

Mach limits in analytic spaces on exterior domains

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ABSTRACT. We address the Mach limit problem for the Euler equations in an exterior domain with analytic boundary. We first prove the existence of tangential analytic vector fields for the exterior domain with constant analyticity radii, and introduce an analytic norm in which we distinguish derivatives taken from different directions. Then we prove the uniform boundedness of the solutions in the analytic space on a time interval independent of the Mach number, and Mach limit holds in the analytic norm. The results extends more generally to Gevrey initial data with convergence in a Gevrey norm.

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1. Introduction

In this paper, we consider the Mach number limit for the compressible Euler equations for non-isentropic fluids with initial data and domains in an analytic or a Gevrey class. We address the problem of convergence in these strong norms under general assumptions on the data and the domain.

The incompressible limit concerns the passage from compressible fluids to incompressible fluids as the Mach number tends to zero. The first rigorous result on this singular limit problem can be traced back to Klainerman and Majda [KM1] (see also Ebin [E]), with a great deal of activity and progress in recent decades [A1, A3, A2, D1, D2, DG, DM, F, FN, H, LMa, M, As, FKM, I, Is1, Is2, Is3, KM2, MS, S1, S2, U]. A general approach to this problem is to first prove the existence of solutions on some time interval independent of the Mach number and then to show the convergence to solutions of the limiting equations when the Mach number tends to zero. A key ingredient in establishing a uniform time interval is a uniform upper bound, while in showing the convergence, the most critical issue is the vanishing of the acoustic waves. It is well-known that the analysis depends on various settings such as isentropic vs. non-isentropic, inviscid vs. viscous, well-prepared data vs. general data, or the whole space vs. domain with a boundary.

For the non-isentropic problem with general data, the non-isentropic Euler flows feature intriguing wave-transport structure, and the coefficients of governing equations (for instance, see E in (2.4) and (2.5)) depend on the

dependent variables, which makes the low Mach number limit a difficult problem. Métivier and Schochet in [MS] gave the first satisfactory answer by making use of the microlocal defect measure, and Alazard in [A1] extended the existence result to the case of domains with boundary and the convergence result for exterior domains. Both results were obtained in Sobolev spaces. In a recent work [JKL], we studied the non-isentropic problem with general analytic or Gevrey initial data in $\mathbb{R}^3$ and proved that the convergence holds in these strong norms.

The goal of this paper is to extend our previous results on analytic/Gevrey convergence in the zero Mach limit from [JKL] to the more difficult case of the domain with boundary. We adopt the configuration of an exterior domain since it is the domain where the Mach limit was established with convergence holding in Sobolev spaces (cf. [A1]). Some of the results in this paper, in particular the uniform bounds, however hold also for a bounded domain. One of the main difficulties in our approach is the existence of a complete family of analytic vector fields which respect to the boundary condition $v^\epsilon \cdot \nu|_{\partial\Omega} = 0$. In Theorem 3.1 below, we construct a family of tangential analytic vector fields for the exterior domain with constant analyticity radii. The existence of a family of complete tangential analytic vector fields was established by Komatsu in [K] for the case of a bounded domain. The completeness refers to the fact that the vector fields span the tangential space in the closure of the domain. In our proof of analytic boundedness, we require for the tangential fields to span only in a neighborhood of the boundary. In order to construct the necessary tangential fields, we use interior analytic hypoellipticity for the Dirichlet problem (cf. [LM]) and analytic regularization by the heat kernel. As an additional benefit, we obtain a low number of tangential vector fields (three) needed for the construction. In order to obtain uniform boundedness in analytic spaces, we first establish boundedness of the entropy, divergence, followed by then normal and tangential derivative reduction schemes for the velocity. The normal and tangential derivatives can be reduced by using elliptic regularity, which leads to the estimates of divergence component, curl component, pure time derivatives, commutators, and lower-order terms associated to the boundary. In order to obtain that the Mach limit holds, we need to change the approach from [JKL, Section 7] since the interpolation inequality used there results in boundary terms which cannot be handled. Instead, we use a simpler interpolation inequality (8.1) and a discrete dominated convergence theorem. As a byproduct of the Mach convergence, we obtain the analyticity of solutions of the stratified incompressible Euler equation in an exterior (or bounded) domain. For the unstratified version of the Euler equations, the analyticity was proven in [KV2] using different methods.

The paper is organized as follows. In Section 2, we recall from [MS, A1] symmetrization of the compressible Euler equations. In Section 3, we construct the tangential vector fields for the exterior domain and state the main results. The a priori estimate needed for the uniform analytic boundedness, stated in Lemma 3.5, is given at the end of Section 6 and relies on the bounds on entropy and the velocity given in previous sections. The proof of the second main theorem is given in Section 8. We emphasize that the construction of the system of analytic tangential fields also applies to the case of a bounded domain; only when proving the Mach limit convergence, we rely on the fact that the domain is external. All considerations also apply in the Gevrey norm, or more generally to the spaces used in the book by Lions and Magenes [LM]. The approach in this paper benefits from ideas in [CKV]. For other approaches to analyticity, we refer to [B, BB, Bi, BF, BGK, BoGK, DE, DL, FT, G, GK, KV1, KP, LO, OT].

2. The setting and notation

We address the incompressible limit for classical solutions of the compressible Euler equations for non-isentropic fluids in an exterior domain $\Omega \subseteq \mathbb{R}^3$ with analytic boundary $\partial\Omega$. More specifically, we consider the compressible Euler equations for an inviscid, non-isentropic fluid

$$\partial_t \rho + v \cdot \nabla \rho + \rho \nabla \cdot v = 0, \quad (2.1)$$

$$\rho (\partial_t v + v \cdot \nabla v) + \nabla P = 0, \quad (2.2)$$

$$\partial_t S + v \cdot \nabla S = 0, \quad (2.3)$$

where $\rho: \Omega \times [0, T) \rightarrow \mathbb{R}_+$ is the density, $v: \Omega \times [0, T) \rightarrow \mathbb{R}^3$ is the velocity, $P: \Omega \times [0, T) \rightarrow \mathbb{R}_+$ is the pressure, and $S: \Omega \times [0, T) \rightarrow \mathbb{R}$ is the entropy of the fluid. To close the system (2.1)–(2.3), we assume the equation of state

$$P = P(\rho, S).$$

For instance, in the case of ideal gas the equation of state reads

$$P(\rho, S) = \rho^\gamma e^S,$$

where $\gamma > 1$ is the adiabatic exponent. After some rescalings and a change of variables (cf. [MS, A1]), we consider the symmetrized version of the compressible Euler equations for non-isentropic fluids

$$E(S, \epsilon u)(\partial_t u + v \cdot \nabla u) + \frac{1}{\epsilon} L(\partial_x)u = 0, \quad (2.4)$$

$$\partial_t S + v \cdot \nabla S = 0, \quad (2.5)$$

where $u = (p, v)^T$ and

$$E(S, \epsilon u) = \begin{pmatrix} a(S, \epsilon u) & 0 \\ 0 & r(S, \epsilon u)I_3 \end{pmatrix}, \quad L(\partial_x) = \begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{pmatrix}.$$

The parameter $\epsilon > 0$ represents the Mach number. In agreement with the equation of state of the ideal gas, we adopt the assumption

$$a(S, \epsilon u) = f_1(S)g_1(\epsilon u)$$

and

$$r(S, \epsilon u) = f_2(S)g_2(\epsilon u),$$

where f_1, f_2, g_1 , and g_2 are positive entire real-analytic functions. We impose the impermeability boundary condition

$$v \cdot \nu|_{\partial\Omega} = 0. \quad (2.6)$$

Due to presence of the boundary, we also require some compatibility conditions. Since the matrix $E(S, \epsilon u)$ is invertible, we obtain

$$\partial_t v = -v \cdot \nabla v + \frac{1}{\epsilon r} \nabla p. \quad (2.7)$$

Differentiating (2.7) with respect to time recursively, we get

$$\partial_t^k v(0) = A_k(u(0), S(0)),$$

for some functions A_k , where $k \in \mathbb{N}_0$. We say that the initial data satisfy the compatibility condition of all orders if

$$\nu \cdot A_k(u(0), S(0))|_{\partial\Omega} = 0, \quad k \in \mathbb{N}_0.$$

One may readily check that the above condition is satisfied in the smooth or non-analytic Gevrey case for initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ vanishing in a neighborhood of the boundary $\partial\Omega$.

3. Analytic vector field in an exterior domain and the main results

Assume that $\Omega \subseteq \mathbb{R}^3$ is an exterior domain, located on one side of its nonempty, compact, and analytic boundary $\partial\Omega$. Denote by $d = d(x)$ the signed distance function to the boundary $\partial\Omega$, taking positive values inside Ω and negative values outside Ω . Since $\partial\Omega$ is smooth, we have

$$\nabla d = -\nu(x) \quad \text{on } \partial\Omega, \quad (3.1)$$

where ν is the unit outward normal vector. Moreover, the signed distance function is a real-analytic function in a neighborhood of the boundary $\partial\Omega$. Namely, we may extend ν with the formula (3.1) to a neighborhood

$$\Omega_{\delta_0} = \{x \in \Omega : d(x) < \delta_0\}$$

of the boundary $\partial\Omega$ such that

$$\sum_{l=0}^{\infty} \sum_{|\alpha|=l} \frac{\eta^l}{(l-3)!} \|\partial^\alpha \nu\|_{L^\infty(\Omega_{\delta_0})} \lesssim 1, \quad (3.2)$$

for some constants $\delta_0, \eta > 0$.

3.1. Analytic vector fields. A vector field X is *tangential* to $\partial\Omega$ if $Xd = 0$ on $\partial\Omega$. Such X may be restricted to $\partial\Omega$ by $Xf = (X\tilde{f})|_{\partial\Omega}$ for $f \in C^\infty(\partial\Omega)$, where $\tilde{f} \in C^\infty(\bar{\Omega})$ is an arbitrary extension of f . In fact, if $\tilde{f}$ vanishes on $\partial\Omega$, then there exists $\tilde{g} \in C^\infty(\bar{\Omega})$ such that $\tilde{f} = \tilde{g}d$ near $\partial\Omega$.

Existence of global analytic vector fields in a bounded domain in $\mathbb{R}^3$ has been proven in Komatsu [K]. Here we construct a family of tangential fields for an exterior domain $\Omega \subseteq \mathbb{R}^3$. An additional benefit from the construction is that we only need three tangential vector fields (which is the minimum possible by the hairy ball theorem).

THEOREM 3.1. *There exist analytic vector fields X_0, T_1, T_2, T_3 defined globally on $\bar{\Omega}$ with the following properties.*

1. *The fields T_1, T_2, T_3 are tangential to $\partial\Omega$, with*

$$T_j = \sum_{i=1}^3 b_{ij}(x) \partial_i, \quad 1 \leq j \leq 3, \quad (3.3)$$

where the coefficients $b_{ij}(x)$ are real-analytic functions with constant analyticity radii, i.e.,

$$|\partial^\alpha b_{ij}(x)| \lesssim C^{|\alpha|} |\alpha|!, \quad x \in \bar{\Omega}, \quad \alpha \in \mathbb{N}_0^3,$$

for $1 \leq i, j \leq 3$.

2. *There exists $\eta_0 > 0$, such that the partial derivatives may be expressed as*

$$\frac{\partial}{\partial x_k} = \xi_k(x) X_0 + \sum_{j=1}^3 \eta_{jk}(x) T_j, \quad 1 \leq k \leq 3 \quad (3.4)$$

on $\bar{\Omega}_{\eta_0}$, for some analytic coefficients $\xi_k(x)$ and $\eta_{jk}(x)$, with $1 \leq j, k \leq 3$.

The same proof works in all space dimensions, with a change in the number of vector fields. If the dimension is n , then the number of tangential fields becomes $n(n-1)/2$.

In the proof of the above proposition, we also need the following statement, which follows from [LM, Theorem 8.1.3].

LEMMA 3.2. (Local analytic hypoellipticity) *Let Ω be an exterior domain in $\mathbb{R}^3$ with analytic boundary, and let $R_0 > 0$ be such that*

$$B_{R_0/2} \supseteq \partial\Omega.$$

Assume that f satisfies

$$|\partial^\alpha f(x)| \leq MR^{|\alpha|} |\alpha|!, \quad \alpha \in \mathbb{N}_0^3, \quad x \in B_{2R_0} \cap \bar{\Omega}, \quad (3.5)$$

for some constants $M, R > 0$, and suppose that ψ solves

$$(-\Delta + 1)\psi = f \quad \text{in } \Omega \quad (3.6)$$

$$\psi = 0 \quad \text{on } \partial\Omega. \quad (3.7)$$

Then we have

$$|\partial^\alpha \psi(x)| \leq C(M + \|\psi\|_{L^2(B_{2R_0} \cap \Omega)})(CR)^{|\alpha|} |\alpha|!, \quad \alpha \in \mathbb{N}_0^3, \quad (3.8)$$

for $x \in B_{R_0} \cap \bar{\Omega}$, where C depends on $\partial\Omega$ and R_0 .

This statement also applies to the case $\Omega = \mathbb{R}^3$, in which case we refer to it as the interior hypoellipticity. It follows directly from [LM, Theorem 8.1.3].

We say that $\psi: \bar{\Omega} \rightarrow \mathbb{R}$ is a *globally defining function* if $\psi \in C^1$ and

$$\psi|_{\partial\Omega} = 0 \quad \text{and} \quad \nabla \psi|_{\partial\Omega} \neq 0.$$

PROOF OF THEOREM 3.1. First we construct globally defining functions which have constant analyticity radii. Let $R_0 \geq 4$ be a radius such that $\Omega^c \subseteq B_{R_0/2}$. We pick an arbitrary globally defining function $d_0 \in C^\infty(\mathbb{R}^3)$ which is compactly supported in B_{R_0} and it satisfies

$$\nabla d_0|_{\partial\Omega} \neq 0. \quad (3.9)$$

For $\epsilon \in (0, 1]$, we define $d_\epsilon: \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$d_\epsilon(x) = \int_{\mathbb{R}^3} H(x-y, \epsilon) d_0(y) dy, \quad x \in \mathbb{R}^3, \quad (3.10)$$

where $H: \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is the heat kernel. Consider the Dirichlet problem (3.6)–(3.7) with $f = f_\epsilon = (-\Delta + 1)d_\epsilon$. Since f_ϵ satisfies (3.5), it follows from Lemma 3.2 that the solution ψ_ϵ of (3.6)–(3.7) satisfies (3.8) for $x \in B_{R_0} \cap \bar{\Omega}$; note that $\|\psi_\epsilon\|_{L^2(\Omega)}$ is uniformly bounded in $\epsilon \in (0, 1]$. In order to obtain (3.8) for remaining x , we consider (3.6)–(3.7) with $f_\epsilon = (-\Delta + 1)d_\epsilon$ and the version of Lemma 3.2 with $\Omega = \mathbb{R}^3$. Fix any $y \in \mathbb{R}^3 \setminus B_{R_0}$. Since f_ϵ satisfies (3.5) in $B_2(y)$, Lemma 3.2 implies that the solution ψ_ϵ of (3.6) satisfies (3.8) for all $x \in B_1(y)$. Thus, we obtain

$$|\partial^\alpha \psi_\epsilon(x)| \leq CM(CR)^{|\alpha|} |\alpha|!, \quad \alpha \in \mathbb{N}_0^3, \quad x \in \bar{\Omega}. \quad (3.11)$$

By (3.9)–(3.10) and continuity, we infer that

$$\nabla \psi_\epsilon|_{\partial\Omega} \neq 0,$$

for sufficiently small $\epsilon > 0$, which implies that ψ_ϵ is a globally defining function.

Next, we pick an arbitrary globally defining function ψ satisfying (3.11). Choose $\eta_0 > 0$ so that $\nabla \psi \neq 0$ on $\bar{\Omega}_{\eta_0}$. For $1 \leq j, k \leq 3$, we set

$$X_0 = \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i} \frac{\partial}{\partial x_i} \quad \text{and} \quad T_{jk} = \frac{\partial \psi}{\partial x_j} \frac{\partial}{\partial x_k} - \frac{\partial \psi}{\partial x_k} \frac{\partial}{\partial x_j}. \quad (3.12)$$

Since $\psi \in C^\infty(\bar{\Omega})$ vanishes on $\partial\Omega$, there exists some $\tilde{g} \in C^\infty(\bar{\Omega})$ such that $\psi = \tilde{g}d$ near $\partial\Omega$. For $1 \leq j, k \leq 3$, we compute

$$\begin{aligned} T_{jk}d &= \frac{\partial(\tilde{g}d)}{\partial x_j} \frac{\partial d}{\partial x_k} - \frac{\partial(\tilde{g}d)}{\partial x_k} \frac{\partial d}{\partial x_j} \\ &= d \left(\frac{\partial \tilde{g}}{\partial x_j} \frac{\partial d}{\partial x_k} - \frac{\partial \tilde{g}}{\partial x_k} \frac{\partial d}{\partial x_j} \right) + \tilde{g} \left(\frac{\partial d}{\partial x_j} \frac{\partial d}{\partial x_k} - \frac{\partial d}{\partial x_k} \frac{\partial d}{\partial x_j} \right) = 0, \quad \text{on } \partial\Omega, \end{aligned}$$

which implies that T_{jk} are tangential vector fields, and thus (3.3) is proven. From (3.12), we have

$$\frac{\partial \psi}{\partial x_k} X_0 + \sum_{j=1}^3 \frac{\partial \psi}{\partial x_j} T_{jk} = |\nabla \psi|^2 \frac{\partial}{\partial x_k},$$

for $1 \leq k \leq 3$. Thus (3.4) follows since $\nabla \psi \neq 0$ on $\bar{\Omega}_{\eta_0}$. To show that one may choose only three rather than nine tangential fields, note that $T_{jk} = -T_{kj}$ for $1 \leq j, k \leq 3$, and thus also $T_{11} = T_{22} = T_{33} = 0$. $\square$

3.2. Main results. Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with an analytic boundary $\partial\Omega$. We assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies

$$\|(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{H^s(\Omega)} \leq M_0 \quad (3.13)$$

and

$$\sum_{j=0}^{\infty} \frac{\tau_0^{(j-3)+}}{(j-3)!} \|\partial_x^j (p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{L^2(\Omega)} \leq M_0, \quad (3.14)$$

for some fixed constants $\tau_0, M_0 > 0$. It can be shown, analogously to Lemma 4.7 below, that (3.14) implies

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\bar{\tau}_0^{(j+k-3)+}}{(j+k-3)!} \|\partial_x^j \mathbf{T}^k(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)\|_{L^2(\Omega)} \leq Q(M_0), \quad (3.15)$$

for some function Q and some constant $\bar{\tau}_0 > 0$. In (3.14)–(3.15) and below, we use $\mathbf{T} = (T_1, T_2, T_3)$ to denote the tangential vector fields from Theorem 3.1, and use $\partial_x = (\partial_1, \partial_2, \partial_3)$ to denote the gradient. We adopt the following agreement for the iterative derivatives $\mathbf{T}^k$ and ∂_x^j . The symbol $\mathbf{T}^k$ is understood in the tensorial sense, i.e., $\mathbf{T}^k$ stands for the list of all possible operators $T_{\beta_1} \cdots T_{\beta_k}$, where $\beta_i \in \{1, 2, 3\}$ for $1 \leq i \leq k$. We also adopt the same agreement for ∂_x^j .

For $\tau > 0$, we define the mixed weighted analytic space

$$A(\tau) = \{u \in C^\infty(\Omega) : \|u\|_{A(\tau)} < \infty\},$$

where

$$\|u\|_{A(\tau)} = \sum_{(j,k,i) \in \mathbb{N}_0^3} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau(t)^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u\|_{L^2(\Omega)}; \quad (3.16)$$

here $\tau \in (0, 1]$ represents the mixed space-time analyticity radius, while κ and $\bar{\kappa}$, where $0 < \kappa \leq \bar{\kappa} \leq 1$ are parameters which represent the balances of radii in different directions. In (3.16) and below we use the convention $n! = 1$ for $n \in -\mathbb{N}$. We show in Section 7 that (3.15) implies, regardless of the choices of $\kappa \in (0, 1]$ and $\bar{\kappa} \in (0, 1]$, that we have

$$\|(p_0, v_0, S_0)\|_{A(\bar{\tau}_0)} \leq Q(M_0),$$

for some function Q , where $\bar{\tau}_0 = \bar{\tau}_0/Q(M_0)$ represents the mixed space-time analyticity radius. Note that the time derivatives of the initial data are defined iteratively by differentiating the equations (2.4)–(2.5) and then evaluating at $t = 0$. The analyticity radius function is defined as

$$\tau(t) = \tau(0) - Kt, \quad (3.17)$$

where $\tau(0) \leq \min\{1, \bar{\tau}_0\}$ is a sufficiently small parameter (different from $\bar{\tau}_0$), and $K \geq 1$ is a sufficiently large parameter, both to be determined below.

Our first main theorem provides a uniform in ϵ boundedness of the analytic norm on a time interval independent of ϵ .

THEOREM 3.3. Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with analytic boundary $\partial\Omega$. We assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies (3.13)–(3.14) for some fixed constants $M_0, \tau_0 > 0$. Also, suppose that $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies compatibility condition of all orders. Then there exist sufficiently small constants $\kappa, \bar{\kappa}, \tau(0), \epsilon_0, T_0 > 0$, depending on M_0 , such that

$$\|(p^\epsilon, v^\epsilon, S^\epsilon)(t)\|_{A(\tau)} \leq M, \quad 0 < \epsilon \leq \epsilon_0, \quad t \in [0, T_0],$$

where τ is as in (3.17) and K and M are sufficiently large constants depending on M_0 .

We note that the same statement and the proof apply to the case of a bounded domain.

The second main result states that the solutions of (2.4)–(2.5) converge to the solution of the stratified incompressible Euler equations

$$r(S, 0)(\partial_t v + v \cdot \nabla v) + \nabla \pi = 0, \quad (3.18)$$

$$\operatorname{div} v = 0, \quad (3.19)$$

$$\partial_t S + v \cdot \nabla S = 0, \quad (3.20)$$

as $\epsilon \rightarrow 0$. We define the spatial analytic space

$$X(\delta) = \{u \in C^\infty(\Omega) : \|u\|_{X(\delta)} < \infty\},$$

where

$$\|u\|_{X(\delta)} = \sum_{j=0}^{\infty} \frac{\delta^{(j-3)_+}}{(j-3)!} \|\partial_x^j u\|_{L^2(\Omega)},$$

for some constant $\delta > 0$.

THEOREM 3.4. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with an analytic boundary. Assume that the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfy (3.13)–(3.14) uniformly for fixed $\tau_0, M_0 > 0$ and the compatibility condition of all orders. Also, suppose that $(v_0^\epsilon, S_0^\epsilon)$ converge to (v_0, S_0) in $H^3(\Omega)$, with S_0^ϵ decaying in the sense of*

$$|S_0^\epsilon(x)| \lesssim |x|^{-1-\zeta}, \quad \text{and} \quad |\nabla S_0^\epsilon(x)| \lesssim |x|^{-2-\zeta},$$

for $0 < \epsilon \leq \epsilon_0$ and some constant $\zeta > 0$. Then $(v^\epsilon, p^\epsilon, S^\epsilon)$ converges to $(v^{(\text{inc})}, 0, S^{(\text{inc})}) \in L^\infty([0, T_0], X(\delta))$ in $L^2([0, T_0], X(\delta))$, where $\delta \in (0, \tau_0]$ is a sufficiently small constant and $(v^{(\text{inc})}, S^{(\text{inc})})$ is the solution to (3.18)–(3.20) with the initial data (w_0, S_0) , and w_0 is the unique solution of

$$\begin{aligned} \operatorname{div} w_0 &= 0, \\ \operatorname{curl}(r_0 w_0) &= \operatorname{curl}(r_0 v_0), \end{aligned}$$

with $r_0 = r(S_0, 0)$.

In the rest of the paper, we omit the superscript ϵ , and write S , u , and v for S^ϵ , u^ϵ , and v^ϵ . The constant C and the function Q denote a generic constant and a positive increasing function, respectively, which depend only on M_0, τ_0 , and the exterior domain Ω ; they may vary from an inequality to an inequality. The domain of dependence in the norms is understood to be Ω unless stated otherwise. We write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$.

In order to prove Theorem 3.3, we establish analytic a priori estimates of the entropy S and the (modified) velocity u . The a priori estimate needed to prove Theorem 3.3 is the following.

LEMMA 3.5. *Let $M_0, \tau_0 > 0$. For sufficiently small parameters κ and $\bar{\kappa}$ satisfying $0 < \kappa \leq \bar{\kappa} \leq 1$, there exist constants $\tau_1, \epsilon_0, T_0 > 0$, and a nonnegative continuous function Q such that for all $\epsilon \in (0, \epsilon_0]$, the norm*

$$M_{\epsilon, \kappa, \bar{\kappa}}(T) = \sup_{t \in [0, T]} (\|S(t)\|_{A(\tau(t))} + \|u(t)\|_{A(\tau(t))})$$

satisfies the estimate

$$M_{\epsilon, \kappa, \bar{\kappa}}(t) \lesssim 1 + (t + \kappa + \bar{\kappa} + \epsilon + \tau(0)) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)),$$

for $t \in (0, T_0]$ and $\tau(0) \in (0, \tau_1]$, provided

$$K \geq Q(M_{\epsilon, \kappa, \bar{\kappa}}(T_0)) \tag{3.21}$$

holds, where τ and K are as in (3.17).

The constant K in (3.21) depends on M , which implies that K eventually depends on M_0 . In the rest of the paper, we work on an interval of time such that

$$T_0 \leq \frac{\tau(0)}{2K}. \tag{3.22}$$

Thus from (3.17) we have $\tau(0)/2 \leq \tau(t) \leq \tau(0)$ for $t \in [0, T_0]$.

For the proof of Theorem 3.3 given Lemma 3.5, cf. [JKL]. Sections 4–6 are devoted to the proof of Lemma 3.5, thus completing the proof of Theorem 3.3.

By [A1, Theorem 1.1] the H^5 norm of (p, v, S) can be estimated by a constant on a time interval $[0, T_0]$, where T_0 only depends on the H^5 norm of the initial data. Thus we may assume

$$\sup_{(j, k, i) \in \mathbb{N}_0^3, 0 \leq j+k+i \leq 5} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i (p, v, S)(t)\|_{L^2} \lesssim 1, \quad t \in [0, T_0], \quad \epsilon \in (0, 1]. \tag{3.23}$$

In particular, if F is a smooth function of u and S , then there exists a constant $C > 0$ depending on the function F such that

$$\|F(\epsilon u(t), S(t))\|_{L^\infty} \lesssim 1, \quad t \in [0, T_0], \quad \epsilon \in (0, 1].$$

In the rest of the paper, we work on the time interval $[0, T]$ where $0 < T \leq T_0$.

4. Derivative reductions for the velocity

Here we introduce the normal and tangential derivative reduction schemes for the velocity. Throughout this section, we assume that Ω is an exterior domain with analytic boundary and the vector field v satisfies

$$v \cdot \nu|_{\partial\Omega} = 0, \quad (4.1)$$

where ν is the unit outward normal vector to the boundary $\partial\Omega$.

4.1. Normal derivative reduction. We start with the normal derivative reduction.

LEMMA 4.1. *For $j \geq 2$, we have*

$$\begin{aligned} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2} &\lesssim \|\partial_x^{j-1} \mathbf{T}^k(\epsilon \partial_t)^i \operatorname{div} v\|_{L^2} + \|\partial_x^{j-1} \mathbf{T}^k(\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2} + \|\partial_x^{j-2} \mathbf{T}^{k+1}(\epsilon \partial_t)^i v\|_{H^1} \\ &\quad + \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2} + \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{div}](\epsilon \partial_t)^i v\|_{L^2} \\ &\quad + \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{curl}](\epsilon \partial_t)^i v\|_{L^2} + \|[\mathbf{T}, \partial_x^{j-2}] \mathbf{T}^k(\epsilon \partial_t)^i v\|_{H^1}. \end{aligned} \quad (4.2)$$

For $j = 1$ and $k = 0$, we have

$$\|\partial_x(\epsilon \partial_t)^i v\|_{L^2} \lesssim \|(\epsilon \partial_t)^i \operatorname{div} v\|_{L^2} + \|(\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2} + \|(\epsilon \partial_t)^i v\|_{L^2}. \quad (4.3)$$

In order to prove this statement, we use a form of H^2 regularity for the Laplacian.

LEMMA 4.2. *Let Ω be an exterior domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Then*

$$\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)} + \|\mathbf{T}v\|_{H^1(\Omega)} + \|v\|_{L^2(\Omega)}, \quad (4.4)$$

for all $v \in H^2(\Omega)$.

In the proof of Lemma 4.1, we also need the following classical div-curl estimate due to Bourguignon and Brezis.

LEMMA 4.3. ([BoB]) *Let Ω be an exterior domain in $\mathbb{R}^3$ with analytic boundary $\partial\Omega$ and outward unit normal vector ν . Then there exists a constant $C > 0$ such that*

$$\|\nabla v\|_{L^2(\Omega)} \lesssim \|\operatorname{div} v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|v \cdot \nu\|_{H^{1/2}(\partial\Omega)} + \|v\|_{L^2(\Omega)}, \quad (4.5)$$

for all $v \in H^1(\Omega)$.

PROOF OF LEMMA 4.2. Using the H^2 regularity for the Laplace equation

$$\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)} + \|v\|_{H^{3/2}(\partial\Omega)},$$

combined with the trace theorem, we arrive at

$$\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)} + \|\mathbf{T}v\|_{H^1(\Omega)} + \|v\|_{L^2(\Omega)},$$

concluding the proof of (4.4). $\square$

PROOF OF LEMMA 4.1. Let $j \geq 2$. Appealing to the H^2 regularity (4.4), we obtain

$$\|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \lesssim \|\Delta \partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} + \|\mathbf{T} \partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{H^1(\Omega)} + \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)},$$

from where we obtain by using the vector calculus identity $\Delta v = \nabla \operatorname{div} v - \operatorname{curl} \operatorname{curl} v$ the inequality

$$\begin{aligned} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} &\lesssim \|\operatorname{div} \partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{\dot{H}^1(\Omega)} + \|\operatorname{curl} \partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{\dot{H}^1(\Omega)} \\ &\quad + \|\mathbf{T} \partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{H^1(\Omega)} + \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \\ &\lesssim \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i \operatorname{div} v\|_{\dot{H}^1(\Omega)} + \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i \operatorname{curl} v\|_{\dot{H}^1(\Omega)} \\ &\quad + \|\partial_x^{j-2} \mathbf{T}^{k+1}(\epsilon \partial_t)^i v\|_{H^1(\Omega)} + \|\partial_x^{j-2} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \\ &\quad + \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{div}](\epsilon \partial_t)^i v\|_{L^2(\Omega)} + \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{curl}](\epsilon \partial_t)^i v\|_{L^2(\Omega)} \\ &\quad + \|[\mathbf{T}, \partial_x^{j-2}] \mathbf{T}^k(\epsilon \partial_t)^i v\|_{H^1(\Omega)}, \end{aligned}$$

and (4.2) follows. On the other hand, the inequality (4.3) follows from the elliptic regularity (4.5) and the boundary condition (4.1). $\square$

4.2. Tangential derivative reduction. The following lemma allows us to reduce the number of tangential derivatives.

LEMMA 4.4. *For $j = 1$ and $k \geq 1$, we have*

$$\begin{aligned} \|\partial_x \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} &\lesssim \|\mathbf{T}^k(\epsilon \partial_t)^i \operatorname{div} v\|_{L^2(\Omega)} + \|\mathbf{T}^k(\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2(\Omega)} + \|\mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \\ &\quad + \sum_{l=1}^k \binom{k}{l} (\|\partial_x \mathbf{T}^{k-l}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\mathbf{T}^l v\|_{L^\infty(\Omega_{\delta_0})} \\ &\quad + \|\mathbf{T}^{k-l}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\partial_x \mathbf{T}^l v\|_{L^\infty(\Omega_{\delta_0})} + \|\mathbf{T}^{k-l}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\mathbf{T}^l v\|_{L^\infty(\Omega_{\delta_0})}) \\ &\quad + \|[\mathbf{T}^k, \operatorname{div}](\epsilon \partial_t)^i v\|_{L^2(\Omega)} + \|[\mathbf{T}^k, \operatorname{curl}](\epsilon \partial_t)^i v\|_{L^2(\Omega)}. \end{aligned} \quad (4.6)$$

Note that when $j = 0$ and $k \geq 1$, we may simply use

$$\|\mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \lesssim \|\partial_x \mathbf{T}^{k-1}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \quad (4.7)$$

and apply (4.6) with the reduced k .

PROOF OF LEMMA 4.4. For $j = 1$ and $k \geq 1$, we appeal to (4.5) and obtain

$$\begin{aligned} \|\partial_x \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} &\lesssim \|\operatorname{div} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} + \|\operatorname{curl} \mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \\ &\quad + \|\mathbf{T}^k(\epsilon \partial_t)^i v \cdot \nu\|_{H^{1/2}(\partial\Omega)} + \|\mathbf{T}^k(\epsilon \partial_t)^i v\|_{L^2(\Omega)}. \end{aligned} \quad (4.8)$$

Now, note that

$$\mathbf{T}^k(\epsilon \partial_t)^i v \cdot \nu = \mathbf{T}^k(\epsilon \partial_t)^i (v \cdot \nu) - \sum_{l=1}^k \binom{k}{l} \mathbf{T}^{k-l}(\epsilon \partial_t)^i v \cdot \mathbf{T}^l \nu,$$

with the first term on the right side vanishing by (4.1). Thus we get

$$\|\mathbf{T}^k(\epsilon \partial_t)^i v \cdot \nu\|_{H^{1/2}(\partial\Omega)} \leq \sum_{l=1}^k \binom{k}{l} \|\mathbf{T}^{k-l}(\epsilon \partial_t)^i v \cdot \mathbf{T}^l \nu\|_{H^{1/2}(\partial\Omega)}.$$

Extending the unit normal vector to Ω_{δ_0} as in Section 3, and using the trace theorem, we obtain

$$\|\mathbf{T}^{k-l}(\epsilon \partial_t)^i v \cdot \mathbf{T}^l \nu\|_{H^{1/2}(\partial\Omega)} \lesssim \|\mathbf{T}^{k-l}(\epsilon \partial_t)^i v \cdot \mathbf{T}^l \nu\|_{H^1(\Omega_{\delta_0})}.$$

By the Leibniz rule and Hölder's inequality, we conclude that

$$\begin{aligned} &\|\mathbf{T}^{k-j}(\epsilon \partial_t)^i v \cdot \mathbf{T}^j \nu\|_{H^1(\Omega_{\delta_0})} \\ &\leq \|\partial_x \mathbf{T}^{k-j}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\mathbf{T}^j \nu\|_{L^\infty(\Omega_{\delta_0})} + \|\mathbf{T}^{k-j}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\partial_x \mathbf{T}^j \nu\|_{L^\infty(\Omega_{\delta_0})} \\ &\quad + \|\mathbf{T}^{k-j}(\epsilon \partial_t)^i v\|_{L^2(\Omega)} \|\mathbf{T}^j \nu\|_{L^\infty(\Omega_{\delta_0})}. \end{aligned} \quad (4.9)$$

Combining (4.8)–(4.9), we then obtain (4.6). $\square$

4.3. Commutator estimates. Here we recall a Leibniz formula for k -folded commutators. Given two linear differential operators Y, Z , the adjoint operator $\text{ad } Y(Z)$ is defined as

$$\text{ad } Y(Z) = [Y, Z] = YZ - ZY.$$

Recall a Leibniz-type formula

$$[\mathbf{T}^k, Z] = \sum_{m=1}^k \binom{k}{m} ((\text{ad } \mathbf{T})^m(Z)) \mathbf{T}^{k-m}, \quad k \in \mathbb{N} \quad (4.10)$$

from [CKV, Lemma 3.4], which holds for any differential operator Z . From [K, Lemma 5.3], we also recall an analytic estimate for the adjoint operator.

LEMMA 4.5 ([K]). *Let $Y_1, \dots, Y_m$ and Y_0 be analytic vector fields defined on $\Omega \subseteq \mathbb{R}^3$ such that*

$$Y_n = \sum_{i=1}^3 a_n^i \partial_i, \quad n = 0, 1, \dots, m,$$

where

$$\max_{|\alpha|=k} |\partial^\alpha a_n^i| \lesssim k! K_1^k, \quad i = 1, 2, 3, \quad n = 0, 1, \dots, m, \quad k \in \mathbb{N}_0,$$

for some $K_1 \geq 1$. Then there exists $\bar{K}_1, \bar{K}_2 \geq 1$ such that

$$(\text{ad } Y_m \dots \text{ad } Y_1)(Y_0) = \sum_{i=1}^3 b_m^i \partial_i,$$

where

$$\max_{|\alpha|=k} |\partial^\alpha b_m^i| \lesssim (k+m)! \bar{K}_1^k \bar{K}_2^m,$$

for $i = 1, 2, 3$ and $k \in \mathbb{N}_0$.

In the following lemma, we derive commutator estimates for various spatial derivative operators.

LEMMA 4.6. *There exists a constant $\bar{K} \geq 1$ such that*

$$\|\partial_x^j [\mathbf{T}^k, \text{div}] (\epsilon \partial_t)^i v\|_{L^2} \lesssim \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} (j' + k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v\|_{L^2}, \quad (4.11)$$

$$\|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i v\|_{L^2} \lesssim \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} (j' + k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v\|_{L^2}, \quad (4.12)$$

$$\|\partial_x^j [\mathbf{T}^k, \text{curl}] (\epsilon \partial_t)^i v\|_{L^2} \lesssim \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} (j' + k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v\|_{L^2}, \quad (4.13)$$

$$\|\partial_x^j [\partial_x^k, \mathbf{T}] \mathbf{T}^l (\epsilon \partial_t)^i v\|_{L^2} \lesssim \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} (j' + k')! \bar{K}^{j'+k'} \|\partial_x^{k-k'+j-j'+1} \mathbf{T}^l (\epsilon \partial_t)^i v\|_{L^2}, \quad (4.14)$$

for $i, j, l \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

PROOF OF LEMMA 4.6. By (4.10), we have the expansion

$$[\mathbf{T}^k, \text{div}] v = \sum_{s=1}^3 [\mathbf{T}^k, \partial_s] v_s = \sum_{s=1}^3 \sum_{k'=1}^k \binom{k}{k'} ((\text{ad } \mathbf{T})^{k'} (\partial_s)) \mathbf{T}^{k-k'} v_s.$$

Using Theorem 3.1 and Lemma 4.5, we obtain

$$(\text{ad } \mathbf{T})^{k'}(\partial_s) = \sum_{l=1}^3 b_{k',s}^l \partial_l, \quad k' \in \mathbb{N}, \quad s \in \{1, 2, 3\}, \quad (4.15)$$

where

$$\max_{|\beta|=k} |\partial^\beta b_{k',s}^l| \lesssim (k+k')! \bar{K}_1^k \bar{K}_2^{k'}, \quad k \in \mathbb{N}_0, \quad k' \in \mathbb{N}, \quad s, l \in \{1, 2, 3\}, \quad (4.16)$$

for some constants $\bar{K}_1, \bar{K}_2 \geq 1$. From the expression (4.15) and the Leibniz rule, we arrive at

$$\begin{aligned} \partial_x^j [\mathbf{T}^k, \text{div}] (\epsilon \partial_t)^i v &= \partial_x^j \sum_{s=1}^3 \sum_{k'=1}^k \sum_{l=1}^3 \binom{k}{k'} b_{k',s}^l \partial_l \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v_s \\ &= \sum_{s=1}^3 \sum_{k'=1}^k \sum_{l=1}^3 \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} \partial_x^{j'} b_{k',s}^l \partial_x^{j-j'} \partial_l \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v_s. \end{aligned}$$

Using (4.16), we obtain

$$\begin{aligned} \|\partial_x^j [\mathbf{T}^k, \text{div}] (\epsilon \partial_t)^i v\|_{L^2} &\lesssim \sum_{s=1}^3 \sum_{k'=1}^k \sum_{j'=0}^j \sum_{l=1}^3 \binom{k}{k'} \binom{j}{j'} \|\partial_x^{j'} b_{k',s}^l \partial_x^{j-j'+1} T^{k-k'} (\epsilon \partial_t)^i v_s\|_{L^2} \\ &\lesssim \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} (j'+k')! \bar{K}_1^{j'} \bar{K}_2^{k'} \|\partial_x^{j-j'+1} T^{k-k'} (\epsilon \partial_t)^i v\|_{L^2}, \end{aligned}$$

and (4.11) follows by setting $\bar{K} \geq \max(\bar{K}_1, \bar{K}_2)$.

The proofs of (4.12)–(4.14) are analogous. $\square$

The next lemma provides an analytic estimate for the unit normal vector to the boundary.

LEMMA 4.7. There exists a constant $\tilde{\eta} > 0$ such that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\tilde{\eta}^{j+k}}{(j+k-3)!} \|\partial_x^j \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \lesssim 1, \quad (4.17)$$

where ν is the unit normal vector to the boundary satisfying (3.2).

PROOF OF LEMMA 4.7. We claim that there exists a constant $\bar{\eta} > 0$, such that for all $k \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} \frac{\bar{\eta}^{j+k}}{(j+k-3)!} \|\partial_x^j \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \leq C_0^{k+1}, \quad (4.18)$$

for some constant $C_0 \geq 1$. In (4.18), we then choose $\tilde{\eta} = \bar{\eta}/2C_0$ to get

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\tilde{\eta}^{j+k}}{(j+k-3)!} \|\partial_x^j \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\bar{\eta}^{j+k}}{2^k C_0^k (j+k-3)!} \|\partial_x^j \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \lesssim 1,$$

obtaining (4.17). In the remainder of the proof, we proceed by induction to prove (4.18) for all $k \in \mathbb{N}_0$.

Firstly, we use (3.2) to obtain (4.18) for $k = 0$ by taking $\bar{\eta} = \eta$. Now we assume that (4.18) holds for some $k \in \mathbb{N}_0$ and aim to show that it also holds for $k+1$. Using (3.3), the Leibniz rule, and Hölder's inequality, we arrive

at

$$\begin{aligned}
& \sum_{j=0}^{\infty} \frac{\bar{\eta}^{j+k+1}}{(j+k-2)!} \|\partial_x^j \mathbf{T}^{k+1} \nu\|_{L^\infty(\Omega_{\delta_0})} \\
& \leq C \sum_{i=1}^3 \sum_{l=1}^3 \sum_{j=0}^{\infty} \sum_{j'=0}^j \binom{j}{j'} \frac{\bar{\eta}^{j+k+1}}{(j+k-2)!} \|\partial_x^{j'} b_{il}\|_{L^\infty(\Omega_{\delta_0})} \|\partial_x^{j-j'+1} \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \\
& \leq C \sum_{j=0}^{\infty} \sum_{j'=0}^j \bar{\eta}^{j'} C^{j'} \frac{j!(j-j'+k-2)!}{(j-j')!(j+k-2)!} \times \left(\frac{\bar{\eta}^{j-j'+k+1}}{(j-j'+k-2)!} \|\partial_x^{j-j'+1} \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \right) \\
& \leq C \sum_{j''=1}^{\infty} \frac{\bar{\eta}^{j''+k}}{(j''+k-3)!} \|\partial_x^{j''} \mathbf{T}^k \nu\|_{L^\infty(\Omega_{\delta_0})} \times \sum_{j=j''-1}^{\infty} (C_1 \bar{\eta})^{j+1-j''} \frac{j!(j''+k-3)!}{(j''-1)!(j+k-2)!},
\end{aligned} \tag{4.19}$$

where $C_1 \geq 1$ is a constant. Note that the sum in j is dominated by a constant C uniformly for all $j'', k \in \mathbb{N}_0$, by taking $\bar{\eta} \leq 1/2C_1$. Thus from (4.19) and the induction hypothesis for k , we have

$$\sum_{j=0}^{\infty} \frac{\bar{\eta}^{j+k+1}}{(j+k-2)!} \|\partial_x^j \mathbf{T}^{k+1} \nu\|_{L^\infty(\Omega_{\delta_0})} \leq C C_0^{k+1},$$

concluding the proof of (4.18) for $k+1$ by simply taking $C_0 \geq C$. $\square$

5. Analytic estimate of the entropy

The following lemma provides an analytic estimate for the entropy S .

LEMMA 5.1. *Let $M_0 > 0$. For any $\kappa, \bar{\kappa} \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ such that if $0 < \tau(0) \leq \tau_1$, then*

$$\|S(t)\|_{A(\tau(t))} \lesssim 1 + tQ(M_{\epsilon, \kappa, \bar{\kappa}}(t)), \quad t \in (0, T_0], \tag{5.1}$$

for all $\epsilon \in (0, 1]$, provided K in (3.17) satisfies

$$K \geq Q(M_{\epsilon, \kappa, \bar{\kappa}}(T_0)),$$

where $T_0 > 0$ is a sufficiently small constant depending on M_0 .

PROOF OF LEMMA 5.1. Let

$$\tau_1 = \kappa^4. \tag{5.2}$$

Now, we fix $(j, k, i) \in \mathbb{N}_0^3$, apply $\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i$ to the equation (2.5), and take the L^2 -inner product with $\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S$, obtaining

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S\|_{L^2}^2 + \langle v \cdot \nabla \partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S, \partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S \rangle = \langle [v \cdot \nabla, \partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i] S, \partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in L^2 . Using the Cauchy-Schwarz inequality and the boundary condition (2.6), we obtain

$$\frac{d}{dt} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S\|_{L^2} \lesssim \|\nabla v\|_{L_x^\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S\|_{L^2} + \|[v \cdot \nabla, \partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i] S\|_{L^2}. \tag{5.3}$$

Using the notation (3.16) and

$$\|u\|_{\tilde{A}(\tau)} = \sum_{j+k+i \geq 4} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u\|_{L^2} \frac{\kappa^{(j-1)+\bar{\kappa}^k(j+k+i-3)} \tau(t)^{j+k+i-4}}{(j+k+i-3)!},$$

the estimate (5.3) implies

$$\begin{aligned}
\frac{d}{dt} \|S\|_{A(\tau)} &= \dot{\tau}(t) \|S\|_{\tilde{A}(\tau)} + \sum_{(j,k,i) \in \mathbb{N}_0^3} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \frac{d}{dt} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i S\|_{L^2} \\
&\lesssim \dot{\tau}(t) \|S\|_{\tilde{A}(\tau)} + \|\nabla v\|_{L_x^\infty} \|S\|_{A(\tau)} + \sum_{(j,k,i) \in \mathbb{N}_0^3} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ j'+k'+i' \geq 1}} \mathcal{C}_{j,k,i,j',k',i'} \\
&\quad + \|v\|_{L_x^\infty} \sum_{(j,k,i) \in \mathbb{N}_0^3} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2},
\end{aligned} \tag{5.4}$$

where

$$\mathcal{C}_{j,k,i,j',k',i'} = \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \binom{j}{j'} \binom{k}{k'} \binom{i}{i'} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v \cdot \partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{L^2}.$$

We split the third term on the far right side of (5.4) according to the low and high values of $j' + k' + i'$. We claim that there exists a constant $C > 0$ such that

$$I_1 = \sum_{0 \leq j+k+i \leq 4} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ j'+k'+i' \geq 1}} \mathcal{C}_{j,k,i,j',k',i'} \lesssim 1, \tag{5.5}$$

$$I_2 = \sum_{j+k+i \geq 5} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ 1 \leq j'+k'+i' \leq [(j+k+i)/2]}} \mathcal{C}_{j,k,i,j',k',i'} \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)}, \tag{5.6}$$

$$I_3 = \sum_{j+k+i \geq 5} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ [(j+k+i)/2]+1 \leq j'+k'+i' \leq j+k+i-3}} \mathcal{C}_{j,k,i,j',k',i'} \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)}, \tag{5.7}$$

$$I_4 = \sum_{j+k+i \geq 5} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ j+k+i-2 \leq j'+k'+i'}} \mathcal{C}_{j,k,i,j',k',i'} \lesssim \|v\|_{A(\tau)}, \tag{5.8}$$

$$I_5 = \sum_{(j,k,i) \in \mathbb{N}_0^3} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2} \lesssim 1 + \|S\|_{\tilde{A}(\tau)}. \tag{5.9}$$

The proofs of (5.5)–(5.8) are analogous to those in Section 4 in [JKL]. Here we only outline necessary modifications.

Proof of (5.5): We apply Hölder and Sobolev inequalities on the factor $\|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v \cdot \partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{L^2}$ and then use (4.12) to estimate $\|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i \nabla S\|_{L^2}$. Then (5.5) follows by appealing to (3.23).

Proof of (5.6): Using Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
&\mathcal{C}_{j,k,i,j',k',i'} \mathbb{1}_{\{j+k+i \geq 5\}} \mathbb{1}_{\{1 \leq j'+k'+i' \leq [(j+k+i)/2]\}} \\
&\lesssim \left(\frac{\kappa^{j'+1} \bar{\kappa}^{k'} \tau^{(j'+k'+i'-1)+}}{(j'+k'+i'-1)!} \|\partial_x^{j'+2} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v\|_{L^2} \right)^{3/4} \\
&\quad \times \left(\frac{\kappa^{(j'-1)+} \bar{\kappa}^{k'} \tau^{(j'+k'+i'-3)+}}{(j'+k'+i'-3)!} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v\|_{L^2} \right)^{1/4} \\
&\quad \times \left(\frac{\kappa^{j-j'} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-i'-3)+}}{(j+k+i-j'-k'-i'-3)!} \|\partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{L^2} \right).
\end{aligned} \tag{5.10}$$

In (5.10), we bounded the rest of the powers of κ , $\bar{\kappa}$, and τ by C , which is possible by (5.2), and bounded the rest of the factors involving combinatorial symbols by C since $j' + k' + i' \leq [(j+k+i)/2]$. From (5.10), we use the

discrete Hölder and Young inequalities to get

$$I_2 \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)} \sum_{j+k+i \geq 3} \frac{\kappa^j \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2}. \quad (5.11)$$

Appealing to (4.12), the sum in the second term on the right side of above may be estimated by

$$\begin{aligned} & \sum_{j+k+i \geq 3} \sum_{k'=1}^k \sum_{j'=0}^j \binom{k}{k'} \binom{j}{j'} \frac{\kappa^j \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} (j'+k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i S\|_{L^2} \\ & \lesssim \sum_{j+k+i \geq 3} \sum_{k'=1}^k \sum_{j'=0}^j (\kappa \bar{K})^{j'} (\bar{\kappa} \bar{K})^{k'} \mathbb{1}_{\{i \geq 3\}} \\ & \quad \times \left(\frac{\kappa^{j-j'} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-3)+}}{(j+k+i-j'-k'-3)!} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i S\|_{L^2} \right) \mathcal{A}_{j,k,i,j',k'} \\ & + \sum_{j+k+i \geq 3} \sum_{k'=1}^k \sum_{j'=0}^j (\kappa \bar{K})^{j'} (\bar{\kappa} \bar{K})^{k'} \mathbb{1}_{\{0 \leq i \leq 2\}} \\ & \quad \times \left(\frac{\kappa^{j-j'} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-3)+}}{(j+k+i-j'-k'-3)!} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i S\|_{L^2} \right) \mathcal{B}_{j,k,i,j',k'}. \end{aligned} \quad (5.12)$$

In (5.12), we denote

$$\mathcal{A}_{j,k,i,j',k'} = \binom{k}{k'} \binom{j}{j'} \frac{(j+k+i-j'-k'-3)!(j'+k')!}{(j+k+i-3)!} \mathbb{1}_{\{i \geq 3\}} \quad (5.13)$$

and

$$\mathcal{B}_{j,k,i,j',k'} = \binom{k}{k'} \binom{j}{j'} \frac{(j+k+i-j'-k'-3)!(j'+k')!}{(j+k+i-3)!} \mathbb{1}_{\{0 \leq i \leq 2\}}.$$

Applying the combinatorial inequality

$$\binom{k}{k'} \binom{j}{j'} \leq \binom{k+j}{k'+j'}$$

to (5.13), we obtain

$$\mathcal{A}_{j,k,i,j',k'} \lesssim \frac{(j+k+i-j'-k'-3)!(j+k)!}{(j+k+i-3)!(j+k-j'-k')!} \lesssim 1, \quad (5.14)$$

since $i \geq 3$. Similarly,

$$\mathcal{B}_{j,k,i,j',k'} \lesssim \frac{(j+k+i-j'-k'-3)!(j+k)!}{(j+k+i-3)!(j+k-j'-k')!} \mathbb{1}_{\{0 \leq i \leq 2\}}.$$

If $j' + k' \geq [(j+k)/2] + 1$, then there exists a constant $C \geq 1$ such that

$$\mathcal{B}_{j,k,i,j',k'} \lesssim C^{j'+k'},$$

uniformly for all $i \in \{0, 1, 2\}$; if $1 \leq j' + k' \leq [(j+k)/2]$, then we have

$$\mathcal{B}_{j,k,i,j',k'} \lesssim 1, \quad (5.15)$$

uniformly for all $i \in \{0, 1, 2\}$. Combining (5.12), (5.14)–(5.15), and switching indices from $j - j'$ to j'' and $k - k'$ to k'' , we obtain

$$\begin{aligned}
& \sum_{j+k+i \geq 3} \frac{\kappa^j \bar{\kappa}^k \tau^{j+k+i-3}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2} \\
& \lesssim \sum_{j''=0}^{\infty} \sum_{k''=0}^{\infty} \sum_{i=3}^{\infty} \left(\frac{\kappa^{j''} \bar{\kappa}^{k''} \tau^{(j''+k''+i-3)_+}}{(j''+k''+i-3)!} \|\partial_x^{j''+1} \mathbf{T}^{k''} (\epsilon \partial_t)^i S\|_{L^2} \right) \\
& \quad \times \sum_{j=j''}^{\infty} \sum_{k=k''+1}^{\infty} (\kappa \bar{\kappa})^{j-j''} (\bar{\kappa} \bar{\kappa})^{k-k''} \\
& \quad + \sum_{j''=0}^{\infty} \sum_{k''=0}^{\infty} \sum_{i=0}^2 \left(\frac{\kappa^{j''} \bar{\kappa}^{k''} \tau^{(j''+k''+i-3)_+}}{(j''+k''+i-3)!} \|\partial_x^{j''+1} \mathbf{T}^{k''} (\epsilon \partial_t)^i S\|_{L^2} \right) \\
& \quad \times \sum_{j=j''}^{\infty} \sum_{k=k''+1}^{\infty} (C \kappa \bar{\kappa})^{j-j''} (C \bar{\kappa} \bar{\kappa})^{k-k''}.
\end{aligned} \tag{5.16}$$

Note that the sums in j and k in (5.16) are bounded by C for $\kappa \leq 1/C\bar{\kappa}$ and $\bar{\kappa} \leq 1/C\bar{\kappa}$, uniformly for all $j', k', i \in \mathbb{N}_0$. Therefore, by (5.11) and (5.16) we conclude

$$I_2 \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)} \sum_{1 \leq j+k+i \leq 3} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i S\|_{L^2} \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)},$$

where the last inequality follows from (3.23).

Proof of (5.7): Using Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
& \mathcal{C}_{j,k,i,j',k',i'} \mathbb{1}_{\{j+k+i \geq 5\}} \mathbb{1}_{\{[(j+k+i)/2]+1 \leq j'+k'+i' \leq j+k+i-3\}} \\
& \lesssim \left(\frac{\kappa^{(j'-1)_+} \bar{\kappa}^{k'} \tau^{(j'+k'+i'-3)_+}}{(j'+k'+i'-3)!} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v\|_{L^2} \right) \\
& \quad \times \left(\frac{\kappa^{j-j'+2} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-i'-1)_+}}{(j+k+i-j'-k'-i')!} \|\partial_x^{j-j'+2} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{L^2} \right)^{3/4} \\
& \quad \times \left(\frac{\kappa^{j-j'} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-i'-3)_+}}{(j+k+i-j'-k'-i'-3)!} \|\partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{L^2} \right)^{1/4}.
\end{aligned} \tag{5.17}$$

Similarly to (5.10), by (5.2), the remaining factors of κ , $\bar{\kappa}$, and τ are bounded by C ; the product of factors involving combinatorial symbols may be bounded by C since $j' + k' + i' \geq [(j+k+i)/2] + 1$. By (5.17), using the discrete Hölder and the discrete Young inequalities we arrive at

$$I_3 \lesssim \|v\|_{A(\tau)} \|S\|_{\tilde{A}(\tau)} + \|v\|_{A(\tau)} \sum_{j+k+i \geq 3} \frac{\kappa^j \bar{\kappa}^k \tau^{(j+k+i-3)_+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2}.$$

Proceeding as in (5.12)–(5.16), we obtain (5.7).

Proof of (5.8): Using Hölder and Sobolev inequality, we obtain

$$\begin{aligned}
I_4 & \lesssim \sum_{j+k+i \geq 5} \sum_{\substack{(j',k',i') \leq (j,k,i) \\ j+k+i-2 \leq j'+k'+i'}} \left(\frac{\kappa^{(j'-1)_+} \bar{\kappa}^{k'} \tau^{j'+k'+i'-3}}{(j'+k'+i'-3)!} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} v\|_{L^2} \right) \\
& \quad \times \|\partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla S\|_{H^2} \\
& \lesssim \|v\|_{A(\tau)},
\end{aligned}$$

where the last inequality follows from (3.23).

Proof of (5.9): For low values of $j + k + i$, from (3.23) and (4.12), we obtain

$$\sum_{0 \leq j+k+i \leq 4} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2} \lesssim 1. \quad (5.18)$$

For high values of $j + k + i$, by (4.12), we have

$$\begin{aligned} & \sum_{j+k+i \geq 5} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2} \\ & \lesssim \sum_{j+k+i \geq 5} \sum_{k'=1}^k \sum_{j'=0}^j \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \binom{k}{k'} \binom{j}{j'} (j'+k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i S\|_{L^2} \\ & \lesssim \sum_{j+k+i \geq 5} \sum_{k'=1}^k \sum_{j'=0}^j \frac{(j+k)! (j+k+i-j'-k'-3)!}{(j+k-j'-k')! (j+k+i-3)!} (\kappa \bar{K})^{j'+k'} \\ & \quad \times \left(\frac{\kappa^{j-j'} \bar{\kappa}^{k-k'} \tau^{(j+k+i-j'-k'-3)+}}{(j+k+i-j'-k'-3)!} \|\partial_x^{j-j'+1} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i S\|_{L^2} \right), \end{aligned}$$

where the last inequality follows from (5.2). We then proceed as in (5.12)–(5.16) to obtain

$$\sum_{j+k+i \geq 5} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i S\|_{L^2} \lesssim 1 + \|S\|_{\bar{A}(\tau)}. \quad (5.19)$$

Thus (5.9) follows from (5.18) and (5.19).

Combining (5.4) and (5.5)–(5.9), we obtain

$$\frac{d}{dt} \|S\|_{A(\tau)} \lesssim \|S\|_{\bar{A}(\tau)} (\dot{\tau}(t) + \|v\|_{A(\tau)} + 1) + \|S\|_{A(\tau)} + \|v\|_{A(\tau)} + 1, \quad (5.20)$$

where we used (3.23) to bound $\|\nabla v\|_{L_x^\infty}$ and $\|v\|_{L_x^\infty}$. Now, we choose K in (3.17) to be sufficiently large so that the term next to $\|S\|_{\bar{A}(\tau)}$ is less than or equal to 0. The lemma then follows by integrating (5.20) on $[0, T_0]$ and using the Gronwall lemma. $\square$

6. Velocity estimates

In this section, we use derivative reductions for the divergence, curl, and pure time derivative components of the velocity. First, we split $\|v\|_{A(\tau)}$ as

$$\|v\|_{A(\tau)} = \sum_{l=1}^5 \sum_{J_l} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2} = \sum_{l=1}^5 U_l, \quad (6.1)$$

where

$$U_l = \sum_{J_l} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2}, \quad l = 1, \dots, 5,$$

and

$$\begin{aligned} J_1 &= \{(j, k, i) \in \mathbb{N}_0^3 : j \geq 2\}, & J_2 &= \{(j, k, i) \in \mathbb{N}_0^3 : j = 1, k \geq 1\}, \\ J_3 &= \{(j, k, i) \in \mathbb{N}_0^3 : j = 1, k = 0\}, & J_4 &= \{(j, k, i) \in \mathbb{N}_0^3 : j = 0, k \geq 1\}, \\ J_5 &= \{(j, k, i) \in \mathbb{N}_0^3 : j = 0, k = 0\}. \end{aligned}$$

For simplicity of notation, we abbreviate, for $(j, k, i) \in \mathbb{N}_0^3$,

$$U_{j,k,i} = \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2}$$

for the velocity coefficients arising in expansions below. Similarly, we denote $E = \{(j, k, i) \in \mathbb{N}_0^3 : j \geq 1\}$, and we write

$$D_{j,k,i} = \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i \operatorname{div} v\|_{L^2}$$

for the coefficients containing the divergence and

$$C_{j,k,i} = \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2},$$

for those with the curl.

The term U_1 : First we estimate the sum U_1 . By (4.2), we have

$$\begin{aligned} U_1 &\lesssim \frac{\kappa}{\bar{\kappa}} U_1 + \kappa U_1 + \sum_E D_{j,k,i} + \sum_E C_{j,k,i} \\ &\quad + \sum_{J_1} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{div}] (\epsilon \partial_t)^i v\|_{L^2} \\ &\quad + \sum_{J_1} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^{j-1} [\mathbf{T}^k, \operatorname{curl}] (\epsilon \partial_t)^i v\|_{L^2} \\ &\quad + \sum_{J_1} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|[\mathbf{T}, \partial_x^{j-2}] \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2} \\ &\quad + \sum_{J_1} \frac{\kappa^{(j-1)+} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x [\mathbf{T}, \partial_x^{j-2}] \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2} \\ &= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16} + I_{17} + I_{18}. \end{aligned} \tag{6.2}$$

For the term I_{15} , we use (4.11) to write

$$\begin{aligned} I_{15} &\lesssim \sum_{J_1} \sum_{k'=1}^k \sum_{j'=0}^{j-1} \binom{k}{k'} \binom{j-1}{j'} \frac{\kappa^{j-1} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \\ &\quad \times (j' + k')! \bar{K}^{j'+k'} \|\partial_x^{j-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v\|_{L^2} \\ &\lesssim \sum_{i=0}^{\infty} \sum_{k''=0}^{\infty} \sum_{j''=0}^{\infty} \frac{\kappa^{j''} \bar{\kappa}^{k''} \tau^{(i+j''+k''-2)+}}{(i+j''+k''-2)!} \|\partial_x^{j''+1} \mathbf{T}^{k''} (\epsilon \partial_t)^i v\|_{L^2} \\ &\quad \times \sum_{j=j''+1}^{\infty} \sum_{k=k''+1}^{\infty} \frac{(k+j-1)!(i+j''+k''-2)!}{(j''+k'')!(i+j+k-3)!} \kappa^{j-j''-1} \bar{\kappa}^{k-k''} \bar{K}^{k-k''+j-j''-2}. \end{aligned} \tag{6.3}$$

Note that the sum in j, k is dominated by

$$\bar{\kappa} \bar{K} \sum_{j=j''+1}^{\infty} \sum_{k=k''+1}^{\infty} \frac{(k+j-1)!(i+j''+k''-2)!}{(j''+k'')!(i+j+k-3)!} \kappa^{j-j''-1} \bar{\kappa}^{k-k''-1} \bar{K}^{k-k''+j-j''-2} \lesssim \bar{\kappa}, \tag{6.4}$$

uniformly for all $i, k'', j'' \in \mathbb{N}_0$, by taking $\bar{\kappa} \leq 1/CK$. Thus, from (6.3)–(6.4) we obtain

$$I_{15} \lesssim \bar{\kappa} U_1 + \bar{\kappa} U_2 + \bar{\kappa} U_3. \tag{6.5}$$

The term I_{16} can be treated analogously as I_{15} . Namely, using (4.13), we get

$$I_{16} \lesssim \bar{\kappa} U_1 + \bar{\kappa} U_2 + \bar{\kappa} U_3. \tag{6.6}$$

For the term I_{17} , we appeal to (4.14) obtaining

$$\begin{aligned}
I_{17} &\lesssim \sum_{J_1} \sum_{j'=1}^{j-2} \binom{j-2}{j'} j'! \bar{K}^{j'} \frac{\kappa^{j-1} \bar{\kappa}^k \tau^{(i+j+k-3)+}}{(i+j+k-3)!} \|\partial_x^{j-j'-1} \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2} \\
&\lesssim \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j''=0}^{\infty} \left(\frac{\kappa^{j''} \bar{\kappa}^k \tau^{(i+j''+k-2)+}}{(i+j''+k-2)!} \|\partial_x^{j''+1} \mathbf{T}^k (\epsilon \partial_t)^i v\|_{L^2} \right) \\
&\quad \times \sum_{j=j''+3}^{\infty} \frac{(i+j''+k-2)!(j-2)!}{(i+j+k-3)!j''!} \kappa^{j-j''-1} \bar{K}^{j-j''-2}.
\end{aligned} \tag{6.7}$$

Note that the sum in j is dominated by

$$\kappa \sum_{j=j''+3}^{\infty} \frac{(i+j''+k-2)!(j-2)!}{(i+j+k-3)!j''!} \kappa^{j-j''-2} \bar{K}^{j-j''-2} \lesssim \kappa, \tag{6.8}$$

uniformly for all $i, k, j'' \in \mathbb{N}_0$, by taking $\kappa \leq 1/C\bar{K}$. From (6.7)–(6.8), we obtain

$$I_{17} \lesssim \kappa U_1 + \kappa U_2 + \kappa U_3. \tag{6.9}$$

The term I_{18} can be estimated analogously as I_{17} , and we arrive at

$$I_{18} \lesssim \kappa U_1 + \kappa U_2 + \kappa U_3. \tag{6.10}$$

Collecting the estimates (6.2), (6.5)–(6.6), and (6.9)–(6.10), we obtain

$$U_1 \lesssim \left(\frac{\kappa}{\bar{\kappa}} + \kappa + \bar{\kappa} \right) U_1 + (\kappa + \bar{\kappa}) U_2 + (\kappa + \bar{\kappa}) U_3 + \sum_E D_{j,k,i} + \sum_E C_{j,k,i}. \tag{6.11}$$

The term U_2 : Next, we estimate the sum U_2 . By (4.6), we have

$$\begin{aligned}
U_2 &\lesssim \sum_E D_{j,k,i} + \sum_E C_{j,k,i} + U_4 + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|\partial_x \mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|\mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \|\partial_x \mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|\mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|[\mathbf{T}^k, \text{div}](\epsilon \partial_t)^i v\|_{L^2} \\
&\quad + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|[\mathbf{T}^k, \text{curl}](\epsilon \partial_t)^i v\|_{L^2} \\
&= I_{21} + I_{22} + I_{23} + I_{24} + I_{25} + I_{26} + I_{27} + I_{28}.
\end{aligned} \tag{6.12}$$

We split the sum I_{24} according to the values of k and i , obtaining

$$\begin{aligned}
I_{24} &= \sum_{k=1}^{\infty} \sum_{i=2}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k+i-2)+}}{(k+i-2)!} \|\partial_x \mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{k=2}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k-2)+}}{(k-2)!} \|\partial_x \mathbf{T}^{k-l} v\|_{L^2} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{k=2}^{\infty} \sum_{l=1}^k \binom{k}{l} \frac{\bar{\kappa}^k \tau^{(k-1)+}}{(k-1)!} \|\partial_x \mathbf{T}^{k-l} (\epsilon \partial_t) v\|_{L^2} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&\quad + \sum_{i=0}^1 \frac{\bar{\kappa} \tau^{(i-1)+}}{(i-1)!} \|\partial_x (\epsilon \partial_t)^i v\|_{L^2} \|\mathbf{T} \nu\|_{L^\infty(\Omega_{\delta_0})} \\
&= I_{241} + I_{242} + I_{243} + I_{244}.
\end{aligned}$$

For the term I_{241} , we estimate

$$\begin{aligned}
I_{241} &\lesssim \sum_{k=1}^{\infty} \sum_{i=2}^{\infty} \sum_{l=1}^k \frac{\bar{\kappa}^l \tau^l}{\tilde{\eta}^l} \left(\frac{\bar{\kappa}^{k-l} \tau^{(k+i-l-2)+}}{(k+i-l-2)!} \|\partial_x \mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \right) \left(\frac{\tilde{\eta}^l}{(l-3)!} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \right) \\
&\quad \times \frac{k!(k+i-l-2)!(l-3)!}{l!(k-l)!(k+i-2)!} \\
&\lesssim \bar{\kappa} \sum_{k=1}^{\infty} \sum_{i=2}^{\infty} \sum_{l=1}^k \frac{\bar{\kappa}^{l-1}}{\tilde{\eta}^{l-1}} \left(\frac{\bar{\kappa}^{k-l} \tau^{(k+i-l-2)+}}{(k+i-l-2)!} \|\partial_x \mathbf{T}^{k-l} (\epsilon \partial_t)^i v\|_{L^2} \right) \left(\frac{\tilde{\eta}^l}{(l-3)!} \|\mathbf{T}^l \nu\|_{L^\infty(\Omega_{\delta_0})} \right).
\end{aligned}$$

We choose $\bar{\kappa} \leq \tilde{\eta}$ and use (4.17) to get

$$I_{241} \lesssim \bar{\kappa} U_2 + \bar{\kappa} U_3.$$

The terms I_{242} , I_{243} , and I_{244} can be estimated analogously to I_{241} . Thus we arrive at

$$I_{24} \lesssim \bar{\kappa} U_2 + \bar{\kappa} U_3. \quad (6.13)$$

Similarly, the terms I_{25} and I_{26} are treated analogously to I_{24} , obtaining

$$I_{25} + I_{26} \lesssim \bar{\kappa} U_2 + \bar{\kappa} U_3. \quad (6.14)$$

For the term I_{27} , we use (4.11) and obtain

$$\begin{aligned}
I_{27} &\lesssim \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \sum_{k'=1}^k \binom{k}{k'} \frac{\bar{\kappa}^k \tau^{(i+k-2)+}}{(k+i-2)!} k'! \bar{K}^{k'} \|\partial_x \mathbf{T}^{k-k'} (\epsilon \partial_t)^i v\|_{L^2} \\
&\lesssim \sum_{i=0}^{\infty} \sum_{k''=0}^{\infty} \frac{\bar{\kappa}^{k''} \tau^{(i+k''-2)+}}{(i+k''-2)!} \|\partial_x \mathbf{T}^{k''} (\epsilon \partial_t)^i v\|_{L^2} \times \sum_{k=k''+1}^{\infty} \frac{k!(i+k''-2)!}{k''!(k+i-2)!} \bar{\kappa}^{k-k''} \bar{K}^{k-k''}.
\end{aligned} \quad (6.15)$$

Note that the sum in k is bounded by

$$\bar{\kappa} \bar{K} \sum_{k=k''+1}^{\infty} \frac{k!(i+k''-2)!}{k''!(k+i-2)!} \bar{\kappa}^{k-k''-1} \bar{K}^{k-k''-1} \lesssim \bar{\kappa}, \quad (6.16)$$

uniformly for all $k'', i \in \mathbb{N}$, by taking $\bar{\kappa} \leq 1/CK$. Thus from (6.15)–(6.16) we estimate

$$I_{27} \lesssim \bar{\kappa} U_2 + \bar{\kappa} U_3. \quad (6.17)$$

The term I_{28} can be estimated analogously as I_{27} by using (4.13), and we arrive at

$$I_{28} \lesssim \bar{\kappa} U_2 + \bar{\kappa} U_3. \quad (6.18)$$

Collecting the estimates (6.12), (6.13)–(6.14), and (6.17)–(6.18), we conclude

$$U_2 \lesssim \bar{\kappa}U_2 + \bar{\kappa}U_3 + U_4 + \sum_E D_{j,k,i} + \sum_E C_{j,k,i},$$

from where we arrive at

$$U_2 \lesssim \bar{\kappa}U_3 + U_4 + \sum_E D_{j,k,i} + \sum_E C_{j,k,i}, \quad (6.19)$$

by taking $\bar{\kappa} \leq 1/C$.

The terms U_3 and U_4 : For U_3 , we have by (4.3),

$$U_3 \lesssim \sum_E D_{j,k,i} + \sum_E C_{j,k,i} + U_5. \quad (6.20)$$

For U_4 , we have by (4.7),

$$U_4 \lesssim \bar{\kappa} \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{\bar{\kappa}^{k-1} \tau^{(k+i-3)+}}{(k+i-3)!} \|\partial_x \mathbf{T}^{k-1}(\epsilon \partial_t)^i v\|_{L^2} = \bar{\kappa}U_2 + \bar{\kappa}U_3.$$

The term U_5 : We claim that there exists $t_0 > 0$ sufficiently small depending only on M_0 and $\epsilon_1 > 0$ sufficiently small depending on $M_{\epsilon, \kappa, \bar{\kappa}}(T)$, such that

$$\sum_{i=0}^{\infty} \frac{\tau^{(i-3)+}}{(i-3)!} \|(\epsilon \partial_t)^i(p, v)\|_{L^2} \lesssim 1 + tQ(M_{\epsilon, \kappa, \bar{\kappa}}(t)), \quad (6.21)$$

for all $\epsilon \in (0, \epsilon_1)$. The proof of (6.21) relies on the energy estimate of the partially linearized equation as in the case $\Omega = \mathbb{R}^3$, cf. [JKL, Lemma 6.2]. Here we only outline the modification needed for the presence of boundary. Instead of using the elliptic regularity, we appeal to Lemma 4.3 to estimate the dissipative term. Namely, we have

$$\begin{aligned} \|(\epsilon \partial_t)^i \nabla v\|_{L^2} &\lesssim \|(\epsilon \partial_t)^i \operatorname{div} v\|_{L^2} + \|(\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2} + \|(\epsilon \partial_t)^i v \cdot \nu\|_{H^{1/2}(\partial\Omega)} + \|(\epsilon \partial_t)^i v\|_{L^2} \\ &\lesssim \|(\epsilon \partial_t)^i \operatorname{div} v\|_{L^2} + \|(\epsilon \partial_t)^i \operatorname{curl} v\|_{L^2} + \|(\epsilon \partial_t)^i v\|_{L^2}, \end{aligned}$$

since $v \cdot \nu = 0$ on $\partial\Omega$. Then we may proceed as in [JKL, Lemma 6.2] since the term $\|(\epsilon \partial_t)^i v\|_{L^2}$ can be absorbed in the A norm. It then follows from (6.21) that

$$U_5 \lesssim 1 + tQ(M_{\epsilon, \kappa, \bar{\kappa}}(t)), \quad t \in (0, t_0), \quad (6.22)$$

6.1. Nonhomogeneous transport equation. As shown in [JKL] and [A1], the curl component of the velocity satisfies the non-homogeneous transport equation. Thus, we consider

$$\partial_t \tilde{S} + v \cdot \nabla \tilde{S} = G,$$

where $\tilde{S} = \tilde{S}(t, x)$, $v = v(t, x)$ and $G = G(t, x)$. Denote

$$\|u\|_{B(\tau)} = \sum_{(j,k,i) \in \mathbb{N}_0^3} \frac{\kappa^j \bar{\kappa}^k \tau^{(j+k+i-2)+}}{(j+k+i-2)!} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u\|_{L^2},$$

with the corresponding dissipative analytic norm

$$\|u\|_{\tilde{B}(\tau)} = \sum_{j+k+i \geq 3} \frac{\kappa^j \bar{\kappa}^k (j+k+i-2) \tau^{j+k+i-3}}{(j+k+i-2)!} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u\|_{L^2}.$$

LEMMA 6.1. *For any $\kappa, \bar{\kappa} \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ such that if $0 < \tau(0) \leq \tau_1$, then*

$$\|\tilde{S}(t)\|_{A(\tau)} \lesssim \|\tilde{S}(0)\|_{A(\tau)} + \int_0^t (\|G(s)\|_{A(\tau)} + \|v(s)\|_{A(\tau)}) + t,$$

for some constant $C > 0$ and sufficiently small $T_0 > 0$, provided K in (3.17) satisfies

$$K \geq C\|v\|_{A(\tau)}, \quad t \in [0, T_0], \quad (6.23)$$

where $C \geq 1$ is a sufficiently large constant, and T_0 is chosen so that (3.22) holds. Similarly, for any $\kappa, \bar{\kappa} \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ such that if $0 < \tau(0) \leq \tau_1$, then

$$\|\tilde{S}(t)\|_{B(\tau)} \lesssim \|\tilde{S}(0)\|_{B(\tau)} + \int_0^t (\|G(s)\|_{B(\tau)} + \|v(s)\|_{B(\tau)}) + t,$$

for some constant $C > 0$ and sufficiently small $T_0 > 0$, provided K in (3.17) satisfies

$$K \geq C\|v\|_{B(\tau)}, \quad t \in [0, T_0], \quad (6.24)$$

for $C \geq 1$ sufficiently large and where T_0 is chosen so that (3.22) holds.

Note that $\|v\|_{A(\tau)} \geq \|v\|_{B(\tau)}$ for all v , and thus (6.23) implies (6.24).

6.2. Product rule. In this section, we introduce the product rule which is needed in the estimates of divergence, curl, and pure time derivative components of the velocity. We denote

$$\|u\|_Y = \sup_{(j,k,i) \in \mathbb{N}_0^3, 0 \leq j+k+i \leq 4} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i u\|_{L^2}.$$

The following lemma provides a product rule for the analytic B -norm.

LEMMA 6.2. *Let $k \in \{2, 3, \dots\}$ and $\tau > 0$. For $f_1, \dots, f_k \in B(\tau)$, and any $\kappa, \bar{\kappa} \in (0, 1]$, there exists $\tau_1 \in (0, 1]$ and $T_0 > 0$ such that if $0 < \tau(0) \leq \tau_1$, then*

$$\left\| \prod_{i=1}^k f_i \right\|_{B(\tau)} \leq C^k \sum_{i=1}^k \left(\|f_i\|_{B(\tau)} \prod_{1 \leq j \leq k; j \neq i} (\|f_j\|_{B(\tau)} + \|f_j\|_Y) \right).$$

The next statement provides an analytic estimate for composition of functions.

LEMMA 6.3. *Assume that f is an entire real-analytic function. Then for $\tau > 0$ and $w \in A(\tau)$, we have*

$$\|f(w)\|_{A(\tau)} \lesssim Q(\|w\|_{A(\tau)} + \|w\|_Y)$$

and

$$\|f(w)\|_{B(\tau)} \lesssim Q(\|w\|_{B(\tau)} + \|w\|_Y),$$

for some function Q .

Recall that E is assumed to be the product of two entire real-analytic functions. Suppose that $\tilde{e}$ is one of the components of the matrix E . Thus,

$$\tilde{e}(S, \epsilon u) = f(S)g(\epsilon u). \quad (6.25)$$

In the next statement, we provide an analytic estimates of $\tilde{e}$.

LEMMA 6.4. *Let $M_0 > 0$ and assume that $\tilde{e}$ satisfies (6.25). Then*

$$\|\partial_t \tilde{e}\|_{B(\tau)} \lesssim Q(\|u\|_{A(\tau)} + \|u\|_Y, \|S\|_{A(\tau)} + \|S\|_Y), \quad \epsilon \in (0, 1]$$

and

$$\|\tilde{e}\|_{A(\tau)} \lesssim Q(\|u\|_{A(\tau)} + \|u\|_Y, \|S\|_{A(\tau)} + \|S\|_Y), \quad \epsilon \in (0, 1],$$

where Q is a function.

The proofs of Lemmas 6.1, 6.2, 6.3, and 6.4 are analogous to Lemmas 4.2, 5.1, 5.2, and 5.3 in [JKL]. Thus we simply state them and refer to [JKL] for details.

6.3. The divergence components. In this section, we estimate the sum of terms involving the divergence of the velocity. First, we rewrite the equation (2.4) as

$$L(\partial_x)u = -E(S, \epsilon u)(\epsilon \partial_t u + \epsilon v \cdot \nabla u). \quad (6.26)$$

For $j \geq 2$ and $k, i \in \mathbb{N}_0$, we commute $\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i$ with (6.26), obtaining

$$\begin{aligned} & \|\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i L(\partial_x)u\|_{L^2} \\ & \lesssim \sum_{j'=0}^{j-1} \sum_{k'=0}^k \sum_{i'=0}^i \binom{j-1}{j'} \binom{k}{k'} \binom{i}{i'} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} E \partial_x^{j-1-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'+1} u\|_{L^2} \\ & \quad + \epsilon \|Ev\|_{L^\infty} \|\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i \nabla u\|_{L^2} \\ & \quad + \epsilon \sum_{j'=0}^{j-1} \sum_{k'=0}^k \sum_{i'=0}^i \binom{j-1}{j'} \binom{k}{k'} \binom{i}{i'} \|\partial_x^{j'} \mathbf{T}^{k'} (\epsilon \partial_t)^{i'} (Ev) \partial_x^{j-1-j'} \mathbf{T}^{k-k'} (\epsilon \partial_t)^{i-i'} \nabla u\|_{L^2}. \end{aligned} \quad (6.27)$$

Multiplying the above estimate with appropriate weights and following the arguments from [JKL, Lemma 6.3], which is justified since we have Lemmas 6.2, 6.3, and 6.4, we obtain

$$\sum_{j=2}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} D_{j,k,i} \lesssim (\kappa + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.28)$$

For $j = 1$, $k = 0$, and $i \in \mathbb{N}_0$, we proceed as in [JKL, Lemma 6.3] to get

$$\sum_{i=0}^{\infty} D_{1,0,i} \lesssim 1 + (t + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.29)$$

For $j = 1$, $k \in \mathbb{N}$, and $i \in \mathbb{N}_0$, from (4.7), we have

$$\|\mathbf{T}^k (\epsilon \partial_t)^i L(\partial_x)u\|_{L^2} \lesssim \|\partial_x \mathbf{T}^{k-1} (\epsilon \partial_t)^i L(\partial_x)u\|_{L^2},$$

from where we proceed as in (6.27)–(6.28) to obtain

$$\sum_{k=1}^{\infty} \sum_{i=0}^{\infty} D_{1,k,i} \lesssim (\bar{\kappa} + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.30)$$

Combining (6.28)–(6.29) and (6.30), we arrive at

$$\sum_E D_{j,k,i} \lesssim 1 + (\kappa + \bar{\kappa} + t + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.31)$$

6.4. The curl components. As in [JKL, Section 6.1], we use Lemmas 6.2, 6.3, and 6.4 to obtain

$$\sum_E C_{j,k,i} \lesssim 1 + (t + \tau) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)), \quad (6.32)$$

for all $t \in [0, T_0]$,

6.5. The pressure estimates. The analytic norm of the pressure can be recovered by the mixed space-time derivatives and pure time derivatives. Namely, for $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have

$$\|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i p\|_{L^2} \lesssim \|\partial_x^{j-1} \mathbf{T}^k (\epsilon \partial_t)^i L(\partial_x)u\|_{L^2} + \|\partial_x^{j-1} [\mathbf{T}^k, \nabla] (\epsilon \partial_t)^i p\|_{L^2}. \quad (6.33)$$

The first term on the right side is estimated in Section 6.3, while the second term is estimated analogously to (5.9). Thus, we arrive at

$$\sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\kappa^{(j-1)+\bar{\kappa}k} \tau^{(j+k+i-3)+}}{(j+k+i-3)!} \|\partial_x^j \mathbf{T}^k (\epsilon \partial_t)^i p\|_{L^2} \lesssim 1 + (\kappa + \bar{\kappa} + t + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.34)$$

For $j = 0$ and $k \in \mathbb{N}$, we may use (4.7) and (6.33)–(6.34) to get

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{\bar{\kappa}^k \tau^{(k+i-3)_+}}{(k+i-3)!} \|\mathbf{T}^k(\epsilon \partial_t)^i p\|_{L^2} &\lesssim \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{\bar{\kappa}^k \tau^{(k+i-3)_+}}{(k+i-3)!} \|\partial_x \mathbf{T}^{k-1}(\epsilon \partial_t)^i p\|_{L^2} \\ &\lesssim 1 + (\kappa + \bar{\kappa} + t + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \end{aligned} \quad (6.35)$$

Combining (6.34)–(6.35) with the pure time derivative estimates of the pressure obtained in (6.21), we arrive at

$$\|p\|_{A(\tau)} \lesssim 1 + (\kappa + \bar{\kappa} + t + \epsilon) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)). \quad (6.36)$$

6.6. Proof of the main lemma. Here we combine the results of the previous section to prove Lemma 3.5.

PROOF OF LEMMA 3.5. Combining the estimates (6.1), (6.11), (6.19), (6.20)–(6.22), and (6.31)–(6.32), we arrive at

$$\|v\|_{A(\tau)} \lesssim 1 + (t + \kappa + \epsilon + \tau(0) + \bar{\kappa}) Q(M_{\epsilon, \kappa, \bar{\kappa}}(t)), \quad (6.37)$$

by taking $\bar{\kappa} \leq 1/C$ and $\kappa \leq \bar{\kappa}/C$. Thus Lemma 3.5 follows by combining (6.36)–(6.37) with (5.1). $\square$

7. Analyticity assumptions on the initial data

In this section, we assume the initial data satisfies (3.14), and show that for low values of i ,

$$\sum_{i=0}^3 \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u(0)\|_{L^2} \frac{\tau_0^{(j+k+i-3)_+}}{(j+k+i-3)!} \leq \Gamma, \quad (7.1)$$

and

$$\sum_{i=0}^3 \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S(0)\|_{L^2} \frac{\tau_0^{(j+k+i-3)_+}}{(j+k+i-3)!} \leq \Gamma, \quad (7.2)$$

where $\Gamma > 0$ is a sufficiently large constant depending on M_0 ; for high values of i , there exists a sufficiently large constant $C > 1$ such that for all $n \geq 4$ we have

$$\sum_{i=4}^n \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i u(0)\|_{L^2} \frac{\tau_0^{j+k+i-3}}{C^{i-3}(j+k+i-3)!} \leq 1, \quad (7.3)$$

and

$$\sum_{i=4}^n \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S(0)\|_{L^2} \frac{\tau_0^{j+k+i-3}}{C^{i-3}(j+k+i-3)!} \leq 1. \quad (7.4)$$

In (7.3) and (7.4) we then choose $\tilde{\tau}_0 = \tau_0/C$ and using (7.1)–(7.2), we obtain

$$\begin{aligned} \|(p_0, v_0, S_0)\|_{A(\tilde{\tau}_0)} &\leq \sum_{(j,k,i) \in \mathbb{N}_0^3} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i (u, S)(0)\|_{L^2} \frac{\tilde{\tau}_0^{(j+k+i-3)_+}}{(j+k+i-3)!} \\ &\leq \sum_{i=0}^3 \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S(0)\|_{L^2} \frac{\tau_0^{(j+k+i-3)_+}}{(j+k+i-3)!} \\ &\quad + \sum_{i=4}^{\infty} \sum_{j,k=0}^{\infty} \|\partial_x^j \mathbf{T}^k(\epsilon \partial_t)^i S(0)\|_{L^2} \frac{\tau_0^{j+k+i-3}}{C^{i-3}(j+k+i-3)!} \\ &\leq \Gamma + 1, \end{aligned}$$

and we conclude as in [JKL, Section 8]. The proofs of (7.1)–(7.4) are analogous to those in [JKL, Section 8] by using the commutator estimate (4.12), and thus we omit the details.

8. Proof of the convergence theorem

The following lemma is needed in the proof of the Theorem 3.4.

LEMMA 8.1. *There exists a constant $C_1 > 1$ such that for any $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = j$ where $j \in \mathbb{N}_0$, we have*

$$\|\partial_x^j u\|_{L^2} \leq C_1^j \|u\|_{L^2}^{1/(j+1)} \|\partial_x^{j+1} u\|_{L^2}^{j/(j+1)}, \quad (8.1)$$

for all $u \in H^{j+1}(\Omega)$.

We emphasize that C_1 is j -independent.

PROOF OF LEMMA 8.1. We proceed by induction on $j \in \mathbb{N}_0$. The case $j = 0$ is clear. Now we assume that (8.1) holds for some $j \in \mathbb{N}_0$ and aim to prove it for the case $j+1$. Let $\alpha \in \mathbb{N}_0^3$ be any multiindex with $|\alpha| = j+1$. We choose a multiindex $\alpha_- \in \mathbb{N}_0^3$ such that there exists $\xi \in \mathbb{N}_0^3$ for which $\alpha_- + \xi = \alpha$ and $|\xi| = 1$. Let $\alpha_+ = 2\alpha - \alpha_-$. Using integration by parts, we obtain

$$\|\partial^\alpha u\|_{L^2}^2 = \int_{\partial\Omega} \partial^{\alpha_-} u \cdot \partial^\alpha u \nu^i d\sigma - \int_{\Omega} \partial^{\alpha_-} u \cdot \partial^{\alpha_+} u dx = I_1 + I_2, \quad (8.2)$$

for some $i \in \{1, 2, 3\}$. For the term I_1 , we use the Cauchy-Schwarz inequality to get

$$\begin{aligned} I_1 &\leq \left(\int_{\partial\Omega} |\partial^{\alpha_-} u|^2 d\sigma \right)^{1/2} \left(\int_{\partial\Omega} |\partial^\alpha u|^2 d\sigma \right)^{1/2} \\ &\leq C \|\partial^{\alpha_-} u\|_{L^2}^{1/2} \|\nabla \partial^{\alpha_-} u\|_{L^2}^{1/2} \|\partial^\alpha u\|_{L^2}^{1/2} \|\nabla \partial^\alpha u\|_{L^2}^{1/2}, \end{aligned} \quad (8.3)$$

where the last inequality follows from the trace theorem. For the term I_2 , using the Cauchy-Schwarz inequality, we arrive at

$$I_2 \leq \|\partial^{\alpha_-} u\|_{L^2} \|\partial^{\alpha_+} u\|_{L^2}. \quad (8.4)$$

Summing (8.2) over $|\alpha| = j+1$ and using (8.3)–(8.4), we obtain

$$\begin{aligned} \|\partial_x^{j+1} u\|_{L^2}^2 &\leq C \sum_{|\alpha|=j+1} \|\partial^{\alpha_-} u\|_{L^2}^{1/2} \|\nabla \partial^{\alpha_-} u\|_{L^2}^{1/2} \|\partial^\alpha u\|_{L^2}^{1/2} \|\nabla \partial^\alpha u\|_{L^2}^{1/2} + \sum_{|\alpha|=j+1} \|\partial^{\alpha_-} u\|_{L^2} \|\partial^{\alpha_+} u\|_{L^2} \\ &\leq C \|\partial_x^j u\|_{L^2}^{1/2} \|\partial_x^{j+1} u\|_{L^2} \|\partial_x^{j+2} u\|_{L^2}^{1/2} + C \|\partial_x^j u\|_{L^2} \|\partial_x^{j+2} u\|_{L^2}. \end{aligned}$$

Using the Cauchy-Schwarz inequality to absorb $\|\partial_x^{j+1} u\|_{L^2}$ from the first factor and then using the induction hypothesis for the case j , we obtain

$$\|\partial_x^{j+1} u\|_{L^2}^2 \leq C \|\partial_x^j u\|_{L^2} \|\partial_x^{j+2} u\|_{L^2} \leq C C_1^j \|u\|_{L^2}^{1/(j+1)} \|\partial_x^{j+1} u\|_{L^2}^{j/(j+1)} \|\partial_x^{j+2} u\|_{L^2},$$

from where we arrive at

$$\begin{aligned} \|\partial_x^{j+1} u\|_{L^2} &\leq C^{(j+1)/(j+2)} C_1^{j(j+1)/(j+2)} \|u\|_{L^2}^{1/(j+2)} \|\partial_x^{j+2} u\|_{L^2}^{(j+1)/(j+2)} \\ &\leq C_1^{j+1} \|u\|_{L^2}^{1/(j+2)} \|\partial_x^{j+2} u\|_{L^2}^{(j+1)/(j+2)}, \end{aligned}$$

by taking $C_1 \geq C$. Therefore, (8.1) is proven for $j+1$ for some sufficiently large constant $C_1 > 1$. $\square$

Next we prove the second main result on convergence of the Mach limit.

PROOF OF THEOREM 3.4. Denote the spatial analyticity radius

$$\delta = \frac{\kappa \tau(0)}{C_0},$$

where $\tau(0), \kappa \in (0, 1]$ are fixed constants as in Theorem 3.3 and $C_0 \geq 2$ is a sufficiently large constant to be determined below. It is proved in Theorem 3.3 that if the initial data $(p_0^\epsilon, v_0^\epsilon, S_0^\epsilon)$ satisfies (3.13)–(3.14) for some constants $M_0, \tau_0 > 0$, as well as the compatibility condition of all orders, then

$$\|(p^\epsilon, v^\epsilon, S^\epsilon)(t)\|_{X(\delta)} \lesssim M \kappa^{-2}, \quad \epsilon \in (0, \epsilon_0], \quad t \in [0, T_0],$$

for some parameters $\kappa, \tau(0), \epsilon_0$, and $T_0 > 0$.

We proceed differently than in [JKL, Section 7]. For the sake of contradiction, we assume that $(v^\epsilon, p^\epsilon, S^\epsilon)$ does not converge to $(v^{(\text{inc})}, 0, S^{(\text{inc})})$ in $L^2([0, T], X(\delta))$. Then there exists a sequence $\{\epsilon_n\} \rightarrow 0$ such that $\{(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})\}$ does not converge to $(v^{(\text{inc})}, 0, S^{(\text{inc})})$ in $L^2([0, T], X(\delta))$. Recall from [A1, Theorem 1.2] that $(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})$ converges to $(v^{(\text{inc})}, 0, S^{(\text{inc})})$ in $L^2([0, T], L^2(\Omega))$ as $\epsilon_n \rightarrow 0$. We define $v_{kn}(t) = v^{\epsilon_k}(t) - v^{\epsilon_n}(t)$, for $k, n \in \mathbb{N}$. Using Lemma 8.1, we obtain

$$\phi_{kn}(t) = \sum_{m=4}^{\infty} \sum_{|\alpha|=m} \frac{\delta^{m-3}}{(m-3)!} \|\partial^\alpha v_{kn}\|_{L_x^2} \leq \sum_{m=4}^{\infty} \frac{C_1^m \delta^{m-3}}{(m-3)!} \|v_{kn}\|_{L_x^2}^{1/(m+1)} \|\partial_x^{m+1} v_{kn}\|_{L_x^2}^{m/(m+1)}, \quad (8.5)$$

and thus, using the Minkowski and Hölder inequalities

$$\|\phi_{kn}(t)\|_{L_t^2} \lesssim \sum_{m=4}^{\infty} \frac{C_1^m \delta^{m-3}}{(m-3)!} \|v_{kn}\|_{L_{x,t}^2}^{1/(m+1)} \|\partial_x^{m+1} v_{kn}\|_{L_{x,t}^2}^{m/(m+1)}, \quad (8.6)$$

where the space and time domains are understood to be Ω and $[0, T]$, respectively. To show convergence to zero, we shall apply the discrete dominated convergence theorem. To get a uniform bound, we use the discrete Young inequality, we get

$$\begin{aligned} & \sum_{m=4}^{\infty} \frac{(C_1 \delta)^{m-3}}{(m-3)!} \|\partial_x^{m+1} v_{kn}\|_{L_{x,t}^2}^{m/(m+1)} \\ &= \sum_{m=4}^{\infty} \left(\frac{(C_1 \delta)^{m-3}}{(m-3)!} \|\partial_x^{m+1} v_{kn}\|_{L_{x,t}^2} \right)^{m/(m+1)} \left(\frac{(C_1 \delta)^{m-3}}{(m-3)!} \right)^{1/(m+1)} \\ &\leq \sum_{m=4}^{\infty} \frac{m(C_1 \delta)^{m-3}}{(m+1)(m-3)!} \|\partial_x^{m+1} v_{kn}\|_{L_{x,t}^2} + \sum_{m=4}^{\infty} \frac{(C_1 \delta)^{m-3}}{(m+1)(m-3)!}. \end{aligned}$$

Now, choose $\delta = \kappa\tau(0)/C_1 C_2$, where C_1 is the constant from Lemma 8.1 and $C_2 \geq 1$. We obtain

$$\sum_{m=4}^{\infty} \frac{(C_1 \delta)^{m-3}}{(m-3)!} \|\partial_x^{m+1} v_{kn}\|_{L_{x,t}^2}^{m/(m+1)} \leq C M \kappa^{-2} \delta^{-1} C_1^{-1} + C. \quad (8.7)$$

Thus the discrete dominated convergence theorem applies, and it thus follows from (8.5)–(8.7) that

$$\sum_{m=4}^{\infty} \sum_{|\alpha|=m} \frac{\delta^{(m-3)_+}}{(m-3)!} \|\partial^\alpha v_{kn}\|_{L_{x,t}^2} \rightarrow 0 \quad \text{as } k, n \rightarrow \infty \quad (8.8)$$

since $\|v_{kn}(t)\|_{L_{x,t}^2} \rightarrow 0$ as $k, n \rightarrow \infty$. Note that from (3.23) we have

$$\|v_{kn}\|_{L_t^2 H^3}^2 \leq C \|v_{kn}\|_{H^4}^{3/2} \|v_{kn}\|_{L_{x,t}^2}^{1/2} + C \|v_{kn}\|_{L_{x,t}^2}^2 \leq C \|v_{kn}\|_{L_{x,t}^2}^{1/2} + C \|v_{kn}\|_{L_{x,t}^2}^2 \rightarrow 0 \quad \text{as } k, n \rightarrow \infty. \quad (8.9)$$

From (8.8)–(8.9) and analogous inequalities for p_{kn} and S_{kn} , we infer that the sequence $\{(v^{\epsilon_n}, p^{\epsilon_n}, S^{\epsilon_n})\}$ is Cauchy in $L^2([0, T_0], X(\delta))$, which along with $(v^\epsilon, p^\epsilon, S^\epsilon) \rightarrow (v^{(\text{inc})}, 0, S^{(\text{inc})})$ in $L^2([0, T], L^2(\Omega))$ leads to a contradiction. Therefore, $\{(v^\epsilon, p^\epsilon, S^\epsilon)\}$ is convergent and thus goes to $(v^{(\text{inc})}, 0, S^{(\text{inc})})$ in $L^2([0, T_0], X(\delta))$ as $\epsilon \rightarrow 0$. $\square$

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