up to canonical homotopy.

With this modification, the three maps $\underline{\tau}(t)$, $\underline{\zeta}(t^{-1})$ and $\underline{\chi}(t)$ are all described as maps $K(\mathrm{End}_A^S) \rightarrow K(\mathrm{Aut}_{A[t]_S})$ induced by exact functors,

followed by coassembly. After applying the forgetful map $K(\text{Aut}_{A[t]_S}) \rightarrow K(A[t]_S)$, all three of the exact functors send (P, f) to $P[t]_S$. So we get a commuting diagram

$$\begin{array}{ccccc} K(\text{End}_A^S) & \xrightarrow{\cong} & K(\text{Aut}_{A[t]_S}) & \xrightarrow{c} & LK(A[t]_S) \\ & \searrow (P, f) \mapsto P[t]_S & \downarrow \text{forget} & & \downarrow \\ & & K(A[t]_S) & \xlongequal{\quad} & K(A[t]_S) \end{array}$$

This proves that the loop composition of any two of these maps is well-defined. In fact, as a result of Lemma 2.8, we can evaluate this loop product after coassembly, by taking the composition product in $K(\text{Aut}_{A[t]_S})$ before coassembly. So now it is enough to prove that the composition product of the functors

$$(P, f) \mapsto (P[t]_S, (t - f)), \quad (P, f) \mapsto (P[t]_S, (t - f)^{-1}t)$$

gives the functor

$$(P, f) \mapsto (P[t]_S, t).$$

In other words, we need to show that the self-maps of $P[t]_S$ given by $(t - f)$ and by $(t - f)^{-1}t$ compose to give the self-map t . This is clearly the case, and that finishes the proof. $\square$

Projecting from free loops to based loops, this proves Theorem A from the introduction:

Corollary 5.2. *There is a homotopy*

$$\zeta(t^{-1}) * \tau(t) \sim \chi(t) \quad \text{as maps} \quad K(\text{End}_A^S) \rightarrow \Omega K(A[t]_S).$$

Taking π_0 gives the formula

$$\zeta(t^{-1}) + \tau(t) = \chi(t) \quad \text{as maps} \quad K_0(\text{End}_A^S) \rightarrow K_1(A[t]_S),$$

where $+$ is the abelian group structure on K_1 . Applying the determinant map $K_1(R) \rightarrow R^\times$ makes our definitions of $\zeta(t^{-1})$ and $\tau(t)$ agree with their original definitions, namely the determinant of $(1 - ft^{-1})^{-1}$ and $t - f$, respectively. We recover Milnor's original identity (1),

$$\zeta(t^{-1})\tau(t) = t^{\chi(P)} \quad \text{as maps} \quad K_0(\text{End}_A^S) \rightarrow (A[t]_S)^\times.$$

Note that Milnor only considers the case when A is a field and S consists of all monic polynomials. In this case the terms of the above formula all lie in the units of the ring of rational functions $A(t)$.

Remark 5.3. The notation $t^{\chi(P)} = \det(\chi(t))$ is slightly misleading because in general, P is not necessarily a free module, so the determinant of $\cdot t$ on $P[t]$ might not be a monomial of the form $t^{\chi(P)}$. However it is

guaranteed to be a monomial as soon as A is an integral domain and therefore has no nontrivial idempotents; see [12].

For a different simplification, we can define the reduced torsion $\tilde{\tau}(t)$ to be the restriction of $\tau(t)$ to the reduced endomorphisms $\tilde{K}(\text{End}_A^S)$. By a short diagram-chase, this is the same as taking the extended torsion $\underline{\tau}(t)$, noticing that it commutes along forgetful maps with the extension of scalars $K(A) \rightarrow K(A[t]_S)$, and concluding that it induces a map $\tilde{\tau}(t)$ on the homotopy fibers:

$$\begin{array}{ccc} \tilde{K}(\text{End}_A^S) & \xrightarrow{\tilde{\tau}} & \Omega K(A[t]_S) \\ \downarrow & & \downarrow \\ K(\text{End}_A^S) & \xrightarrow{\underline{\tau}} & LK(A[t]_S) \\ \downarrow & & \downarrow \\ K(A) & \longrightarrow & K(A[t]_S) \end{array}$$

We define the reduced zeta function $\tilde{\zeta}(t^{-1})$ and reduced zero-torsion $\tilde{\chi}(t)$ in the same way. The formula from Theorem 5.1 and passes to homotopy fibers, giving

$$\tilde{\zeta}(t^{-1}) * \tilde{\tau}(t) \sim \tilde{\chi}(t) \quad \text{as maps} \quad \tilde{K}(\text{End}_A^S) \rightarrow \Omega K(A[t]_S).$$

Moreover, by (18) the zero-torsion map $\chi(t)$ factors through the forgetful functor $K(\text{End}_A) \rightarrow K(A)$, and therefore its reduced version $\tilde{\chi}(t)$ is nullhomotopic. This proves Corollary B from the introduction:

Corollary 5.4. *The sum of the maps $\tilde{\zeta}(t^{-1})$ and $\tilde{\tau}(t)$ is nullhomotopic – the two maps are additive inverses in the homotopy category.*

6. CHAIN COMPLEXES

In this section we describe a chain complex model for $K(\text{End}_A^S)$. We introduce this model in order to construct endomorphism torsion for topological families of endomorphisms in §8. The construction and proof of Milnor’s identity for the chain complex model is essentially identical to the constructions given above, but passing to chain complexes gives us the ability to relate the higher torsion to the boundary map of the K -theory localization sequence; see Appendix B.

For simplicity, we consider chain complexes in non-negative degrees only, though the results of this section also hold for chain complexes in all degrees.

Again let A be an associative unital ring. Let $\text{Ch}(A)$ denote the category of chain complexes of A -modules and chain maps. Let $\text{Ch}^{A\text{-proj}}(A)$ denote the subcategory of degreewise projective complexes. This has

the structure of a Waldhausen category: a cofibration is a levelwise injective map with projective cokernel, and a weak equivalence is a quasi-isomorphism.

Given a map $f: P. \rightarrow Q.$ of A -chain complexes, we let $T(f).$ denote its mapping cylinder. It is characterized by the property that maps $T(f). \rightarrow D.$ correspond to pairs of maps $g: P. \rightarrow D., h: Q. \rightarrow D.$ and a chain homotopy $g \sim hf$. Similarly let $C(f). = T(f)./P.$ be the mapping cone.

More explicitly, if we let $I. = (\mathbb{Z} \xrightarrow{(-1,1)} \mathbb{Z} \oplus \mathbb{Z})$ denote the interval in chain complexes, then

$$T(f). = P. \cup_P (I. \otimes P.) \cup_P Q.$$

$$C(f). = 0 \cup_P (I. \otimes P.) \cup_P Q.$$

where the two copies of $P.$ are included into $I. \otimes P.$ as the two endpoints of the interval.

If $f: P. \rightarrow P.$ is an endomorphism, let $T^+(f).$ denote its mapping telescope and let $T^-(f).$ its reverse mapping telescope. These are also characterized by universal properties: a map $T^+(f) \rightarrow D.$ consists of maps $g^i: P. \rightarrow D.$ for $i \geq 0$ and chain homotopies $g^i \sim fg^{i-1}$. A map $T^-(f) \rightarrow D.$ consists of the same except the maps are indexed over $i \leq 0$. More explicitly, they are given by

$$T^+(f). = P. \cup_P (I. \otimes P.) \cup_P (I. \otimes P.) \cdots$$

$$T^-(f). = \cdots \cup_P (I. \otimes P.) \cup_P (I. \otimes P.) \cup_P P.$$

where each map $I. \otimes P. \leftarrow P.$ is the inclusion of the right end and each map $P. \rightarrow I. \otimes P.$ is the composition of f and the inclusion of the left end.

Note that the inclusion of the far left end $P. \rightarrow T^+(f).$ is a localization by f . The inclusion of the far right end $P. \rightarrow T^-(f).$ and the projection $T^-(f). \rightarrow P.$ are both quasi-isomorphisms.

Remark 6.1. The reverse mapping telescope $T^-(f)$ has an endomorphism sh^{-1} that shifts one slot to the left. With respect to this $A[t]$ -action the collapse $T^-(f). \rightarrow P.$ is $A[t]$ -linear, in fact it is essentially a rewriting of the characteristic sequence (11) for chain complexes. Similarly, $T^+(f)$ has an endomorphism that shifts one slot to the right, making it a chain complex version of the dual characteristic sequence (15) that we used to define the zeta function.

We say a complex of A -modules is *strictly perfect* if it is bounded (i.e. nonzero in only finitely many degrees) and finitely generated projective over A at each level. A complex is *perfect* if it is quasi-isomorphic to

a strictly perfect complex. This condition is equivalent to being compact in the derived category, so it is preserved by extensions, retracts, kernels, and cokernels. The perfect complexes define a Waldhausen subcategory

$$\mathrm{Ch}_{\mathrm{perf}}(A) := \mathrm{Ch}_{A\text{-perf}}^{A\text{-proj}}(A) \subseteq \mathrm{Ch}^{A\text{-proj}}(A).$$

Next we consider chain complexes with an endomorphism. This is the same thing as an $A[t]$ -chain complex, but a priori, there is more than one way to characterize when such a complex is perfect over A , or is S -torsion. The following lemmas eliminate this ambiguity.

Lemma 6.2. *The following two conditions are equivalent. We say an $A[t]$ -complex P is A -perfect if either one holds.*

- There is a zig-zag of $A[t]$ -linear quasi-isomorphisms to an A -strictly perfect complex.
- There is a zig-zag of A -linear quasi-isomorphisms to an A -strictly perfect complex.

Proof. Clearly the first implies the second. Suppose P is an $A[t]$ -chain complex satisfying the second condition, and let $f: P \rightarrow P$ be the action of the endomorphism $t \in A[t]$. Without loss of generality the zig-zag is a single quasi-isomorphism $g: C \rightarrow P$ where C is a strictly perfect A -chain complex. Lift f along this equivalence to an A -linear endomorphism $f': C \rightarrow C$ such that the following square commutes up to a chosen chain homotopy.

$$\begin{array}{ccc} C & \xrightarrow{f'} & C \\ g \downarrow & & \downarrow g \\ P & \xrightarrow{f} & P \end{array}$$

Let $T^-(f')$ be the reverse mapping telescope as an $A[t]$ -chain complex (Remark 6.1). Let $T^-(f') \rightarrow C$ be the collapse onto the end. Define a map $T^-(f') \rightarrow P$ using the universal property of T^- , sending the i th copy of C to P by $f^{|i|} \circ g$, and filling in the intervals using the chosen chain homotopy for the square above (composed with iterates of f). This gives the desired zig-zag of $A[t]$ -linear quasi-isomorphisms

$$C \xleftarrow{\sim} T^-(f') \xrightarrow{\sim} P.$$

from P to a strictly perfect A -chain complex. □

Lemma 6.3. *If P is A -perfect then it is $A[t]$ -perfect.*

Proof. In light of the previous lemma, without loss of generality P is A -strictly perfect and we must show it admits a quasi-isomorphism from an $A[t]$ -strictly perfect complex. Simply take the reverse mapping telescope $T^-(f). \rightarrow P$; it is obtained from the bicomplex

$$P.[t] \xrightarrow{t-f} P.[t]$$

(cf. (11)) and is therefore $A[t]$ -strictly perfect. $\square$

Now we turn to torsion. Fix a multiplicative central set of monic polynomials $S \subseteq A[t]$. We do not require that $t \in S$.

Lemma 6.4. *For an $A[t]$ -perfect complex P ., the following five conditions are equivalent. We say P is S -torsion if any of them hold.*

- There exists $p(t) \in S$ such that $p(f)$ is equivalent to 0 in the $A[t]$ -derived category.
- There exists $p(t) \in S$ such that $p(f)$ is equivalent to 0 in the A -derived category.
- There exists $p(t) \in S$ such that $p(f) = 0$ on $H_*(P)$.
- There exists $p(t) \in S$ such that $p(f)^{-1}H_*(P) = 0$.
- There exists $p(t) \in S$ such that $p(f)^{-1}P$ is acyclic.

Proof. Clearly each one implies the next. So assume that $p(f)^{-1}P$ is acyclic. Without loss of generality P is $A[t]$ -strictly perfect. The localization $p(f)^{-1}P$ can be modeled as the mapping telescope $T^+(p(f))$., so this telescope is acyclic. Therefore the inclusion $P. \rightarrow T^+(p)$. admits a chain homotopy to zero. This extends the map to $C(0). \rightarrow T^+(p)$., where 0 denotes the inclusion $0. \rightarrow P$ and $C(0).$ is its mapping cone. The mapping cone $C(0).$ is also $A[t]$ -strictly perfect, so this extension factors through a finite stage of the mapping telescope

$$T^n(f). = P. \cup_P (I. \otimes P.) \cdots \cup_P (I. \otimes P.) \cup_P P.$$

This finite stage admits a quasi-isomorphism to P by collapsing onto the end, and the composite $P. \rightarrow T^n. \rightarrow P$ is $p(f)^n$. However this factors through the acyclic complex $C(0).$, so it is zero in the $A[t]$ -derived category. Therefore $p(t)^n \in S$ agrees with the zero endomorphism of P in the $A[t]$ -derived category. $\square$

Lemma 6.5. *If P is $A[t]$ -perfect and S -torsion then P is A -perfect.*

Proof. Let $p(t) \in S$ be a monic central polynomial such that $p(f): P. \rightarrow P$ is zero in the $A[t]$ -derived category. Let $R.$ be the two-term complex $A[t] \xrightarrow{p} A[t]$, whose $A[t]$ -bimodule structure extends in a unique way to the structure of a DGA. Note that $R.$ is equivalent as an A -complex to the one-term complex $A[t]/p \cong A^{\oplus \deg p}$, and therefore $R.$ is perfect

over A . Since P is perfect over $A[t]$, the extension of scalars $P \otimes_{A[t]} R$ is perfect over A as well. This latter complex is isomorphic to the mapping cone $C(p(f))$, so this mapping cone is perfect over A .

Since P is $A[t]$ -projective and $p(f)$ is zero in the $A[t]$ -derived category, $p(f): P \rightarrow P$ is chain null homotopic. Picking a null homotopy defines an $A[t]$ -linear map $C(p(f)) \rightarrow P$ that splits the inclusion $P \rightarrow C(p(f))$ of the back end. Therefore P is an A -linear retract of the A -perfect complex $C(p(f))$, so P is A -perfect as well. $\square$

Corollary 6.6. *If P is S -torsion then it is $A[t]$ -perfect iff it is A -perfect.*

Now that the ambiguity has been removed from the definitions, we explain how these chain complex models are equivalent to the ones of the previous sections. Recall that the functor that sends each module to the corresponding chain complex supported in degree 0 gives an equivalence

$$K(A) = \Omega |iS.P(A)| \xrightarrow{\sim} \Omega |wS.Ch_{A\text{-perf}}^{A\text{-proj}}(A)|.$$

This is by an application of Waldhausen's "sphere theorem" [29, thm. 1.71]. The main hypothesis can be verified by a method described in Appendix C (Proposition C.4). By a standard abuse of notation we let $K(A)$ refer to either of these spaces.

Once we consider endomorphisms and torsion, we have more choices of model, illustrated in the diagram below.

$$\begin{array}{ccccccc} Ch_{A[t]\text{-perf}}^{A[t]\text{-proj}}(A[t]) & \longleftarrow & Ch_{A\text{-perf}}^{A[t]\text{-proj}}(A[t]) & \xrightarrow{\sim} & Ch_{A\text{-perf}}^{A\text{-proj}}(A[t]) & \longleftarrow & \text{End}_A \\ & & & \searrow \sim & & & \\ Ch_{A[t]\text{-perf}}^{A[t]\text{-proj}}(A[t])^S & \xlongequal{\sim} & Ch_{A\text{-perf}}^{A[t]\text{-proj}}(A[t])^S & \xrightarrow{\sim} & Ch_{A\text{-perf}}^{A\text{-proj}}(A[t])^S & \longleftarrow & \text{End}_A^S \end{array}$$

The straight maps are all inclusions. The curved map is functor that takes (P, f) to the reverse mapping telescope $(T^-(f), \text{sh}^{-1})$, as in Remark 6.1 and the proof of Lemma 6.3. The S -superscripts denote the subcategory of S -torsion complexes in the sense of Lemma 6.4.

Proposition 6.7. *Every map marked $\sim$ induces an equivalence on K -theory.*

Proof. The leftmost equivalence is an equality of categories by Corollary 6.6. The arrows in the center are inverses up to equivalence by the proof of Lemma 6.3. Alternatively, the straight arrows are equivalences using the approximation theorem. The remaining inclusions on

the right-hand side are by Waldhausen's sphere theorem [29, thm. 1.71]. In each case the main hypothesis of the theorem is not obvious because the modules are not finitely generated projective over $A[t]$, but the hypothesis does hold (Propositions C.7 and C.8). $\square$

Corollary 6.8. *The exact functor sending (P, f) to the $A[t]$ -complex $P[t] \xrightarrow{t-f} P[t]$ induces an equivalence*

$$K(\text{End}_A^S) \xrightarrow{\sim} K(\text{Ch}_{\text{perf}}(A[t])^S).$$

The construction of higher torsion τ and the zeta function ζ can be defined as in the previous sections, only taking P to be an A -perfect complex with an endomorphism, instead of a finitely generated projective A -module with an endomorphism.

Theorem 6.9. *The definitions of $\underline{\tau}$ and $\underline{\zeta}$ extend in a natural way along the inclusions of categories*

$$\text{End}_A^S \subseteq \text{Ch}_{A\text{-perf}}^{A\text{-proj}}(A[t])^S, \quad P(A[t]_S) \subseteq \text{Ch}_{\text{perf}}(A[t]_S)$$

and the composition

$$K(\text{Ch}_{A\text{-perf}}^{A\text{-proj}}(A[t])^S) \xrightarrow{\underline{\zeta}(t^{-1}) * \underline{\tau}(t)} LK(\text{Ch}_{\text{perf}}(A[t])_S)$$

is homotopic to the map $\underline{\chi}(t)$.

Proof. The most significant change in defining the maps is that the map $t - f$, respectively $1 - ft$, is a quasi-isomorphism rather than an isomorphism, because its cofiber is acyclic, not identically zero. When proving this for $1 - ft$, it is illuminating to refer to $f^{-1}P$ as the extension of scalars $P \otimes_{A[t]} A[t, t^{-1}]$, and then to observe that inverting $p(f)$ and $\tilde{p}(f^{-1})$ are the same localization. Then we define the automorphism category to consist of chain complexes and self-quasi-isomorphisms.

As a result, $(1 - ft)$ is not an isomorphism, so it can't be inverted. To correct for this, instead of defining ζ using the map $(1 - ft)^{-1}$, we define it using $(1 - ft)$ and at the very end apply the flip map to the free loop space to accomplish the inversion.

It is possible to modify the proof of the identity accommodate this weaker setup, but it is faster to deduce the identity directly from Theorem 5.1, since all the K -theory spaces involved are equivalent to those appearing in Theorem 5.1. $\square$

7. THE NON-LINEAR SETTING

For a space X , we define a Waldhausen category End_X . We construct a linearization map on K -theory $K(\text{End}_X) \rightarrow K(\text{End}_{\mathbb{Z}[\pi]})$, where $\pi = \pi_1(X)$.

For a fixed ring homomorphism $\mathbb{Z}[\pi] \rightarrow A$ and any multiplicative system of monic centric polynomials $S \subset A[t]$, we associate a full subcategory $\text{End}_X^S \subset \text{End}_X$ of S -torsion objects. We then exhibit a linearization map $K(\text{End}_X^S) \rightarrow K(\text{End}_A^S)$.

7.1. Endomorphisms of retractive spaces. For each topological space X , let $T(X)$ be the category of retractive spaces Y over X . There is a Quillen model category structure on $T(X)$ in which the weak equivalences are weak homotopy equivalences and the cofibrations are the retracts of relative cell complexes. This makes the subcategory $R(X) \subset T(X)$ of cofibrant objects into a Waldhausen category.

An object $Y \in T(X)$ is *finite* if it is built up from the zero object by a finite number of cell attachments. It is *homotopy finite* if it is weakly equivalent to a finite object, and *finitely dominated* if it is a retract of a homotopy finite object.

Let $R_{\text{fd}}(X) \subset R(X)$ be the full subcategory on the finitely dominated objects. Then $R_{\text{fd}}(X)$ together with the above cofibrations $\text{co}R_{\text{fd}}(X)$ and weak equivalences $vR_{\text{fd}}(X)$ forms a Waldhausen category.

Let $\text{End}_X = \text{End}(R_{\text{fd}}(X))$ denote the category of endomorphisms of objects in $R_{\text{fd}}(X)$. As before, this inherits a Waldhausen structure in which a weak equivalence or cofibration is determined by forgetting the endomorphisms.

7.2. The T construction. We will define a linearization map back to $\text{End}_{\mathbb{Z}[\pi_1(X)]}$, but for this we need a variant of the S construction due to Thomason [29, p. 334]. To each a Waldhausen category $(C, \text{co}C, wC)$ we may associate a simplicial category $wT_k C$. The objects of $wT_k C$ are sequences of cofibrations

$$A_\bullet := A_0 \rightharpoonup A_1 \rightharpoonup \cdots \rightharpoonup A_k.$$

A morphism $A_\bullet \rightarrow B_\bullet$ consists of maps $A_i \rightarrow B_i$ for $0 \leq i \leq k$ such that each of the squares

$$\begin{array}{ccc} A_i & \rightharpoonup & A_{i+1} \\ \downarrow & & \downarrow \\ B_i & \rightharpoonup & B_{i+1} \end{array}$$

is a homotopy cocartesian square, i.e. the square commutes and the map $B_i \cup_{A_i} A_{i+1} \rightarrow B_{i+1}$ is a weak equivalence. The i -th face operator drops A_i from the sequence and the i -th degeneracy operator inserts

the identity map. There is a chain of homotopy equivalences

$$wT.C \xleftarrow{\sim} wT^+C \xrightarrow{\sim} wS.C$$

where the middle simplicial category is defined in a way similar to $wT.C$ but where quotient data is included. The left equivalence is given by forgetting quotient data. The right-hand map is given by mapping the sequence $A_\bullet$ to the sequence $A_1/A_0 \rightarrow A_2/A_0 \rightarrow \dots$. It is an equivalence by the argument in [29, p. 334].

7.3. Linearization. Now assume that X is based, path-connected and has a universal cover $\tilde{X} \rightarrow X$. Let $\pi = \pi_1(X)$ be the fundamental group of X . Under these assumptions we define a linearization map

$$\mathcal{L}: \text{End}_X \rightarrow \text{End}_{\mathbb{Z}[\pi]}.$$

For each $Y \in R_{\text{fd}}(X)$, let $\tilde{Y}$ denote the pullback

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & X. \end{array}$$

Then the reduced singular chain complex $C.(\tilde{Y}, \tilde{X})$ is a perfect $\mathbb{Z}[\pi]$ -chain complex. The functor $\mathcal{L}$ is then the operation that assigns (Y, f) to $(C.(\tilde{Y}, \tilde{X}), f_*)$.

The functor $\mathcal{L}$ preserves cofibrations and weak equivalences but does not preserve pushouts, so it does not induce a map on the S . construction. However $\mathcal{L}$ preserves homotopy cocartesian squares, hence it induces a map of simplicial categories

$$wT.\text{End}_X \rightarrow wT.\text{End}_{\mathbb{Z}[\pi]}$$

Taking realization and loops, we obtain the linearization map

$$\mathcal{L}: K(\text{End}_X) \rightarrow K(\text{End}_{\mathbb{Z}[\pi]}).$$

7.4. Torsion endomorphisms. Let $\mathbb{Z}[\pi] \rightarrow A$ be any ring homomorphism and let $S \subset A[t]$ be a multiplicative subset of monic centric polynomials. To define S -torsion for endomorphisms of retractive spaces we simply apply linearization.

Definition 7.1. An object $(Y, f) \in \text{End}_X$ is S -torsion if $\mathcal{L}(Y, f) \in \text{End}_{\mathbb{Z}[\pi]}$ becomes, after extending scalars to A , an S -torsion endomorphism of chain complexes in the sense of Lemma 6.4. Let

$$\text{End}_X^S \subset \text{End}_X$$

be the full subcategory consisting of the S -torsion objects.

It is readily verified that End_X^S inherits the structure of a Waldhausen category. Then linearization followed by extension of scalars induces a functor

$$\mathcal{L}: \text{End}_X^S \rightarrow \text{End}_A^S,$$

and as above this induces a map on K -theory,

$$(20) \quad \mathcal{L}: K(\text{End}_X^S) \rightarrow K(\text{End}_A^S).$$

Remark 7.2. Our definition of End_X^S differs from that of Levikov [20], who defines S -torsion at the level of suspension spectra rather than at the chain level.

We define the endomorphism torsion τ and zeta function ζ for any object $(Y, f) \in \text{End}_X^S$ by applying linearization (20). These give classes in $\Omega K(A[t]_S)$ satisfying the same identity as before. When A is a field, we recover the original form of Milnor's identity [25, p. 123].

8. FAMILIES OF ENDOMORPHISMS

In this section we give a few examples of higher classes in endomorphism K -theory $K(\text{End}_X^S)$, which by the previous section produce higher classes in $K(\text{End}_A^S)$ and therefore have higher endomorphism torsion.

8.1. Higher torsion for families. The essential idea is that to give a map $B \rightarrow K(\text{End}_X)$, where B is some topological space, it is enough to give a map

$$B \rightarrow |w.\text{End}_X|,$$

and such maps correspond to fibrations of the following form.

Definition 8.1. A B -family of endomorphisms over X is a retractive space E over $B \times X$, and an endomorphism $f: E \rightarrow E$ of retractive spaces, such that the projection $p: E \rightarrow B$ is a fibration and each fiber is a finitely dominated retractive space over X . We sometimes abbreviate this by (p, f) or just E , but the other data is understood.

Proposition 8.2. *Such families, up to the evident notion of weak equivalence, correspond to homotopy classes of maps $B \rightarrow |w.\text{End}_X|$.*

We omit the proof, which is by classical arguments as in [23]. However we describe the correspondence itself. Given a map to $|w.\text{End}_X|$, we pull back the universal family $U \rightarrow |w.\text{End}_X|$ to produce a family $E \rightarrow B$. The space U is defined by taking the two-sided bar construction $U = B(\iota, w\text{End}_X, *)$ where $\iota: w\text{End}_X \rightarrow \text{End}_X$ is the

functor taking every retractive space Y over X to itself, and the projection to $|w.\text{End}_X|$ is given by the evident map $B(\iota, w\text{End}_X, *) \rightarrow B(*, w\text{End}_X, *)$.

By the proof of Quillen's Theorem B, the projection $U \rightarrow |w.\text{End}_X|$ is a quasifibration, every fiber of which is a retractive space over X with an endomorphism. In other words, once we replace the map to $X \times |w.\text{End}_X|$ by a fibration, it is a $|w.\text{End}_X|$ -family. Then this is the universal family in the sense that any other family is the pullback of this one along a map $B \rightarrow |w.\text{End}_X|$.

Remark 8.3. We can also give an explicit inverse to this operation. Given a B -family of endomorphisms, we can regard it as a point in the classifying space of all B -families up to weak equivalence. This classifying space, as a functor of B , is a contravariant homotopy functor and therefore has a coassembly map, see e.g. [5, §5] and [22, §5]. This coassembly map sends the given family back to a map $B \rightarrow |w.\text{End}_X|$. Using the universal property of coassembly, we identify this as the inverse to the above correspondence.

Let S a multiplicative subset of monic polynomials in $A[T]$ that contains t , and let T be the associated multiplicative subset of renormalized polynomials as in §3.1. Then for each B -family of endomorphisms $E \rightarrow B$, we call the resulting compositions

$$B \longrightarrow K(\text{End}_X^S) \xrightarrow{\mathcal{L}_A} K(\text{End}_A^S) \begin{array}{l} \xrightarrow{\tau} \Omega K(A[t]_S) \\ \searrow \zeta \Omega K(A[t]_T) \end{array}$$

the *higher endomorphism torsion* $\tau(E)$ and *higher zeta function* $\zeta(E)$ of the family.

8.2. Examples. We give a few examples of higher endomorphism torsion where the base space $B = S^1$ is the circle and $X = *$ is a point. In what follows, we take $A = \mathbb{Q}$ and $S \subset A[t]$ to be the multiplicative subset consisting of all monic polynomials. Then $A[t]_T = \mathbb{Q}(t)$ is the ring of rational functions. In this case the interesting part of the endomorphism torsion lies in the group $K_2(\mathbb{Q}(t))$. To give a map $S^1 \rightarrow |w.\text{End}_*^S|$ it suffices to select a based space Y with two commuting self-maps f and θ , such that θ is a weak equivalence.

Example 8.4. Let $n \geq 1$ and consider any based self-equivalence

$$\theta: S^n \vee S^n \rightarrow S^n \vee S^n$$

whose action on n th homology is the invertible matrix

$$(21) \quad Q = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Set $f = \theta$, so that f and θ trivially commute. This determines the S^1 -family of endomorphisms

$$p: E \rightarrow S^1, \quad f: E \rightarrow E.$$

To be more precise, E is the mapping torus of θ , made into a fibration with fiber $S^n \vee S^n$. In fact it is a non-trivial homology circle with $\pi_1(E) \cong \mathbb{Z}$ (when $n \geq 2$). Let u denote the resulting homotopy class

$$u: S^1 \rightarrow K(\text{End}_*).$$

An unraveling of the construction shows that the composition

$$S^1 \xrightarrow{u} K(\text{End}_*) \xrightarrow{L_{\mathbb{Q}}} K(\text{End}_{\mathbb{Q}}),$$

where $L_{\mathbb{Q}}$ is the linearization map of §7.3, is homotopic to the class represented by the loop

$$(22) \quad \theta_*: (\tilde{C} \cdot (S^n \vee S^n), f_*) \xrightarrow{\cong} (\tilde{C} \cdot (S^n \vee S^n), f_*),$$

where $\tilde{C} \cdot (S^n \vee S^n)$ is the reduced total singular complex over $\mathbb{Q}$ of $S^n \vee S^n$. We then simplify this by the zig-zag of quasi-isomorphisms

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{C}_{n+1}(S^n \vee S^n) & \xleftarrow{=} & \tilde{C}_{n+1}(S^n \vee S^n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{C}_n(S^n \vee S^n) & \xleftarrow{\quad} & \tilde{Z}_n(S^n \vee S^n) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{C}_{n-1}(S^n \vee S^n) & \xleftarrow{\quad} & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

to the chain complex $(\mathbb{Z} \oplus \mathbb{Z})[n]$ with θ_*, f_* acting by the matrix (21). Alternatively, we can write this chain complex as $(\mathbb{Q}[t]/(t^2 - t + 1))[n]$ with θ_*, f_* acting by multiplication by t . The resulting class $\theta_* \in K_1(\text{End}_{\mathbb{Q}})$ is therefore the non-trivial unit

$$(-1)^n t \in (\mathbb{Q}[t]/(t^2 - t + 1))^{\times} \cong K_1(\mathbb{Q}[t]/(t^2 - t + 1))$$

by §3.3. By Proposition 3.3, we infer that the loop (22) is non-trivial, and conclude the following.

Theorem 8.5. *The endomorphism torsion invariant of the above pair (p, f) is non-trivial in $K_2(\mathbb{Q}(t))$.*

Example 8.6. Another example is given by the matrix

$$R := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this case the associated fibration $p: E \rightarrow S^1$ with fiber $S^n \vee S^n$ is a rational homology circle:

$$H_*(E) \cong \begin{cases} \mathbb{Z} & * = 0, 1, \\ \mathbb{Z}/2 & * = n, \\ 0 & \text{otherwise.} \end{cases}$$

As in the previous example, we use the matrix R for both the clutching data θ and the fiberwise endomorphism f . The unit is then $t \in \mathbb{Q}[t]/(t^2 + 1)$, giving rise to a non-trivial element of $K_2(\mathbb{Q}(t))$ which is the endomorphism torsion of (p, f) .

In this example, the fibration can be chosen as a fiber bundle with monodromy $\theta: S^n \vee S^n \rightarrow S^n \vee S^n$ defined by $\theta(x, y) = (r(y), x)$, in which $r: S^n \rightarrow S^n$ is the reflection $r(y_0, \dots, y_n) = (-y_0, \dots, y_n)$. Observe that θ is 4-periodic.

Example 8.7. Consider the trefoil knot $K: S^1 \subset S^3$. This is a fibered knot, so one has a smooth fiber bundle

$$p: E \rightarrow S^1$$

where E is the knot complement and the fiber V , a Seifert surface for K , is a torus with an open disk removed. The geometric monodromy $\theta: V \rightarrow V$ is a diffeomorphism which restricts to the identity on $\partial V = S^1$. Moreover, θ is 6-periodic [13]. With respect to a suitable choice of basis for $H_1(V; \mathbb{Z})$, the homological monodromy matrix is given by (21). Since θ commutes with itself, it induces a fiberwise diffeomorphism $f: E \rightarrow E$ that covers the identity map of S^1 . Again, the endomorphism torsion $\tau(p, f) \in K_2(\mathbb{Q}(t))$ is non-trivial, by the argument appearing in Example 8.4.

Example 8.8. There is a variant of the Example 8.6 in which the fibration $E \rightarrow S^1$ is a smooth fiber bundle over the circle with fiber $S^n \times S^n$ and monodromy map $\theta: S^n \times S^n \rightarrow S^n \times S^n$, defined by $\theta(x, y) = (r(y), x)$. The endomorphism torsion invariant is again non-trivial; we omit the details.

8.3. Higher dimensional examples. We briefly sketch how one can indirectly construct higher dimensional examples with non-trivial endomorphism torsion in $K_*(\mathbb{Q}(t))$ in every even degree ≥ 4 . The details of this construction depend on a recent result due to Andrew Salch and

go well beyond the scope of the current work, and we plan to publish the details elsewhere.

Let $\mathbb{S}$ be the sphere spectrum. Let $\mathbb{S}[i]$ be a *Gaussian sphere*, i.e. a structured (A_∞) , connective $\mathbb{S}$ -algebra such that

- The ring $\pi_0(\mathbb{S}[i])$ is isomorphic to the Gaussian integers $\mathbb{Z}[i]$;
- the homology $H_*(\mathbb{S}[i]; \mathbb{Z})$ is concentrated in degree zero;
- $\mathbb{S}[i]$ is weakly equivalent to $\mathbb{S} \vee \mathbb{S}$ as an $\mathbb{S}$ -module.

Such an $\mathbb{S}$ -algebra exists [28] but we assume this without proof.

For a structured ring spectrum R we let $M_k(R)$ be the derived R -module endomorphisms of a wedge of k -copies of R . Then $\mathrm{GL}_k(R)$ is defined by as a pullback

$$\begin{array}{ccc} \mathrm{GL}_k(R) & \longrightarrow & M_k(R) \\ \downarrow & & \downarrow \\ \mathrm{GL}_k(\pi_0(R)) & \longrightarrow & M_k(\pi_0(R)) \end{array}$$

i.e., those endomorphisms which are invertible up to homotopy. Then $\mathrm{GL}_k(R)$ is a topological monoid and one has stabilization maps $\mathrm{GL}_k(R) \rightarrow \mathrm{GL}_{k+1}(R)$. Let $B\mathrm{GL}(R)$ be the colimit of $B\mathrm{GL}_k(R)$ under stabilization. The algebraic K -theory of R is

$$K(R) = K_0(\pi_0(R)) \times B\mathrm{GL}(R)^+$$

(cf. [2, p. 67]). When $R = \mathbb{S}[i]$ the map

$$K(\mathbb{S}[i]) \rightarrow K(\mathbb{Z}[i])$$

is a rational homotopy equivalence [19, lem. 2.4].

According to [3], the abelian group $K_{2n+1}(\mathbb{Z}[i]) \cong K_{2n+1}(\mathbb{Q}[i])$ has rank one for $n \geq 1$. Consequently, the abelian group $K_{2n+1}(\mathbb{S}[i])$ has rank one for $n \geq 1$. Represent a non-torsion element $x \in K_{2n+1}(\mathbb{S}[i])$ by a map $S^n \rightarrow B\mathrm{GL}_k(\mathbb{S}[i])^+$ for k sufficiently large, and form the homotopy pullback

$$\begin{array}{ccc} \Sigma & \longrightarrow & B\mathrm{GL}_k(\mathbb{S}[i]) \\ \downarrow & & \downarrow \\ S^{2n+1} & \longrightarrow & B\mathrm{GL}_k(\mathbb{S}[i])^+ . \end{array}$$

Then Σ is a homology $(2n+1)$ -sphere (cf. [16]). The top horizontal map is represented unstably by a fibration $p: E \rightarrow \Sigma$ whose fiber is a wedge $\vee_{2k}(S^j \vee S^j)$ for j sufficiently large and the $\mathbb{S}[i]$ -module structure on $\vee_k \mathbb{S}[i]$ gives rise to a fiberwise endomorphism $f: E \rightarrow E$ which we may heuristically think of as multiplication by i . By construction, the

endomorphism torsion $\tau(E) \in [\Sigma, \Omega K(\mathbb{Q}(t))] = K_{2n+2}(\mathbb{Q}(t))$ is non-trivial by Theorem B.1.

APPENDIX A. FUNDAMENTAL THEOREMS OF ENDOMORPHISM K-THEORY

The following result involves ideas of Quillen [26], Grayson [15] and Waldhausen [29]. We make no claim to originality. As before, A is an associative ring, $S \subseteq A[t]$ is a multiplicative subset of monic centric polynomials containing t , $T \subseteq A[t^{-1}]$ is the subset $\{\tilde{p}(t^{-1}) \mid p(t) \in S\}$ defined in §3.1, and $\tilde{K}(\text{End}_A^S)$ the homotopy fiber of the forgetful map $K(\text{End}_A^S) \rightarrow K(A)$.

Theorem A.1. *There are natural homotopy equivalences*

- (1) $\Omega K(A[t]_S) \simeq \Omega K(A[t]) \times K(\text{End}_A^S)$, and
- (2) $\Omega K(A[t^{-1}]_T) \simeq \Omega K(A) \times \tilde{K}(\text{End}_A^S)$.

Remark A.2. Part (1) actually does not require that $t \in S$. We aren't aware of a proof of it in the literature and so we provide a sketch below. The equivalence is in fact the endomorphism torsion map $\tau(t)$ on the second factor (Theorem B.1). Part (2) is due to Grayson [15]; a non-linear version was also proved by Levikov [20]. It appears the equivalence in (2) is the zeta function $\zeta(t^{-1})$ on the second factor, but the proof requires a translation of [15] to Waldhausen categories and goes beyond the scope of this paper.

Taking $S = \{t^n : n \geq 0\}$, part (2) gives

$$(23) \quad \Omega K(A[t]) \simeq \Omega K(A) \times \tilde{K}(\text{Nil}_A),$$

where $\text{Nil}_A \subseteq \text{End}_A$ is the full subcategory of nilpotent endomorphisms. Substituting this formula and the splitting

$$K(\text{Nil}_A) \simeq K(A) \times \tilde{K}(\text{Nil}_A)$$

into part (1) gives

$$\Omega K(A[t, t^{-1}]) \simeq \Omega K(A) \times \tilde{K}(\text{Nil}_A) \times K(A) \times \tilde{K}(\text{Nil}_A).$$

Therefore the usual formulation of the fundamental theorem of algebraic K -theory [14, p. 236] is a special case of Theorem A.1.

Combining (23) with Theorem A.1 gives a splitting relating the two localizations in general:

$$\Omega K(A[t]_S) \simeq \Omega K(A[t^{-1}]_T) \times K(\text{Nil}_A).$$

When A is a regular ring the space $\tilde{K}(\text{Nil}_A)$ is contractible, so

$$\Omega K(A[t]_S) \simeq \Omega K(A[t^{-1}]_T) \times K(A).$$

Proof of (1). We apply the generic fibration theorem of Waldhausen [29, thm. 1.64.] to the category of perfect $A[t]$ -chain complexes. Let the v -notion of weak equivalence be quasi-isomorphism, and let the w -notion be those maps whose homotopy cofiber is S -torsion in the sense of Lemma 6.4. In particular, a chain complex is w -acyclic iff it is S -torsion.

By Corollary 6.8, the K -theory of the w -acyclic objects is identified with $K(\text{End}_A^S)$. Therefore the sequence in the generic fibration theorem becomes

$$K(\text{End}_A^S) \rightarrow K(A[t]) \rightarrow K(A[t]; w),$$

the first map sending (P, f) to the two-term $A[t]$ -chain complex $P[t] \xrightarrow{t-f} P[t]$. By the additivity theorem this map is null-homotopic, hence we get an equivalence

$$\Omega K(A[t]; w) \simeq K(\text{End}_A^S) \times \Omega K(A[t]).$$

As $A[t]_S$ is flat over $A[t]$, tensoring is exact, so it induces a map

$$K(A[t]; w) \rightarrow K(A[t]_S).$$

This is well-known to give an equivalence after looping (though not an isomorphism on π_0), which finishes the proof. $\square$

APPENDIX B. ENDOMORPHISM TORSION AND BOUNDARY MAP

In the previous appendix we recalled a fiber sequence of the form

$$K(\text{End}_A^S) \xrightarrow{0} K(A[t]) \rightarrow K(A[t]_S).$$

In this appendix we show the endomorphism torsion splits the boundary map. We use this for nonvanishing results for τ (e.g. Proposition 3.3), and to show that τ gives the equivalence in part (1) of the fundamental theorem of endomorphism K -theory (Theorem A.1).

This is not the only situation in which a torsion invariant splits the boundary map of a K -theory long exact sequence. In general, the boundary map of such a sequence sends every class represented by a w -weak equivalence to its homotopy cofiber, which is w -acyclic. This is the case, for example, with the parametrized Reidemeister torsion invariants of Dwyer-Weiss-Williams [7] and Igusa-Klein [18]. However, the splitting of the invariant in those cases have indeterminacy. By contrast, in the case of endomorphism torsion, the splitting occurs at the K -theory space level (and therefore holds for every group K_i), so there is no indeterminacy.

As in the proof of Theorem A.1, let C be the category of perfect $A[t]$ -chain complexes, let v be the quasi-isomorphisms and let w be the maps whose homotopy cofibers are S -torsion in the sense of Lemma 6.4. Let $C^w \subset C$ be the full subcategory of w -acyclic objects. Waldhausen's generic fibration then fits into a commuting diagram of the form

$$\begin{array}{ccccccc} \Omega K(C, w) & \xrightarrow{\partial} & K(C^w, v) & \longrightarrow & K(C, v) & \longrightarrow & K(C, w) \\ \downarrow \sim & & \sim \uparrow & & \parallel & & \downarrow \\ \Omega K(A[t]_S) & & K(\text{End}_A^S) & \xrightarrow{0} & K(A[t]) & \longrightarrow & K(A[t]_S). \end{array}$$

The second vertical map sends (P, f) to the two-term complex $P[t] \xrightarrow{t-f} P[t]$, while the outside vertical maps extend scalars along $A[t] \rightarrow A[t]_S$.

We observe that the endomorphism torsion naturally lifts along this extension of scalars. The lifted torsion map comes from the categorical operation that sends (P, f) to the one-term complex $P[t]$ with self- w -equivalence $t - f$, instead of the complex $P[t]_S$ with self-isomorphism $t - f$. Composing this with coassembly and then projecting to based loops gives a map

$$\tau: K(\text{End}_A^S) \rightarrow LK(C, w) \rightarrow \Omega K(C, w)$$

that agrees with the endomorphism torsion along the equivalence

$$\Omega K(C, w) \xrightarrow{\sim} \Omega K(A[t]_S).$$

Theorem B.1. *The following commutes in the homotopy category*

$$\begin{array}{ccc} \Omega K(C, w) & \xrightarrow{\partial} & K(C^w, v) \\ & \nwarrow \tau & \uparrow \sim \\ & & K(\text{End}_A^S), \end{array}$$

i.e., the endomorphism torsion map is a section to the boundary map.

Before giving the proof, we first recall from [29] the definition of the boundary map. Let $\overline{w}C$ denote the subcategory of maps that are both cofibrations and weak equivalences. Let $\overline{w}.C$ denote the simplicial category that at simplicial level k is length k flags of maps in $\overline{w}$,

$$B_\bullet := B_0 \xrightarrow{\sim} B_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} B_k.$$

Let $F.(C, C^w)$ denote the same construction but equipped additionally with choices of quotient B_j/B_i for all $i \leq j$ (see [29, p.344] for details). This forms a simplicial Waldhausen category. The forgetful

maps $F_k(C, C^w) \rightarrow \bar{w}_k C$ are equivalences of categories, hence equivalences on the v .-nerve. Together with the equivalence $v\bar{w}.C \simeq vw.C$ and the swallowing lemma, this gives an equivalence of spaces

$$(24) \quad |v.F.(C, C^w)| \xrightarrow{\sim} |v.\bar{w}.C| \xrightarrow[\sigma]{\sim} |w.C|.$$

Here the map σ takes a square grid of objects (A_{ij}) and maps to the flag of diagonal objects (A_{ii}) and the maps between them. Equivalently, it is the most obvious map from the double realization of $v.\bar{w}.C$ to the single realization of $w.C$, which for each $p \times q$ grid of morphisms, subdivides the corresponding copy of $\Delta^p \times \Delta^q$ and maps each of the simplices to the evident corresponding simplex in $|w.C|$.

On the other hand, the operation

$$(25) \quad B_\bullet \mapsto B_\bullet/B_0$$

induces a map $F.(C, C^w) \xrightarrow{\partial} S.C^w$.

Lemma B.2. *Adding a copy of S . and Ω to the maps (24) and (25), the resulting composite*

$$\Omega|w.S.C| \xleftarrow{\sim} \Omega|v.S.F.(C, C^w)| \xrightarrow{\partial} \Omega|v.S.S.C^w|$$

is the connecting homomorphism for the fiber sequence of the generic fibration theorem,

$$|v.S.C^w| \rightarrow |v.S.C| \rightarrow |w.S.C|.$$

Proof. We have a commutative square

$$(26) \quad \begin{array}{ccc} F.(C, C^w) & \xrightarrow{\partial} & S.C^w \\ \downarrow & & \downarrow \\ F.(C, C) & \longrightarrow & S.C \end{array}$$

where the horizontal maps are induced by (25). After passing to S . constructions, the square is homotopy cartesian [29, cor. 1.56.] and $vS.F.(C, C)$ is contractible. Combining this with the generic fibration theorem, we have two homotopy pullback squares

$$\begin{array}{ccccccc} |w.S.C| & \xleftarrow{\sim} & |v.S.F.(C, C^w)| & \xrightarrow{\partial} & |v.S.S.C^w| & \longrightarrow & |w.S.S.C^w| \\ & & \downarrow & & \downarrow & & \downarrow \\ & & |v.S.F.(C, C)| & \longrightarrow & |v.S.S.C| & \longrightarrow & |w.S.S.C|. \end{array}$$

The bottom-left and top-right terms are contractible, so we get two fiber sequences stitched together, in other words a four-term Barratt-Puppe fiber sequence

$$(27) \quad (|w.S.C| \simeq |vS.F.(C, C^w)|) \xrightarrow{\partial} |vS.S.C^w| \rightarrow |vS.S.C| \rightarrow |wS.S.C|.$$

Therefore the zig-zag in the statement of the lemma is *equivalent* to the boundary map of the desired fiber sequence.

To show that it is the *same* map in the homotopy category, we need to check that the identification $|w.S.C| \simeq \Omega|w.S.S.C|$ arising from the Barratt-Puppe sequence (27) agrees with the usual one (5). This follows from the commutativity of the following diagram, because the commuting square at the very bottom gives the usual map $|w.C| \rightarrow \Omega|w.S.C|$ by [29, Lemma 1.5.2], and the composite along the left-hand side agrees in the homotopy category with the identification $|w.S.C| \simeq |vS.F.(C, C^w)|$ from above. For simplicity we have suppressed one copy of S on every term.

$$\begin{array}{ccccc}
 v.F.(C, C^w) & \xrightarrow{\quad} & w.S.C^w & \searrow & \\
 \downarrow \sim & \searrow & \parallel & \searrow & \\
 & v.F.(C, C) & \xrightarrow{\quad} & w.S.C & \\
 w.F.(C, C^w) & \xrightarrow{\quad} & w.S.C^w & \searrow & \\
 \uparrow \sim & \searrow & \parallel & \searrow & \\
 & w.F.(C, C) & \xrightarrow{\quad} & w.S.C & \\
 w.C & \xrightarrow{\quad} & * & \searrow & \\
 & \parallel & \uparrow \sim & \searrow & \\
 & w.F.(C, C) & \xrightarrow{\quad} & w.S.C &
 \end{array}$$

□

Proof of Thm B.1. To prove that $\partial \circ \tau$ is the desired equivalence, we extend the domain of ∂ by adding on the zero map $K(C, w) \rightarrow K(C^w, v)$, giving a map out of the free loop space as indicated below.

$$\begin{array}{ccc}
 LK(C, w) & \xrightarrow{\partial} & K(C^w, v) \\
 & \swarrow \tau & \uparrow \sim \\
 & & K(\text{End}_A^S)
 \end{array}$$

It suffices to prove that this modified diagram commutes. To prove this we rewrite the modified boundary map as the topmost route in

the following diagram.

$$\begin{array}{ccccccc}
 L|w.S.C| & \xleftarrow{\sim} & L|v.S.F.(C, C^w)| & \xrightarrow{\partial} & L|v.S.S.C^w| & \xleftarrow{\sim} & \Omega|v.S.S.C^w| \xleftarrow{\sim} |v.S.C^w| \\
 & \nwarrow \tau & \uparrow \downarrow & & \nearrow \sim & & \\
 & & |i.S.\text{End}_A^S| & & & &
 \end{array}$$

Then we define a dotted map making both triangular regions in the diagram commute up to homotopy. It arises from an exact functor

$$\text{End}_A^S \rightarrow F_1(C, C^w)$$

with the following description. Given (P, f) we regard $P[t] \xrightarrow{t-f} P[t]$ as a map of one-term $A[t]$ -complexes. We take its reverse mapping cone $T^-(t-f)$ as in §6, though in contrast to that section we use the existing t -action, not the action through sh^{-1} . This makes $T^-(t-f)$ a two-term complex of $A[t]$ -modules, with an acyclic cofibration sh^{-1} whose quotient is the two-term complex $P[t] \xrightarrow{t-f} P[t]$. This describes the desired exact functor to $F_1(C, C^w)$:

$$(P, f) \mapsto \left(T^-(t-f) \xrightarrow{\text{sh}^{-1}} T^-(t-f), T^-(t-f)/\text{sh}^{-1} \cong (P[t] \xrightarrow{t-f} P[t]) \right).$$

It produces a map from End_A^S to simplicial level 1 of $F.(C, C^w)$, with the property that the two faces give the same map to $F_0(C, C^w)$, hence on realizations it gives a map to the free loop space.

Now we check the two triangular regions commute. The one on the right is fairly straightforward: both routes arise in the same way from the exact functor

$$\text{End}_A^S \rightarrow S_1 C^w \cong C^w$$

taking (P, f) to the complex $P[t] \xrightarrow{t-f} P[t]$. For the region on the left, we recall that the weak equivalence is a composite of the two maps

$$L|w.S.C| \xleftarrow{\simeq_\sigma} L|w.w.S.C| \xleftarrow{\simeq} L|v.S.F.(C, C^w)|.$$

Unraveling the definition of σ , the composite arises from the natural transformation of exact functors

$$\text{End}_A^S \times [1] \rightarrow C$$

sending (P, f) to the map of chain complexes

$$\text{sh}^{-1}: T^-(t-f) \rightarrow T^-(t-f).$$

We collapse these mapping cylinders onto their ends to get a simpler map sending (P, f) to the map of chain complexes

$$P[t] \xrightarrow{t-f} P[t].$$

This matches precisely the definition of $\underline{\tau}$. The collapsing operation modifies our original map by a homotopy, so we conclude the triangular region on the left commutes up to homotopy. $\square$

APPENDIX C. THE SPHERE THEOREM FOR End_A AND End_A^S

In this appendix we recall a construction in homological algebra that allows us to factor each map $X \rightarrow Y$ of perfect chain complexes as

$$X \rightarrow X_{m+1} \rightarrow X_{m+2} \rightarrow \dots \rightarrow X_n \xrightarrow{\sim} Y$$

where each X_q/X_{q-1} has homology that is finitely generated projective and concentrated in degree q . We then generalize this construction so that it applies to A -perfect chain complexes over $A[t]$, i.e. to chain complexes of endomorphisms, with or without S -torsion. This is used in §6 to show that the chain complex models for $K(\text{End}_A)$ and $K(\text{End}_A^S)$ are equivalent to the classical models (Proposition 6.7).

Let A be an associative ring. Let $f: X \rightarrow Y$ be a map of chain complexes of A -modules, P a projective A -module, and $k: P \rightarrow H_n(Y, X)$ any A -linear map.

Lemma C.1. *There is a factorization $X \rightarrow X' \rightarrow Y$ in which $H_q(X', X)$ is P in degree n and 0 otherwise, and along this identification the induced map $P \rightarrow H_n(Y, X)$ agrees with k .*

Proof. Recall that $H_*(Y, X) = H_*(C(f))$ where $C(f)$ is the mapping cone. We have $C(f)_n \cong Y_n \oplus X_{n-1}$ with boundary map given by the boundary map of Y , the negated boundary of X , and $f: X_{n-1} \rightarrow Y_{n-1}$. Therefore an n -cycle consists of an $(n-1)$ -cycle $x \in Z_{n-1}(X)$ and an n -chain $y \in Y_n$ such that $\partial y = f(x)$. In other words, a map from a “ $(n-1)$ -sphere” into X and an extension to an “ n -disc” in Y .

Since P is projective, k lifts to a map $P \rightarrow Z_n(C(f))$, giving a commuting diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & Y_n \\ \downarrow h & & \downarrow \partial \\ Z_{n-1}(X) & \xrightarrow{f} & Z_{n-1}(Y) \end{array} .$$

Set $X'_n = X_n \oplus P$, with the boundary map on P given by h , and $X'_* = X_*$ in all other degrees. It is immediate that this is a chain complex and there is a factorization of f into maps of chain complexes $X \rightarrow X' \rightarrow Y$, the latter map defined on P by g . The relative homology of (X', X) is also clearly P , and the isomorphism is split by

$$P \rightarrow C(X \rightarrow X')_n \cong (X_n \oplus P) \oplus X_{n-1}$$

$$p \mapsto (0, p, h(p)) .$$

(The latter term is needed for this map to land in the cycles, and therefore induce a map to homology.) Therefore the resulting map $P \rightarrow C(X \rightarrow X')_n \rightarrow C(f)_n$ sends p to $(g(p), h(p))$ as desired. $\square$

As a result, by the long exact sequence

$$0 \rightarrow H_{n+1}(Y, X) \rightarrow H_{n+1}(Y, X') \rightarrow P \xrightarrow{k} H_n(Y, X) \rightarrow H_n(Y, X') \rightarrow 0 ,$$

if we replace X by X' , the new relative homology $H_*(Y, X')$ agrees with the old relative homology $H_*(Y, X)$ in all degrees except n , where it is the cokernel of k , and degree $n + 1$, where it is an extension of $H_{n+1}(Y, X)$ by the kernel of k . In particular, if k is surjective then this procedure kills $H_n(Y, X)$ and only changes $H_{n+1}(Y, X)$ in the process.

We recall two more preliminaries. Let $H(A)$ the category of A -modules that admit a finite resolution by finitely-generated projective modules.

Lemma C.2. *A module is in $H(A)$ iff it occurs as the sole nonzero homology group of a perfect complex.*

Proof. The forward implication is obvious. If our module is the lone homology group of a strictly perfect complex, and the lowest nonzero group of the complex P_0 is beneath the lowest homology group, we include a copy of the two-term complex $P_0 \rightarrow P_0$ into the bottom of the complex and take the quotient. This gives a new perfect complex that is shorter but has the same homology. After performing this move finitely many times, we arrive at the desired resolution for our module. $\square$

Lemma C.3. [31, II.7.7.1] *If H has a length $(n + 1)$ resolution by finitely generated projective A -modules and $0 \rightarrow K \rightarrow P \rightarrow H \rightarrow 0$ is short exact with P finitely generated projective, then K has a resolution of finitely generated projective A -modules of length $\max(0, n)$.*

Now we may verify the main hypothesis of the sphere theorem. Recall that a map of chain complexes $X \rightarrow Y$ is m -connected if $H_q(Y, X) = 0$ for $q \leq m$.

Proposition C.4. *Given an m -connected map $X \rightarrow Y$ of A -perfect A -chain complexes, there is a factorization*

$$X \rightarrow X_{m+1} \rightarrow X_{m+2} \rightarrow \dots \rightarrow X_n \xrightarrow{\sim} Y$$

where each X_q/X_{q-1} has homology that is finitely generated projective over A and concentrated in degree q .

Proof. The mapping cone has lowest homology in degree $(m+1)$. As the mapping cone is perfect, its lowest homology group is finitely generated, therefore we have a surjective map $P \rightarrow H_n(Y, X)$ where P is finitely generated free. The procedure of Lemma C.1 produces X_{m+1} such that X_{m+1}/X has homology P concentrated in degree $(m+1)$ and $X_{m+1} \rightarrow Y$ is $(m+1)$ -connected. We repeat until we have exhausted all of the relative homology groups of the original map. Then $C(f)$ has only one homology group remaining, so it lies in $H(A)$ by Lemma C.2. By Lemma C.3, if we continue the procedure then after finitely many steps the homology group will be finitely generated projective, in which case we apply Lemma C.1 once more with P as the last remaining homology group. $\square$

Now we modify this argument to work for endomorphisms, and endomorphisms with torsion. In each case, an additional trick is needed.

Lemma C.5. *Lemma C.1 remains true if, instead of being projective, P has a length-one resolution by projective modules*

$$0 \rightarrow P_1 \xrightarrow{i} P_0 \xrightarrow{j} P \rightarrow 0 .$$

Proof. We lift the map $P_0 \xrightarrow{j} P \xrightarrow{k} H_n(Y, X)$ to $P_0 \rightarrow C(f)_n$ and apply the procedure of Lemma C.1 to this map. This gives the lower square below. We then add P_1 as shown and define the map b to Y_{n+1} by noting that the composite to Y_n is zero on homology, therefore must land in the boundaries, and therefore has a lift because P_1 is projective. The total modified complex and its map to Y now look like

$$\begin{array}{ccc} X_{n+1} \oplus P_1 & \xrightarrow{(f,b)} & Y_{n+1} \\ \partial \oplus i \downarrow & & \downarrow \partial \\ X_n \oplus P_0 & \xrightarrow{(f,g)} & Y_n \\ (\partial, h) \downarrow & & \downarrow \partial \\ X_{n-1} & \xrightarrow{f} & Y_{n-1} \end{array}$$

This gives a factorization $X \rightarrow X'' \rightarrow Y$ of A -perfect $A[t]$ -chain complexes where $H_*(X'', X)$ is P concentrated in degree n , and the resulting map to $H_n(Y, X)$ is k . So we again get the same conclusions as in Lemma C.1 for the effect of this operation on relative homology. $\square$

Remark C.6. This argument breaks down if the resolution of P is longer, because one begins to encounter obstructions in the homology groups of Y .

Proposition C.7. *Given an m -connected map $X \rightarrow Y$ of A -perfect $A[t]$ -chain complexes, there is a factorization in $A[t]$ -chain complexes*

$$X \rightarrow X_{m+1} \rightarrow X_{m+2} \rightarrow \dots \rightarrow X_n \xrightarrow{\sim} Y$$

where each X_q/X_{q-1} has homology that is finitely generated projective over A and concentrated in degree q . (Therefore every term of the factorization is A -perfect.)

Proof. The mapping cone has lowest homology in degree $(m+1)$, and is finitely generated over A . Therefore there is a surjective A -linear map $k: P \rightarrow H_{m+1}(Y, X)$ where P is a finitely generated projective A -module (with no $A[t]$ -action).

The key observation is that we can always endow such a P with an $A[t]$ -action such that k is $A[t]$ -linear. Without loss of generality P is free, and then we define the action one basis element at a time, by applying k , applying the t action in $H_{m+1}(Y, X)$, then taking a lift along k . We then extend to the rest of P to form an A -linear map $\alpha: P \rightarrow P$ such that $k\alpha = tk$. This makes P into an $A[t]$ -module such that k is $A[t]$ -linear.

Then by the characteristic sequence, P has a length-one resolution by projective $A[t]$ -modules. By Lemma C.5, we can therefore form the desired factorization through X_{m+1} . The rest of the proof now proceeds just as in Proposition C.4. $\square$

Proposition C.8. *Given an m -connected map $X \rightarrow Y$ of A -perfect, S -torsion $A[t]$ -chain complexes, there is a factorization in $A[t]$ -chain complexes*

$$X \rightarrow X_{m+1} \rightarrow X_{m+2} \rightarrow \dots \rightarrow X_n \xrightarrow{\sim} Y$$

where each X_q/X_{q-1} has homology that is S -torsion, finitely generated projective over A and concentrated in degree q . (Therefore every term of the factorization is A -perfect and S -torsion.)

Proof. Since $H_{m+1}(Y, X)$ is $A[t]$ -finitely generated and S -torsion, there is a single element $p(t) \in S$ such that $p(t)$ acts by zero on $H_{m+1}(Y, X)$. Then we get a surjective $A[t]$ -linear map $(A[t]/p(t))^{\oplus i} \rightarrow H_{m+1}(Y, X)$. Observe that $(A[t]/p(t))^{\oplus i} \cong A^{\oplus i(\deg p)}$ is finitely generated free as an A -module, and S -torsion as an $A[t]$ -module, so we may kill it as before using Lemma C.5 without leaving the category of perfect S -torsion complexes. We continue this trick for the remaining steps, except the final step where the last remaining homology group is finitely generated projective, in which case (just as in Proposition C.4) we take P to be that last remaining homology group. $\square$

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