

KINETIC DESCRIPTION OF STABLE WHITE DWARFS

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ABSTRACT. In this paper, we study fermion ground states of the relativistic Vlasov-Poisson system arising in the semiclassical limit from relativistic quantum theory of white dwarfs. We show that fermion ground states of the three dimensional relativistic Vlasov-Poisson system exist for subcritical mass, the mass density of such fermion ground states satisfies the Chandrasekhar equation for white dwarfs, and that they are orbitally stable as long as solutions exist.

1. Introduction. White dwarfs are compact stars with high mean density supported by the pressure of degenerate electron gas and they are considered to be the final stage in the evolution of stars of which mass is not so large, while the more massive stars are expected to become neutron stars or black holes [14]. Given their physical importance, white dwarfs have been studied across different disciplines for the last century [1, 2, 14, 20, 22, 24, 25]. Notably, using the theory of special relativity and the Pauli exclusion principle for fermions from quantum mechanics, Chandrasekhar [1] in 1931 derived the equation of state for white dwarfs

$$P(\rho) = Cf(\sqrt[3]{\rho/D}), \quad f(x) = \int_0^x \frac{u^4}{\sqrt{1+u^2}} du, \quad C, D : \text{const.} \quad (1)$$

The equation of gravitational hydrostatic equilibrium

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho \quad (2)$$

together with (1) leads to the Chandrasekhar's white dwarf equation. Among others, Chandrasekhar showed the existence of the critical mass $m_c > 0$, the so-called

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Chandrasekhar limit, predicting the gravitational collapse: the solutions having finite density and compact support with the prescribed mass m to (2) with (1) exist only if $m < m_c$, and as $m \rightarrow m_c$, the mean density tends to infinity and the star radius tends to 0.

It is evident that quantum mechanics is important in the theory of white dwarfs. In particular, Lieb-Yau in [22] showed that starting with a relativistic linear N -body Schrödinger Hamiltonian H_N for neutral gravitating fermions, the Chandrasekhar equation (2) with (1) for white dwarfs is obtained as the semiclassical limit when the particle number $N \rightarrow \infty$ and the gravitational constant $G \rightarrow 0$ with $GN^{\frac{2}{3}}$ fixed, and that any solution of the Chandrasekhar equation with finite mean density is a global minimizer for the semiclassical energy functional.

Moreover, approximating the time evolution of N -body Schrödinger equation described by the unitary group $\{e^{-itH_N}\}$ by Slater determinants $\psi_1(t) \wedge \psi_2(t) \wedge \cdots \wedge \psi_N(t)$, one is led to the Hartree-Fock equations (cf. [4]),

$$i\partial_t \psi_k = \sqrt{-\Delta + m^2} \psi_k - \sum_{l=1}^N \left(\frac{1}{|x|} * |\psi_l|^2 \right) \psi_k + \sum_{l=1}^N \psi_l \left(\frac{1}{|x|} * \{\bar{\psi}_l \psi_k\} \right), \quad k = 1, \dots, N$$

with the energy functional

$$\mathcal{E}(\Psi) = \sum_{k=1}^N \langle \psi_k, \sqrt{-\Delta + m^2} \psi_k \rangle - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho_\Psi(x)\rho_\Psi(y) - |\rho_\Psi(x, y)|^2}{|x - y|} dx dy \quad (\text{HF})$$

where $\Psi = \{\psi_k\}_{k=1}^N$ and ρ_Ψ denotes the particle density $\sum_{k=1}^N |\psi_k(x)|^2$. Then, in this regime, white dwarfs are described as a minimizer of the Hartree-Fock energy $\mathcal{E}(\Psi)$ subject to the given mass $\int_{\mathbb{R}^3} \rho_\Psi(x) dx = M$.

Having the connections between quantum N -body problem and Chandrasekhar theory, it is natural to ask if other connections and descriptions of white dwarfs are possible at different hierarchies. In this paper, we are interested in the kinetic formulation of white dwarfs standing between the relativistic mean-field quantum theory and the Chandrasekhar theory. As we shall see in Section 1.1, starting from the Hartree-Fock energy functional in relativistic quantum theory, the minimization problem for fermion ground states for the relativistic Vlasov-Poisson system (cf. (3)) naturally emerges via the semiclassical limit. The goal of this paper is to prove the existence of such fermion ground states to the relativistic Vlasov-Poisson system featuring the critical mass phenomenon analogous to the Chandrasekhar limit, to show that the associated mass density satisfies the Chandrasekhar's white dwarf equation, and to study their orbital stability for general perturbations.

Before we introduce the kinetic equation and main results, we briefly discuss a formal connection (semiclassical limit) between Hartree-Fock energy functional and the semiclassical energy functional of the relativistic Vlasov-Poisson system.

1.1. From relativistic quantum theory to relativistic kinetic theory. We begin with the equivalent operator version of Hartree-Fock energy [20]

$$\mathcal{E}(\gamma) := \text{Tr}(T\gamma) - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{1}{|x - y|} \rho_\gamma(x)\rho_\gamma(y) dx dy + \frac{1}{2} \iint_{\mathbb{R}^6} \frac{1}{|x - y|} |\gamma(x, y)|^2 dx dy \quad (\text{HF})$$

where $T := (\sqrt{-\Delta + 1} - 1)$ is a relativistic kinetic operator and γ is a nonnegative compact self-adjoint operator acting on $L^2(\mathbb{R}^3)$ with the kernel representation

$\gamma(x, y)$, i.e.,

$$\gamma(\phi)(x) = \int_{\mathbb{R}^3} \gamma(x, y) \phi(y) dy, \quad \phi \in L^2.$$

We define the density function ρ_γ of γ by $\rho_\gamma(x) := \gamma(x, x)$. In relativistic quantum theory, white dwarfs are understood as a minimizer γ_0 of the following minimization problem

$$\min \{ \mathcal{E}(\gamma) \mid \gamma \in \mathcal{O}^1, \gamma = \gamma^*, 0 \leq \gamma \leq I, \text{Tr}(\gamma) = m \}$$

where $\mathcal{O}^1$ denotes the space of trace class operators,

$$\mathcal{O}^1 := \{ \gamma : L^2 \rightarrow L^2 \mid \text{Tr}(|\gamma|) < \infty \}.$$

In particular, the density function ρ_γ of γ is interpreted as the mass density of a white dwarf star.

To derive a semi-classical formulation, we introduce the Planck constant $\hbar$ in Hartree-Fock energy functional as follows

$$\mathcal{E}^\hbar(\gamma) :=$$

$$\text{Tr}^\hbar(T^\hbar \gamma) - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{1}{|x-y|} \rho_\gamma^\hbar(x) \rho_\gamma^\hbar(y) dx dy + \frac{1}{2} \iint_{\mathbb{R}^6} \frac{1}{|x-y|} |(2\pi\hbar^3)\gamma(x, y)|^2 dx dy$$

where

$$T^\hbar = \sqrt{-\hbar^2 \Delta + 1} - 1, \quad \text{Tr}^\hbar = (2\pi\hbar)^3 \text{Tr}, \quad \rho_\gamma^\hbar = (2\pi\hbar)^3 \rho_\gamma.$$

This is obtained by replacing T and γ with $T^\hbar$ and $(2\pi\hbar)^3 \gamma$ respectively.

Define the Töplitz quantization $\text{Op}_\hbar^T$ by

$$\text{Op}_\hbar^T[f] := \frac{1}{(2\pi\hbar)^3} \iint_{\mathbb{R}^6} |\varphi_{(q,p)}^\hbar\rangle \langle \varphi_{(q,p)}^\hbar| f(q, p) dq dp$$

where f is a distribution function defined on the phase space $\mathbb{R}^6$ and

$$\varphi_{(q,p)}^\hbar(x) := \frac{1}{(\pi\hbar)^{\frac{3}{4}}} e^{-\frac{|x-q|^2}{2\hbar}} e^{\frac{ip \cdot x}{\hbar}}.$$

Then one has the following semiclassical convergences:

- (i) (Mass) $\text{Tr}^\hbar(\text{Op}_\hbar^T[f]) = \|f\|_{L^1(\mathbb{R}^6)}$
- (ii) (Kinetic energy) If $f, |p|f \in L^1(\mathbb{R}^6)$, then

$$\lim_{\hbar \rightarrow 0} \text{Tr}^\hbar(\sqrt{-\hbar^2 \Delta + 1} \text{Op}_\hbar^T[f]) = \iint_{\mathbb{R}^6} (\sqrt{|p|^2 + 1}) f(q, p) dq dp$$

- (iii) (Potential energy) If $\rho_f = \int_{\mathbb{R}^3} f dp \in L^{6/5}(\mathbb{R}^3)$, then

$$\lim_{\hbar \rightarrow 0} \iint_{\mathbb{R}^6} \frac{\rho_{\text{Op}_\hbar^T[f]}^\hbar(x) \rho_{\text{Op}_\hbar^T[f]}^\hbar(y)}{|x-y|} dx dy \rightarrow \iint_{\mathbb{R}^6} \frac{\rho_f(x) \rho_f(y)}{|x-y|} dx dy.$$

Dropping the exchange term which is negligible in the classical regime, we formally obtain the semiclassical kinetic formulation for the variational problem:

$$\min \left\{ H(f) \mid f \in L^1(\mathbb{R}^6), 0 \leq f \leq 1, \int f(q, p) dq dp = m \right\}$$

where

$$H(f) = \iint_{\mathbb{R}^6} (\sqrt{|p|^2 + 1} - 1) f(q, p) dq dp - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho_f(x) \rho_f(y)}{|x-y|} dx dy.$$

More details for semiclassical and mean-field limits including the Töplitz quantization can be found in [6].

1.2. Relativistic gravitational Vlasov-Poisson system and main results.

The dynamics of a stellar system subject to self-gravitation is commonly described by the Vlasov-Poisson system [27]. When high velocities occur, however, special relativistic corrections should be introduced [28], and more accurate model is given by the relativistic gravitational Vlasov-Poisson system:

$$\begin{cases} \partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \nabla_x \Phi_f \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \\ \Delta \Phi_f = 4\pi \rho_f(t, x) \end{cases} \quad (3)$$

where $f \geq 0$ is the density distribution function in the phase space, the mass density $\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv$, the speed of light and the gravitational constant have been normalized to unity. The system (3) is Hamiltonian and sufficiently regular solutions enjoy the conservation of all L^q norms. In particular, the total mass and total energy are conserved as long as solutions exist:

$$M(f) = \int_{\mathbb{R}^3} \rho_f(t, x) dx = \iint_{\mathbb{R}^6} f(t, x, v) dv dx \quad (\text{Mass})$$

$$H(f) = \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f(t, x, v) dx dv - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho_f(t, x) \rho_f(t, y)}{|x - y|} dx dy. \quad (\text{Energy})$$

While for the non-relativistic Vlasov-Poisson system the Cauchy problem is globally well-posed both for the classical solutions and for weak solutions [23, 26], the relativistic problem is critical in the energy space and in fact there exists a critical threshold for global existence and finite time blowup for solutions to (3). Glassey-Schaeffer in [5] showed that radially symmetric classical solutions to (3) with negative total energy $\iint \sqrt{1+|v|^2} f dx dv - \frac{1}{2} \|\Phi_f\|_{L^2}^2$ blow up in finite time (see also Kiessling-Tahvildar-Zadeh [13]). In [18], Lemou-Méhats-Raphaël gave a detailed description of a stable self similar blow up dynamics. In a recent work [15], Körner-Rein proved the existence and uniqueness of strong Lagrangian solutions to (3) which preserve all the conserved quantities and allow initial data to be discontinuous under radial symmetry assumption. Up to our best knowledge, global existence question for the Cauchy problem with general initial data without symmetry is still open; we refer to [15] for more details on the existence theories.

In this paper, inspired by the relativistic quantum theory described in Section 1.1, we consider the following constrained minimization problem

$$\tilde{E} = \min_{f \in \mathcal{A}} H(f) \quad (\text{K})$$

where

$$\mathcal{A} = \{f \in L^1(\mathbb{R}^6) \mid M(f) = m, 0 \leq f \leq 1, \text{supp}(f) \text{ is bdd}\}.$$

The point-wise constraint $0 \leq f \leq 1$ inherits the quantum feature of fermions (quantum white dwarfs) in contrast to the energy-Casimir functional approach closely related to bosons [17]. In Section 2, we will prove the existence of the minimizer in the form of

$$f_0(x, v) = \chi_{\{\sqrt{1+|v|^2} - 1 + \Phi_{f_0} \leq E_0\}} \quad (4)$$

for some $E_0 < 0$ by adapting Guo's variational method [7]. In Section 3, the connection to white dwarfs will be given at the level of the mass density $\rho_{f_0}(x) = \int f_0(x, v) dv$ which enjoys the minimization problem for the Chandrasekhar theory

of white dwarfs. In particular, the equation of state for white dwarfs (1) will be directly obtained through the reduction from (4):

$$\iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f_0(x, v) dx dv = \int_{\mathbb{R}^3} A(\rho_{f_0}) dx$$

where $A(\rho)$ is related to $P(\rho)$ in (1) through $A''(\rho) = \frac{P'(\rho)}{\rho}$ (cf. Section 3). More generally, we will prove that for any distribution function $f \in \mathcal{A}$,

$$\iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f(x, v) dx dv \geq \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) G_f(x, v) dx dv = \int_{\mathbb{R}^3} A(\rho_{f_0}) dx$$

where G_f is the so-called Gibbs state of f defined by $G_f(x, v) := \chi_{\{|v| \leq (\frac{3}{4\pi} \rho_f(x))^{1/3}\}}$. For these reasons, we call such a minimizer f_0 fermion ground states or kinetic white dwarfs. Lastly in Section 4, based on the reduction method [12, 27] we will show an orbital stability of fermion ground states (4) for general perturbations as long as solutions exist. We will see that the topology of stability reflects the reduction from the fermion ground state to the Chandrasekhar theory of white dwarfs, which is known to be (conditionally) orbitally stable under the Euler-Poisson dynamics [24]. More precisely, the measure of distance for stability consists of two parts; one measures the distance between a distribution function f and its Gibbs state G_f , the other measures the relative distance of mass densities ρ_f for f and ρ_0 for the white dwarf.

There exist other interesting ground states obtained by minimizing the energy-Casimir functionals. We refer to [7, 8, 9, 10, 11, 16, 17, 19, 27] and references therein for the existence and orbital stability for the relativistic and non-relativistic Vlasov-Poisson system. We also mention that the variational problem with the point-wise constraint $0 \leq f \leq 1$ has been considered for the non-relativistic Vlasov-Poisson system, for instance in [3, 16]. The ground state in [3] is obtained by applying the symmetric rearrangement technique so that the corresponding orbital stability result appears to be rather restrictive. And the stability result for the fermion type ground state in [16] is absent because of the lack of compactness of minimizing sequences in $L^1 \cap L^\infty$. It is worth pointing out that in our analysis, we bypass this difficulty by taking into account the reduction process from the fermion ground state to the Chandrasekhar theory of white dwarfs.

2. Existence of Fermion Ground States. The aim of this section is to show the existence of a minimizer of the variational problem (K) as well as the critical mass phenomenon. The critical mass will be obtained via the interpolation estimate. Consider the energy space

$$\mathcal{E} := \{f \in L^1(\mathbb{R}^6) \mid f \geq 0, \|f\|_{L^1} + \|f\|_{L^\infty} + \| |v| f \|_{L^1} < \infty\}. \quad (5)$$

Lemma 2.1. *For any $f \in \mathcal{E}$, there exists $C > 0$ independent of f such that*

$$\|\nabla \Phi_f\|_{L^2}^2 \leq C \| |v| f \|_{L^1} \|f\|_{L^1}^{\frac{2}{3}} \|f\|_{L^\infty}^{\frac{1}{3}}. \quad (6)$$

Proof. Let $f \in \mathcal{E}$ be given. First note

$$\begin{aligned} \rho(t, x) &= \int_{|v| \leq R} f(t, x, v) dv + \int_{|v| \geq R} f(t, x, v) dv \\ &\leq \|f\|_\infty \frac{4\pi}{3} R^3 + \frac{1}{R} \int_{\mathbb{R}^3} |v| f dv \end{aligned}$$

for any $R > 0$. By optimizing R , we obtain

$$\rho(t, x) \leq C_1 \left(\int_{\mathbb{R}^3} |v| f dv \right)^{\frac{3}{4}} \|f\|_{L^\infty}^{\frac{1}{4}}$$

for some constant C_1 so that

$$\|\rho\|_{\frac{4}{3}} \leq C_1 \|v|f\|_{L^1}^{\frac{3}{4}} \|f\|_{L^\infty}^{\frac{1}{4}}. \quad (7)$$

On the other hand, Hardy-Littlewood-Sobolev inequality leads to

$$\|I_2 \rho\|_{12} = \left\| \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy \right\|_{12} \leq C_2 \|\rho\|_{\frac{4}{3}}. \quad (8)$$

By (7) and (8) and using the interpolation inequality

$$\|\nabla \Phi_f\|_2^2 \leq \|\rho\|_{\frac{12}{11}} \|I_2 \rho\|_{12} \lesssim \|\rho\|_1^{\frac{2}{3}} \|\rho\|_{\frac{4}{3}}^{\frac{1}{3}} \|\rho\|_{\frac{4}{3}} \lesssim \|v|f\|_{L^1} \|f\|_{L^1}^{\frac{2}{3}} \|f\|_{L^\infty}^{\frac{1}{3}}.$$

□

Let

$$K := \inf_{f \in \mathcal{E} \setminus \{0\}} \frac{\|v|f\|_{L^1} \|f\|_{L^1}^{\frac{2}{3}} \|f\|_{L^\infty}^{\frac{1}{3}}}{\|\nabla \Phi_f\|_{L^2}^2}. \quad (9)$$

Then by Lemma 2.1, $0 < K < \infty$.

Lemma 2.2. *Let $m > 0$ satisfy the following*

$$m^{\frac{2}{3}} < 2K. \quad (10)$$

Let $\tilde{E}$ be the infimum of $H(f)$:

$$\tilde{E} = \tilde{E}(m) = \inf_{f \in \mathcal{A}} H(f). \quad (11)$$

Then we have

$$-\infty < \tilde{E} < 0 \quad (12)$$

and there holds the nondichotomy condition: for all $0 < \alpha < 1$,

$$\tilde{E}(\alpha m) + \tilde{E}((1-\alpha)m) > \tilde{E}(m). \quad (13)$$

Proof. We first show that the infimum is negatively finite. Since $f \in \mathcal{A}$, $f \in \mathcal{E}$ with $\|f\|_{L^1} = m$, $\|f\|_{L^\infty} \leq 1$. Now using the definition of K ,

$$\begin{aligned} H(f) &= \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f dx dv - \frac{1}{2} \|\nabla \Phi_f\|_{L^2}^2 \\ &\geq \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f dx dv - \frac{1}{2K} \|v|f\|_{L^1} m^{\frac{2}{3}} \\ &\geq \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1 - |v|) f dx dv \geq - \iint_{\mathbb{R}^6} f dx dv = -m \end{aligned}$$

which shows that $\tilde{E}(m)$ is bounded from below. A standard rescaling argument shows that for $f_\lambda(x, v) := f(\frac{x}{\lambda}, \lambda v)$,

$$H(f_\lambda) = \frac{1}{\lambda} \iint_{\mathbb{R}^6} \frac{|v|^2 f}{\sqrt{\lambda^2 + |v|^2} + \lambda} dx dv - \frac{1}{2\lambda} \|\nabla \Phi_f\|_{L^2}^2$$

and the second term dominates as $\lambda \rightarrow +\infty$. Hence the infimum should be negative.

We next claim that for all $0 < \alpha \leq 1$

$$\tilde{E}(\alpha m) \geq \alpha^{\frac{5}{3}} \tilde{E}(m). \quad (14)$$

To this end, take any $f \in \mathcal{A}(\alpha m)$ such that $\|f\|_{L^1} = \alpha m$ and $0 \leq f \leq 1$, and let $\tilde{f}(x, v) = f(\alpha^{\frac{1}{3}}x, v)$ so that $\|\tilde{f}\|_{L^1} = m$ and $0 \leq \tilde{f} \leq 1$. Then

$$\tilde{E}(m) \leq H(\tilde{f}) = \alpha^{-1} \iint_{\mathbb{R}^6} (\sqrt{1 + |v|^2} - 1) f dx dv - \alpha^{-\frac{5}{3}} \frac{1}{2} \|\nabla \Phi_f\|_{L^2}^2 \leq \alpha^{-\frac{5}{3}} H(f)$$

where we have used $\alpha^{-1} \leq \alpha^{-\frac{5}{3}}$ since $0 < \alpha \leq 1$. Therefore, (14) follows.

Applying (14) with α and $1 - \alpha$ for $0 < \alpha < 1$, we first see

$$\tilde{E}(\alpha m) + \tilde{E}((1 - \alpha)m) \geq (\alpha^{\frac{5}{3}} + (1 - \alpha)^{\frac{5}{3}}) \tilde{E}(m). \quad (15)$$

Since $0 < \alpha < 1$, $\alpha^{\frac{5}{3}} + (1 - \alpha)^{\frac{5}{3}} < 1$ and moreover since $\tilde{E}(m) < 0$, (13) follows. $\square$

Remark 2.3. We remark that if $m^{\frac{2}{3}} > 2K$, $\tilde{E}(m) = -\infty$. In fact, the condition (10) prevents the total energy $(\iint_{\mathbb{R}^6} \sqrt{1 + |v|^2} dx dv - \frac{1}{2} \|\Phi_f\|_{L^2}^2)$ from being negative, while we know that smooth radial data solutions with negative energy blow up in finite time by Glassey-Schaeffer [5].

Lemma 2.4 (Radial mass density). *Let the mass $m > 0$ satisfy the bound (10). There exists a minimizing sequence $\{f_j\}_{j \in \mathbb{N}} \in \mathcal{A}$ of the energy functional H so that for any $j \in \mathbb{N}$, the mass density $\rho_j(x) = \int_{\mathbb{R}^3} f_j(x, v) dv$ is a radial non-increasing function.*

Proof. Let $\{f_j\}_{j \in \mathbb{N}} \in \mathcal{A}$ be any minimizing sequence of H . Let f_j^{*x} be the non-increasing symmetric rearrangement of f_j with respect to the variable x . Then $f_j^{*x} \in \mathcal{A}$ and moreover by Riesz's rearrangement inequality (cf. Theorem 3.7 of [21]), $\|\nabla \Phi_{f_j}\|_{L^2}^2 \leq \|\nabla \Phi_{f_j^{*x}}\|_{L^2}^2$ and hence f_j^{*x} is a minimizing sequence whose mass density is a radial non-increasing function. $\square$

Lemma 2.5 (Splitting estimate). *Let $f \in \mathcal{A}$ with radial mass density ρ_f . For given $R > 0$, let*

$$m - \lambda = \int_{|x| < R} \int f(x, v) dv dx$$

for some $\lambda \in [0, m]$. Then

$$H(f) - \tilde{E}(m) \geq - \left(\frac{5}{3} \frac{\tilde{E}(m)}{m^2} + \frac{1}{4\pi R} \right) (m - \lambda) \lambda. \quad (16)$$

Proof. The proof follows from the splitting argument of Guo [7]. We split the potential function $\Phi_f = \Phi_1 + \Phi_2$ so that

$$\Delta \Phi_1 = \int_{\mathbb{R}^3} \chi_{B_R}(x) f(x, v) dv, \quad \Delta \Phi_2 = \int_{\mathbb{R}^3} (1 - \chi_{B_R}(x)) f(x, v) dv.$$

Then it is easy to see that

$$\begin{aligned} H(f) &= H(\chi_{B_R} f) + H((1 - \chi_{B_R}) f) - \int_{\mathbb{R}^3} \nabla \Phi_1 \cdot \nabla \Phi_2 dx \\ &\geq \tilde{E}(m - \lambda) + \tilde{E}(\lambda) + \int_{\mathbb{R}^3} \Phi_2 \Delta \Phi_1 dx. \end{aligned}$$

By (15),

$$H(f) - \tilde{E}(m) \geq \left[\left(1 - \frac{\lambda}{m}\right)^{\frac{5}{3}} + \left(\frac{\lambda}{m}\right)^{\frac{5}{3}} - 1 \right] \tilde{E}(m) + \int_{\mathbb{R}^3} \Phi_2 \Delta \Phi_1 dx.$$

Now (16) follows from $(1-x)^{\frac{5}{3}} + x^{\frac{5}{3}} - 1 \leq -\frac{5}{3}x(1-x)$ for $x \in [0, 1]$, and

$$\left| \int_{\mathbb{R}^3} \Phi_2 \Delta \Phi_1 dx \right| \leq \max |\Phi_2| (m - \lambda) \leq \frac{\lambda}{4\pi R} (m - \lambda)$$

where we have used the fact that $\tilde{E}(m) < 0$ and Φ_2 is radially symmetric:

$$\max |\Phi_2| = |\Phi_2(0)| \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|y|} \int_{\mathbb{R}^3} (1 - \chi_{B_R}(y)) f(y, v) dv dy \leq \frac{\lambda}{4\pi R}.$$

□

Proposition 2.6 (Existence of minimizers). *Let the mass $m > 0$ satisfy the bound (10). Then*

- (i) *Every minimizing sequence with radial non-increasing mass density of the variational problem (11), up to a subsequence, converges to a minimizer $f_0 \in \mathcal{A}$ such that $\tilde{E} = H(f_0)$ and $\text{supp} f_0 \subset B_{R_0} \times \mathbb{R}^3$ where $R_0 = \frac{3m^2}{20\pi|\tilde{E}(m)|}$.*
- (ii) *There exists $E_0 < 0$ such that a minimizer f_0 takes the following form*

$$f_0(x, v) = \chi_{\{\sqrt{1+|v|^2} - 1 + \Phi_{f_0} \leq E_0\}}. \quad (17)$$

Proof. *Proof of (i).* We first claim that any minimizing sequence with radial non-increasing mass density does not vanish:

$$\limsup_{j \rightarrow \infty} \int_{|x| \geq R_1} \int_{\mathbb{R}^3} f_j dv dx = 0 \quad (18)$$

for any $R_1 > R_0 = \frac{3m^2}{20\pi|\tilde{E}(m)|}$. If not, there exist $\lambda \in (0, m]$ and a subsequence denoting f_j again such that $\lim_{j \rightarrow \infty} \int_{|x| \geq R_1} \int_{\mathbb{R}^3} f_j dv dx = \lambda$. Choose $R(j) > R_1$ such that $\frac{\lambda}{2} = \int_{|x| \geq R(j)} \int_{\mathbb{R}^3} f_j dv dx$. Applying Lemma 2.5,

$$H(f_j) - \tilde{E}(m) \geq - \left(\frac{5}{3} \frac{\tilde{E}(m)}{m^2} + \frac{1}{4\pi R_1} \right) \left(m - \frac{\lambda}{2} \right) \frac{\lambda}{2} > 0$$

which is a contradiction to the assumption that f_j is a minimizing sequence. Hence, we deduce that

$$\lim_{j \rightarrow \infty} \int_{B_{R_0}} \int_{\mathbb{R}^3} f_j dv dx = m. \quad (19)$$

We next show that $\|\nabla \Phi_{f_j} - \nabla \Phi_{f_0}\|_{L^2} \rightarrow 0$. For any $R > 0$ we have

$$\int_{\mathbb{R}^3} |\nabla \Phi_{f_j} - \nabla \Phi_{f_0}|^2 dx \leq \int_{B_R} |\nabla \Phi_{f_j} - \nabla \Phi_{f_0}|^2 dx + \frac{m^2}{2\pi R}$$

where we have used the radial symmetry for the second bound. In the same spirit of Theorem 1 and Lemma 2 of [7], we now use the compactness theorem for Sobolev spaces and regularity of radial potential functions to deduce that $\|\nabla \Phi_{f_j} - \nabla \Phi_{f_0}\|_{L^2(B_R)} \rightarrow 0$. By choosing R sufficiently large, we obtain $\|\nabla \Phi_{f_j} - \nabla \Phi_{f_0}\|_{L^2} \rightarrow 0$ up to a subsequence.

Since f_j is a minimizing sequence, this also shows that $\iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f_j dx dv$ is bounded above. Thus one has

$$\begin{aligned} & \int_{B_R^c} \int_{\mathbb{R}^3} R f_j dx dv \\ & \leq \int_{B_R^c} \int_{\mathbb{R}^3} (\sqrt{1+|R|^2} - 1) f_j dx dv \leq \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f_j dx dv < C \end{aligned}$$

which implies that

$$\int_{B_R^c} \int_{\mathbb{R}^3} f_j dx dv \leq \frac{C}{R}.$$

Moreover, since f_j is bounded in $L^1(\mathbb{R}^6)$ and $L^\infty(\mathbb{R}^6)$, $\{f_j\}$ satisfies the hypothesis of the Dunford-Pettis theorem and therefore, there exists a function $f_0 \in L^1(\mathbb{R}^6)$ and a weakly convergent subsequence in $L^1(\mathbb{R}^6)$ such that $\|f_0\|_{L^1} = m$ and $f_0 \geq 0$ a.e. as a weak limit of nonnegative functions. Moreover, $\{f_j\}$ converges to f_0 (weak* L^∞ topology) up to a subsequence so that $\|f_0\|_{L^\infty} \leq 1$. Hence $f_0 \in \mathcal{A}$ and $\text{supp} f_0 \subset B_{R_0} \times \mathbb{R}^3$. The weak convergence in L^1 implies

$$\iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f dx dv \leq \liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f_j dx dv$$

from which we deduce that $H(f_0) = \tilde{E}(m)$.

Proof of (ii). First we show that $\|f_0\|_{L^\infty} = 1$. Suppose not: $\|f_0\|_{L^\infty} = \frac{1}{\alpha} < 1$. Consider $\tilde{f}(x, v) = \alpha f_0(\alpha^{\frac{2}{3}} x, \alpha^{-\frac{1}{3}} v)$ so that $\|\tilde{f}\|_{L^1} = m$, $\|\tilde{f}\|_{L^\infty} = 1$ so that $\tilde{f} \in \mathcal{A}$. However, since $\alpha > 1$ and $\tilde{E} < 0$,

$$\begin{aligned} H(\tilde{f}) &= \alpha^{\frac{2}{3}} \left(\iint_{\mathbb{R}^6} \frac{|v|^2}{\sqrt{1+|\alpha^{\frac{1}{3}} v|^2} + 1} f_0(x, v) dx dv - \frac{1}{2} \|\nabla \Phi_{f_0}\|_{L^2}^2 \right) \\ &\leq \alpha^{\frac{2}{3}} H(f_0) = \alpha^{\frac{2}{3}} \tilde{E} < \tilde{E} \end{aligned}$$

which is a contradiction to the fact that f_0 is a minimizer.

Next we show that $f_0 = 1$ on $\text{supp} f_0$ based on the Euler-Lagrange multiplier method. To this end, let $\epsilon \in (0, 1)$ be arbitrarily given and consider the set

$$S_\epsilon = \{(x, v) : \epsilon \leq f_0(x, v) \leq 1 - \epsilon\}.$$

We want to show that S_ϵ has a set of measure 0 for any $\epsilon > 0$. Consider a test function $g \in L^1(\mathbb{R}^6) \cup L^\infty(\mathbb{R}^6)$ such that $g \geq 0$ a.e. in $\mathbb{R}^6 \setminus \text{supp} f_0$ with compact support contained inside $(\text{supp} f_0 \setminus S_\epsilon)^c = (\mathbb{R}^6 \setminus \text{supp} f_0) \cup S_\epsilon$. Let

$$g(t) = m \frac{tg + f_0}{\|tg + f_0\|_{L^1}}, \quad t \in \left[0, \frac{m\epsilon}{(m\|g\|_{L^\infty} + \|g\|_{L^1})}\right].$$

One can readily check that $g(t) \in \mathcal{A}$ for $t \in \left[0, \frac{m\epsilon}{(m\|g\|_{L^\infty} + \|g\|_{L^1})}\right]$ and $g(0) = f_0$. Note that g is smooth in t from the right. In fact,

$$\begin{aligned} g'(t) &= \frac{mg}{\|tg + f_0\|_{L^1}} - m \frac{(tg + f_0) \iint_{\mathbb{R}^6} g dx dv}{\|tg + f_0\|_{L^1}^2} \\ g''(t) &= -2m \frac{g \iint_{\mathbb{R}^6} g dx dv}{\|tg + f_0\|_{L^1}^2} + 2m \frac{(tg + f_0) (\iint_{\mathbb{R}^6} g dx dv)^2}{\|tg + f_0\|_{L^1}^3} \end{aligned}$$

and hence

$$g(t) - f_0 = g'(0)t + \frac{g''(t_*)t^2}{2} = \left(g - \frac{\iint_{\mathbb{R}^6} g dx dv}{m} f_0\right)t + \frac{g''(t_*)t^2}{2}$$

for some $t_* \in (0, t)$ and $|g''(t_*)| \lesssim |f_0| + |g|$. Now let $h(t) := H(g(t)) - H(f_0)$ so that

$$\begin{aligned} h(t) &= \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1)(g(t) - f_0) dx dv - \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \Phi_{g(t)}|^2 - |\nabla \Phi_{f_0}|^2) dx \\ &= \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1)(g(t) - f_0) dx dv \\ &\quad + \iint_{\mathbb{R}^6} \Phi_{f_0}(g(t) - f_0) dx dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi_{g(t)} - \nabla \Phi_{f_0}|^2 dx. \end{aligned}$$

Therefore, we have

$$\begin{aligned} h(t) &= t \iint_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1 + \Phi_{f_0} \right) \left(g - \frac{\iint g dx dv}{m} f_0 \right) dx dv \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi_{g(t)} - \nabla \Phi_{f_0}|^2 dx + O(t^2) \\ &= t \iint_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1 + \Phi_{f_0} - \frac{\tilde{E}(m)}{m} \right) g dx dv \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \Phi_{g(t)} - \nabla \Phi_{f_0}|^2 dx + O(t^2). \end{aligned}$$

Since f_0 minimizes $H(f)$ on $\mathcal{A}$, $h(t) \geq 0$ for all $t \in \left[0, \frac{m\epsilon}{(m\|g\|_{L^\infty} + \|g\|_{L^1})}\right]$, and in turn we have

$$\iint_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1 + \Phi_{f_0} - \frac{\tilde{E}(m)}{m} \right) g dx dv \geq 0$$

for all g specified above and ϵ . Since $g \geq 0$ in $\mathbb{R}^6 \setminus \text{supp} f_0$, we deduce that $\sqrt{1+|v|^2} - 1 + \Phi_{f_0} \geq \frac{\tilde{E}(m)}{m}$ for all $(x, v) \in \mathbb{R}^6 \setminus \text{supp} f_0$, and hence

$$\left\{ (x, v) : \sqrt{1+|v|^2} - 1 + \Phi_{f_0} \leq \frac{\tilde{E}(m)}{m} \right\} \subset \text{supp} f_0.$$

On the other hand, g doesn't have a definite sign on S_ϵ , and thus $\sqrt{1+|v|^2} - 1 + \Phi_{f_0} = \frac{\tilde{E}(m)}{m}$ for all $(x, v) \in S_\epsilon \cap \text{supp} f_0$. The Lebesgue measure in $\mathbb{R}^6$ of the set satisfying this identity is zero and so is the measure of S_ϵ . Since ϵ is arbitrary, we deduce $f_0 = 1$ on $\text{supp} f_0$ and $\text{supp} f_0 = \left\{ (x, v) : \sqrt{1+|v|^2} - 1 + \Phi_{f_0} \leq \frac{\tilde{E}(m)}{m} \right\}$. $\square$

Remark 2.7. A minimizer f_0 given in (17) is a steady state of the relativistic Vlasov-Poisson system (in a weak sense). Also, it is unique with radial mass density. In fact, we will see in Section 3 that ρ_{f_0} is a minimizer of the variational problem (F) in Proposition 3.2, which is proved to be unique in the class of radial mass density [24]. Then the form (17) satisfied by f_0 says that a minimizer f_0 is also unique in the radial class with respect to x .

Remark 2.8. One may apply the general concentration compactness principle to general minimizing sequences without radial assumption on mass density. As in [16] one can show that there exists shift vectors a_j in $\mathbb{R}^3$ such that $f_j(x + a_j, v)$, up to a subsequence, converges to f weakly in L^1 and L^∞ weak* sense but without strong compactness. This lack of strong compactness at the level of f motivates us to adopt the reduction process via the mass density ρ_f to obtain an orbital stability result for white dwarfs in Section 4.

3. Reduction and white dwarfs. In this section, we show that the mass density ρ_0 associated with our original variational problem (11) is a minimizer of the reduced variational problem for ρ and that it gives a variational description of the equilibrium white dwarf star.

Define $A(\rho)$ as

$$A(\rho) = \int_0^\rho \sqrt{\left(\frac{3}{4\pi}u\right)^{\frac{2}{3}} + 1} - 1 \, du. \quad (20)$$

It is then easy to check that

$$(A'_+)^{-1}(x) = \frac{4\pi}{3}((x+1)^2 - 1)_+^{\frac{3}{2}}. \quad (21)$$

Lemma 3.1. *Let f_0 be a minimizer obtained in Proposition 2.6. Then $\rho_0(x) = \int f_0(x, v) dv$ and $\Phi_{\rho_0} = -|\cdot|^{-1} * \rho_0$ satisfy*

$$\rho_0(x) = (A'_+)^{-1}(E_0 - \Phi_{\rho_0}(x)). \quad (22)$$

Moreover,

$$f_0(x, v) = \chi_{\{|v| \leq (\frac{3}{4\pi}\rho_0(x))^{\frac{1}{3}}\}}. \quad (23)$$

Proof. We note that for x such that $(E_0 + 1 - \Phi_{f_0}(x))^2 - 1 \geq 0$

$$\begin{aligned} \rho_0(x) &= \int_{\mathbb{R}^3} f_0(x, v) dv = \int_{\mathbb{R}^3} \chi_{\{\sqrt{1+|v|^2} - 1 + \Phi_{f_0} \leq E_0\}} dv \\ &= \int_0^{\sqrt{(E_0+1-\Phi_{f_0}(x))^2-1}} 4\pi r^2 dr = \frac{4\pi}{3}((E_0 + 1 - \Phi_{\rho_0}(x))^2 - 1)^{\frac{3}{2}} \end{aligned}$$

so that

$$\rho_0(x) = \frac{4\pi}{3}((E_0 + 1 - \Phi_{\rho_0}(x))^2 - 1)_+^{\frac{3}{2}}.$$

With $A(\rho)$ given in (20), it is easy to see

$$\rho_0(x) = (A'_+)^{-1}(E_0 - \Phi_{\rho_0}(x)).$$

Since $(E_0 + 1 - \Phi_{\rho_0}(x))^2 - 1 = (\frac{3}{4\pi}\rho_0(x))^{\frac{2}{3}}$, the assertion (23) easily follows. $\square$

Proposition 3.2. *Let f_0 be a minimizer obtained in Proposition 2.6. Then $\rho_0(x) = \int f_0(x, v) dv$ is a minimizer of the variational problem*

$$(F) \quad \min_{\rho \in \mathcal{B}} \left\{ \int_{\mathbb{R}^3} A(\rho) dx - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|} dx dy \right\}$$

where

$$\mathcal{B} = \left\{ \rho \in L^1(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \rho = m, \rho \geq 0, \text{ supp}(\rho) \text{ is bdd} \right\}.$$

And the associated Euler-Lagrange equation is given by (22).

Proof. Let $\rho \in L^1(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3)$ be a nonnegative function so that $\|\rho\|_{L^1} = m$. Define

$$f_\rho(x, v) = \begin{cases} 1, & \text{if } |v| \leq (\frac{3}{4\pi}\rho(x))^{\frac{1}{3}} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $f_\rho \in \mathcal{A}$. Since f_0 minimizes H on $\mathcal{A}$ and f_0 takes the form of (23), f_0 also minimizes the problem

$$\min \{H(f_\rho) : \rho \in \mathcal{B}\}. \quad (24)$$

Direct computations show that

$$\begin{aligned} \int_{\mathbb{R}^3} (\sqrt{1+|v|^2} - 1) f_\rho dv &= \int_{|v| \leq (\frac{3}{4\pi} \rho(x))^{\frac{1}{3}}} \sqrt{1+|v|^2} - 1 dv \\ &= \int_0^{(\frac{3}{4\pi} \rho(x))^{\frac{1}{3}}} (\sqrt{1+r^2} - 1) 4\pi r^2 dr \\ &= \int_0^{\rho(x)} \sqrt{\left(\frac{3}{4\pi} u\right)^{\frac{2}{3}} + 1} - 1 du = A(\rho(x)) \end{aligned}$$

and $\|\nabla \Phi_{f_\rho}\|_{L^2} = \|\nabla \Phi_\rho\|_{L^2}$. Therefore, the variational problem (24) can be reformulated as

$$\min_{\rho \in \mathcal{B}} \{F(\rho) : \rho \in \mathcal{B}\} \quad (25)$$

where

$$F(\rho) = \int_{\mathbb{R}^3} A(\rho) dx - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|} dxdy. \quad (26)$$

The mass density ρ_0 of f_0 is a minimizer of (25) and therefore, it satisfies the Euler-Lagrange equations: $A'(\rho_0) + \Phi_{\rho_0}(x) - \mu = E_0$ where E_0 is a Lagrange multiplier associated with the constraint $\|\rho\|_{L^1} = m$ and $\mu = 0$ if $\rho_0(x) > 0$ and $\mu \geq 0$ if $\rho_0(x) = 0$. Therefore, we obtain (22). $\square$

From (20), we have

$$A''(\rho) = \frac{1}{2} \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \frac{1}{\rho^{\frac{1}{3}} \sqrt{\left(\frac{3}{4\pi}\rho\right)^{\frac{2}{3}} + 1}}. \quad (27)$$

Thus $A(\rho)$ is a pressure energy for white dwarfs (cf. (1)) so that

$$A''(\rho) = \frac{P'(\rho)}{\rho}, \quad P(\rho) = Cf(\sqrt[3]{\rho/D}), \quad f(x) = \int_0^x \frac{u^4}{\sqrt{1+u^2}} du, \quad C, D : \text{const.}$$

and also we see that the mass density ρ_0 of the fermion ground state f_0 is a radial steady state of the Euler-Poisson system with the pressure law of white dwarfs. The above Proposition shows that the white dwarf equilibrium stars have the variational characterization through reduction from fermion ground states of the relativistic Vlasov-Poisson system.

4. Orbital stability of white dwarfs. In this section, we study orbital stability of kinetic white dwarf solutions constructed in the previous sections. We begin with the following compactness result whose proof can be found in [24].

Proposition 4.1. *Let $\rho_0 \in \mathcal{B}$ and $\{\rho_n\} \subset \mathcal{B}$ be a minimizer and a minimizing sequence of the variational problem (25) respectively. Then there exists a sequence $\{x_n\} \subset \mathbb{R}^3$ such that*

$$\lim_{n \rightarrow \infty} \|\nabla \Phi_{\rho_n} - \nabla \Phi_{\rho_0(\cdot - x_n)}\|_{L^2(\mathbb{R}^3)} = 0.$$

Proposition 4.2. *For any given $\rho_0 \in \mathcal{B}$, one has*

$$\int_{\mathbb{R}^3} A(\rho_0) dx = \min \left\{ \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1) f(x, v) dxdv \mid f \in \mathcal{A}, \rho_f = \rho_0 \right\}.$$

Moreover, the unique minimizer is given by

$$f_0(x, v) = \chi_{\{\sqrt{1+|v|^2}-1 \leq \lambda(\rho_0(x))\}}, \quad \lambda(\rho) = \sqrt{\left(\frac{3}{4\pi}\rho\right)^{\frac{2}{3}} + 1} - 1$$

and equivalently

$$f_0(x, v) = \chi_{\{|v| \leq (\frac{3}{4\pi}\rho_0(x))^{\frac{1}{3}}\}}.$$

Proof. Consider the minimization problem

$$\min \left\{ \int_{\mathbb{R}^3} (\sqrt{1+|v|^2} - 1)g(v) dv \mid 0 \leq g \leq 1, \int_{\mathbb{R}^3} g(v) dv = \rho_0 \right\}$$

where ρ_0 is a given positive real number. By following arguments of proof of Proposition 2.6, we see that the problem admits a minimizer $g_0(v) = \chi_{\{\sqrt{1+|v|^2}-1 \leq \lambda\}}$ for some $\lambda = \lambda(\rho_0) \in \mathbb{R}$. From the assumption $\int g(v) dv = \rho_0$, $\lambda(\rho_0)$ should be determined by $\lambda(\rho_0) = \sqrt{\left(\frac{3}{4\pi}\rho_0\right)^{\frac{2}{3}} + 1} - 1$. Also one has from direct computation

$$A(\rho_0) = \int_{\mathbb{R}^3} (\sqrt{1+|v|^2} - 1)g_0(v) dv.$$

Then integrating with respect to x , we see that the proposition holds true. $\square$

For given $f \in \mathcal{A}$, we define a Gibbs state G_f of f by

$$G_f(x, v) := \chi_{\{|v| \leq (\frac{3}{4\pi}\rho_f(x))^{\frac{1}{3}}\}}.$$

Since $\rho_{G_f} = \rho_f$, Proposition 4.2 implies that for any $f \in \mathcal{A}$,

$$\int_{\mathbb{R}^3} A(\rho_f) dx = \iint_{\mathbb{R}^6} (\sqrt{1+|v|^2} - 1)G_f(x, v) dx dv. \quad (28)$$

We next introduce a distance function which measures the difference between f and its Gibbs state G_f :

$$\delta(f, G(f)) = \iint_{\mathbb{R}^6} (\sqrt{|v|^2 + 1} - 1)(f - G_f) dx dv$$

and a relative distance function which measures the difference between the mass density ρ_f of f and the mass density ρ_0 for the white dwarf star:

$$d(\rho, \rho_0) = \int_{\mathbb{R}^3} A(\rho) - A(\rho_0) + \Phi_{\rho_0}(\rho - \rho_0) dx.$$

The following lemma asserts that they are distance functions.

Lemma 4.3. *For any $f \in \mathcal{A}$, one has*

- (i) $\delta(f, G(f)) \geq 0$ and $\delta(f, G(f)) = 0$ if and only if $f = G_f$
- (ii) $d(\rho_f, \rho_0) \geq 0$ and $d(\rho_f, \rho_0) = 0$ if and only if $\rho_f = \rho_0$ where $\rho_0 \in \mathcal{B}$ is a minimizer of the variational problem (25).

Proof. The statement (i) follows from Proposition 4.2, while the statement (ii) is a consequence of Proposition 3.2 and the convexity of $A(\rho)$ (cf. (27)). $\square$

Proposition 4.4. *Let $\rho_0 \in \mathcal{B}$ be a minimizer of the variational problem (25). For $f \in \mathcal{A}$, there holds*

$$H(f) - F(\rho_0) = \delta(f, G(f)) + d(\rho_f, \rho_0) - \frac{1}{8\pi} \|\nabla \Phi_{\rho_f} - \nabla \Phi_{\rho_0}\|_{L^2(\mathbb{R}^3)}^2.$$

Proof. Observe from (28) that

$$\begin{aligned}
H(f) - F(\rho_0) &= H(f) - H(G_f) + H(G_f) - F(\rho_0) \\
&= \delta(f, G_f) + \int_{\mathbb{R}^3} A(\rho_f) - A(\rho_0) dx + \frac{1}{2} \int_{\mathbb{R}^3} \Phi_{\rho_f} \rho_f - \Phi_{\rho_0} \rho_0 dx \\
&= \delta(f, G_f) + d(\rho_f, \rho_0) + \int_{\mathbb{R}^3} \Phi_{\rho_0} (\rho_0 - \rho_f) dx + \frac{1}{2} \int_{\mathbb{R}^3} \Phi_{\rho_f} \rho_f - \Phi_{\rho_0} \rho_0 dx \\
&= \delta(f, G_f) + d(\rho_f, \rho_0) + \frac{1}{2} \int_{\mathbb{R}^3} \Phi_{\rho_f} \rho_f - 2\Phi_{\rho_0} \rho_f + \Phi_{\rho_0} \rho_0 dx \\
&= \delta(f, G_f) + d(\rho_f, \rho_0) + \frac{1}{2} \int_{\mathbb{R}^3} (\Phi_{\rho_f} - \Phi_{\rho_0}) (\rho_f - \rho_0) dx \\
&= \delta(f, G_f) + d(\rho_f, \rho_0) - \frac{1}{8\pi} \|\nabla \Phi_{\rho_f} - \nabla \Phi_{\rho_0}\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

□

Theorem 4.5. *Let f_0 be a minimizer of the variational problem (11). For any $\varepsilon > 0$, there exists some $\delta > 0$ such that if $f \in \mathcal{A}$ satisfies*

$$\delta(f, G(f)) + d(\rho_f, \rho_{f_0}) + \frac{1}{8\pi} \|\nabla \Phi_{\rho_f} - \nabla \Phi_{\rho_{f_0}}\|_{L^2(\mathbb{R}^3)}^2 < \delta,$$

then there exists $\{x(t)\} \subset \mathbb{R}^3$ such that the global solution $f(t)$ of IVP (3) with initial data f preserving the total mass, energy and L^∞ norm, provided it exists, satisfies

$$\delta(f(t), G(f(t))) + d(\rho_{f(t)}, \rho_{f_0}(\cdot - x(t))) + \frac{1}{8\pi} \|\nabla \Phi_{\rho_{f(t)}} - \nabla \Phi_{\rho_{f_0}(\cdot - x(t))}\|_{L^2(\mathbb{R}^3)}^2 < \varepsilon$$

for all $t \geq 0$.

Proof. To the contrary, suppose not. Then there are $\varepsilon_0 > 0$ and sequences $\{f_n\} \subset \mathcal{A}$ and $\{t_n\} \subset \mathbb{R}_+$ such that

$$\delta(f_n, G(f_n)) + d(\rho_{f_n}, \rho_{f_0}) + \frac{1}{8\pi} \|\nabla \Phi_{\rho_{f_n}} - \nabla \Phi_{\rho_{f_0}}\|_{L^2(\mathbb{R}^3)}^2 < \frac{1}{n} \quad (29)$$

but the global solution $f_n(t)$ of IVP (3) with initial data f satisfies

$$\delta(f_n(t_n), G(f_n(t_n))) + d(\rho_{f_n(t_n)}, \rho_{f_0}(\cdot - x)) + \frac{1}{8\pi} \|\nabla \Phi_{\rho_{f_n(t_n)}} - \nabla \Phi_{\rho_{f_0}(\cdot - x)}\|_{L^2(\mathbb{R}^3)}^2 \geq \varepsilon_0 \quad (30)$$

for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^3$. Since ρ_{f_0} coincides with ρ_0 , a minimizer of the variational problem (25), Proposition 4.4 combined with (29) and conservation of Hamiltonian under the flow of (3) says that $H(f_n) \rightarrow F(\rho_0)$ as $n \rightarrow \infty$. Since the flow of (3) also preserves the total mass and L^∞ norm, we see from Proposition 4.2 that $\{\rho_{f_n(t_n)}\}$ is a minimizing sequence of the problem (25). Then we may apply Proposition 4.1 to see that there exists a sequence $\{x_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n \rightarrow \infty} \|\nabla \Phi_{\rho_{f_n(t_n)}} - \nabla \Phi_{\rho_0(\cdot - x_n)}\|_{L^2(\mathbb{R}^3)} = 0.$$

Then we again invoke Proposition 4.4 to conclude that as $n \rightarrow \infty$

$$\delta(f_n(t_n), G(f_n(t_n))) + d(\rho_{f_n(t_n)}, \rho_{f_0}(\cdot - x_n)) + \frac{1}{8\pi} \|\nabla \Phi_{\rho_{f_n(t_n)}} - \nabla \Phi_{\rho_{f_0}(\cdot - x_n)}\|_{L^2(\mathbb{R}^3)}^2 \rightarrow 0,$$

which makes a contradiction to (30). This completes the proof. □

Remark 4.6. In Theorem 4.5, perturbations f need not to be spherically symmetric, while the existence of global solutions preserving the total mass, energy and L^∞ norm is assumed, in which sense our stability result is a conditional one. On the other hand, radially symmetric perturbations are dynamically accessible thanks to the existence theory of strong Lagrangian solutions [15] which allows discontinuous initial data such as fermion ground states f_0 .

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