ON THE GLOBAL CLASSICAL SOLUTION TO COMPRESSIBLE EULER SYSTEM WITH SINGULAR VELOCITY ALIGNMENT[*]

LI CHEN[†], CHANGHUI TAN[‡], AND LINING TONG[§]

Dedicated to Professor Ling Hsiao's 80th birthday

Abstract. We consider a compressible Euler system with singular velocity alignment, known as the *Euler-alignment system*, describing the flocking behaviors of large animal groups. We establish a local well-posedness theory for the system, as well as a global well-posedness theory for small initial data. We also show the asymptotic flocking behavior, where solutions converge to a constant steady state exponentially in time.

1. Introduction. In this paper, we study the following Cauchy problem

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = -\beta \rho u - \rho \int_\Omega \phi(x-y)(u(x)-u(y))\rho(y)\mathrm{d}y, \tag{1.2}$$

in $(0,\infty) \times \Omega$ with initial conditions

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \quad x \in \Omega. \tag{1.3}$$

The spatial domain Ω can be either the whole space $\mathbb{R}^N$ or the torus $\mathbb{T}^N$, where N denotes the dimension. ρ and u are the unknown density and velocity, respectively. The pressure is given by the power law $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$, and the damping coefficient $\beta \geq 0$. The last term in (1.2) represents the nonlocal velocity alignment, where ϕ is called the *communication weight*, measuring the strength of the alignment interactions.

System (1.1)-(1.2) can be formally derived from a mean field M-particle Newtonian interaction system of the type

$$\begin{cases} \dot{X}_i(t) = V_i(t), \quad 1 \leq i \leq M, \\ \dot{V}_i(t) = \dfrac{1}{M} \displaystyle\sum_{j \neq i} F\Big(t, X_i(t) - X_j(t), V_i(t) - V_j(t)\Big) - \beta V_i(t), \end{cases} \tag{1.4}$$

with the interacting force $F(t,x,v) = -\phi(x)v$. It is known as the Cucker-Smale model [10] which describes the flocking phenomenon for animal groups. Other celebrated models that lie in the framework of (1.4) range from the classical mechanics with Coulomb force in 3D [17] $F(t,x,v) = x/|x|^3$, to modeling the social behavior of agents, for example, wealth distribution in [11] and pedestrian flow in [13].

[†]Department of Mathematics, University of Mannheim, 68131 Mannheim, Germany (chen@math.uni-mannheim.de).

[‡]Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA (tan@math.sc.edu). The research of CT is supported by the NSF grant DMS 1853001.

[§]Corresponding author. Department of Mathematics, Shanghai University, 200244, China (tongln@shu.edu.cn). The research of LT is supported by the NSFC grants No. 11771274, 11901379.

Taking the mean field limit of the particle system (1.4), one obtains, in the meso-scopic level, the Vlasov type kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \Big(\iint_{\Omega \times \Omega} F(t, x-y, v-v') f(t, y, v') \mathrm{d}y \mathrm{d}v' f(t, x, v) \Big) - \beta \nabla_v \cdot (vf) = 0.$$

The rigorous derivation of the mean field model with different backgrounds has been studied extensively in the last decades, for example in [1, 2, 3, 5, 7, 16, 17, 20], to name a few.

The compressible Euler systems like (1.7)-(1.8) serve as hydrodynamic limits of the kinetic equations. Different choices of Ansatz lead to different pressure laws. For instance, mono-kinetic Ansatz $f(t, x, v) = \rho(t, x)\delta_{u(t,x)}(v)$ implies the pressure-less system $p(\rho) \equiv 0$; local Maxwellian $f(t, x, v) = \rho(t, x)\frac{1}{(2\pi)^{\frac{N}{2}}} e^{-|v-u(t,x)|^2/2}$ leads to linear pressure $p(\rho) = \rho$; and the Ansatz $f(t, x, v) = \chi_{|v-u(t,x)|^2 \leq \rho^{2/N}(t,x)}(x, v)$, which comes from the minimization of kinetic energy $\iint \frac{v^2}{2} f(t, x, v) \mathrm{d}x \mathrm{d}v$ under the restric-tion $\|f(t, x, v)\|_{L^\infty} \leq 1$, would yield the nonlinear pressure $p(\rho) = \rho^{\frac{N+2}{N}}$. Rigorous justifications of hydrodynamic limits on Cucker-Smale model can be found in [15, 18].

It is well-known that for the compressible Euler system (1.7)-(1.8) with $\beta = \phi = 0$, only local existence of smooth solutions could be expected, and shock waves will form in finite time. Velocity damping helps in preventing the formation of shocks (see [27, 31, 34]).

When the communication weight ϕ is bounded and has a positive lower bound (or decays sufficiently slow at infinity in the whole space case), the nonlocal velocity alignment has a damping effect, which could restrain shock formation, for a class of subcritical initial data. See [4, 28, 30] for threshold conditions for the pressureless systems, and [8] for isothermal pressure $p(\rho) = \rho$ with small initial data. For the case where the communication weight is not positive, such as in the pedestrian and material flow case, an additional damping effect is required to prove the global smooth solution for small initial data in [6, 32, 33].

We are interested in the case when the communication weight ϕ is singular at the origin. A prototype choice of ϕ would be

$$\phi(x) = \frac{c_\alpha}{|x|^{N+2\alpha}}, \quad c_\alpha = \frac{2^{2\alpha}\Gamma(\alpha + \frac{N}{2})}{\pi^{\frac{N}{2}}|\Gamma(-\alpha)|}. \tag{1.5}$$

for $\alpha \in (0, 1)$, where the constant c_α is related to the fractional Laplacian operator, which in $\mathbb{R}^N$ reads

$$\mathcal{L} = (-\Delta)^{\frac{1}{2}}, \quad \mathcal{L}^{2\alpha} f = c_\alpha P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2\alpha}} \mathrm{d}y, \quad 0 < \alpha < 1. \tag{1.6}$$

Note that (1.6) holds in $\Omega = \mathbb{T}^N$ by viewing f as a periodic function in $\mathbb{R}^N$.

By considering a purturbation of the constant solution $\rho \equiv 1$, $u \equiv 0$, the system (1.1)-(1.2) with ϕ defined in (1.5) can be reformulated into

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \tag{1.7}$$
$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) = -\beta \rho u - \rho \mathcal{L}^{2\alpha} u - \rho \mathcal{L}^{2\alpha}((\rho - 1)u) + \rho u \mathcal{L}^{2\alpha} \rho. \tag{1.8}$$

On can observe a linear fractional viscosity term $-\mathcal{L}^{2\alpha}u$ in (1.8), which has a regular-ization effect. Such effect has been captured beautifully in 1D, where global regularity

can be shown for all smooth initial data away from vacuum (see [29] for the effect of the vacuum), for the pressureless system with $\alpha \in (0,1)$ [12, 19, 25, 26], and for the isentropic system ($p(\rho) = \rho^\gamma$) with $\alpha \in (\frac{5}{6}, 1)$ [9]. The multi-dimensional system, however, is much less understood, due to the lack of an auxiliary quantity, first introduced in [4], that nicely captures the commutator structure (or cancelation property) in $-\mathcal{L}^{2\alpha}((\rho - 1)u) + u\mathcal{L}^{2\alpha}\rho$, so that it is dominated by the linear dissipation. To our best knowledge, the only global result in multi-dimension is for the pressureless system with small initial data [24].

The main goal of this paper is to establish a global theory for the system (1.1)-(1.2) in multi-dimension with pressure. First, we establish a local well-posedness theory, together with a regularity criterion. The singular kernel leads to a regularization effect for the velocity u. Next, we prove global regularity for small initial data. The main subtlety is to make use of the pressure to generate dissipation for the density (the idea was introduced in [27]), and to control the last two terms in (1.8) together with the velocity dissipation. Finally, we show an exponential convergence of the solution towards the constant steady state $\rho \equiv 1$ and $u \equiv 0$, when $\Omega = \mathbb{T}^N$. Unlike the pressureless dynamics, where the asymptotic density profile is not necessarily uniformly distributed (e.g. [24]), the presence of pressure enforces the steady state to be a constant $\rho \equiv 1$. The exponential decay is obtained by a careful examination on the physical energy.

The arrangement of this paper is the following. In section 2, we state the main results, together with several preliminary lemmas. In section 3, we establish the local well-posedness theory and regularity criterion, using energy method. In section 4, we show global well-posedness for small initial data. Finally, in section 5, we prove the exponential decay of the solution when the domain is a torus.

Here in the following we introduce several notations used throughout the paper. We will repeatedly use C as a generic positive constant. Unless specified, C can depend on parameters γ, β, s, N, etc, but is independent of t and data (ρ, u). Denote $C^\lambda := C^\lambda(\Omega)$ be the Hölder space, for $\lambda \in (0, 1)$, and $C^\lambda = C^{1,\lambda-1}$ for $\lambda \in (1, 2)$. For $\lambda = 1$, we use C^1 to represent $C^{1,\epsilon}$ for simple notations. See Remark 2.4 for more details. For simplicity, we write $\int f\mathrm{d}x := \int_\Omega f\mathrm{d}x$.

2. Preliminaries and main results.

2.1. Reformulation of the problem. We start by reformulating the Cauchy problem of the compressible Euler system (1.1)-(1.3) with respect to the constant solution $\rho \equiv 1$ and $u \equiv 0$, following the idea in [27].

Introduce a new variable σ, defined as follows

$$\sigma = \sigma(\rho) := \begin{cases} \ln \rho & \gamma = 1, \\ \frac{2\sqrt{\gamma}}{\gamma-1}(\rho^{\frac{\gamma-1}{2}} - 1) & \gamma > 1. \end{cases} \tag{2.1}$$

Inversely, ρ can be expressed by

$$\rho = \rho(\sigma) := \begin{cases} e^\sigma & \gamma = 1 \\ \left(\frac{\gamma-1}{2\sqrt{\gamma}}\sigma + 1\right)^{\frac{2}{\gamma-1}} & \gamma > 1 \end{cases} \tag{2.2}$$

The equation (1.7)-(1.8) are transformed into the following equivalent system:

$$\partial_t \sigma + u \cdot \nabla \sigma + \left(\frac{\gamma - 1}{2} \sigma + \sqrt{\gamma} \right) \nabla \cdot u = 0, \tag{2.3}$$

$$\partial_t u + u \cdot \nabla u + \left(\frac{\gamma - 1}{2} \sigma + \sqrt{\gamma} \right) \nabla \sigma$$
$$= - \beta u - \mathcal{L}^{2\alpha} u - \mathcal{L}^{2\alpha}((\rho(\sigma) - 1)u) + u \mathcal{L}^{2\alpha}(\rho(\sigma) - 1). \tag{2.4}$$

subject to the initial data

$$\sigma|_{t=0} = \sigma_0(x) = \sigma(\rho_0(x)) \quad u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{2.5}$$

One can check that the C^1 solution of (2.3)-(2.5) is equivalent to the solution of (1.1)-(1.3), if the density is positive, namely

$$\rho_{\min}(t) := \inf_{x \in \Omega} \rho(t, x) = \inf_{x \in \Omega} \rho(\sigma(t, x)) > 0.$$

2.2. Main results. We study local and global well-posedness of the system (2.3)-(2.5).

The first result concerns the local well-posedness of the system.

THEOREM 2.1 (Local well-posedness). *Let $s > \frac{N}{2} + \max\{1, 2\alpha\}$, assume that $(\sigma_0(x), u_0(x)) \in (H^s(\Omega))^{N+1}$ and $\inf_{x \in \Omega} \rho(\sigma_0(x)) > 0$. Then, there exist a unique classical solution (σ, u) of the Cauchy problem (2.3)-(2.5) satisfying*

$$\sigma \in C([0, T], H^s(\Omega)), \quad u \in C([0, T], (H^s(\Omega))^N) \cap L^2([0, T], (H^{s+\alpha}(\Omega))^N) \tag{2.6}$$

for some finite $T > 0$. Moreover, (2.6) holds for any time T if and only if

$$\int_0^T \left(\|\nabla u(t, \cdot)\|_{L^\infty} + \|\sigma(t, \cdot)\|_{C^{\max\{1, 2\alpha\}}} \right) dt < +\infty. \tag{2.7}$$

REMARK 2.1. Theorem 2.1 holds for any smooth initial data, as long as density stays away from vacuum, in which case the linear dissipation $-\mathcal{L}^{2\alpha} u$ plays a dominate role in the alignment force. Note that the last two terms in (2.4) can be viewed as a commutator

$$\mathcal{L}^{2\alpha}((\rho(\sigma) - 1)u) - u \mathcal{L}^{2\alpha}(\rho(\sigma) - 1) = [\mathcal{L}^{2\alpha}, u](\rho(\sigma) - 1). \tag{2.8}$$

One needs to make good use of this commutator structure in order to obtain the desired dissipative estimates. With the singular alignment force, the solution u gains regularity instantly. This is a major difference compared with the regular (bounded) alignment force.

Next, we turn to the global well-posedness theory. One standard approach is to show that the Beale-Kato-Majda type regularity criterion (2.7) holds in all finite time. However, it is generally difficult to validate such criterion (except in 1D with the aid of an additional structure [9]).

We focus on global regularity for small initial data. We will show that the smallness propagates in time, and hence condition (2.7) holds in all finite time.

THEOREM 2.2 (Global well-posedness for small data). *Let* $s >$ $\max\left\{\frac{N}{2} + \max\{1, 2\alpha\}, 2 - \alpha\right\}$. *Assume either* $\Omega = \mathbb{R}^N$ *and* $\beta > 0$, *or* $\Omega = \mathbb{T}^N$ *and* $\beta \geq 0$. *There exists a small parameter* $\delta_0 > 0$, *such that if*

$$\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 \leq \delta_0^2, \tag{2.9}$$

then the Cauchy problem (2.3)-(2.5) *has a unique global classical solution* (σ, u).

REMARK 2.2. The damping is needed when $\Omega = \mathbb{R}^N$ in order to provide enough control on $\|u\|_{L^2}$. It is not required when $\Omega = \mathbb{T}^N$ as $\|u\|_{L^2}$ can be controlled by dissipation via Poincaré inequality. See Remark 4.1 for discussions on asymptotic flocking behaviors and convergence rate.

Our final result is on the asymptotic behavior of the system. We show the flocking phenomenon with fast alignment in the case $\Omega = \mathbb{T}^N$.

THEOREM 2.3 (Large-time behavior). *Assume* $\Omega = \mathbb{T}^N$, $\beta \geq 0$, *and* (ρ, u) *be the classical solution to the system* (1.1)-(1.3) *satisfying* $(\rho, u) \in L^\infty((0, +\infty) \times \mathbb{T}^N)$. *We further assume the initial data* (ρ_0, u_0) *satisfies*

$$\frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \rho_0(x) \mathrm{d}x = 1, \tag{2.10}$$

and for the case $\beta = 0$, *we assume additionally*

$$\int_{\mathbb{T}^N} \rho_0(x) u_0(x) \mathrm{d}x = 0. \tag{2.11}$$

Then there exists constants $\mu = \mu(\alpha, \|\rho\|_{L^\infty}, \|u\|_{L^\infty})$ *and* $C = C(\alpha, \rho_0, u_0, \|\rho\|_{L^\infty}, \|u\|_{L^\infty})$ *such that*

$$\int \rho(t, x) |u(t, x)|^2 \mathrm{d}x + \int (\rho(t, x) - 1)^2 \mathrm{d}x \leq C e^{-\mu t}. \tag{2.12}$$

Moreover, if (2.9) *is satisfied for the corresponding* (σ, u) *system* (2.3)-(2.5), *then there exists constants* $\mu = \mu(\alpha, \delta_0)$ *and* $C = C(\alpha, \delta_0)$ *such that*

$$\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 \leq C e^{-\mu t}. \tag{2.13}$$

REMARK 2.3. Conditions (2.10) and (2.11) is naturally required to obtain (2.12), due to the conservation of mass, and the conservation of momentum (when $\beta = 0$), respectively. These two conditions can be easily removed by scaling on ρ and shifting on u, and the estimate (2.12) will be replaced by

$$\int \rho(t, x) |u(t, x) - \bar{u}|^2 \mathrm{d}x + \int (\rho(t, x) - \bar{\rho})^2 \mathrm{d}x \leq C e^{-\mu t},$$

where $\bar{\rho}$ is the average density, and $\bar{u}$ is the average velocity.

2.3. Elementary estimates. Next, we state the fractional Leibniz rule and commutator estimates that will be used. We refer to e.g. [22, Theorem 1.2], [23, Lemma 6.1] for more details.

LEMMA 2.1 (Fractional Leibniz rule). *For $\lambda > 0$, there exists a constant $C = C(\lambda, N)$ such that*

$$\|\mathcal{L}^\lambda(fg)\|_{L^2} \leq C(\|\mathcal{L}^\lambda f\|_{L^2}\|g\|_{L^\infty} + \|\mathcal{L}^\lambda g\|_{L^2}\|f\|_{L^\infty}), \qquad (2.14)$$

LEMMA 2.2 (Commutator estimates). *For $\lambda > 1$, there exists a constant $C = C(\lambda, N)$ such that*

$$\|[\mathcal{L}^\lambda, f]g\|_{L^2} \leq C(\|\mathcal{L}^\lambda f\|_{L^2}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|\mathcal{L}^{\lambda-1}g\|_{L^2}), \qquad (2.15)$$

$$\|[\mathcal{L}^\lambda, f, g]\|_{L^2} \leq C(\|\mathcal{L}^{\lambda-1}f\|_{L^2}\|\nabla g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|\mathcal{L}^{\lambda-1}g\|_{L^2}), \qquad (2.16)$$

where

$$[\mathcal{L}^\lambda, f]g = \mathcal{L}^\lambda(fg) - f\mathcal{L}^\lambda g, \quad [\mathcal{L}^\lambda, f, g] = \mathcal{L}^\lambda(fg) - f\mathcal{L}^\lambda g - g\mathcal{L}^\lambda f.$$

For $\lambda \in (0,1]$, there exists a constant $C = C(\lambda, N)$ such that

$$\|[\mathcal{L}^\lambda, f]g\|_{L^2} \leq C\|\mathcal{L}^\lambda f\|_{L^\infty}\|g\|_{L^2}, \qquad (2.17)$$

The following composition estimate is useful for handling the nonlinear mappings between σ and ρ. It indicates that σ and ρ have the same regularity if ρ is away from zero.

LEMMA 2.3 (Composition estimates). *Let $\lambda > 0$. There exists a constant $C = C(\gamma, \lambda, \|\sigma\|_{L^\infty}, \rho_{\min}^{-1})$ such that*

$$\|\mathcal{L}^\lambda(\rho(\sigma) - 1)\|_{L^2} \leq C(\gamma, \lambda, \|\sigma\|_{L^\infty}, \rho_{\min}^{-1})\|\mathcal{L}^\lambda\sigma\|_{L^2}. \qquad (2.18)$$

Proof. We make use of the following composition estimate, the proof of which can be found, for instance, in [21, Theorem A.1.].

$$\|\mathcal{L}^\lambda(\rho(\sigma))\|_{L^2} \leq C\|\rho\|_{C^{\lceil\lambda\rceil}(\text{supp}(\sigma))}(1 + \|\sigma\|_{L^\infty}^\lambda)\|\mathcal{L}^\lambda\sigma\|_{L^2}.$$

The term $\|\rho\|_{C^{\lceil\lambda\rceil}(\text{supp}(\sigma))}$ can be estimated via definition (2.2). For $\gamma = 1$, $\frac{d^{\lceil\lambda\rceil}}{d\sigma^{\lceil\lambda\rceil}}\rho = e^\sigma$. Then,

$$\|\rho\|_{C^{\lceil\lambda\rceil}(\text{supp}(\sigma))} \leq \exp(\|\sigma\|_{L^\infty}).$$

For $\gamma > 1$, we have

$$\left|\frac{d^{\lceil\lambda\rceil}}{d\sigma^{\lceil\lambda\rceil}}\rho\right| \leq C(\gamma, \lambda)(\rho(\sigma))^{1-\lceil\lambda\rceil\frac{\gamma-1}{2}} \leq \begin{cases} C(\gamma, \lambda)(1 + \|\sigma\|_{L^\infty})^{\frac{2}{\gamma-1}-\lceil\lambda\rceil} & \lceil\lambda\rceil \leq \frac{2}{\gamma-1} \\ C(\gamma, \lambda)\rho_{\min}^{-(\lceil\lambda\rceil-\frac{2}{\gamma-1})} & \lceil\lambda\rceil > \frac{2}{\gamma-1} \end{cases}$$

Therefore, we conclude that

$$\|\rho\|_{C^{\lceil\lambda\rceil}(\text{supp}(\sigma))} \leq C(\gamma, \lambda, \|\sigma\|_{L^\infty}, \rho_{\min}^{-1}). \quad \square$$

REMARK 2.4. A similar estimate holds when we replace L^2 by L^∞ in (2.18).

$$\|\mathcal{L}^\lambda(\rho(\sigma) - 1)\|_{L^\infty} \leq C(\gamma, \lambda, \|\sigma\|_{L^\infty}, \rho_{\min}^{-1})\|\sigma\|_{C^\lambda}. \qquad (2.19)$$

We will make use of this estimate for $\lambda \in (0, 2)$. Note that (2.19) needs to be slightly modified when $\lambda = 1$, where $\|\sigma\|_{C^1}$ should be replaced by $\|\sigma\|_{C^{1+\epsilon}}$ for any $\epsilon > 0$ (and the constant depends on ϵ). For the sake of simplicity, we will keep the compact notation $\|\sigma\|_{C^1}$ throughout the paper.

3. Local well-posedness. The local well-posedness theory of the system (2.3)-(2.5) can be established by using standard iteration scheme and the compactness argument. For simplicity, in this section we will give only the *a priori* energy estimates, and omit the detailed construction of approximation solutions.

We start with the L^2 energy estimate.

LEMMA 3.1 (L^2 estimate). *Let (σ, u) be the classical solution of $(2.3) - (2.4)$. Then the following estimate holds*

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \beta\|u\|_{L^2}^2 + \frac{\rho_{\min}}{2}\|\mathcal{L}^\alpha u\|_{L^2}^2 \tag{3.1}$$
$$\leq C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})(\|\nabla\cdot u\|_{L^\infty} + \|\sigma\|_{C^{2\alpha}})(\|u\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2).$$

Proof. Firstly, by multiplying (2.3) and (2.4) by σ and u respectively, summing up and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(|\sigma|^2 + |u^2|\right)\mathrm{d}x + \beta\|u\|_{L^2}^2 = -\int\mathcal{L}^{2\alpha}u\cdot u\mathrm{d}x \tag{3.2}$$
$$-\int\left(\sigma u\cdot\nabla\sigma + u\cdot\nabla u\cdot u\right)\mathrm{d}x - \int\left(\frac{\gamma-1}{2}\sigma + \sqrt{\gamma}\right)\nabla\cdot(\sigma u)\mathrm{d}x$$
$$-\int\mathcal{L}^{2\alpha}((\rho(\sigma)-1)u)\cdot u\mathrm{d}x + \int u\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\cdot u\mathrm{d}x = \sum_{i=1}^5 I_i.$$

We estimate I_i item by item. The commutator estimate (2.17) is used for I_4, and the composition estimate (2.19) is used for I_4 and I_5.

$$I_1 = -\int\mathcal{L}^{2\alpha}u\cdot u\mathrm{d}x = -\int|\mathcal{L}^\alpha u|^2\mathrm{d}x;$$

$$I_2 = -\int\left(\sigma u\cdot\nabla\sigma + u\cdot\nabla u\cdot u\right)\mathrm{d}x = -\int\sigma u\cdot\nabla\sigma\mathrm{d}x + \frac{1}{2}\int|u|^2\nabla\cdot u\mathrm{d}x$$
$$\leq \|\sigma\|_{L^\infty}\|u\|_{L^2}\|\nabla\sigma\|_{L^2} + \frac{1}{2}\|\nabla\cdot u\|_{L^\infty}\|u\|_{L^2}^2$$
$$\leq C(\|\sigma\|_{L^\infty} + \|\nabla\cdot u\|_{L^\infty})(\|\nabla\sigma\|_{L^2}^2 + \|u\|_{L^2}^2);$$

$$I_3 = \frac{\gamma-1}{2}\int\sigma u\cdot\nabla\sigma\mathrm{d}x \leq C\|\sigma\|_{L^\infty}(\|\nabla\sigma\|_{L^2}^2 + \|u\|_{L^2}^2);$$

$$I_4 = -\int\mathcal{L}^{2\alpha}((\rho(\sigma)-1)u)\cdot u\mathrm{d}x = -\int\mathcal{L}^\alpha((\rho(\sigma)-1)u)\cdot\mathcal{L}^\alpha u\mathrm{d}x$$
$$= -\int(\rho(\sigma)-1)\mathcal{L}^\alpha u\cdot\mathcal{L}^\alpha u\mathrm{d}x - \int[\mathcal{L}^\alpha,\rho(\sigma)-1]u\cdot\mathcal{L}^\alpha u\mathrm{d}x$$
$$\leq (1-\rho_{\min})\|\mathcal{L}^\alpha u\|_{L^2}^2 + C\|\mathcal{L}^\alpha(\rho(\sigma)-1)\|_{L^\infty}\|u\|_{L^2}\|\mathcal{L}^\alpha u\|_{L^2}$$
$$\leq \left(1-\frac{\rho_{\min}}{2}\right)\|\mathcal{L}^\alpha u\|_{L^2}^2 + C(\|\sigma\|_{L^\infty},\rho_{\min}^{-1})\|\sigma\|_{C^\alpha}\|u\|_{L^2}^2;$$

$$I_5 = \int u\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\cdot u\mathrm{d}x \leq \|\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\|_{L^\infty}\|u\|_{L^2}^2$$
$$\leq C(\|\sigma\|_{L^\infty},\rho_{\min}^{-1})\|\sigma\|_{C^{2\alpha}}\|u\|_{L^2}^2.$$

Collecting the above estimates into (3.2), we obtain the L^2-estimate of (σ, u) (3.1). $\square$

Next,we provide the $\dot{H}^s$ energy estimate.

LEMMA 3.2 ($\dot{H}^s$ energy estimate). *Let (σ, u) be a classical solution of* $(2.3)-(2.4)$. *Then the following estimate holds for* $s > \frac{N}{2} + \max\{1, 2\alpha\}$

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2\right) + \beta\|\mathcal{L}^s u\|_{L^2}^2 + \frac{\rho_{\min}}{2}\|\mathcal{L}^{s+\alpha}u\|_{L^2}^2 \qquad (3.3)$$
$$\leq C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}})(\|u\|_{H^s}^2 + \|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^{s+\alpha-1}\sigma\|_{L^2}^2).$$

Proof. We apply $\mathcal{L}^s$ to (2.3), (2.4), multiply the resulting identities by $\mathcal{L}^s\sigma$, $\mathcal{L}^s u$ respectively, and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int\left(|\mathcal{L}^s\sigma|^2 + |\mathcal{L}^s u|^2\right)dx + \beta\|\mathcal{L}^s u\|_{L^2}^2$$
$$= -\int \mathcal{L}^{s+2\alpha}u \cdot \mathcal{L}^s u dx - \int\left(\mathcal{L}^s(u \cdot \nabla\sigma)\mathcal{L}^s\sigma + \mathcal{L}^s(u \cdot \nabla u) \cdot \mathcal{L}^s u\right)dx$$
$$- \frac{\gamma - 1}{2}\int\left(\mathcal{L}^s(\sigma\nabla \cdot u)\mathcal{L}^s\sigma + \mathcal{L}^s(\sigma\nabla\sigma) \cdot \mathcal{L}^s u\right)dx$$
$$- \int \mathcal{L}^{s+2\alpha}((\rho(\sigma) - 1)u) \cdot \mathcal{L}^s u dx + \int \mathcal{L}^s(u\mathcal{L}^{2\alpha}(\rho(\sigma) - 1)) \cdot \mathcal{L}^s u dx$$
$$= \sum_{i=1}^{5} J_i. \qquad (3.4)$$

We directly get

$$J_1 = -\int \mathcal{L}^{s+2\alpha}u \cdot \mathcal{L}^s u dx = -\int \left|\mathcal{L}^{s+\alpha}u\right|^2 dx. \qquad (3.5)$$

Applying the commutator estimate (2.15), we get

$$J_2 = -\int\left(\mathcal{L}^s(u \cdot \nabla\sigma)\mathcal{L}^s\sigma + \mathcal{L}^s(u \cdot \nabla u) \cdot \mathcal{L}^s u\right)dx$$
$$= -\int\left(u \cdot \mathcal{L}^s\nabla\sigma\mathcal{L}^s\sigma + [\mathcal{L}^s, u] \cdot \nabla\sigma\mathcal{L}^s\sigma + u \cdot \mathcal{L}^s\nabla u \cdot \mathcal{L}^s u + [\mathcal{L}^s, u] \cdot \nabla u \cdot \mathcal{L}^s u\right)dx$$
$$= \int\left(\frac{1}{2}\nabla \cdot u|\mathcal{L}^s\sigma|^2 - [\mathcal{L}^s, u] \cdot \nabla\sigma\mathcal{L}^s\sigma + \frac{1}{2}\nabla \cdot u|\mathcal{L}^s u|^2 - [\mathcal{L}^s, u] \cdot \nabla u \cdot \mathcal{L}^s u\right)dx$$
$$\leq \frac{1}{2}\|\nabla \cdot u\|_{L^\infty}(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2) + \|[\mathcal{L}^s, u] \cdot \nabla\sigma\|_{L^2}\|\mathcal{L}^s\sigma\|_{L^2}$$
$$\quad + \|[\mathcal{L}^s, u] \cdot \nabla u\|_{L^2}\|\mathcal{L}^s u\|_{L^2}$$
$$\leq \frac{1}{2}\|\nabla \cdot u\|_{L^\infty}(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2) + C(\|\mathcal{L}^s u\|_{L^2}\|\nabla\sigma\|_{L^\infty}$$
$$\quad + \|\nabla u\|_{L^\infty}\|\mathcal{L}^{s-1}\nabla\sigma\|_{L^2})\|\mathcal{L}^s\sigma\|_{L^2}$$
$$\quad + C(\|\mathcal{L}^s u\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\mathcal{L}^{s-1}\nabla u\|_{L^2})\|\mathcal{L}^s u\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty} + \|\nabla\sigma\|_{L^\infty})(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2). \qquad (3.6)$$

Similarly we can do the estimates for J_3,

$$
\begin{aligned}
J_3 &= -\frac{\gamma-1}{2}\int\Big(\mathcal{L}^s(\sigma\nabla\cdot u)\mathcal{L}^s\sigma + \mathcal{L}^s(\sigma\nabla\sigma)\cdot\mathcal{L}^s u\Big)\mathrm{d}x \\
&= -\frac{\gamma-1}{2}\int\Big(\sigma\nabla\cdot(\mathcal{L}^s u\mathcal{L}^s\sigma) + [\mathcal{L}^s,\sigma](\nabla\cdot u)\mathcal{L}^s\sigma + [\mathcal{L}^s,\sigma]\nabla\sigma\cdot\mathcal{L}^s u\Big)\mathrm{d}x \\
&\leq C\|\nabla\sigma\|_{L^\infty}(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2) + C(\|\mathcal{L}^s\sigma\|_{L^2}\|\nabla\cdot u\|_{L^\infty} \\
&\quad + \|\nabla\sigma\|_{L^\infty}\|\mathcal{L}^{s-1}\nabla\cdot u\|_{L^2})\|\mathcal{L}^s\sigma\|_{L^2} \\
&\quad + C(\|\mathcal{L}^s\sigma\|_{L^2}\|\nabla\sigma\|_{L^\infty} + \|\nabla\sigma\|_{L^\infty}\|\mathcal{L}^{s-1}\nabla\sigma\|_{L^2})\|\mathcal{L}^s u\|_{L^2} \\
&\leq C(\|\nabla\cdot u\|_{L^\infty} + \|\nabla\sigma\|_{L^\infty})(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^s u\|_{L^2}^2). \tag{3.7}
\end{aligned}
$$

The estimates for J_4 and J_5 have to be handled together, applying commutator estimates (2.15) and (2.16), one obtains

$$
\begin{aligned}
&J_4 + J_5 \\
&= -\int\mathcal{L}^{s+2\alpha}((\rho(\sigma)-1)u)\cdot\mathcal{L}^s u\,\mathrm{d}x + \int\mathcal{L}^s(u\mathcal{L}^{2\alpha}(\rho(\sigma)-1))\cdot\mathcal{L}^s u\,\mathrm{d}x \\
&= -\int\Big(\mathcal{L}^{s+\alpha}((\rho(\sigma)-1)u) - \mathcal{L}^{s-\alpha}(u\mathcal{L}^{2\alpha}(\rho(\sigma)-1))\Big)\cdot\mathcal{L}^{s+\alpha}u\,\mathrm{d}x \\
&= -\int(\rho(\sigma)-1)\mathcal{L}^{s+\alpha}u\cdot\mathcal{L}^{s+\alpha}u\,\mathrm{d}x \\
&\quad + \int\Big([\mathcal{L}^{s+\alpha},u,\rho(\sigma)-1] - [\mathcal{L}^{s-\alpha},u]\big(\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\big)\Big)\cdot\mathcal{L}^{s+\alpha}u\,\mathrm{d}x \\
&\leq (1-\rho_{\min})\|\mathcal{L}^{s+\alpha}u\|_{L^2}^2 \\
&\quad + \Big(\|\mathcal{L}^{s+\alpha-1}u\|_{L^2}\|\nabla(\rho(\sigma)-1)\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\mathcal{L}^{s+\alpha-1}(\rho(\sigma)-1)\|_{L^2}\Big)\|\mathcal{L}^{s+\alpha}u\|_{L^2} \\
&\quad + \Big(\|\mathcal{L}^{s-\alpha}u\|_{L^2}\|\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\mathcal{L}^{s+\alpha-1}(\rho(\sigma)-1)\|_{L^2}\Big)\|\mathcal{L}^{s+\alpha}u\|_{L^2}.
\end{aligned}
$$

We remark that in the third equality, there is a cancelation of the term

$$
\int u\mathcal{L}^{s+\alpha}(\rho(\sigma)-1)\cdot\mathcal{L}^{s+\alpha}u\,\mathrm{d}x, \tag{3.8}
$$

which itself can not be controlled. This is the place where the commutator structure (2.8) is crucially used.

Next, we apply the composition estimates (2.18), (2.19), and use the facts $s-\alpha < s$, $s+\alpha-1 < s$ to get

$$
\begin{aligned}
J_4 + J_5 &\leq \Big(1-\frac{\rho_{\min}}{2}\Big)\|\mathcal{L}^{s+\alpha}u\|_{L^2}^2 + C(\|\sigma\|_{L^\infty},\rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}}) \\
&\quad \times (\|\mathcal{L}^{s+\alpha-1}u\|_{L^2}^2 + \|\mathcal{L}^{s-\alpha}u\|_{L^2}^2 + \|\mathcal{L}^{s+\alpha-1}\sigma\|_{L^2}^2) \\
&\leq \Big(1-\frac{\rho_{\min}}{2}\Big)\|\mathcal{L}^{s+\alpha}u\|_{L^2}^2 + C(\|\sigma\|_{L^\infty},\rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}}) \\
&\quad \times (\|u\|_{H^s}^2 + \|\mathcal{L}^{s+\alpha-1}\sigma\|_{L^2}^2) \tag{3.9}
\end{aligned}
$$

Finally, we plug in the estimates (3.5)-(3.9) to (3.4) and finish the $\dot{H}^s$ estimate (3.3). $\square$

With the energy estimates above, we are ready to prove the local well-posedness theory.

Proof of Theorem 2.1. We combine the estimates (3.1) and (3.3) for the full H^s estimate

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2\right) + \beta\|u\|_{H^s}^2 + \frac{\rho_{\min}}{2}\|u\|_{H^{s+\alpha}}^2 \qquad (3.10)$$
$$\leq C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}})(\|u\|_{H^s}^2 + \|\sigma\|_{H^s}^2).$$

For $s > \frac{N}{2} + \max\{1, 2\alpha\}$, Sobolev embedding implies

$$\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}} \leq C(\|u\|_{H^s} + \|\sigma\|_{H^s}) \leq \sqrt{2}C(\|u\|_{H^s}^2 + \|\sigma\|_{H^s}^2)^{1/2}.$$

As $\|\sigma_0\|_{L^\infty}$ and $\rho_{\min}^{-1}(0)$ are bounded, there exists a time T_0 such that $\|\sigma(t,\cdot)\|_{L^\infty}$ and $\rho_{\min}^{-1}(t)$ are uniformly bounded for $t \in [0, T_0]$. Then, there is a universal constant $C(T_0)$ such that

$$C(\|\sigma(t,\cdot)\|_{L^\infty}, \rho_{\min}^{-1}(t)) \leq C(T_0).$$

Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2\right) \leq C(T_0)\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2\right)^{3/2}, \quad \forall\, t \in [0, T_0].$$

Standard ODE theory implies the existence of time $T \leq T_0$ such that $\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2$ is bounded for $t \in [0, T]$.

Integrating (3.10) in time, we get

$$\int_0^T \|u(t,\cdot)\|_{H^{s+\alpha}}^2\,dt$$
$$\leq \max_{t\in[0,T]}\rho_{\min}^{-1}(t)\left(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + 2C(T_0)\int_0^T\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2\right)dt\right) < +\infty,$$

which implies $u \in L^2([0,T], (H^{s+\alpha})^N)$.

Finally, we show that the regularity (2.6) holds as long as condition (2.7) is satisfied. Applying the Gronwall's inequality on (3.10), we get

$$\|\sigma(t,\cdot)\|_{H^s}^2 + \|u(t,\cdot)\|_{H^s}^2$$
$$\leq \left(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2\right)\exp\left[\int_0^t C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}})dt\right].$$

Therefore, the solution exists up to time T as long as

$$\int_0^T C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})(\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1,2\alpha\}}})dt < +\infty. \qquad (3.11)$$

From (1.1), we get $(\partial_t + u\cdot\nabla)\rho = -(\nabla\cdot u)\rho$. Therefore, we can bound $\rho(t,x)$ by

$$\rho_{\min}(0)\exp\left[-\int_0^t \|\nabla\cdot u(\tau,\cdot)\|_{L^\infty}d\tau\right] \leq \rho(t,x) \leq \|\rho_0\|_{L^\infty}\exp\left[\int_0^t \|\nabla\cdot u(\tau,\cdot)\|_{L^\infty}d\tau\right].$$

Hence, if (2.7) holds, $\rho_{\min}^{-1}(t)$ is bounded for $t \in [0, T]$. Also, using the relation (2.1), we have that $\|\sigma(t,\cdot)\|_{L^\infty}$ is bounded for $t \in [0, T]$. Hence, $C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})$ is bounded by a universal constant, and condition (3.11) is reduced to (2.7). This finishes the proof. □

4. Global regularity for small data. In this section, we discuss the global well-posedness of the system (2.3)-(2.5), with small initial data

$$\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 \le \delta_0^2,$$

for a small parameter $\delta_0 > 0$ to be chosen.

We will show that the solution stays small. In particular, the following lemma suffices to show Theorem 2.2.

LEMMA 4.1 (Propagation of smallness). *Assume* (σ, u) *is a classical solution of* (2.3)-(2.5), *such that*

$$\sup_{0 \le t \le T} \left(\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 \right) \le \delta^2, \tag{4.1}$$

where $\delta > 0$ *is sufficiently small. Then the solution stays small, namely, there exists a universal constant* $C_0 > 1$, *such that*

$$\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 \le C_0^2(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2), \quad \forall\, t \in [0, T]. \tag{4.2}$$

Proof of Theorem 2.2. Let δ be the small parameter in Lemma 4.1. Let $T_{\max}$ be the maximum time that the solution stays small, namely

$$T_{\max} := \inf \left\{ t \ge 0 \;:\; \|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 > \delta^2 \right\}.$$

Suppose $T_{\max} < \infty$, then by continuity argument,

$$\|\sigma(T_{\max}, \cdot)\|_{H^s}^2 + \|u(T_{\max}, \cdot)\|_{H^s}^2 = \delta^2..$$

However, if we pick $\delta_0 < \frac{\delta}{C_0}$, from Lemma 4.1 we get

$$\|\sigma(T_{\max}, \cdot)\|_{H^s}^2 + \|u(T_{\max}, \cdot)\|_{H^s}^2 \le C_0^2(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2) \le C_0^2 \delta_0^2 < \delta^2.$$

This leads to a contradiction. Therefore, $T_{\max} = +\infty$, finishing the proof. $\square$

We are left to show Lemma 4.1. Recall the H^s estimate (combination of Lemmas 3.1 and 3.2)

$$\frac{d}{dt}\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2 \right) + \beta\|u\|_{H^s}^2 + \frac{1}{4}\|\mathcal{L}^\alpha u\|_{H^s}^2 \le C\delta(\|u\|_{H^s}^2 + \|\nabla\sigma\|_{H^{s-1}}^2). \tag{4.3}$$

It is slightly different from (3.10), as we will comment in the following.

First, the smallness condition (4.1) and Sobolev embedding implies

$$\|\nabla u\|_{L^\infty} + \|\sigma\|_{C^{\max\{1, 2\alpha\}}} < C\delta.$$

Second, picking δ small enough, we have $\rho_{\min} > \frac{1}{2}$ and $\|\sigma\|_{L^\infty} < C\delta$. Then, $C(\|\sigma\|_{L^\infty}, \rho_{\min}^{-1})$ has a uniform bound. Therefore, we can drop its dependence on σ. Finally, as we assume $s > 2 - \alpha$ so that $s + \alpha - 1 \in (1, s)$, the term $\|\mathcal{L}^{s+\alpha-1}\sigma\|_{L^2}$ in (3.3) can be controlled by $\|\nabla\sigma\|_{H^{s-1}}$. It does not depend on $\|\sigma\|_{L^2}$. This is important as we do not have dissipation estimate on $\|\sigma\|_{L^2}$.

To show (4.2), we need to control the right hand side of (4.3).

The term $\|u\|_{H^s}^2$ can be controlled by the damping or the dissipation. If $\beta > 0$, then we can pick $\delta < \frac{\beta}{2C}$, such that

$$C\delta\|u\|_{H^s}^2 \leq \frac{\beta}{2}\|u\|_{H^s}^2.$$

If $\beta = 0$, we can use the dissipation term to control $\|\nabla u\|_{H^{s-1}}^2$. The $\|u\|_{L^2}^2$ can only be controlled in the case $\Omega = \mathbb{T}^N$ by Poincare inequality $\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$, which implies $\|u\|_{H^s} \leq C\|\mathcal{L}^\alpha u\|_{H^s}$. Similarly, we can pick δ small enough so that

$$C\delta\|u\|_{H^s}^2 \leq \frac{1}{8}\|\mathcal{L}^\alpha u\|_{H^s}^2.$$

To sum up, there exists a positive number $\nu > 0$, such that

$$\frac{d}{dt}\left(\|\sigma\|_{H^s}^2 + \|u\|_{H^s}^2\right) + \nu\|u\|_{H^{s+\alpha}}^2 \leq C\delta\|\nabla\sigma\|_{H^{s-1}}^2. \tag{4.4}$$

To control the remaining term $\|\nabla\sigma\|_{H^{s-1}}^2$, we adopt the idea introduced in [27], using cross terms to obtain dissipation estimates for σ.

LEMMA 4.2 (Dissipation estimates for σ).

$$\frac{d}{dt}\int u \cdot \nabla\sigma\mathrm{d}x + \frac{3\sqrt{\gamma}}{4}\|\nabla\sigma\|_{L^2}^2 \leq C\|u\|_{H^s}^2, \tag{4.5}$$

$$\frac{d}{dt}\int \mathcal{L}^{s-1}u \cdot \nabla\mathcal{L}^{s-1}\sigma\mathrm{d}x + \frac{\sqrt{\gamma}}{2}\|\mathcal{L}^s\sigma\|_{L^2}^2 \leq C\|u\|_{H^{s+\alpha}}^2 + C\delta\|\nabla\sigma\|_{L^2}^2. \tag{4.6}$$

Proof. First, we can directly calculate to obtain

$$\frac{d}{dt}\int u \cdot \nabla\sigma\mathrm{d}x = -\int \sigma_t\nabla \cdot u\mathrm{d}x + \int u_t \cdot \nabla\sigma\mathrm{d}x.$$

The two terms separately are

$$-\int \sigma_t\nabla \cdot u\mathrm{d}x = \int \left(u \cdot \nabla\sigma + \left(\frac{\gamma-1}{2}\sigma + \sqrt{\gamma}\right)\nabla \cdot u\right)(\nabla \cdot u)\mathrm{d}x$$

$$\leq \|u\|_{L^\infty}\|\nabla\sigma\|_{L^2}\|\nabla \cdot u\|_{L^2} + \left(\frac{\gamma-1}{2}\|\sigma\|_{L^\infty} + \sqrt{\gamma}\right)\|\nabla \cdot u\|_{L^2}^2$$

$$\leq C\|\nabla \cdot u\|_{L^2}^2 + C\delta\|\nabla\sigma\|_{L^2}^2$$

and

$$\int u_t \cdot \nabla \sigma \mathrm{d}x = -\sqrt{\gamma} \|\nabla \sigma\|_{L^2}^2 - \beta \int u \cdot \nabla \sigma \mathrm{d}x$$
$$- \int u \cdot \nabla u \cdot \nabla \sigma \mathrm{d}x - \frac{\gamma - 1}{2} \int \sigma \nabla \sigma \cdot \nabla \sigma \mathrm{d}x - \int \mathcal{L}^{2\alpha} u \cdot \nabla \sigma \mathrm{d}x$$
$$- \int \mathcal{L}^{2\alpha}((\rho(\sigma) - 1)u) \cdot \nabla \sigma \mathrm{d}x + \int u \mathcal{L}^{2\alpha}(\rho(\sigma) - 1) \cdot \nabla \sigma \mathrm{d}x$$
$$\leq -\frac{7\sqrt{\gamma}}{8} \|\nabla \sigma\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|u \cdot \nabla u\|_{L^2}^2 + \|\mathcal{L}^{2\alpha} u\|_{L^2}^2)$$
$$+ C(\|\mathcal{L}^{2\alpha}((\rho(\sigma) - 1)u)\|_{L^2}^2 + \|u\mathcal{L}^{2\alpha}(\rho(\sigma) - 1)\|_{L^2}^2)$$
$$\leq -\frac{7\sqrt{\gamma}}{8} \|\nabla \sigma\|_{L^2}^2 + C\|u\|_{H^{2\alpha}}^2$$
$$+ C\left(\|\mathcal{L}^{2\alpha} u\|_{L^2}\|\rho(\sigma) - 1\|_{L^\infty} + \|u\|_{L^\infty}\|\mathcal{L}^{2\alpha}(\rho(\sigma) - 1)\|_{L^2}\right)^2$$
$$\leq -\frac{7\sqrt{\gamma}}{8} \|\nabla \sigma\|_{L^2}^2 + C\|u\|_{H^s}^2.$$

The fractional Leibniz rule (2.14) is used in the second last inequality, followed by the estimates

$$\|\mathcal{L}^{2\alpha}(\rho(\sigma) - 1)\|_{L^2} \leq C\|\sigma\|_{\dot{H}^{2\alpha}} < \delta, \quad \|u\|_{L^\infty} \leq C\|u\|_{H^s}.$$

Sum up the above estimates, and choose δ small enough so that $C\delta\|\nabla\sigma\|_{L^2}^2$ in the first term is absorbed by $\frac{\sqrt{\gamma}}{8}\|\nabla\sigma\|_{L^2}^2$ in the second term. We end up with (4.5).

Now we prove (4.6), by a direct computation we obtain

$$\frac{d}{dt} \int \mathcal{L}^{s-1} u \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x = -\int \mathcal{L}^{s-1}\sigma_t \mathcal{L}^{s-1}\nabla \cdot u \mathrm{d}x + \int \mathcal{L}^{s-1} u_t \cdot \nabla \mathcal{L}^{s-1}\sigma \mathrm{d}x.$$

For the first term, apply the Leibniz rule (2.14) and get

$$- \int \mathcal{L}^{s-1}\sigma_t \mathcal{L}^{s-1}\nabla \cdot u \mathrm{d}x$$
$$= \sqrt{\gamma} \int |\mathcal{L}^{s-1}\nabla \cdot u|^2 \mathrm{d}x + \int \mathcal{L}^{s-1}(u \cdot \nabla\sigma)\mathcal{L}^{s-1}\nabla \cdot u \mathrm{d}x$$
$$+ \frac{\gamma - 1}{2} \int \mathcal{L}^{s-1}(\sigma\nabla \cdot u)\mathcal{L}^{s-1}\nabla \cdot u \mathrm{d}x$$
$$\leq C\|\mathcal{L}^s u\|_{L^2}^2 + C\|\mathcal{L}^s u\|_{L^2}\left(\|\mathcal{L}^{s-1}u\|_{L^2}\|\nabla\sigma\|_{L^\infty} + \|\mathcal{L}^{s-1}\nabla\sigma\|_{L^2}\|u\|_{L^\infty}\right)$$
$$+ C\|\mathcal{L}^s u\|_{L^2}\left(\|\mathcal{L}^{s-1}\nabla \cdot u\|_{L^2}\|\sigma\|_{L^\infty} + \|\mathcal{L}^{s-1}\sigma\|_{L^2}\|\nabla \cdot u\|_{L^\infty}\right)$$
$$\leq C\|\mathcal{L}^s u\|_{L^2}^2 + C\delta(\|\mathcal{L}^{s-1}u\|_{L^2}^2 + \|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\mathcal{L}^{s-1}\sigma\|_{L^2}^2).$$
$$\leq C\|u\|_{H^s}^2 + C\delta(\|\mathcal{L}^s\sigma\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2).$$

For the second term,

$$\int \mathcal{L}^{s-1} u_t \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x$$

$$= - \sqrt{\gamma} \|\nabla \mathcal{L}^{s-1} \sigma\|_{L^2}^2 - \beta \int \mathcal{L}^{s-1} u \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x - \int \mathcal{L}^{s-1+2\alpha} u \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x$$

$$- \int \mathcal{L}^{s-1} (u \cdot \nabla u) \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x - \frac{\gamma-1}{2} \int \mathcal{L}^{s-1} (\sigma \nabla \sigma) \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x$$

$$- \int \mathcal{L}^{s-1+2\alpha} ((\rho(\sigma)-1)u) \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x + \int \mathcal{L}^{s-1} (u \mathcal{L}^{2\alpha} (\rho(\sigma)-1)) \cdot \nabla \mathcal{L}^{s-1} \sigma \mathrm{d}x$$

$$\le - \frac{3\sqrt{\gamma}}{4} \|\mathcal{L}^s \sigma\|_{L^2}^2 + C \|\mathcal{L}^{s-1} u\|_{L^2}^2 + C \|\mathcal{L}^{s-1+2\alpha} u\|_{L^2}^2$$

$$+ C(\|\mathcal{L}^{s-1}(u \cdot \nabla u)\|_{L^2}^2 + \|\mathcal{L}^{s-1}(\sigma \nabla \sigma)\|_{L^2}^2)$$

$$+ C \left\|\mathcal{L}^{s-1+2\alpha}((\rho(\sigma)-1)u) - \mathcal{L}^{s-1}(u\mathcal{L}^{2\alpha}(\rho(\sigma)-1))\right\|_{L^2}^2 =: \sum_{i=1}^{3} K_i.$$

Since $s - 1 + 2\alpha < s + \alpha$, K_1 can be simply estimated by

$$K_1 \le -\frac{3\sqrt{\gamma}}{4} \|\mathcal{L}^s \sigma\|_{L^2}^2 + C\|u\|_{H^{s+\alpha}}^2,$$

We estimate K_2 by using the fractional Leibniz rule (2.14),

$$K_2 \le C \Big(\|\mathcal{L}^{s-1} u\|_{L^2}^2 \|\nabla u\|_{L^\infty}^2 + \|\mathcal{L}^{s-1} \nabla u\|_{L^2}^2 \|u\|_{L^\infty}^2$$

$$+ \|\mathcal{L}^{s-1} \sigma\|_{L^2}^2 \|\nabla \sigma\|_{L^\infty}^2 + \|\mathcal{L}^{s-1} \nabla \sigma\|_{L^2}^2 \|\sigma\|_{L^\infty}^2 \Big)$$

$$\le C\delta(\|u\|_{H^s}^2 + \|\mathcal{L}^s \sigma\|_{L^2}^2 + \|\nabla \sigma\|_{L^2}^2).$$

For K_3, we use a similar cancelation as (3.8), and apply commutator estimates

$$K_3 \le C\|[\mathcal{L}^{s-1+2\alpha}, u](\rho(\sigma)-1)\|_{L^2}^2 + C\|[\mathcal{L}^{s-1}, u]\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\|_{L^2}^2$$

$$\le C \left(\|\mathcal{L}^{s-1+2\alpha} u\|_{L^2} \|\rho(\sigma)-1\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\mathcal{L}^{s-2+2\alpha}(\rho(\sigma)-1)\|_{L^2} \right)^2$$

$$+ C \left(\|\mathcal{L}^{s-1} u\|_{L^2} \|\mathcal{L}^{2\alpha}(\rho(\sigma)-1)\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\mathcal{L}^{s-2+2\alpha}(\rho(\sigma)-1)\|_{L^2} \right)^2$$

$$\le C\delta \|u\|_{H^s}^2 + C\|\nabla u\|_{L^\infty}^2 \|\mathcal{L}^{s-2+2\alpha} \sigma\|_{L^2}^2 \le C\delta \|u\|_{H^s}^2.$$

Note that as $s > \frac{N}{2} + \max\{1, 2\alpha\}$ and $s > 2 - \alpha$, one can easily check $0 < s - 2 + 2\alpha < s$. In the last inequality, we have used $\|\mathcal{L}^{s-2+2\alpha} \sigma\|_{L^2} < \|\sigma\|_{H^s} < \delta$, and $\|\nabla u\|_{L^\infty} \le C\|u\|_{H^s}$.

   Collect all the estimates of K_i, $i = 1, 2, 3$, and choose δ small so that the $\|\mathcal{L}^s \sigma\|_{L^2}$ term in K_2 be absorbed in to K_1, We end up with the desired estimate (4.6). $\square$

   Adding (4.5) and (4.6) and pick δ small enough, we get

$$\frac{d}{dt} \int \left(u \cdot \nabla \sigma + \mathcal{L}^{s-1} u \cdot \nabla \mathcal{L}^{s-1} \sigma \right) \mathrm{d}x + \frac{\sqrt{\gamma}}{2} \|\nabla \sigma\|_{H^{s-1}}^2 \le C\|u\|_{H^s}^2, \qquad (4.7)$$

which can be used to control the term $\|\nabla \sigma\|_{H^{s-1}}$ in (4.4). Indeed, multiplying (4.7) by $\frac{4C\delta}{\sqrt{\gamma}}$, adding it to (4.4), and choosing δ small, we obtain

$$\frac{d}{dt} Y(t) + \frac{\nu}{2} \|u\|_{H^{s+\alpha}}^2 + C\delta \|\nabla \sigma\|_{H^{s-1}}^2 \le 0, \qquad (4.8)$$

where

$$Y(t) := \|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 + \frac{4C\delta}{\sqrt{\gamma}} \int (u \cdot \nabla\sigma + \mathcal{L}^{s-1}u \cdot \nabla\mathcal{L}^{s-1}\sigma) \mathrm{d}x.$$

Note that if δ is small enough, $Y(t) \geq 0$ and is equivalent to $\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2$. Indeed, there exists a constant $C_0 > 1$ such that

$$C_0^{-1}\left(\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2\right) \leq Y(t) \leq C_0\left(\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2\right). \qquad (4.9)$$

Integrating (4.8) directly in time, we have

$$Y(t) + \int_0^t \left(\frac{\nu}{2}\|u(\tau, \cdot)\|_{H^{s+\alpha}}^2 + C\delta\|\nabla\sigma(\tau, \cdot)\|_{H^{s-1}}^2\right) \mathrm{d}\tau \leq Y(0). \qquad (4.10)$$

With the help of (4.9), we obtain that for any $t \in [0, T]$,

$$\|\sigma(t, \cdot)\|_{H^s}^2 + \|u(t, \cdot)\|_{H^s}^2 \leq C_0 Y(t) \leq C_0 Y(0) \leq C_0^2\left(\|\sigma_0\|_{H^s}^2 + \|u_0\|_{H^s}^2\right).$$

This ends the proof of Lemma 4.1.

REMARK 4.1. Another outcome of (4.10) is that

$$\int_0^\infty \left(\|u(t, \cdot)\|_{H^{s+\alpha}}^2 + \|\nabla\sigma(t, \cdot)\|_{H^{s-1}}^2\right)\mathrm{d}t \leq Y(0) < +\infty.$$

Therefore, $\|u\|_{H^s}$ and $\|\nabla\sigma\|_{H^{s-1}}$ decay to zero as $t \to \infty$. In particular, $\|u\|_{L^\infty} \to 0$ implies the flocking phenomenon (to be more precise, velocity alignment). Note that our estimates does not imply any decay for $\|\sigma\|_{L^2}$. The convergence rate of $\|u\|_{H^s}$ and $\|\nabla\sigma\|_{H^{s-1}}$ will also depend on the long time behavior of $\|\sigma\|_{L^2}$. More discussions and results for the case $\Omega = \mathbb{T}^N$ will be offered in the next section.

5. Long time behavior. In this part, we study the large-time behavior of global classical solutions (ρ, u) to the system (1.1)-(1.3) for the case $\Omega = \mathbb{T}^N$. Without lose of generality we assume $|\mathbb{T}^N| = 1$.

Firstly, in order to obtain the dissipation estimate of ρ, we denote the function

$$h(\rho) = \int_1^\rho \frac{z^\gamma - 1}{z^2}\mathrm{d}z = \begin{cases} \rho\ln\rho - \rho + 1, & \gamma = 1, \\ \dfrac{1}{\gamma - 1}\rho^\gamma - \dfrac{\gamma}{\gamma - 1}\rho + 1, & \gamma > 1, \end{cases}$$

so that $h(1) = h'(1) = 0$ and $p'(\rho) = h''(\rho)\rho$. One can easily check that when ρ is close to 1, $h(\rho) \sim (\rho - 1)^2$. Indeed, we state the following lemma.

LEMMA 5.1 ([14]). *Let $\delta > 0$ be a small parameter, and $|\rho - 1| < \delta$. Then, there exists constants c_1, c_2, depending only on δ, such that*

$$c_1(\rho - 1)^2 \leq h(\rho) \leq c_2(\rho - 1)^2. \qquad (5.1)$$

Now, we are ready to prove Theorem 2.3.

Step 1: Dissipation of the physical energy. Let (ρ, u) be a global classical solutions to (1.1)-(1.2), we will establish the decay estimate of the physical energy:

$$\frac{d}{dt}\int\Big(\frac{1}{2}\rho|u|^2+h(\rho)\Big)dx+\beta\int\rho|u|^2dx+\frac{1}{2}\iint\phi(x-y)(u(x)-u(y))^2\rho(x)\rho(y)dxdy\leq 0.$$
(5.2)

Using the symmetry of $\phi(\cdot)$ and equation (1.1)-(1.2), it is obtained that

$$\frac{d}{dt}\int\rho|u|^2dx=\int\rho_t|u|^2+2\int\rho u\cdot u_t dx$$

$$=-\int\mathrm{div}(\rho u)|u|^2dx-2\int\rho u\Big(u\cdot\nabla u+\frac{\nabla p(\rho)}{\rho}+\beta u+\int\phi(x-y)(u(x)-u(y))\rho(y)dy\Big)$$

$$=-2\int u\cdot\nabla p(\rho)dx-2\beta\int\rho|u|^2dx-2\iint\rho(x)u(x)\phi(x-y)(u(x)-u(y))\rho(y)dxdy$$

$$=-2\int u\cdot\nabla p(\rho)dx-2\beta\int\rho|u|^2dx-\iint\phi(x-y)(u(x)-u(y))^2\rho(x)\rho(y)dxdy.$$
(5.3)

Furthermore, applying the definition of $h(\rho)$, we can compute that

$$\frac{d}{dt}\int h(\rho)dx=\int h'(\rho)\rho_t dx=-\int h'(\rho)\mathrm{div}(\rho u)dx=\int h''(\rho)\rho\nabla\rho\cdot u dx=\int u\cdot\nabla p(\rho)dx.$$
(5.4)

Combining (5.3)-(5.4), we obtain (5.2).

Step 2: Dissipation of $\|\rho-1\|_{L^2}$. We first define a stream function ψ which solves the Poisson equation

$$-\Delta\psi=\rho-1,\qquad\int_{\mathbb{T}^N}\psi dx=0.$$

It is uniquely defined on $\mathbb{T}^N$ since $\rho-1$ has zero mean (condition (2.10)).

Similar as Lemma 4.2, we introduce a small cross-term to the physical energy, and define

$$V_\varepsilon=\int\Big(\frac{1}{2}\rho|u|^2+h(\rho)+\varepsilon\rho u\cdot\nabla\psi\Big)dx.$$

Note that the cross term can be controled by

$$\left|\int\rho u\cdot\nabla\psi dx\right|\leq\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^2}\|\nabla\psi\|_{L^2}\leq\frac{1}{2}\int\rho|u|^2dx+\frac{C\|\rho\|_{L^\infty}}{2}\int(\rho-1)^2dx,$$
(5.5)

where Poincaré inequality is used so that $\|\nabla\psi\|_{L^2}\leq C\|\nabla^{\otimes 2}\psi\|_{L^2}\leq C\|\rho-1\|_{L^2}$. Then, it follows from (5.1), (5.5) and for a sufficient small ε that there exists a constant $C_1>0$ which depends on ε and $\|\rho\|_{L^\infty}$, such that

$$\frac{1}{C_1}\Big(\int\rho|u|^2dx+\int|\rho-1|^2dx\Big)\leq V_\varepsilon\leq C_1\Big(\int\rho|u|^2dx+\int(\rho-1)^2dx\Big),$$
(5.6)

namely, V_ε is equivalent to the physical energy.

Applying (5.2), It is easy to check that

$$\frac{d}{dt}V_\varepsilon+W_\varepsilon\leq 0,$$
(5.7)

where

$$W_\varepsilon = \beta \int \rho |u|^2 \mathrm{d}x + \frac{1}{2} \iint \phi(x-y)(u(x)-u(y))^2 \rho(x)\rho(y)\mathrm{d}x\mathrm{d}y + \varepsilon \int \nabla\psi \cdot \nabla p(\rho)\mathrm{d}x$$

$$+ \varepsilon \int \nabla\psi \cdot \nabla(\rho u \otimes u)\mathrm{d}x - \varepsilon \int \rho u \cdot \nabla\psi_t \mathrm{d}x$$

$$+ \varepsilon \iint \nabla\psi(x) \cdot \phi(x-y)(u(x)-u(y))\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{6} L_i.$$

We investigate further for the terms L_1 to L_6 in W_ε. Notice that L_1 and L_2 are positive. The positivity of L_3 can be obtained in the following:

$$L_3 = -\varepsilon \int \Delta\psi(p(\rho)-p(1))\mathrm{d}x = \varepsilon \int (\rho-1)(p(\rho)-p(1))\mathrm{d}x \geq \varepsilon \int (\rho-1)^2\mathrm{d}x,$$

where we have used the fact that $\frac{\rho^\gamma-1}{\rho-1} \geq 1$ for any $\rho \geq 0$ and $\gamma \geq 1$. This term produces dissipation for $\|\rho-1\|_{L^2}$.

L_4 can be controlled by the dissipation as follows

$$|L_4| = \varepsilon \left| \int \nabla \otimes \nabla\psi : (\rho u \otimes u)\mathrm{d}x \right| \leq \varepsilon \|\rho-1\|_{L^2} \|\rho u \otimes u\|_{L^2}$$

$$\leq \frac{\varepsilon}{4} \|\rho-1\|_{L^2}^2 + \varepsilon \|\rho u \otimes u\|_{L^2}^2 \leq \frac{\varepsilon}{4} \|\rho-1\|_{L^2}^2 + \varepsilon \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \int \rho |u|^2 \mathrm{d}x.$$

The term L_5 can be estimated in the following

$$|L_5| = \left| \varepsilon \int \rho u \cdot \nabla(-\Delta)^{-1}\nabla \cdot (\rho u)\mathrm{d}x \right| \leq \varepsilon \|\rho u\|_{L^2} \|\nabla(-\Delta)^{-1}\nabla \cdot (\rho u)\|_{L^2}$$

$$\leq \varepsilon \|\rho u\|_{L^2}^2 \leq \varepsilon \|\rho\|_{L^\infty} \int \rho |u|^2 \mathrm{d}x.$$

Finally, for L_6 we have

$$|L_6| = \frac{\varepsilon}{2} \left| \iint (\nabla\psi(x)-\nabla\psi(y))\phi(x-y) \cdot (u(x)-u(y))\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y \right|$$

$$\leq \theta\varepsilon \iint \frac{|\nabla\psi(x)-\nabla\psi(y)|^2}{|x-y|^{n+2\alpha}}\mathrm{d}x\mathrm{d}y$$

$$+ \frac{C}{\theta}\varepsilon \|\rho\|_{L^\infty}^2 \iint \phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y,$$

where the first part

$$\iint \frac{|\nabla\psi(x)-\nabla\psi(y)|^2}{|x-y|^{n+2\alpha}}\mathrm{d}x\mathrm{d}y = \|\nabla\psi\|_{\dot{H}^\alpha}^2 \leq C\|\psi\|_{\dot{H}^2}^2 \leq C\|\rho-1\|_{L^2}^2, \quad \forall\, \alpha \in (0,1).$$

Then, choosing θ appropriately, we obtain that

$$|L_6| \leq \frac{\varepsilon}{4} \|\rho-1\|_{L^2}^2 + C\varepsilon \|\rho\|_{L^\infty}^2 \iint \phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y.$$

Collecting all the estimates of L_i, and picking ε small enough (which depends on $\|\rho\|_{L^\infty}$ and $\|u\|_{L^\infty}$), we get a lower bound on W_ε

$$W_\varepsilon \geq \left(\beta - C\varepsilon\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2 - \varepsilon\|\rho\|_{L^\infty}\right)\int \rho|u|^2\mathrm{d}x$$

$$+ \left(\frac{1}{2} - C\varepsilon\|\rho\|_{L^\infty}^2\right)\iint \phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y + \frac{\varepsilon}{2}\int(\rho-1)^2\mathrm{d}x.$$

$$\geq \frac{\beta}{2}\int\rho|u|^2\mathrm{d}x + \frac{1}{4}\iint\phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y + \frac{\varepsilon}{2}\int(\rho-1)^2\mathrm{d}x.$$

$$\tag{5.8}$$

When there is no damping, namely $\beta = 0$, we can use the second term in (5.8) to produce dissipation for the kinetic energy.

LEMMA 5.2. *There exists a constant $\tilde{C} > 0$, depending on α, such that*

$$\frac{1}{4}\iint\phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y \geq \tilde{C}\int\rho|u|^2\mathrm{d}x. \tag{5.9}$$

Proof. Write the integral on the left hand side of (5.9) in $\mathbb{T}^{2N}$

$$\frac{1}{4}\iint_{\mathbb{T}^{2N}}\phi(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{4}\iint_{\mathbb{T}^{2N}}\phi_P(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y$$

with periodic kernel ϕ_P

$$\phi_P(x) = \sum_{k\in\mathbb{Z}^N}\phi(x+k), \quad \forall\, x\in\mathbb{T}^N.$$

Clearly, ϕ_P has a positive lower bound (e.g. $\phi_P(x) > \min_{\|x\|_\infty < \frac{1}{2}}\phi(x) > 0$). Let us denote the bound by ϕ_m. It only depends on α (and $\mathbb{T}^N$ as well, but we have set $|\mathbb{T}^N| = 1$ here). Then, we have

$$\frac{1}{4}\iint_{\mathbb{T}^{2N}}\phi_P(x-y)\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y$$

$$\geq \frac{\phi_m}{4}\iint_{\mathbb{T}^{2N}}\rho(x)\rho(y)(u(x)-u(y))^2\mathrm{d}x\mathrm{d}y$$

$$= \frac{\phi_m}{2}\left(\int_{\mathbb{T}^N}\rho(x)\mathrm{d}x\int_{\mathbb{T}^N}\rho(x)|u(x)|^2\mathrm{d}x - \left|\int_{\mathbb{T}^N}\rho(x)u(x)\mathrm{d}x\right|^2\right) = \frac{\phi_m}{2}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x,$$

where we have used (2.10) and (2.11). This implies (5.9) with $\tilde{C} = \frac{\phi_m}{2}$. $\square$

Thus, we deduce that

$$W_\varepsilon \geq \left(\frac{\beta}{2} + \tilde{C}\right)\int\rho|u|^2\mathrm{d}x + \frac{\varepsilon}{2}\int(\rho-1)^2\mathrm{d}x. \tag{5.10}$$

Step 3: The decay estimate. Combining the inequalities in (5.6), (5.7) and (5.10), we deduce that there exists a constant $\mu > 0$, which depend on γ and ε, such that

$$\frac{d}{dt}V_\varepsilon + \mu V_\varepsilon \leq 0.$$

Applying Gronwall's inequality and (5.6), we get the desired exponential decay

$$\int \rho|u|^2 \mathrm{d}x + \int (\rho-1)^2 \mathrm{d}x \le C_1 V_\varepsilon(t) \le C_1 V_\varepsilon(0) e^{-\mu t}$$
$$\le C_1^2 \left(\int \rho_0 |u|_0^2 \mathrm{d}x + \int (\rho_0-1)^2 \mathrm{d}x \right) e^{-\mu t}.$$

This finishes the proof of (2.12),

Step 4: Decay on higher order norms. To obtain exponential decay for $\|\sigma\|_{H^s}$ and $\|u\|_{H^s}$, we start with the control on $\|\sigma\|_{L^2}$. From the relation (2.1), we have

$$\|\sigma\|_{L^2}^2 = \int (\sigma(\rho) - \sigma(1))^2 \mathrm{d}x \le \|\sigma'\|_{L^\infty(\mathrm{supp}(\rho))}^2 \int (\rho-1)^2 \mathrm{d}x \le C\|\rho-1\|_{L^2}^2,$$

where $\sigma'(\rho) = \sqrt{\gamma}\rho^{\frac{\gamma-3}{2}}$ is bounded provided that ρ is bounded and $\rho_{\min} > 0$. This is guaranteed by Theorem 2.2 with a small enough δ_0. Therefore, we have

$$\|\sigma(t,\cdot)\|_{L^2}^2 \le C e^{-\mu t}.$$

Next, recall the estimate (4.8) and add $C\delta\|\sigma\|_{L^2}^2$ on both sides of the inequality

$$\frac{d}{dt}Y(t) + \frac{\nu}{2}\|u\|_{H^{s+\alpha}}^2 + C\delta\|\sigma\|_{H^s}^2 \le C\delta\|\sigma\|_{L^2}^2,$$

Apply (4.9) and get

$$\frac{d}{dt}Y(t) + \mu_1 Y(t) \le C\delta\|\sigma\|_{L^2}^2 \le C\delta e^{-\mu t},$$

where we can choose $\mu_1 = C_0^{-1}\min\{\frac{\nu}{2}, C\delta\}$. This implies

$$Y(t) \le \big(Y(0) + C\delta\big)e^{-\min\{\mu_1,\mu\}t}.$$

Using (4.9) again, we conclude

$$\|\sigma(t,\cdot)\|_{H^s}^2 + \|u(t,\cdot)\|_{H^s}^2 \le C_0 Y(t) \le C_0\big(Y(0) + C\delta\big)e^{-\min\{\mu_1,\mu\}t} \le C\delta_0 e^{-\min\{\mu_1,\mu\}t}.$$

This ends the proof of (2.13).

REFERENCES

[1] N. BOERS AND P. PICKL, *On mean field limits for dynamical systems,* J. Stat. Phys., 164:1 (2016), pp. 1–16.

[2] J. A. CANIZO, J. A. CARRILLO AND J. ROSADO, *A well-posedness theory in measures for some kinetic models of collective motion,* Math. Mod. Meth. Appl. Sci., 21 (2011), pp. 515–539.

[3] J. A. CARRILLO, Y.-P. CHOI AND S. P. PEREZ, *A review on attractiverepulsive hydrodynamics for consensus in collective behavior*, In Active Particles, Volume 1, pp. 259–298. Birkhuser, Cham, 2017.

[4] J. A. CARRILLO, Y.-P. CHOI, E. TADMOR AND C. TAN, *Critical thresholds in 1D Euler equations with nonlocal forces,* Math. Mod. Meth. Appl. Sci., 26:1 (2016), pp. 185–206.

[5] J. A. CARRILLO, M. FORNASIER, G. TOSCANI AND F. VECIL, *Particle, kinetic, and hydrodynamic models of swarming,* In Mathematical modeling of collective behavior in socio-economic and life sciences, pp. 297–336. Birkhuser, Boston, 2010.

[6] J. CHE, L. CHEN, S. GÖTTLICH AND J. WANG, *Existence of a classical solution to complex material flow problems,* Math. Meth. Appl. Sci., 39:14 (2016), pp. 4069–4081.

[7] L. Chen, S. Göttlich and Q. Yin, *Mean field limit and propagation of chaos for a pedestrian flow model,* J. Stat. Phys., 166:14 (2017), pp. 211–229.

[8] Y. P. Choi, *The global Cauchy problem for compressible Euler equations with a nonlocal dissipation,* Math. Mod. Meth. Appl. Sci., 29:1 (2019), pp. 185–207.

[9] P. Constantin, T. D. Drivas and R. Shvydkoy, *Entropy Hierarchies for equations of compressible fluids and self-organized dynamics,* SIAM J. Math. Anal., 52:3 (2020), pp. 3073–3092.

[10] F. Cucker and S. Smale, *Emergent behavior in flocks,* IEEE Trans. Automat. Control, 52 (2007), pp. 852–862.

[11] P. Degond, J. G. Liu and C. Ringhofer, *Evolution of the distribution of wealth in an economic environment driven by local Nash equilibria,* J. Stat. Phys., 154:3 (2014), pp. 751–780.

[12] T. Do, A. Kiselev, L. Ryzhik and C. Tan, *Global regularity for the fractional Euler alignment system,* Arch. Ration. Mech. Anal., 228:1 (2018), pp. 1–37.

[13] R. Etikyala, S. Goettlich, A. Klar and S. Tiwari, *Particle methods for pedestrian flow models: from microscopic to nonlocal continuum models,* Math. Mod. Meth. Appl. Sci., 24:12 (2014), pp. 2503–2523.

[14] D. Fang, R. Zi, and T. Zhang, *Decay estimates for isentropic compressible Navier-Stokes equations in bounded domain,* J. Math. Anal. Appl., 386 (2012), pp. 939–947.

[15] A. Figalli and M-J. Kang, *A rigorous derivation from the kinetic CuckerSmale model to the pressureless Euler system with nonlocal alignment,* Anal. PDE, 12:3 (2018), pp. 843–866.

[16] S.-Y. Ha and J.-G. Liu, *A simple proof of the Cucker-Smale flocking dynamics and mean-field limit,* Commun. Math. Sci., 7:2 (2009), pp. 297–325.

[17] M. Hauray and P. E. Jabin, *N-particles approximation of the Vlasov equations with singular potential,* Arch. Ration. Mech. Anal., 183:3 (2007), pp. 489–524.

[18] T. Karper, A. Mellet and K. Trivisa, *Hydrodynamic limit of the kinetic Cucker-Smale flocking model,* Math. Mod. Meth. Appl. Sci., 25 (2015), pp. 131–163.

[19] A. Kiselev and C. Tan, *Global regularity for 1D Eulerian dynamics with singular interaction forces,* SIAM J. Math. Anal., 50:6 (2018), pp. 6208–6229.

[20] D. Lazarovici and P. Pickl, *A mean field limit for the Vlasov-Poisson system,* Arch. Ration. Mech. Anal., 225:3 (2017), pp. 1201–1231.

[21] Y. Lee and C. Tan, *A sharp critical threshold for a traffic flow model with look-ahead dynamics,* arXiv preprint arXiv:1905.05090, 2019.

[22] D. Li, *On Kato-Ponce and fractional Leibniz,* Rev. Mat. Iberoam., 35:1 (2019), pp. 23–100.

[23] Q. Miao, C. Tan and L. Xue, *Global regularity for a 1D Euler-alignment system with misalignment,* arXiv preprint arXiv:2004.03652, 2020.

[24] R. Shvydkoy, *Global existence and stability of nearly aligned flocks,* J. Dyna. Diff. Equa., 31:4 (2019), pp. 2165–2175.

[25] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing,* Trans. Math. Appl., 1:1 (2017), tnx001.

[26] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing III. Fractional diffusion of order $0 < \alpha < 1$,* Physica D, 376-377 (2018), pp. 131–137.

[27] T. C. Sideris, B. Thomases and D. Wang, *Long time behavior of solutions to the 3D compressible Euler equations with damping,* Commun. Part. Diff. Eq,, 28:3-4 (2003), pp. 795–816.

[28] E. Tadmor and C. Tan, *Critical thresholds in flocking hydrodynamics with non-local alignment,* Philos. Trans. Roy. Soc. A, 372 (2014), 20130401.

[29] C. Tan, *Singularity formation for a fluid mechanics model with nonlocal velocity,* Commun. Math. Sci., 17:7 (2019), pp. 1779–1794.

[30] C. Tan, *On the Euler-alignment system with weakly singular communication weights,* Nonlinearity, 33:4 (2020), pp. 1907–1924.

[31] Z. Tan and Y. Wang, *Global solution and large-time behavior of the 3D compressible Euler equations with damping,* J. Diff. Equa., 254:4 (2013), pp. 1686–1704.

[32] L. Tong and L. Chen, *Decay estimate to a compressible Euler system with non-local velocity alignment,* Commun. Math. Sci., (2020), to appear.

[33] L. Tong, L. Chen, S. Göttlich, and S. Wang, *Global classical solution to compressible Euler system with velocity alignment,* arXiv preprint arXiv:1912.01374, 2019.

[34] W. Wang and T. Yang, *The pointwise estimates of solutions for Euler equations with damping in multi-dimensions,* J. Diff. Equa., 173:2 (2001), pp. 410–450.