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On uniformly rotating binary stars and galaxies

Juhi Jang · Jinmyoung Seok

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Abstract In this paper, we study the asymptotic profiles, uniqueness and orbital stability of McCann's uniformly rotating binary stars [40] governed by the Euler-Poisson system. A new uniqueness result will be importantly used in stability analysis. Moreover, we apply our framework to the study of uniformly rotating binary galaxies of the Vlasov-Poisson system through Rein's reduction [43].

Keywords Euler-Poisson · Vlasov-Poisson · binary stars and galaxies · dynamical stability

Mathematics Subject Classification (2010) 35Q31 · 35Q83 · 35J60

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1 Introduction

In astrophysical fluid dynamics, stars are considered as isolated fluid masses subject to self-gravity and a fundamental hydrodynamic model describing the dynamics of Newtonian stars is given by the Euler-Poisson system

$$\text{(EP)}\quad \begin{aligned} &\partial_t \rho + \nabla\cdot(\rho u) = 0,\\ &\rho\partial_t u + \rho(u\cdot\nabla)u + \nabla(K\rho^\gamma) = -\rho\nabla\Phi,\\ &\Delta\Phi = 4\pi\rho, \end{aligned} \tag{1.0.1}$$

Juhi Jang
Department of Mathematics
University of Southern California
Los Angeles, CA 90089 USA
E-mail: juhijang@usc.edu

Jinmyoung Seok
Department of Mathematics
Kyonggi University
Suwon 16227, South Korea
E-mail: jmseok@kgu.ac.kr

where $\rho(x,t) \geq 0$ is the density of fluids at position $x \in \mathbb{R}^3$ and time $t \geq 0$, $u(x,t) \in \mathbb{R}^3$ the velocity, and $\Phi(x,t) \in \mathbb{R}$ the gravitational potential. We have taken the equation of states as the polytropic law $p = K\rho^\gamma$ for $1 < \gamma < 2$. On the other hand, galaxies containing billions of stars or globular clusters are described by the Vlasov-Poisson system of the kinetic theory:

$$\text{(VP)} \quad \begin{aligned} &\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = 0, \\ &\Delta \Phi = 4\pi \rho_f, \end{aligned} \tag{1.0.2}$$

where $\rho_f(x,t) = \int_{\mathbb{R}^3} f(x,v,t)\,dv$. Here $f(x,v,t) \geq 0$ is the density distribution function of particles in the phase space $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ at time $t \geq 0$. In this article, we are concerned with the dynamics of rotating binary stars and binary galaxies with small uniform angular velocity, located far away from each other, governed by (EP) and (VP) respectively.

Stellar rotation is a classical subject in celestial mechanics, astrophysics and mathematics going back to Newton. The study of rotating stars has been of a great interest in both mathematics and physics communities. Early developments can be tracked back to Maclaurin, Jacobi, Poincaré, Liapounov et al., who studied incompressible stars with homogenous or almost homogenous density; more general cases including compressible stars were considered by Lichtenstein [33] and Chandrasekhar [9]. See [10,26] for a historical account of the topic. In the case of gas-like fluids or distribution of a large number of stars, the inhomogeneity of the density has to be taken into account, and a lot of progress has been made for a single rotating star problem. As for the existence of rotating stars, two modern approaches are available: one is based on variational methods [2,3,6,11,32] and the other is a perturbative approach relying on the implicit function theorem around non-rotating Lane-Emden stars [24,25,27,46,47]. Nonlinear dynamical stability of rotating stars were shown in [38,39] based on variational approach, while nonlinear stability theory for non-rotating stars can be found in [12,23,42]. For rotating galaxies, a single rotating galaxy was constructed in [45,46] in the spirit of Lichtenstein and Heilig [27,33], while there is a vast literature on non-rotating galaxies: see [4,5,18,43] and references therein. The orbital stability of stationary solutions has seen a great deal of activity and progress over the last two decades [14–17,19,20,28–31,43,49]. See also a recent work [21] for the study of linearly oscillating galaxies.

If we consider more than one stellar object such as binary stars and galaxies or more generally N distinct stars and galaxies, there are fewer mathematical works available. The construction of rotating binary stars can be tracked back to Lichtenstein [33]. In [40] McCann constructed binary stars whose supports are separated and determined by the Kepler problem by formulating a minimization problem with given mass ratio. In [7] Campos, del Pino and Dolbeault constructed N-body rotating galaxies by making the connection to the relative equilibria in N-body dynamics with small uniform angular velocity and by perturbing radial equilibria. To the best of our knowledge, the stability question of these multi-body stellar configurations has not been addressed yet. The goal of this article is to study the asymptotic profiles, uniqueness and orbital stability of uniformly rotating binary stars governed by the Euler-Poisson system (1.0.1) for $\frac{4}{3} < \gamma < 2$ and the corresponding rotating binary galaxies modeled by the Vlasov-Poisson system (1.0.2).

One special feature exhibited by McCann's binary solution is while it has a variational characterization as a Hamiltonian minimizer under some conservative constraint, it can also be understood as a perturbation of simpler objects, for example, the non-rotating Lane-Emden star and the relative equilibria for point masses (cf. Section 3.2). This characterization plays a crucial role for our uniqueness and stability analysis. On the other hand, we remark that N-body rotating solutions by Campos et al [7] are perturbative in nature and the framework is not best suited for stability analysis. Furthermore, when $N \geq 3$, uniformly rotating N-body stellar objects do not retain a variational characterization analogous to the binary case and they are not expected to be stable in general.

In what follows, we discuss the main results and methodologies of the paper. For the rest of the paper, we fix the range of γ to $\frac{4}{3} < \gamma < 2$.

1.1 Uniqueness and orbital stability of McCann's binary stars

We first briefly discuss McCann's construction of binary star solutions by a constrained minimization method [40]. Specifically, McCann introduced the following effective Hamiltonian for density ρ,

$$E^J(\rho) := \int_{\mathbb{R}^3} \frac{K\gamma}{\gamma-1}\rho^\gamma(x)\,dx + \frac{J^2}{2I(\rho)} - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy, \tag{1.1.1}$$

where $J > 0$ is the total angular momentum of the system and $I(\rho)$ denotes the second moment of inertia:

$$I(\rho) := \int (x_1^2 + x_2^2)\rho(x)\,dx.$$

For each $m^\pm \in (0,1)$ such that $m^+ + m^- = 1$, $E^J(\rho)$ is minimized subject to the constraint

$$\mathcal{W}_J := \left\{\rho = \rho^- + \rho^+ \in L^\gamma(\mathbb{R}^3) \mid \rho \geq 0, \int \rho^+ = m^+, \int \rho^- = m^-, \operatorname{spt}(\rho^\pm) \subset \Omega^\pm\right\},$$

where $\Omega^\pm \subset \mathbb{R}^3$ are separated closed balls whose centers lie in the plane $x_3 = 0$ with radii and separation of scale J^2. By careful analysis based on the separation of $\Omega^\pm$ determined by the Kepler problem for two body masses m^+ and m^-, McCann proved that for sufficiently large J, there exists a minimizer $\tilde{\rho}^J$ continuous on $\mathbb{R}^3$ such that

$$(\tilde{\rho}(t,x), \tilde{u}(t,x)) := (\tilde{\rho}^J(R_{-\omega t}x),\, \omega(-x_2, x_1, 0)), \quad \omega = \frac{J}{I(\tilde{\rho}^J)} \tag{1.1.2}$$

gives a uniformly rotating binary star solution to (EP). Here R_θ is a rotation map about x_3 axis (cf. (2.0.1)). He also showed that the minimizer $\tilde{\rho}^J$ is a local minimizer of E^J with respect to the Wasserstein L^∞ metric, which indicates a structural stability of $\tilde{\rho}^J$ to some extent.

The first aim of this paper is to show the uniqueness of the minimizer $\tilde{\rho}^J$ (up to a translation and a rotation) by determining the asymptotic profiles and asymptotic positions of $\tilde{\rho}^J$. Since $\Omega^\pm$ are separated as scale J^2, we see that $I(\rho) \sim J^4$ so that $J^2/I(\rho) = O(J^{-2})$. This shows as $J \to \infty$, $E^J(\rho)$ in (1.1.1) formally converges to

$$E^\infty(\rho) := \int_{\mathbb{R}^3} \frac{K\gamma}{\gamma-1}\rho^\gamma(x)\,dx - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy,$$

whose minimizer $\tilde{\rho}_m^\infty$ on the constraint

$$\left\{\rho \in L^\gamma(\mathbb{R}^3) \mid \rho \geq 0, \int \rho = m\right\}$$

yields a non-rotating star as a solution to (EP), so-called the Lane-Emden star [42]. Therefore, it is naturally to expect that $\tilde{\rho}_{m^\pm}^\infty$ is an asymptotic profile for $(\tilde{\rho}^J)^\pm$.

In order to determine the asymptotic relative position between two stars $(\tilde{\rho}^J)^+$ and $(\tilde{\rho}^J)^-$, we further expand the Hamiltonian E^J in terms of J. At the next order $O(J^{-2})$ of (1.1.1), the kinetic energy due to uniform rotation $J^2/I(\rho)$ and the tidal energy due to two-body interaction $-\iint \rho^+(x)\rho^-(y)/|x-y|dxdy$ emerge: both energies are $O(J^{-2})$ because of the separation scale J^2, and the balance of these energies brings in the Hamiltonian for two-body relative equilibria, which will determine the asymptotic relative position between two stars. In Section 3, we will show that for $\rho = \rho^- + \rho^+ \in \mathcal{W}_J$ with $\mathrm{spt}(\rho^\pm) \subset B(0,R)$,

$$E^J(\rho) = E^\infty(\rho^+) + E^\infty(\rho^-) + \frac{1}{J^2}\left(H^{\mathrm{re}}\left(\frac{\bar{x}(\rho^+)}{J^2}, \frac{\bar{x}(\rho^-)}{J^2}\right) + O(\frac{1}{J^2})\right) \qquad \text{as } J \to \infty,$$

where $\bar{x}(\rho) := \int x\rho\, dx$ denotes the center of mass for a density ρ and

$$H^{\mathrm{re}}(\zeta_1, \zeta_2) := \frac{1}{2m^+m^-|\zeta_1 - \zeta_2|^2} - \frac{m^+m^-}{|\zeta_1 - \zeta_2|}$$

is the effect Hamiltonian for two-body relative equilibria whose critical point provides the positions of the circular binary stars of point masses (cf. Section 3.2).

The expansion for E^J suggests that the $(J^{-2}\bar{x}((\tilde{\rho}^J)^+), J^{-2}\bar{x}((\tilde{\rho}^J)^-))$ should minimize H^{re} as $J \to \infty$. Since, by Proposition 3.2.1, the global minimizers of H^{re} are characterized as (ζ_1, ζ_2) with $|\zeta_1 - \zeta_2| = (m^+m^-)^{-2}$, we are ready to describe our first main results.

Result A. *Let $\tilde{\rho}^J$ be the density for McCann's binary stars, that is, a minimizer of E^J subject to the constraint $\mathcal{W}_J$ and let $\tilde{\rho}_{m^\pm}^\infty$ be the Lane-Emden star with mass $m^\pm$. Then one has the following:*

(*i*) *(asymptotic profile for $(\tilde{\rho}^J)^\pm$) after taking suitable translations,*

$$\lim_{J\to\infty} \|(\tilde{\rho}^J)^\pm - \tilde{\rho}_{m^\pm}^\infty\|_{L^\infty} = 0;$$

(*ii*) *(asymptotic relative position for $(\tilde{\rho}^J)^\pm$)*

$$\lim_{J\to\infty} \frac{|\bar{x}((\tilde{\rho}^J)^+) - \bar{x}((\tilde{\rho}^J)^-)|}{J^2} = (m^+m^-)^{-2}.$$

We note that the variations of E^J at $\tilde{\rho}^J$ yield a system of equations satisfied by $(\tilde{\rho}^J)^\pm$:

$$(\tilde{\rho}^J)^\pm = \left(\frac{\gamma-1}{K\gamma}\right)^{\frac{1}{\gamma-1}} \left(\frac{J^2}{2I(\tilde{\rho}^J)^2}(x_1^2 + x_2^2) - \Phi_{\tilde{\rho}^J} - C_\pm^J(\tilde{\rho}^J)\right)_+^{\frac{1}{\gamma-1}} \quad \text{in } \Omega^\pm, \qquad (1.1.3)$$

where $C_\pm^J$ is the cut-off chemical potential levels determined by a minimizer $\tilde{\rho}^J$ as Lagrange multipliers. Relying on asymptotic properties in Result A, we shall prove the following uniqueness result for $\tilde{\rho}^J$.

Result B. *Let* $\{\rho^J = (\rho^J)^- + (\rho^J)^+\} \subset \mathcal{W}_J$ *be a family of solutions to* (1.1.3) *satisfying asymptotic properties* (*i*) *and* (*ii*) *in Result A and*

$$\lim_{J\to\infty} C^J_\pm(\tilde{\rho}^J) = \frac{5\gamma - 6}{4 - 3\gamma} E^\infty(\tilde{\rho}^\infty).$$

Then for large J, the family $\{\tilde{\rho}^J\}$ *is unique up to a rigid motion.*

As a consequence of Result B, we also obtain the following:

Result B′ *The density* $\tilde{\rho}^J$ *of McCann's binary stars is unique up to a rigid motion. Moreover, if the masses of constituent stars are equal* ($m^+ = m^-$)*, then the shapes of two stars are the same.*

We provide mathematically rigorous statements of Result A, B and B′ in Section 2.

The second aim of the paper is to study the dynamical stability of the McCann's binary stars (1.1.2) with respect to the flows of the Euler-Poisson system (1.0.1). We will see that the uniqueness of $\tilde{\rho}^J$ in Result B′ plays a fundamental role to this study. Inspired by stability results of steady solutions of (EP) by Rein [42], Luo and Smoller [39] (see also Cazenave and Lions [8] for the orbital stability of standing waves in nonlinear Schrödinger equations and Guo and Rein [16–19] for the stability of galaxy solutions of (VP)), we will show orbital stability of McCann's binary stars by exploiting the variational structure and the uniqueness result above: if the initial data of (EP) is close to McCann's binary star solution in some topology, then the solution of (EP) stays close, up to a rigid motion, to the binary star solution in the same topology, as long as the solution exists and the support of the solution stays bounded. The following three properties are the key ingredients in proving such results [8, 39, 42]:

(i) Relative compactness of any minimizing sequence up to symmetries of E^J
(ii) Uniqueness of a minimizer up to symmetries of E^J
(iii) Admissibility of the solution of (EP) along the dynamics so that any time-dependent solution $(\rho(t), u(t))$ such that $(\rho(0), u(0)) \sim (\tilde{\rho}, \tilde{u})$ belongs to $\mathcal{W}_J$

The property (i) is obtained in the spirit of Lions' concentration compactness principle, as shown in the construction of the binary star solution [40]. The uniqueness result B′ in the above ensures the property (ii). And the property (iii) is achieved by making use of the variational structure and the uniqueness of McCann's solution and the boundedness of the support of the solutions. With these properties in hand, we show the following conditional stability result with respect to some distance functionals naturally arising at the level of the energy space:

Result C. *The McCann's solution of binary stars* $(\tilde{\rho}(t, x), \tilde{u}(x))$ *is orbitally stable for a class of small perturbation* (ρ_0, u_0) *of* $(\tilde{\rho}^J, \tilde{u})$ *that there exists a global weak solution* $(\rho(t), u(t))$ *with the initial data* (ρ_0, u_0) *and the support of* $\rho(t)$ *does not unboundedly spread as* $t \to \infty$.

Rigorous statement and the proof of Result C are given in Section 6.

1.2 Rein's reduction and binary galaxies

The third aim of the paper is to study the existence and stability of rotating binary galaxy solutions. To this end, we adapt Rein's reduction method [42, 43] to lift McCann's binary star solutions and our framework described above to the study of binary galaxies. To explain more in detail, we consider the constrained minimization problem of free energy for (VP):

$$\inf_{f\in\mathcal{A}_J}\mathcal{F}(f),\quad \mathcal{F}(f)=E_{\mathrm{VP}}(f)+C(f) \tag{1.2.1}$$

where the energy functional is

$$E_{\mathrm{VP}}(f)=K_{\mathrm{VP}}(f)+G_{\mathrm{VP}}(f)=\iint_{\mathbb{R}^6}\frac{1}{2}|v|^2f(x,v)\,dxdv-\iint_{\mathbb{R}^6}\frac{1}{2}\frac{\rho_f(x)\rho_f(y)}{|x-y|}\,dxdy,$$

and Casimir functional is

$$C(f)=\iint_{\mathbb{R}^6}\beta(f(x,v))\,dxdv=\iint_{\mathbb{R}^6}\frac{1}{q}\kappa_q^{q-1}(f(x,v))^q\,dxdv,$$

where $q>5/3$ and $\kappa_q=\int_0^1 4\pi(1-s)^{\frac{1}{q-1}}\sqrt{2s}\,ds$. Here the constraint $\mathcal{A}_J$ is characterized by requiring the mean density $\rho_f\in\mathcal{W}_J$ and prescribing total angular momentum for (VP) by J (cf. Section 7). As a key step for the reduction, we will show that for $f\in\mathcal{A}_J$

$$K_{\mathrm{VP}}(G_f)+C(G_f)\le K_{\mathrm{VP}}(f)+C(f)$$

where G_f is the local Gibbs state associated with f involving only the mean density ρ_f and the mean velocity u_f (cf. Definition 7.1.2), the equality holds iff $f=G_f$, and moreover $\int_{\mathbb{R}^3}A(\rho_f)dx+\int_{\mathbb{R}^3}\frac{1}{2}|u_f|^2\rho_f\,dx=K_{\mathrm{VP}}(G_f)+C(G_f)$ where $A(\rho)=\frac{3q-1}{5q-3}\rho^{\frac{5q-3}{3q-1}}$. The constrained minimization problem is then reduced to the binary star problem (ρ_f,u_f) with the prescribed angular momentum J introduced in the previous subsection. By lifting McCann's binary star solutions, we obtain the existence of rotating binary galaxy solutions:

Result D. *For any $m^\pm>0$ and sufficiently large J, there exists a minimizer $\tilde f\in\mathcal{A}_J$ of the problem* (1.2.1) *such that $\tilde f(R_{-\omega t}x,R_{-\omega t}v-(0,0,\omega)^T\times(R_{-\omega t}x))$ solves (VP) and $(\rho_{\tilde f},u_{\tilde f})$ is the McCann's binary star solution.*

As in the non-rotating star problem [42], the stability analysis of binary galaxies can be reduced to the analysis of binary stars through Rein's reduction scheme. In addition to the distances used for Result C, we introduce another distance functional measuring the difference between f and its local Gibbs state $G(f)$ (cf. Definition 7.2.1) to fully measure the deviation of the perturbation f from $\tilde f$. Our final result is on orbital stability of rotating binary galaxies with small uniform angular velocity:

Result E. *The binary galaxy solution obtained in Result D is orbitally stable for a class of small perturbation f_0 of $\tilde f$ that there exists a global weak solution $f(t)$ with the initial data f_0 and the support of the mean density $\rho_f(t)$ does not unboundedly spread as $t\to\infty$.*

Rigorous statements and proofs of Result D and Result E can be found in Section 7.

The paper proceeds as follows. In Section 2, we formulate uniformly rotating 2-body problem and give new results including the crucial uniqueness on McCann's binary stars. In Section 3, we prove Result A: the convergence results of E^J, $\tilde{\rho}^J$ and detailed asymptotic behaviors of minimizers including relative positions. Section 4 is devoted to the proof of the uniqueness result B. In Section 5, we show the symmetry of binary star solutions with equal masses. In Section 6, we prove Result C: orbital stability of McCann's binary star solutions. In Section 7, we show Result D and result E on rotating binary galaxy solutions.

2 Binary galaxies and stars: uniformly rotating 2-body solutions for (VP) and (EP)

We start with basic notations to be used throughout the paper.

Definition 2.0.1 *Notations*

(*i*) *Three dimensional open ball:* $B(a,R) := \{x \in \mathbb{R}^3 \mid |x-a| < R\}$ *for given* $a \in \mathbb{R}^3$.
(*ii*) *k dimensional open ball:* $B_k(a,R) := \{x \in \mathbb{R}^k \mid |x-a| < R\}$ *for given* $a \in \mathbb{R}^k$.
(*iii*) *The projection operator of x to the* x_1x_2 *plane:* $P(x) = P(x_1,x_2,x_3) := (x_1,x_2,0)$.
(*iv*) *Rotation map:*

$$R_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(*v*) *Transformation by rigid motions: for* $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$,

$$T^{\theta,\nu}x := R_\theta x + \nu, \qquad \rho^{\theta,\nu}(x) := \rho(T^{\theta,\nu}x), \qquad f^{\theta,\nu}(x,v) := f(T^{\theta,\nu}x, v).$$

We denote the velocity u and gradients of functions by column vectors.

2.1 Uniformly rotating N-body solutions

We start with the following ansatz for rotating N-body solutions to the Euler-Poisson system (1.0.1):

$$\rho(t,y) = \rho(R_{-\omega t}y), \quad u(t,y) = \omega(-y_2, y_1, 0)^T, \quad \Phi(t,y) = \Phi(R_{-\omega t}y),$$

where $\omega > 0$ is angular velocity, ρ is a nonnegative density function with compact support.

Inserting this ansatz to (1.0.1), we see the first and third equations are automatically satisfied and the second equation becomes

$$-\omega^2\rho(R_{-\omega t}y)Py + K\gamma\rho^{\gamma-1}(R_{-\omega t}y)R^T_{-\omega t}(\nabla\rho)(R_{-\omega t}y) + \rho(R_{-\omega t}y)R^T_{-\omega t}(\nabla\Phi)(R_{-\omega t}y) = 0,$$

where P denotes the projection operator of y to y_1y_2 plane, i.e., $P(y_1,y_2,y_3) = (y_1,y_2,0)$. Then by the change of variable $x = R_{-\omega t}y$, we get

$$-\omega^2\rho(x)Px + K\gamma\rho(x)^{\gamma-1}(\nabla\rho)(x) + \rho(x)(\nabla\Phi)(x) = 0$$

We denote

$$\rho = \sum_{i=1}^{N} \rho_i, \quad \rho_i \geq 0$$

such that $\mathrm{spt}(\rho_i)$ is connected for every $i \in 1, \ldots, N$ and mutually disjoint. Then dividing by ρ_i, one has

$$\nabla(-\frac{1}{2}\omega^2|Px|^2 + K\frac{\gamma}{\gamma-1}\rho_i^{\gamma-1} + \Phi) = 0, \quad \forall x \in \mathrm{spt}(\rho_i).$$

Therefore (1.0.1) reduces to

$$\begin{cases} -\frac{1}{2}\omega^2|Px|^2 + K\frac{\gamma}{\gamma-1}\rho_i^{\gamma-1} + \Phi = -C_i & \text{in } \mathrm{spt}(\rho_i), \\ \Delta\Phi = 4\pi\rho \end{cases} \tag{2.1.1}$$

for some positive constants C_i, $i = 1, \ldots, N$. We note that (2.1.1) is written w.r.t. Φ

$$\Delta\Phi = \begin{cases} 4\pi\left(\frac{\gamma-1}{K\gamma}\right)^{\frac{1}{\gamma-1}}\left(\frac{1}{2}\omega^2|Px|^2 - \Phi - C_i\right)^{\frac{1}{\gamma-1}}, & \text{in } \mathrm{spt}(\rho_i), \\ 0, & \text{in } \mathbb{R}^3 \setminus \cup_{i=1}^{N}\mathrm{spt}(\rho_i). \end{cases} \tag{2.1.2}$$

Uniformly rotating binary star solutions when $N = 2$ were constructed by McCann [40] for sufficiently small ω, namely sufficiently large angular momentum, by solving a constrained minimization problem associated with (2.1.1), while uniformly rotating N-body solutions in the context of galaxies were constructed by Campos et. al. [7] for sufficiently small ω by solving (2.1.2) based on a finite dimensional reduction.

In this paper, we take McCann's approach for the existence of binary star solutions that retain both variational and perturbative structures. For the rest of the paper, we take $N = 2$.

2.2 McCann's construction of binary stars

In this subsection, we review McCann's binary star solutions [40] for $\gamma \in (\frac{4}{3}, 2)$. For a sake of convenience, we introduce a slightly different setting of construction from the McCann's one but they are essentially the same.

We first record the energy, the center of mass, the total angular momentum, and the moment of inertia.

Energy of (EP):

$$E(\rho, u) = \int_{\mathbb{R}^3} A(\rho(x))\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2\rho(x)\,dx - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy, \tag{2.2.1}$$

where $A(\rho) = \frac{K}{\gamma-1}\rho^\gamma, \gamma \in (\frac{4}{3}, 2)$.

The center of mass for ρ:

$$\bar{x}(\rho) := \frac{\int_{\mathbb{R}^3} x\rho(x)\,dx}{\int_{\mathbb{R}^3} \rho(x)\,dx}, \quad \bar{x}_i(\rho) := \frac{\int_{\mathbb{R}^3} x_i\rho(x)\,dx}{\int_{\mathbb{R}^3} \rho(x)\,dx}\; i = 1, 2, 3.$$

The total angular momentum $\mathbf{J}$:

$$\mathbf{J}(\rho, u) := \int_{\mathbb{R}^3} ((x - \bar{x}(\rho)) \times u)\rho(x)\, dx.$$

We denote by J_{x_3} the x_3-component of $\mathbf{J}$, i.e., $J_{x_3}(\rho, u) := \vec{e}_{x_3} \cdot \mathbf{J}(\rho, u)$, where $\vec{e}_{x_3} = (0, 0, 1)$.

The moment of inertia $I(\rho)$ is given by

$$I(\rho) = \int_{\mathbb{R}^3} |P(x - \bar{x}(\rho))|^2 \rho(x)\, dx.$$

Define an admissible class for ρ and u by

$$\mathcal{R} := \left\{\rho \in L^1(\mathbb{R}^3) \,\middle|\, \rho \geq 0, \int A(\rho) < \infty, \text{ spt}(\rho) \text{ is bdd}\right\}, \; \mathcal{V} := \left\{u : \mathbb{R}^3 \to \mathbb{R}^3 \,\middle|\, u \text{ is measurable}\right\}.$$

Definition 2.2.1 *For any given $m_1, m_2 > 0$, let*

$$L = \frac{m_1 + m_2}{(m_1 m_2)^2}.$$

Fix two values r_1, r_2 such that $0 < r_1 < m_2 L/(m_1 + m_2) = 1/(m_1^2 m_2)$ and $0 < r_2 < m_1 L/(m_1 + m_2) = 1/(m_1 m_2^2)$. By denoting

$$\mathbf{x}_1 = \left(\frac{1}{m_1^2 m_2}, 0, 0\right)^T \qquad \mathbf{x}_2 = \left(-\frac{1}{m_1 m_2^2}, 0, 0\right)^T$$

so that

$$|\mathbf{x}_1 - \mathbf{x}_2| = L \quad \textit{and} \quad m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2 = 0$$

we define

$$\overline{\mathcal{W}}^J_{m_1,m_2} :=$$

$$\left\{\rho = \rho_1 + \rho_2 \in \mathcal{R} \,\middle|\, \int \rho_1 = m_1, \int \rho_2 = m_2, \text{ spt}(\rho_1) \subset B(J^2\mathbf{x}_1, J^2 r_1), \text{ spt}(\rho_2) \subset B(J^2\mathbf{x}_2, J^2 r_2)\right\}$$

and

$$\mathcal{W}^J_{m_1,m_2} := \left\{\rho \in \mathcal{R} \,\middle|\, \exists\, \theta, v \in \mathbb{R}^3 \textit{ such that } \rho^{\theta,v} \in \overline{\mathcal{W}}^J_{m_1,m_2}\right\}.$$

Let

$$\mathcal{S}^J_{m_1,m_2} := \left\{(\rho, u) \in \mathcal{W}^J_{m_1,m_2} \times \mathcal{V} \mid J_{x_3}(\rho, u) = J\right\}.$$

Fix two arbitrary masses $m_1, m_2 > 0$. For $J > 0$, consider the following minimization problem

$$E^J_{\min} := \inf_{(\rho,u)\in\mathcal{S}^J_{m_1,m_2}} E(\rho, u). \tag{2.2.2}$$

The following lemma obtained by McCann in [40] shows that the velocity distribution minimizing the kinetic energy is given by uniform rotation.

Lemma 2.2.2 *For given $J > 0$ and $\rho \in \mathcal{R}$, consider the minimization problem*

$$T^J_{min}(\rho) = \inf_{u \in \mathcal{V}(\mathbb{R}^3)} \left\{ \int_{\mathbb{R}^3} |u(x)|^2 \rho(x)\, dx \;\Big|\; J_{x_3}(\rho, u) = J \right\}.$$

Then the minimum value $T^J_{\min}(\rho)$ is uniquely attained by

$$u(x) = \frac{J}{I(\rho)} \vec{e}_{x_3} \times (x - \bar{x}(\rho)). \tag{2.2.3}$$

so that

$$T^J_{min}(\rho) = \frac{J^2}{I(\rho)}.$$

For $J > 0$, we insert the ansatz (2.2.3) into (2.2.1) to get the reduced energy

$$E^J(\rho) = \int_{\mathbb{R}^3} A(\rho(x))\, dx + \frac{J^2}{2I(\rho)} - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy.$$

Consider the following reduced minimization problem:

$$\tilde{E}^J_{\min} := \inf_{\rho \in \mathcal{W}^J_{m_1,m_2}} E^J(\rho) \tag{2.2.4}$$

It is clear from the construction that $\tilde{E}^J_{\min} = E^J_{\min}$ of (2.2.2).

Theorem 2.2.3 (Existence of a minimizer [40]) *For any $J > 0$, the variational problem* (2.2.4) *admits a minimizer $\tilde{\rho} \in \mathcal{W}^J_{m_1,m_2}$. Moreover, any minimizer $\tilde{\rho} \in \mathcal{W}^J_{m_1,m_2}$ satisfies the following self-consistent equations:*

$$\begin{cases} -\dfrac{J^2}{2I(\tilde{\rho})^2} |P(x - \bar{x}(\tilde{\rho}))|^2 + A'(\tilde{\rho}_i) + \Phi_{\tilde{\rho}} = -C_i & \text{in } \{\tilde{\rho}_i > 0\}, \\ -\dfrac{J^2}{2I(\tilde{\rho})^2} |P(x - \bar{x}(\tilde{\rho}))|^2 + A'(\tilde{\rho}_i) + \Phi_{\tilde{\rho}} \geq -C_i & \text{in } (T^{\theta,\nu})^{-1}(B(J^2 \mathbf{x}_i, J^2 r_i)), \end{cases} \quad i = 1, 2, \tag{2.2.5}$$

for any $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$ such that $\tilde{\rho}^{\theta,\nu} \in \overline{\mathcal{W}}^J_{m_1,m_2}$. In particular,

$$\tilde{\rho}_i = (A')^{-1}_+ \left(\frac{J^2}{2I(\tilde{\rho})^2} |P(x - \bar{x}(\tilde{\rho}))|^2 - \Phi_{\tilde{\rho}} - C_i \right) \quad \text{in } (T^{\theta,\nu})^{-1}(B(J^2 \mathbf{x}_i, J^2 r_i)),\ i = 1, 2. \tag{2.2.6}$$

Remark 2.2.4 *We note that* $\left(\tilde{\rho}, \frac{J}{I(\tilde{\rho})} \vec{e}_{x_3} \times (x - \bar{x}(\tilde{\rho}))\right)$ *gives a minimizer of* (2.2.2).

Theorem 2.2.5 (Properties of a minimizer [40]) *For sufficiently large $J > 0$, every minimizer $\tilde{\rho} = \sum_{i=1}^2 \tilde{\rho}_i \in \mathcal{W}^J_{m_1,m_2}$ of* (2.2.4) *enjoys the following properties:*

(*i*) *$\tilde{\rho}$ is continuous on $\mathbb{R}^3$.*
(*ii*) *There exists a constant $R > 0$ independent of J such that $spt(\tilde{\rho}_i)$ is contained in a ball with radius R.*

Moreover, as $J \to \infty$, $\tilde{\rho}$ satisfies

$$\frac{L}{2} \leq \lim_{J \to \infty} \frac{|\bar{x}(\tilde{\rho}_1) - \bar{x}(\tilde{\rho}_2)|}{J^2} \leq L, \quad i = 1, 2. \tag{2.2.7}$$

Remark 2.2.6 *By* (*i*), *we see that* $(\tilde{\rho}(R_{-\omega t}(x - \bar{x}(\tilde{\rho}))), \frac{J}{I(\tilde{\rho})} \vec{e}_{x_3} \times (x - \bar{x}(\tilde{\rho})))$ *solves* (*EP*) *for sufficiently large J.*

2.3 Asymptotic positions, uniqueness and symmetry of binary stars

We introduce the notation of normalized densities.

Definition 2.3.1

(*i*) *We say* $\rho \in \mathcal{W}^J_{m_1,m_2}$ *is normalized if*

$$\bar{x}(\rho) = 0, \quad \bar{x}_1(\rho_1) > 0 \quad \text{and} \quad \bar{x}_2(\rho_1) = \bar{x}_2(\rho_2) = 0.$$

(*ii*) *We say* $\hat{\rho}$ *is a normalization of* ρ *if* $\hat{\rho}$ *is normalized and* $\hat{\rho} = \rho^{\theta,\nu}$ *for some* $\theta \in \mathbb{R}$ *and* $\nu \in \mathbb{R}^3$. *Note that for any* $\rho \in \mathcal{W}^J_{m_1,m_2}$, *a normalization* $\hat{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ *always exists and is unique.*

The statement (*ii*) and (2.2.7) of Theorem 2.2.5 indicate that for any minimizer $\tilde{\rho} \in \mathcal{W}^J_{m_1,m_2}$ of (2.2.4), its normalization $\hat{\tilde{\rho}}$ is contained in $\overline{\mathcal{W}}^J_{m_1,m_2}$. Since the reduced energy functional E^J is invariant under a rigid motion, we see that $\hat{\tilde{\rho}}$ is still a minimizer of (2.2.4). This shows that considering minimizers of (2.2.4), we may only take into account normalized minimizers in $\overline{\mathcal{W}}^J_{m_1,m_2}$. We will also see in Proposition 3.3.1 that if $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ is a normalized minimizer for (2.2.4), then one automatically has $\bar{x}_3(\tilde{\rho}_1) = \bar{x}_3(\tilde{\rho}_1) = 0$.

We are now ready to state our first result of this paper.

Theorem 2.3.2 *Let* $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ *be a normalized minimizer of the variational problem* (2.2.4). *Then it satisfies the following:*

(*i*) *(Relative position of the binary stars):*

$$\lim_{J\to\infty} \frac{\bar{x}_1(\tilde{\rho}_1) - \bar{x}_1(\tilde{\rho}_2)}{J^2} = L, \quad \lim_{J\to\infty} \frac{\bar{x}_1(\tilde{\rho}_1)}{J^2} = \frac{1}{m_1^2 m_2}, \quad \lim_{J\to\infty} \frac{\bar{x}_1(\tilde{\rho}_2)}{J^2} = -\frac{1}{m_1 m_2^2}.$$

(*ii*) *(Uniqueness):* $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ *is unique.*

(*iii*) *(Rotation symmetry for equal mass): If* $m_1 = m_2$ *and* $r_1 = r_2$, *then* $\tilde{\rho}_1(x) = \tilde{\rho}_2(R_\pi x)$.

We shall prove Theorem 2.3.2 throughout Section 3–5. The following corollary is a direct consequence of Theorem 2.3.2.

Corollary 2.3.3 *Let* $\tilde{\rho} \in \mathcal{W}^J_{m_1,m_2}$ *be a minimizer of the variational problem* (2.2.4). *Then it satisfies the following:*

(*i*) *(Relative position of the binary stars):*

$$\lim_{J\to\infty} \frac{|\bar{x}(\tilde{\rho}_1) - \bar{x}(\tilde{\rho}_2)|}{J^2} = L,$$

(*ii*) *(Uniqueness): it has the form*

$$\tilde{\rho}(x) = \hat{\rho}(R_\theta(x-\nu)), \quad u(x) = \frac{J}{I(\tilde{\rho})}\vec{e}_{x_3} \times (x-\nu) \quad \text{for some } \theta \in \mathbb{R},\ \nu \in \mathbb{R}^3,$$

where $\hat{\rho}$ *is a unique normalized minimizer of* (2.2.4) *in* $\overline{\mathcal{W}}^J_{m_1,m_2}$.

(*iii*) *(Rotation symmetry for equal mass): If* $m_1 = m_2$ *and* $r_1 = r_2$, *then*

$$\tilde{\rho}_1(x) = \tilde{\rho}_2(R_\pi(x - \bar{x}(\tilde{\rho}))).$$

3 Convergence results for a family of normalized minimizers

3.1 Non-rotating star

Define

$$E^\infty(\rho) = \int_{\mathbb{R}^3} A(\rho(x))\, dx - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy$$

and consider a minimization problem

$$\tilde{E}^\infty_m := \inf\left\{E^\infty(\rho) \mid \rho \in \mathcal{R}_m(\mathbb{R}^3)\right\}, \tag{3.1.1}$$

where

$$\mathcal{R}_m(\mathbb{R}^3) := \left\{\rho \in L^\gamma(\mathbb{R}^3) \,\middle|\, \rho \ge 0, \int \rho = m, \quad \text{spt}(\rho) \text{ is bdd}\right\}$$

We denote by ρ^∞_m a minimizer of the problem (3.1.1), i.e.,

$$\tilde{E}^\infty_m = E^\infty(\rho^\infty_m), \quad \rho^\infty_m \in \mathcal{R}_m(\mathbb{R}^3),$$

whose existence and properties are listed in the theorem below.

Theorem 3.1.1 *For any $m > 0$, the minimum energy level $\tilde{E}^\infty_m$ is negative and there exists a minimizer ρ^∞_m of the minimization problem* (3.1.1) *satisfying the following properties:*

(i) *ρ^∞_m is radially symmetric up to a translation and strictly decreasing in the radial direction on its support;*

(ii) $\rho^\infty_m \in W^{1,2}(\mathbb{R}^3) \cap C_c(\mathbb{R}^3)$;

(iii) *ρ^∞_m solves the self-consistent equation*

$$\rho^\infty_m = B_\gamma\left(\frac{1}{|x|} * \rho^\infty_m - C^\infty_m\right)_+^{\frac{1}{\gamma-1}} \quad \text{in } \mathbb{R}^3 \tag{3.1.2}$$

where $B_\gamma = (\frac{\gamma-1}{K\gamma})^{\frac{1}{\gamma-1}}$ and $C^\infty_m > 0$ is a Lagrange-multiplier, which is exactly determined by the ground energy level as $mC^\infty_m = \frac{5\gamma-6}{4-3\gamma}\tilde{E}^\infty_m$.

(iv) *ρ^∞_m is unique up to a translation;*

(v) *Let $\mathcal{L}^\infty$ be a linearized operator of* (3.1.2) *at ρ^∞_m, i.e.,*

$$\mathcal{L}^\infty[\eta] = \eta - \frac{B_\gamma}{\gamma-1}\left(\frac{1}{|x|} * \rho^\infty_m - C^\infty_m\right)_+^{\frac{2-\gamma}{\gamma-1}} \frac{1}{|x|} * \eta.$$

Then $ker(\mathcal{L}^\infty) := \{\eta \in L^2(\mathbb{R}^3) \mid \mathcal{L}^\infty[\eta] = 0\}$ is given by

$$span\left\{\partial_1\rho^\infty_m, \partial_2\rho^\infty_m, \partial_3\rho^\infty_m\right\}.$$

Proof For proofs of negativity of $\tilde{E}^\infty_m$ and (i)–(iii), we refer to a comprehensive review article by Rein [43] and references therein. Proofs of (iv) and (v) can be found in [13], where the statement is given in terms of $U^\infty_m = \frac{1}{|\cdot|} * \rho^\infty_m$.

Lemma 3.1.2 *Let $\eta \in ker(\mathcal{L}^\infty)$ satisfy $\int_{\mathbb{R}^3} x\eta = 0$. Then $\eta \equiv 0$.*

Proof By Theorem 3.1.1.(v), we may write $\eta = c_1\partial_1\rho^\infty_m + c_2\partial_2\rho^\infty_m + c_3\partial_3\rho^\infty_m$ for $c_i \in \mathbb{R}$, $i = 1,2,3$. Since $\int_{\mathbb{R}^3} \partial_i\rho x_j\, dx = 0$, $i \ne j$, and $\int_{\mathbb{R}^3} \partial_i\rho x_i\, dx = -m$, we see that

$$0 = \int_{\mathbb{R}^3} x\eta\, dx = -m(c_1, c_2, c_3)$$

so that $c_1 = c_2 = c_3 = 0$.

3.2 Relative equilibria

A system of two body circular orbits $x_1(t) = R_{\omega t}(\xi_1 - \bar{x}(\xi), 0)$, $x_2(t) = R_{\omega t}(\xi_2 - \bar{x}(\xi), 0)$ satisfies Newton's equation if and only if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfies

$$\frac{m_2(\xi^2 - \xi^1)}{|\xi^2 - \xi^1|^3} + \omega^2(\xi^1 - \bar{x}(\xi)) = 0, \quad \frac{m_1(\xi^1 - \xi^2)}{|\xi^1 - \xi^2|^3} + \omega^2(\xi^2 - \bar{x}(\xi)) = 0. \tag{3.2.1}$$

Here we denote by $\bar{x}(\xi)$, the center of mass of two $\mathbb{R}^2$ vectors ξ^1, ξ^2, i.e., $\bar{x}(\xi) := \frac{m_1\xi^1 + m_2\xi^2}{m_1 + m_2}$. A pair of $\mathbb{R}^2$ vectors (ξ^1, ξ^2) is called a relative equilibrium for two body problem if it solves the system of equations (3.2.1) for some angular velocity ω.

For any fixed angular momentum $J > 0$, the effective Hamiltonian for (3.2.1) is given by

$$\begin{aligned} H_J^{\mathrm{re}}(\xi^1, \xi^2) &:= \frac{J^2}{2(m_1|\xi^1 - \bar{x}(\xi)|^2 + m_2|\xi^2 - \bar{x}(\xi)|^2)} - \frac{m_1 m_2}{|\xi^1 - \xi^2|} \\ &= \frac{J^2(m_1 + m_2)}{2m_1 m_2|\xi^1 - \xi^2|^2} - \frac{m_1 m_2}{|\xi^1 - \xi^2|} \end{aligned} \tag{3.2.2}$$

By a direct computation, we see that any critical point of H_J^{re} is a relative equilibrium. More precisely, it is a solution to (3.2.1) with the angular velocity

$$\omega = \frac{J}{m_1|\xi^1 - \bar{x}(\xi)|^2 + m_2|\xi^2 - \bar{x}(\xi)|^2}.$$

It is worth pointing out that H_J^{re} enjoys some nice scaling and invariance properties. If we set $\zeta = \xi/J^2$, then

$$H_J^{\mathrm{re}}(\xi^1, \xi^2) = J^{-2} H^{\mathrm{re}}(\zeta_1, \zeta_2), \tag{3.2.3}$$

where

$$H^{\mathrm{re}}(\zeta_1, \zeta_2) := \frac{m_1 + m_2}{2m_1 m_2|\zeta_1 - \zeta_2|^2} - \frac{m_1 m_2}{|\zeta_1 - \zeta_2|}. \tag{3.2.4}$$

Also, the effective Hamiltonian H^{re} (as well as H_J^{re}) is invariant with respect to a rigid motion. In other word, for any (ζ_1, ζ_2) such that $\zeta_1 \neq \zeta_2$,

$$H^{\mathrm{re}}(\zeta) = H^{\mathrm{re}}((e^{i\theta}\zeta_1, e^{i\theta}\zeta_2) + (\zeta^0, \zeta^0)), \quad \forall \theta \in \mathbb{R},\ \zeta^0 \in \mathbb{R}^2.$$

Consequently, the set of relative equilibria is invariant with respect to a rigid motion. Indeed, it has a simple variational characterization.

Proposition 3.2.1 (Characterization of relative equilibria for H^{re}) *There holds the following:*

(i) *The effective Hamiltonian H^{re} admits the global minimum value*

$$\tilde{H}^{\mathrm{re}}_{\min} := \min\left\{H^{\mathrm{re}}(\zeta_1, \zeta_2) \mid \zeta_1, \zeta_2 \in \mathbb{R}^2, \zeta_1 \neq \zeta_2\right\}$$

as a unique critical value and it is attained by (ζ_1, ζ_2) if and only if $|\zeta_1 - \zeta_2| = L$ where L is defined in Definition 2.2.1.

(*ii*) *Let* $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ *be a global minimum point for* H^{re}*, i.e.,* $|\tilde{\zeta}_1 - \tilde{\zeta}_2| = L$*. Then* $\ker \nabla^2 H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)$ *is spanned by*

$$\{((1,0),\,(1,0)),\quad ((0,1),\,(0,1)),\quad ((0,1),\,(0,-1))\}\,.$$

We note that these kernel elements come from the invariance of H^{re} *with respect to the rigid motions.*

Proof The assertion (*i*) is immediate from the exact form of H^{re}. Let (ϕ_1, ϕ_2) be a kernel element of $\nabla^2 H^{\mathrm{re}}(\zeta_1, \zeta_2)$. We directly compute the Hessian of H^{re} to get

$$\nabla^2 H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\partial^2_{\zeta_1\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) & \partial^2_{\zeta_1\zeta_2} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)\\ \partial^2_{\zeta_2\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) & \partial^2_{\zeta_2\zeta_2} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = 0, \tag{3.2.5}$$

where

$$\partial^2_{\zeta_1\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) = \partial^2_{\zeta_2\zeta_2} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) = -\partial^2_{\zeta_1\zeta_2} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) = -\partial^2_{\zeta_2\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)$$

and $\partial^2_{\zeta_1\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)$ is given by for $\phi \in \mathbb{R}^2$,

$$\partial^2_{\zeta_1\zeta_1} H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2)\phi = -\frac{\phi}{ML^4} + \frac{4(\tilde{\zeta}_1 - \tilde{\zeta}_2)\cdot\phi}{ML^6}(\tilde{\zeta}_1 - \tilde{\zeta}_2) + \frac{m_1 m_2 \phi}{L^3} - \frac{3m_1 m_2(\tilde{\zeta}_1 - \tilde{\zeta}_2)\cdot\phi}{L^5}(\tilde{\zeta}_1 - \tilde{\zeta}_2),$$

where $M = m_1 m_2/(m_1 + m_2)$. Since a global minimum point $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ is unique up to a rigid motion, we may assume $\tilde{\zeta}_1 = (1/m_1^2 m_2, 0)$ and $\tilde{\zeta}_2 = (-1/m_1 m_2^2, 0)$, the normalized one. Then by denoting $\phi_i = ((\phi_i)_1, (\phi_i)_2)$, $i = 1, 2$, we obtain the following two equations from (3.2.5),

$$\begin{cases} -\dfrac{(\phi_1)_1 - (\phi_2)_1}{ML^4} + \dfrac{4L^2((\phi_1)_1 - (\phi_2)_1)}{ML^6} + \dfrac{m_1 m_2((\phi_1)_1 - (\phi_2)_1)}{L^3} - \dfrac{3m_1 m_2 L^2((\phi_1)_1 - (\phi_2)_1)}{L^5} = 0,\\[2ex] -\dfrac{(\phi_1)_2 - (\phi_2)_2}{ML^4} + \dfrac{m_1 m_2((\phi_1)_2 - (\phi_2)_2)}{L^3} = 0. \end{cases}$$

Solving this, we see that

$$\begin{aligned}\ker\nabla^2 H^{\mathrm{re}}(\tilde{\zeta}_1, \tilde{\zeta}_2) &= \left\{(\phi_1, \phi_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (\phi_1)_1 = (\phi_2)_1\right\}\\ &= \mathrm{span}\,\{((1,0),\,(1,0)),\quad ((0,1),\,(0,1)),\quad ((0,1),\,(0,-1))\}\,.\end{aligned}$$

We are now ready to establish the expansion of E^J in $\frac{1}{J^2}$ which identifies the leading order asymptotics as E^∞ and the next order correction with H^{re}. We recall from Definition 2.2.1 that

$$\mathbf{x}_1 = \left(\frac{1}{m_1^2 m_2}, 0, 0\right)^T \quad \mathbf{x}_2 = \left(-\frac{1}{m_1 m_2^2}, 0, 0\right)^T$$

and r_1, r_2 are two fixed numbers satisfying $0 < r_i < |\mathbf{x}_i|$, $i = 1, 2$.

Proposition 3.2.2 (Asymptotic expansion for E^J) *Fix some $R > 0$ and let $\rho_1, \rho_2 \in \mathcal{R}$ and $(\zeta_1, \zeta_2) \in B_2(\mathbf{x}_1, r_1) \times B_2(\mathbf{x}_2, r_2)$ satisfy*

$$\bar{x}(\rho_i) = 0,\ spt(\rho_i) \subset B(0,R) \text{ for } i = 1,2 \quad \text{and} \quad \rho_1(\cdot - (J^2\zeta_1, 0)) + \rho_2(\cdot - (J^2\zeta_2, 0)) \in \overline{\mathcal{W}}^J_{m_1,m_2}.$$

Then one has

$$E^J\left(\rho_1(\cdot - (J^2\zeta_1,0)) + \rho_2(\cdot - (J^2\zeta_2,0))\right) = E^\infty(\rho_1) + E^\infty(\rho_2) + \frac{1}{J^2}\left(H^{re}(\zeta_1,\zeta_2) + \Re(\rho_1,\rho_2,\zeta_1,\zeta_2)\right),$$

where

$$|\Re(\rho_1,\rho_2,\zeta_1,\zeta_2)| \leq CJ^{-2}$$

for some constant $C > 0$ depending only on R, m_1, m_2, r_1, r_2.

Proof Since

$$\bar{x}(\rho_1(\cdot - a) + \rho_2(\cdot - b)) = \frac{m_1 a + m_2 b}{m_1 + m_2} \quad \text{for } a, b \in \mathbb{R}^3,$$

we have

$$\begin{aligned}
&I(\rho_1(\cdot - a) + \rho_2(\cdot - b)) \\
&\quad = \int \left|P\left(x + a - \frac{m_1 a + m_2 b}{m_1 + m_2}\right)\right|^2 \rho_1\,dx + \int \left|P\left(x + b - \frac{m_1 a + m_2 b}{m_1 + m_2}\right)\right|^2 \rho_2\,dx \\
&\quad = \int |Px|^2 \rho_1\,dx + \left|\frac{m_2 P(a-b)}{m_1 + m_2}\right|^2 \int \rho_1\,dx + \int |Px|^2 \rho_2\,dx + \left|\frac{m_1 P(a-b)}{m_1 + m_2}\right|^2 \int \rho_2\,dx \\
&\quad = I(\rho_1) + I(\rho_2) + \frac{m_1 m_2}{m_1 + m_2}|P(a-b)|^2.
\end{aligned}$$

Also, we note that the distance between two balls $B_2(\mathbf{x}_1, r_1)$ and $B_2(\mathbf{x}_2, r_1)$ is strictly positive. In other words,

$$\min\left\{ |\zeta_1 - \zeta_2| \,\middle|\, \zeta_1 \in B_2(\mathbf{x}_1, r_1),\ \zeta_1 \in B_2(\mathbf{x}_1, r_1) \right\} = |\mathbf{x}_1| - r_1 + |\mathbf{x}_2| - r_2 > 0.$$

Then for sufficiently large $J > 0$, $\rho_1(\cdot - (J^2\zeta_1, 0))$ and $\rho_2(\cdot - (J^2\zeta_2, 0))$ have disjoint supports for all $\zeta_1 \in B_2(\mathbf{x}_1, r_1)$, $\zeta_2 \in B_2(\mathbf{x}_2, r_2)$ since $\bar{x}(\rho_i) = 0$, $\text{spt}(\rho_i) \subset B(0,R)$ for $i = 1,2$. By briefly denoting $\tilde{\rho}_i(x) := \rho_i(x - (J^2\zeta_i, 0))$, $i = 1,2$, one then has

$$\begin{aligned}
\int_{\mathbb{R}^3} (\tilde{\rho}_1(x) + \tilde{\rho}_2(x))^{\gamma-1}\,dx &= \int_{\text{spt}(\tilde{\rho}_1)\cup\text{spt}(\tilde{\rho}_2)} (\tilde{\rho}_1(x) + \tilde{\rho}_2(x))^{\gamma-1}\,dx \\
&= \int_{\text{spt}(\tilde{\rho}_1)} (\tilde{\rho}_1(x) + \tilde{\rho}_2(x))^{\gamma-1}\,dx + \int_{\text{spt}(\tilde{\rho}_2)} (\tilde{\rho}_1(x) + \tilde{\rho}_2(x))^{\gamma-1}\,dx \\
&= \int_{\text{spt}(\tilde{\rho}_1)} \tilde{\rho}_1^{\gamma-1}(x)\,dx + \int_{\text{spt}(\tilde{\rho}_2)} \tilde{\rho}_2^{\gamma-1}(x)\,dx \\
&= \int_{\mathbb{R}^3} \tilde{\rho}_1^{\gamma-1}(x)\,dx + \int_{\mathbb{R}^3} \tilde{\rho}_2^{\gamma-1}(x)\,dx,
\end{aligned}$$

which shows

$$\begin{aligned}
E^\infty(\tilde\rho_1+\tilde\rho_2) &= \int_{\mathbb{R}^3} A(\tilde\rho_1+\tilde\rho_2)\,dx - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{(\tilde\rho_1(x)+\tilde\rho_2(x))(\tilde\rho_1(y)+\tilde\rho_2(y))}{|x-y|}\,dxdy \\
&= \int_{\mathbb{R}^3} A(\tilde\rho_1)\,dx + \int_{\mathbb{R}^3} A(\tilde\rho_2)\,dx \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\tilde\rho_1(x)\tilde\rho_1(y)}{|x-y|}\,dxdy - \frac{1}{2}\iint_{\mathbb{R}^6} \frac{\tilde\rho_2(x)\tilde\rho_2(y)}{|x-y|}\,dxdy - \iint_{\mathbb{R}^6} \frac{\tilde\rho_1(x)\tilde\rho_2(y)}{|x-y|}\,dxdy, \\
&= E^\infty(\tilde\rho_1) + E^\infty(\tilde\rho_2) - \iint_{\mathbb{R}^6} \frac{\tilde\rho_1(x)\tilde\rho_2(y)}{|x-y|}\,dxdy.
\end{aligned}$$

Then we see that

$$\begin{aligned}
&E^J(\tilde\rho_1+\tilde\rho_2)) \\
&= E^\infty(\rho_1)+E^\infty(\rho_2) + \frac{J^2}{2I(\rho_1(\cdot-(J^2\zeta_1,0))+\rho_2(\cdot-(J^2\zeta_2,0)))} - \iint \frac{\rho_1(x)\rho_2(y)}{|x-y+J^2(\zeta_1-\zeta_2,0)|}\,dxdy \\
&= E^\infty(\rho_1)+E^\infty(\rho_2) + \frac{1}{J^2}\left(\frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2 + \frac{I(\rho_1)+I(\rho_2)}{J^4}} - \iint \frac{\rho_1(x)\rho_2(y)}{\left|(\zeta_1-\zeta_2,0)+\frac{x-y}{J^2}\right|}\,dxdy \right)
\end{aligned}$$

so that

$$\begin{aligned}
&\mathfrak{R}(\rho_1,\rho_2,\zeta_1,\zeta_2) \\
&= \frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2 + \frac{I(\rho_1)+I(\rho_2)}{J^4}} - \iint \frac{\rho_1(x)\rho_2(y)}{\left|(\zeta_1-\zeta_2,0)+\frac{x-y}{J^2}\right|}\,dxdy - H^{\mathrm{re}}(\zeta_1,\zeta_2) \\
&= \frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2 + \frac{I(\rho_1)+I(\rho_2)}{J^4}} - \frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2} - \left(\iint \frac{\rho_1(x)\rho_2(y)}{\left|(\zeta_1-\zeta_2,0)+\frac{x-y}{J^2}\right|}\,dxdy - \frac{m_1m_2}{|\zeta_1-\zeta_2|} \right).
\end{aligned}$$

By denoting $c := |\mathbf{x}_1| - r_1 + |\mathbf{x}_2| - r_2$, we have seen $|\zeta_1-\zeta_2| \ge c$, from which we have

$$\begin{aligned}
&\left| \frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2 + \frac{I(\rho_1)+I(\rho_2)}{J^4}} - \frac{1}{\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2} \right| \\
&= \frac{I(\rho_1)+I(\rho_2)}{J^4\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2(\frac{2m_1m_2}{m_1+m_2}|\zeta_1-\zeta_2|^2 + \frac{I(\rho_1)+I(\rho_2)}{J^4})} \le \frac{R^2(m_1+m_2)}{J^4(\frac{2m_1m_2}{m_1+m_2})^2c^4}
\end{aligned}$$

and for $J^2 > 4R/c$,

$$\begin{aligned}
&\left| \iint \frac{\rho_1(x)\rho_2(y)}{\left|(\zeta_1-\zeta_2,0)+\frac{x-y}{J^2}\right|}\,dxdy - \frac{m_1m_2}{|\zeta_1-\zeta_2|} \right| \\
&\le \iint \left| \frac{1}{|(\zeta_1-\zeta_2,0)+\frac{x-y}{J^2}|} - \frac{1}{|\zeta_1-\zeta_2|} \right| \rho_1(x)\rho_2(y)\,dxdy \\
&\le \iint | \frac{|x-y|}{J^2\left||\zeta_1-\zeta_2| - |\frac{x-y}{J^2}|\right|\,|\zeta_1-\zeta_2|} \rho_1(x)\rho_2(y)\,dxdy \le \frac{2Rm_1m_2}{J^2c^2/2}.
\end{aligned}$$

This proves the lemma.

Remark 3.2.3 *The asymptotic expansion of the energy in Proposition 3.2.2 reveals that (i) the energy for the binary system with angular momentum J converges to the energy E^∞ of non-rotating Lane-Emden stars as $J \to \infty$, (ii) the effective Hamiltonian H^{re} introduced in* (3.2.4) *naturally emerges at the next order $O(J^{-2})$ of the expansion. In fact, the kinetic energy due to uniform rotation $J^2/I(\rho)$ and the tidal energy due to two-body interaction $-\iint \rho^+(x)\rho^-(y)/|x-y|dxdy$ are both of $O(J^{-2})$ because of the separation scale J^2, and the balance of these energies brings in the effective Hamiltonian for two-body relative equilibria. While the energy convergence result to the one by non-rotating stars is shown in [40], the energy expansion with a second order correction using the effect Hamiltonian is new. This new characterization of the Hamiltonian suggests that the asymptotic relative position vector of two stars minimizes the effective Hamiltonian since binary star solutions minimize the full Hamiltonian (cf. Lemma 3.3.6) and such an asymptotic convergence result will be important in the uniqueness result in Section 4 (cf. Theorem 4.0.4).*

3.3 Uniform estimates and convergences of minimizers

This subsection is devoted to the proof of several asymptotic behaviors of normalized minimizers of (2.2.4). In particular the assertion (*i*) of Theorem 2.3.2 is proved in the proposition below.

Proposition 3.3.1 *Let $\{\tilde\rho^J\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ be a family of normalized minimizers of* (2.2.4) *and $\tilde\rho^\infty_{m_i}$ be a unique minimizer of the limit variational problem* (3.1.1) *with $m = m_i$ such that $\bar x(\tilde\rho^\infty_{m_i}) = 0$. Then the following properties hold true:*

(*i*)

$$\bar x_3(\tilde\rho^J_1) = \bar x_3(\tilde\rho^J_2) = 0, \quad \lim_{J\to\infty} \frac{\bar x(\tilde\rho^J_1)}{J^2} = (\frac{1}{m_1^2 m_2}, 0, 0), \quad \lim_{J\to\infty} \frac{\bar x(\tilde\rho^J_2)}{J^2} = -(\frac{1}{m_1 m_2^2}, 0, 0).$$

(*ii*)

$$\lim_{J\to\infty} \|\tilde\rho^J_i(\cdot + \bar x(\tilde\rho^J_i)) - \tilde\rho^\infty_{m_i}\|_{L^\infty} + |C^J_i - C^\infty_{m_i}| = 0, \quad i = 1, 2,$$

where $\{C^J_i\}$ and $\{C^\infty_{m_i}\}$ are the Lagrange multipliers in (2.2.5) *and* (3.1.2) *with $m = m_i$ respectively.*

Remark 3.3.2 *It is proved in [40] a weaker version of Proposition 3.3.1 saying that every minimizer $\tilde\rho^J \in \mathcal{W}^J_{m_1,m_2}$ of* (2.2.4) *satisfies*

(*i*) $\lim_{J\to\infty} E^\infty(\tilde\rho^J_1) = \tilde E^\infty_{m_1}, \quad \lim_{J\to\infty} E^\infty(\tilde\rho^J_2) = \tilde E^\infty_{m_2}$ (Proposition 6.4),
(*ii*) $C^\infty_{m_i} \le \liminf_{J\to\infty} C^J_i$ (Lemma 6.5),
(*iii*) $\frac{L}{2} \le \lim_{J\to\infty} \frac{|\bar x(\tilde\rho^J_1) - \bar x(\tilde\rho^J_2)|}{J^2} \le L$ (Proposition 6.8).

Remark 3.3.3 *The convergence results of Proposition 3.3.1 are in fact consequences of the fact that the binary star system is a local minimizer, not a global minimizer of the Hamiltonian. Thanks to the local minimizing property of the binary star solutions with respect to the Wasserstein L^∞ metric, two stars become separated away from one another in the scale of $O(J^2)$ as the angular momentum J goes to ∞, and each star asymptotically resembles the associated non-rotating Lane-Emden star with the same mass. As noted in [40], the chemical potential C^J needs not to be constant throughout the binary star system, but* *each star as a local minimizer has its own chemical potential C_i^J, $i = 1, 2$. The result of Proposition 3.3.1 shows that these C_i^J also converge to the chemical potential of non-rotating Lane-Emden stars of the same mass.*

We begin with the proof of Proposition 3.3.1 by stating a slightly different form of (*ii*) of Theorem 2.2.5. Throughout Lemma 3.3.4–Lemma 3.3.10, $\{\tilde\rho^J\}$ denotes a family normalized minimizer of (2.2.4).

Lemma 3.3.4 *There exists a constant $R > 0$ independent of J such that $spt(\tilde\rho_i^J) \subset B(\bar x(\tilde\rho_i^J), R)$ for any $i = 1, 2$ and large J.*

Proof Theorem 2.2.5.(*ii*) says the existence of a constant $R_0 > 0$ such that $\mathrm{spt}(\tilde\rho_i^J)$ is contained in a ball with radius R_0. We note that $\bar x(\tilde\rho_i^J)$ belongs to the same ball because a ball is convex. Then taking $R = 2R_0$, one can see the lemma holds true.

Lemma 3.3.5 *There holds that*

$$\bar x_3(\tilde\rho_1^J) = \bar x_3(\tilde\rho_2^J) = 0.$$

Proof As is asserted in the proof of Theorem 6.2 of in [40], a minimizer $\tilde\rho^J$ is symmetric with respect to a plane $x_3 = c$ by the strict rearrangement inequality (Theorem 3.9 in [34]) and Fubini's theorem. Since $\bar x(\tilde\rho^J) = 0$, we see that $c = 0$. In other words, $\tilde\rho^J$ is even with respect to the axis x_3. We recall the equation (2.2.6),

$$\tilde\rho_i^J = (A')_+^{-1}\left(\frac{J^2}{2I(\tilde\rho^J)^2}|Px|^2 - \Phi_{\tilde\rho^J} - C_i^J\right) \quad \text{in } B(J^2\mathbf{x}_i, J^2 r_i),\ i = 1, 2.$$

Then the lemma follows from the fact that $(A')_+^{-1}\left(\frac{J^2}{2I(\tilde\rho^J)^2}|Px|^2 - \Phi_{\tilde\rho^J} - C_i^J\right)$ is even with respect to the x_3 direction.

Next we show that the relative position of the centers of mass of two stars divided by square angular momentum is inversely proportional to the product of masses.

Lemma 3.3.6 *There holds that*

$$\lim_{J\to\infty}\frac{\bar x(\tilde\rho_1^J)}{J^2} = (\frac{1}{m_1^2 m_2}, 0, 0), \quad \lim_{J\to\infty}\frac{\bar x(\tilde\rho_2^J)}{J^2} = -(\frac{1}{m_1 m_2^2}, 0, 0).$$

Proof Let us define

$$\hat\rho_i^J := \tilde\rho_i^J(\cdot + \bar x(\tilde\rho_i^J)), \quad \tilde\zeta_i^J := \frac{(\bar x_1(\tilde\rho_i^J), \bar x_2(\tilde\rho_i^J))}{J^2}, \quad i = 1, 2.$$

Then recalling the fact that $\bar x_3(\tilde\rho_i^J) = 0$, $i = 1, 2$ (Lemma 3.3.5), we have from Proposition 3.2.2 that

$$E^J(\tilde\rho^J) = E^\infty(\hat\rho_1^J) + E^\infty(\hat\rho_2^J) + \frac{1}{J^2}H^{\mathrm{re}}(\tilde\zeta_1^J, \tilde\zeta_2^J) + o(J^{-2}) \quad \text{as } J \to \infty. \tag{3.3.1}$$

Choose any $(\zeta_1, \zeta_2) \in B_2(\mathbf{x}_1, r_1) \times B_2(\mathbf{x}_2, r_2)$. By Lemma 3.3.4, if J is sufficiently large, $B(J^2\zeta_1, R)$ and $B(J^2\zeta_2, R)$ are included in $B(J^2\mathbf{x}_1, J^2 r_1)$ and $B_2(J^2\mathbf{x}_2, J^2 r_2)$ respectively. Then one has

$$\hat{\rho}^J_{\zeta_1,\zeta_2} := \hat{\rho}^J_1(\cdot - (J^2\zeta_1, 0)) + \hat{\rho}^J_2(\cdot - (J^2\zeta_2, 0)) \in \overline{\mathcal{W}}^J_{m_1,m_2}$$

so that

$$E^J(\tilde{\rho}^J) \leq E^J(\hat{\rho}^J_{\zeta_1,\zeta_2}) = E^\infty(\hat{\rho}^J_1) + E^\infty(\hat{\rho}^J_2) + \frac{1}{J^2} H^{\mathrm{re}}(\zeta_1, \zeta_2) + o(J^{-2}) \quad \text{as } J \to \infty.$$

Combining this with (3.3.1), we get

$$\limsup_{J\to\infty} H^{\mathrm{re}}(\tilde{\zeta}^J_1, \tilde{\zeta}^J_2) \leq H^{\mathrm{re}}(\zeta_1, \zeta_2), \qquad \forall\, (\zeta_1, \zeta_2) \in B_2(\mathbf{x}_1, r_1) \times B_2(\mathbf{x}_2, r_2).$$

Note that since $(\tilde{\zeta}^J_1, \tilde{\zeta}^J_2) \in B_2(\mathbf{x}_1, r_1) \times B_2(\mathbf{x}_2, r_2)$, after choosing a subsequence, it converges to some point $(\tilde{\zeta}^\infty_1, \tilde{\zeta}^\infty_2) \in \overline{B_2(\mathbf{x}_1, r_1)} \times \overline{B_2(\mathbf{x}_2, r_2)}$ as $J \to \infty$. Then one has

$$H^{\mathrm{re}}(\tilde{\zeta}^\infty_1, \tilde{\zeta}^\infty_2) \leq H^{\mathrm{re}}(\zeta_1, \zeta_2), \qquad \forall\, (\zeta_1, \zeta_2) \in B_2(\mathbf{x}_1, r_1) \times B_2(\mathbf{x}_2, r_2),$$

which says that $(\tilde{\zeta}^\infty_1, \tilde{\zeta}^\infty_2)$ achieves a global minimum of H^{re}. Since $\tilde{\rho}^J$ is normalized, one must have

$$(\tilde{\zeta}^\infty_1)_2 = (\tilde{\zeta}^\infty_2)_2 = 0 \quad \text{and} \quad m_1(\tilde{\zeta}^\infty_1)_1 + m_2(\tilde{\zeta}^\infty_2)_1 = 0.$$

Then, using Proposition 3.2.1, we also see $(\tilde{\zeta}^\infty_1)_1 - (\tilde{\zeta}^\infty_2)_1 = L$. Therefore we can determine

$$\tilde{\zeta}^\infty_1 = (\frac{1}{m_1^2 m_2}, 0, 0), \qquad \tilde{\zeta}^\infty_2 = -(\frac{1}{m_1 m_2^2}, 0, 0).$$

Now, we show the convergence of the following continuum limits:

$$\lim_{J\to\infty} \tilde{\zeta}^J_1 = \tilde{\zeta}^\infty_1, \qquad \lim_{J\to\infty} \tilde{\zeta}^J_2 = \tilde{\zeta}^\infty_2$$

to end the proof. Arguing indirectly, suppose not. Then there is a positive number $\varepsilon_0 > 0$ and a sequence $\{J_k\} \to \infty$ such that without loss of generality,

$$|\tilde{\zeta}^{J_k}_1 - \tilde{\zeta}^\infty_1| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \tag{3.3.2}$$

By the same reasoning, after choosing a subsequence, $\{(\tilde{\zeta}^{J_k}_1, \tilde{\zeta}^{J_k}_2)\}$ converges to some point $(\hat{\zeta}^\infty_1, \hat{\zeta}^\infty_2)$, which is a global minimum point of H^{re}. Then as above, $(\hat{\zeta}^\infty_1, \hat{\zeta}^\infty_2)$ is uniquely determined as $(\tilde{\zeta}^\infty_1, \tilde{\zeta}^\infty_2)$ but this gives a contradiction to (3.3.2).

In the following lemma we show that the energy level of the binary star solutions is negative.

Lemma 3.3.7 *For any sufficiently large $J > 0$, one has*

$$\tilde{E}^J_{min} = E^J(\tilde{\rho}^J) < 0.$$

Proof Since $\tilde\rho^J \in \overline{\mathcal{W}}^J_{m_1,m_2}$, there exists a constant $c_1, c_2, c_3 > 0$ independent of J satisfying $c_1 J^2 \le |Px| \le c_2 J^2$ for $x \in B(J^2\mathbf{x}_i, J^2 r_i), i = 1,2$ and $c_3 J^2 \le |x-y|$ for $x \in B(J^2\mathbf{x}_i, J^2 r_i)$, $y \in B(J^2\mathbf{x}_j, J^2 r_j)$, $i \neq j$. Then we see from this and Remark 3.3.2 that

$$\begin{aligned}E^J(\tilde\rho^J) &= E^\infty(\tilde\rho^J_1) + E^\infty(\tilde\rho^J_2) + \frac{J^2}{2I(\tilde\rho^J)} - \iint \frac{\tilde\rho^J_1(x)\tilde\rho^J_2(y)}{|x-y|}\,dxdy \\ &\le \tilde E^\infty_{m_1} + \tilde E^\infty_{m_2} + o(1) + O(J^{-2})\end{aligned}$$

as $J \to \infty$. Then the lemma follow from the fact that $\tilde E^\infty_{m_i} < 0$, $i = 1,2$. (See Theorem 3.1.1.)

We now discuss the uniform regularity of the density profiles.

Lemma 3.3.8 *There holds that*

$$\limsup_{J\to\infty} \|\tilde\rho^J\|_{W^{1,\infty}(\mathbb{R}^3)} < \infty.$$

In particular, $\tilde\rho^J$ is Lipschitz continuous uniformly for J, i.e.,

$$\limsup_{J\to\infty} \sup_{\substack{x,y\in\mathbb{R}^3\\ x\neq y}} \frac{|\tilde\rho^J(x) - \tilde\rho^J(y)|}{|x-y|} < \infty.$$

Proof Since $\tilde\rho^J$ is a normalized minimizer, it satisfies by (2.2.6)

$$\tilde\rho^J_i = (A')^{-1}_+\left(\frac{J^2}{2I(\tilde\rho^J)^2}|Px|^2 - \Phi_{\tilde\rho^J} - C^J_i\right) \quad \text{in } B(J^2\mathbf{x}_i, J^2 r_i), \quad i = 1,2 \tag{3.3.3}$$

and $\tilde\rho^J_i \equiv 0$ outside $B(J^2\mathbf{x}_i, J^2 r_i)$. Using the fact that $(X-Y)_+ \le |X|$ for $X \in \mathbb{R}$ and $Y \ge 0$, one has for any $q \ge 1$,

$$\left\|\tilde\rho^J_i\right\|_{L^q(\mathbb{R}^3)} \le C(\gamma)\left(\left\|\frac{J^2}{2I(\tilde\rho^J)^2}|Px|^2\right\|^{\frac{1}{\gamma-1}}_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))} + \left\|\Phi_{\tilde\rho^J}\right\|^{\frac{1}{\gamma-1}}_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))}\right)$$

for some constant $C(\gamma) > 0$ depending only on γ. Recall that since $\tilde\rho^J \in \overline{\mathcal{W}}^J_{m_1,m_2}$, there exist constant $c_1, c_2, c_3 > 0$ independent of J satisfying $c_1 J^2 \le |Px| \le c_2 J^2$ for $x \in B(J^2\mathbf{x}_i, J^2 r_i)$ and $c_3 J^2 \le |x-y|$ for $x \in B(J^2\mathbf{x}_i, J^2 r_i)$, $y \in B(J^2\mathbf{x}_j, J^2 r_j)$, $i \neq j$. In particular, this shows that as $J \to \infty$,

$$\begin{aligned}\left\|\frac{J^2}{2I(\tilde\rho^J)^2}|Px|^2\right\|_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))} &\le \frac{c_2^2}{2c_1^4(m_1+m_2)^2 J^2}\|\chi_{B(J^2\mathbf{x}_i,J^2 r_i)}\|_{L^{\frac{q}{\gamma-1}}} \le \frac{c_2^2}{2c_1^4(m_1+m_2)^2 J^2}\left(\frac{4\pi}{3}J^6 r_i^3\right)^{\frac{\gamma-1}{q}} \\ &= O(J^{\frac{6(\gamma-1)}{q}-2}), \\ \left\|\Phi_{\tilde\rho^J}\right\|_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))} &\le \left\|\Phi_{\tilde\rho^J_i}\right\|_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))} + \left\|\Phi_{\tilde\rho^J_j}\right\|_{L^{\frac{q}{\gamma-1}}(B(J^2\mathbf{x}_i,J^2 r_i))} \qquad (i \neq j) \\ &\le \left\|\Phi_{\tilde\rho^J_i}\right\|_{L^{\frac{q}{\gamma-1}}(\mathbb{R}^3)} + \frac{m_j}{c_3 J^2}\left(\frac{4\pi}{3}J^6 r_i^3\right)^{\frac{\gamma-1}{q}}, \\ &\le C\left\|\tilde\rho^J_i\right\|_{L^{\frac{3q}{3(\gamma-1)+2q}}(\mathbb{R}^3)} + O(J^{\frac{6(\gamma-1)}{q}-2})\end{aligned}$$

where we used the Hardy-Littlewood-Sobolev inequality in the last line. This implies that as $J \to \infty$,

$$\|\tilde{\rho}^J\|_{L^q(\mathbb{R}^3)} \le C \left\|\tilde{\rho}^J\right\|^{\frac{1}{\gamma-1}}_{L^{\frac{3q}{3(\gamma-1)+2q}}(\mathbb{R}^3)} + O(J^{\frac{6}{q}-\frac{2}{\gamma-1}}). \tag{3.3.4}$$

Observe that $\frac{6}{q} - \frac{2}{\gamma-1} < 0$ for $q \ge 3$ since $4/3 < \gamma < 2$.

We now claim that there exists $p > 3/2$ such that

$$\limsup_{J\to\infty} \|\tilde{\rho}^J\|_{L^p(\mathbb{R}^3)} < \infty.$$

We divide the proof into the cases that (*i*): $\gamma > 3/2$ and (*ii*): $4/3 < \gamma \le 3/2$. We first assume (*i*). In this case, each $\tilde{\rho}^J$ belongs to L^γ, where $\gamma > 3/2$. For the uniform estimate, observe that

$$\int_{\mathbb{R}^3} A(\tilde{\rho}^J)\, dx = -\frac{J^2}{2I(\tilde{\rho}^J)^2} - \int_{\mathbb{R}^3} \Phi_{\tilde{\rho}^J} \tilde{\rho}^J\, dx + E^J(\tilde{\rho}^J).$$

As we have seen earlier, there holds that $J^2/I(\tilde{\rho}^J)^2 = O(J^{-2})$. Then by Lemma 3.3.7 and the Hardy-Littlewood-Sobolev inequality again, we get

$$\frac{1}{\gamma-1}\|\tilde{\rho}^J\|^\gamma_{L^\gamma(\mathbb{R}^3)} \le o(1) + C\|\tilde{\rho}^J\|^2_{L^{6/5}(\mathbb{R}^3)} \tag{3.3.5}$$

for some constant $C > 0$ independent of J. We now apply Hölder inequality to $\int (\tilde{\rho}^J)^{\frac{6}{5}} dx$

$$\int (\tilde{\rho}^J)^{\frac{6}{5}} dx = \int (\tilde{\rho}^J)^{\frac{\gamma}{5(\gamma-1)}} (\tilde{\rho}^J)^{\frac{6}{5}-\frac{\gamma}{5(\gamma-1)}} dx \le \left(\int (\tilde{\rho}^J)^\gamma dx\right)^{\frac{1}{5(\gamma-1)}} \left(\int \tilde{\rho}^J dx\right)^{\frac{5\gamma-6}{5\gamma-5}}$$

and thus

$$\|\tilde{\rho}^J\|^2_{L^{6/5}(\mathbb{R}^3)} \le \left(\int (\tilde{\rho}^J)^\gamma dx\right)^{\frac{1}{3(\gamma-1)}} (m_1 + m_2)^{\frac{5(5\gamma-6)}{3(\gamma-1)}}. \tag{3.3.6}$$

Now if $\gamma > \frac{4}{3}$, $\frac{1}{3(\gamma-1)} < 1$. So by Young's inequality, the last term of (3.3.5) is bounded by

$$C\|\tilde{\rho}^J\|^2_{L^{6/5}(\mathbb{R}^3)} \le \kappa\|\tilde{\rho}^J\|^\gamma_{L^\gamma(\mathbb{R}^3)} + C_{\kappa, m_1+m_2}$$

We now fix $\kappa = \frac{1}{2(\gamma-1)}$ and then from (3.3.5), we deduce that

$$\limsup_{J\to\infty} \|\tilde{\rho}^J\|_{L^\gamma(\mathbb{R}^3)} < \infty.$$

Next, assume (*ii*). By the same reasoning, we also have

$$\limsup_{J\to\infty} \|\tilde{\rho}^J\|_{L^{\gamma'}(\mathbb{R}^3)} < \infty$$

for any $1 \le \gamma' \le \gamma$. We take $q \ge 3$ in (3.3.4) as follows:

$$\frac{3q}{3(\gamma-1)+2q} = \begin{cases} \gamma & \text{if } \gamma < 3/2, \\ 24/19\,(< 3/2) & \text{if } \gamma = 3/2. \end{cases}$$

Then it is easy to see that $q \geq 4$ for any $4/3 < \gamma \leq 3/2$. This with (3.3.4) proves (*ii*) of the claim.

The next step is to obtain the uniform L^∞ estimate of $\Phi_{\tilde\rho^J}$ and $\tilde\rho^J$. The former is immediate from the following estimate:

$$\begin{aligned}|\Phi_{\tilde\rho^J}(x)| &= \int_{B(x,1)} \frac{1}{|x-y|}\tilde\rho^J(y)\,dy + \int_{B(x,1)^c} \frac{1}{|x-y|}\tilde\rho^J(y)\,dy \\ &\leq \left\| \frac{1}{|x-\cdot|} \right\|_{L^{p/(p-1)}(B(x,1))} \|\tilde\rho^J\|_{L^p} + m_1 + m_2.\end{aligned} \tag{3.3.7}$$

Note that since $p > 3/2$, the Hölder conjugate $p/(p-1) < 3$ so that

$$\limsup_{J\to\infty} \|\Phi_{\tilde\rho^J}\|_{L^\infty} < \infty.$$

Then for any $x \in \mathbb{R}^3$ and for sufficiently large $J_0 > 0$ and $J > J_0$, we use the increasing property of $(A')^{-1}$ to see

$$\begin{aligned}\tilde\rho_i^J(x) &\leq \sup\left\{(A')^{-1}(t) \mid t \in [0,\, J^2|Px|^2/(2I^2(\tilde\rho^J)) - \Phi_{\tilde\rho^J}(x)]\right\} \\ &\leq \sup\left\{(A')^{-1}(t) \mid t \in [0,\, 1 + \limsup_{J\to\infty} \|\Phi_{\tilde\rho^J}\|_{L^\infty}]\right\} \\ &= (A')_+^{-1}(1 + \limsup_{J\to\infty} \|\Phi_{\tilde\rho^J}\|_{L^\infty}),\end{aligned}$$

which shows the latter

$$\limsup_{J\to\infty} \|\tilde\rho^J\|_{L^\infty} < \infty.$$

Now, we are ready to obtain the uniform gradient estimate for $\tilde\rho^J$. By denoting $\beta(t) = (A')_+^{-1}(t)$, note that β is continuously differentiable and $\beta'(t)$ is strictly increasing on $t \geq 0$. We again use the equation (3.3.3) to get

$$\|\nabla\tilde\rho_i^J\|_{L^\infty} \leq \beta'(1 + \limsup_{J\to\infty} \|\Phi_{\tilde\rho^J}\|_{L^\infty})\|\nabla\Phi_{\tilde\rho^J}\|_{L^\infty}$$

for sufficiently large J. Thus to end the whole proof of the lemma, it remains to show the uniform L^∞ estimate for $\nabla\Phi_{\tilde\rho^J}$. Arguing similarly with the estimate (3.3.7), we see that

$$\begin{aligned}|\nabla\Phi_{\tilde\rho^J}(x)| &= \int_{B(x,1)} \frac{1}{|x-y|^2}\tilde\rho^J(y)\,dy + \int_{B(x,1)^c} \frac{1}{|x-y|^2}\tilde\rho^J(y)\,dy \\ &\leq \left\| \frac{1}{|x-\cdot|^2} \right\|_{L^1(B(x,1))} \|\tilde\rho^J\|_{L^\infty} + m_1 + m_2.\end{aligned}$$

This completes the proof.

With the uniform bound and continuity result of the density profiles, we are now ready to prove that the asymptotic profile of the density of the binary stars is given by the density of non-rotating Lane-Emden stars.

Lemma 3.3.9 *Define* $\hat{\rho}_i^J := \tilde{\rho}_i^J(\cdot + \bar{x}(\tilde{\rho}_i^J))$. *Then for all* $i = 1, 2$, *one has*

$$\lim_{J\to\infty}\left(\|\hat{\rho}_i^J - \tilde{\rho}_{m_i}^\infty\|_{L^\infty(\mathbb{R}^3)} + \|\Phi_{\hat{\rho}_i^J} - \Phi_{\tilde{\rho}_{m_i}^\infty}\|_{\dot{H}^1(\mathbb{R}^3)}\right) = 0.$$

In particular,

$$\lim_{J\to\infty}\int A(\hat{\rho}_i^J)\,dx = \int A(\tilde{\rho}_{m_i}^\infty)\,dx \quad \textit{and} \quad \lim_{J\to\infty}\int \Phi_{\hat{\rho}_i^J}\hat{\rho}_i^J\,dx = \int \Phi_{\tilde{\rho}_{m_i}^\infty}\tilde{\rho}_{m_i}^\infty\,dx.$$

Proof We invoke Proposition 6.4 in [40] (See also Remark 3.3.2) which asserts that as $J \to \infty$,

$$E^\infty(\hat{\rho}_i^J) = E^\infty(\tilde{\rho}_i^J) \to E^\infty(\tilde{\rho}_{m_i}^\infty), \quad i = 1, 2.$$

Then $\{\hat{\rho}_i^J\}$ is a minimizing sequence of the limit variational problem (3.1.1) with $m = m_i$. By Lemma 3.3.4, we see that $\mathrm{spt}(\hat{\rho}_i^J) \subset B_R$ for large J. Then by following the proof of Theorem 3.1 and Lemma 3.2 in [43], after choosing a subsequence, $\{\hat{\rho}_i^J\}$ weakly converges in $L^\gamma(\mathbb{R}^3)$ to a minimizer $\hat{\rho}_i^\infty$ of the limit variational problem (3.1.1) with $m = m_i$ and $\{\Phi_{\hat{\rho}_i^J}\}$ strongly converges to $\Phi_{\hat{\rho}_i^\infty}$ in $\dot{H}^1(\mathbb{R}^3)$ as $J \to \infty$.

We see from Lemma 3.3.8 that $\{\hat{\rho}^J\}$ is equicontinuous and has a uniform L^∞ bound. Then the Arzela-Ascoli theorem says after choosing a subsequence, $\{\hat{\rho}^J\}$ strongly converges in L^∞ to $\hat{\rho}_i^\infty$. (L^∞ convergent limit of $\{\hat{\rho}_i^J\}$ must be $\hat{\rho}_i^J$ because L^∞ convergence implies L^γ weak convergence on a bounded domain.)

Also, observe from the L^∞ convergence of $\hat{\rho}_i^J$ that up to a subsequence,

$$0 = \bar{x}(\hat{\rho}_i^J) = \frac{1}{m_i}\int_{B_R} x\hat{\rho}_i^J\,dx \to \frac{1}{m_i}\int_{B_R} x\hat{\rho}_i^\infty\,dx = \bar{x}(\hat{\rho}_i^\infty) \quad \text{as } J \to \infty.$$

This shows that $\hat{\rho}_i^\infty$ is the same with $\tilde{\rho}_{m_i}^\infty$ by the uniqueness of a minimizer of the problem (3.1.1) up to a translation.

Finally, it remains to upgrade the above subsequential convergence to the continuum convergence for J. As in the proof of Lemma 3.3.6, this can be done by taking advantage of the uniqueness of $\tilde{\rho}_{m_i}^\infty$. Suppose that there exists a positive number $\varepsilon_0 > 0$ and a sequence $\{J_k\} \to \infty$ such that

$$\|\hat{\rho}_i^{J_k} - \tilde{\rho}_{m_i}^\infty\| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}. \tag{3.3.8}$$

Since $\{\hat{\rho}_i^{J_k}\}$ is still a minimizing sequence of the problem (3.1.1) and has a uniform $W^{1,\infty}$ bound, we can apply the exactly same reasoning to see that after choosing a subsequence, $\{\hat{\rho}_i^{J_k}\}$ converges in L^∞ to a minimizer $\hat{\hat{\rho}}_i^\infty$ of the problem (3.1.1) satisfying $\bar{x}(\hat{\hat{\rho}}_i^\infty) = 0$. However this is a contradiction to the uniqueness of $\tilde{\rho}_{m_i}^\infty$ and (3.3.8).

The contradiction argument above shows $\lim_{J\to\infty}\|\hat{\rho}_i^J - \tilde{\rho}_{m_i}^\infty\|_{L^\infty} = 0$. Combining this with the Gagliardo-Nirenberg inequality and the Hardy-Littlewood-Sobolev inequality, we also see

$$\lim_{J\to\infty}\|\Phi_{\hat{\rho}_i^J} - \Phi_{\tilde{\rho}_{m_i}^\infty}\|_{\dot{H}^1(\mathbb{R}^3)} = 0.$$

This completes the proof.

We finish the proof of Proposition 3.3.1 by proving the following lemma.

Lemma 3.3.10 *For any $i = 1, 2$, there holds that*

$$\lim_{J\to\infty} C_i^J = C_{m_i}^\infty.$$

Proof Multiplying the first equation in (2.2.5) by $\tilde{\rho}_i^J$ and integrating, we get

$$\int_{\{\tilde{\rho}_i^J>0\}} -\frac{J^2}{2I(\tilde{\rho}^J)^2}|Px|^2\tilde{\rho}_i^J + A'(\tilde{\rho}_i^J)\tilde{\rho}_i^J + \Phi_{\tilde{\rho}^J}\tilde{\rho}_i^J\,dx = -m_iC_i^J. \tag{3.3.9}$$

We have already seen that as $J \to \infty$,

$$\int_{\{\tilde{\rho}_i^J>0\}} -\frac{J^2}{2I(\tilde{\rho}^J)^2}|Px|^2\tilde{\rho}_i^J + \Phi_{\tilde{\rho}_j^J}\tilde{\rho}_i^J\,dx = o(1), \quad i \neq j.$$

Then combining (3.3.9) and Lemma 3.3.9, we see that

$$\begin{aligned}-m_iC_i^J &= \gamma\int A(\tilde{\rho}_{m_i}^\infty)\,dx + \Phi_{\tilde{\rho}_{m_i}^\infty}\tilde{\rho}_{m_i}^\infty\,dx + o(1)\\ &= -m_iC_{m_i}^\infty + o(1),\end{aligned}$$

which ends the proof.

4 Uniqueness result

In this section, we prove some uniqueness result (Theorem 4.0.4) a bit more general than Theorem 2.3.2.(*ii*). We will see that Theorem 4.0.4 also play an important role to prove Theorem 2.3.2.(*iii*). To this end, we first give the expression of Lagrange multipliers C_i^J in terms of $\tilde{\rho}_1^J$ and $\tilde{\rho}_2^J$ for $i = 1, 2$.

Define

$$\begin{aligned}F(\phi,\psi) :=& \frac{5\gamma-6}{4-3\gamma}E^\infty(\phi) + \frac{8-5\gamma}{2(4-3\gamma)}\frac{J^2}{I(\phi+\psi)^2}\int|Px|^2\phi\,dx\\ &+ \frac{2-\gamma}{4-3\gamma}\iint\frac{(x-y)\cdot x}{|x-y|^3}\phi(x)\psi(y)\,dxdy + \iint\frac{\phi(x)\psi(y)}{|x-y|}\,dxdy\end{aligned} \tag{4.0.1}$$

for $\phi, \psi \in L^\gamma$ with compact support.

Lemma 4.0.1 *Let $\{\tilde{\rho}^J\} \in \overline{\mathcal{W}}_{m_1,m_2}^J$ be a family of normalized minimizers of the minimization problem* (2.2.4). *Then one has*

$$m_1C_1^J = F(\tilde{\rho}_1^J, \tilde{\rho}_2^J) \quad \textit{and} \quad m_2C_2^J = F(\tilde{\rho}_2^J, \tilde{\rho}_1^J).$$

Proof For a given ρ, we denote by ρ^t a family of functions $\frac{1}{t^3}\rho(\frac{\cdot}{t})$. By Lemmas 3.3.4 and 3.3.6, we see that $(\tilde{\rho}_1^J)^t + \tilde{\rho}_2^J$ belongs to $\overline{\mathcal{W}}_{m_1,m_2}^J$ for $t \sim 1$ so that $\frac{d}{dt}|_{t=1}E^J((\tilde{\rho}_1^J)^t + \tilde{\rho}_2^J) = 0$. To compute this, observe that

$$\bar{x}((\tilde{\rho}_1^J)^t + \tilde{\rho}_2^J) = \frac{m_1t\bar{x}(\tilde{\rho}_1^J) + m_2\bar{x}(\tilde{\rho}_2^J)}{m_1+m_2}$$

and then

$$I((\tilde\rho_1^J)^t+\tilde\rho_2^J) = \int |P(tx-\frac{m_1 t\bar{x}(\tilde\rho_1^J)+m_2\bar{x}(\tilde\rho_2^J)}{m_1+m_2})|^2\tilde\rho_1^J\,dx+\int |P(x-\frac{m_1 t\bar{x}(\tilde\rho_1^J)+m_2\bar{x}(\tilde\rho_2^J)}{m_1+m_2})|^2\tilde\rho_2^J\,dx.$$

Thus one has

$$\begin{aligned}
&\frac{d}{dt}|_{t=1}I((\tilde\rho_1^J)^t+\tilde\rho_2^J)\\
&\quad= 2\int P(x-\bar{x}(\tilde\rho^J))\cdot P(x-\frac{m_1\bar{x}(\tilde\rho_1^J)}{m_1+m_2})\tilde\rho_1^J\,dx + 2\int P(x-\bar{x}(\tilde\rho^J))\cdot P(-\frac{m_1\bar{x}(\tilde\rho_1^J)}{m_1+m_2})\tilde\rho_2^J\,dx\\
&\quad= 2\int |Px|^2\tilde\rho_1^J\,dx,
\end{aligned}$$

where we used the properties $\bar{x}(\tilde\rho^J)=0$ and $m_1\bar{x}(\tilde\rho_1^J)+m_2\bar{x}(\tilde\rho_2^J)=0$. This shows that

$$\begin{aligned}
0 &= \frac{d}{dt}|_{t=1}E^J((\tilde\rho_1^J)^t+\tilde\rho_2^J)\\
&= 3(1-\gamma)\int A(\tilde\rho_1^J)\,dx - \frac{J^2}{I(\tilde\rho^J)^2}\int |Px|^2\tilde\rho_1^J\,dx \qquad (4.0.2)\\
&\quad+\frac{1}{2}\iint \frac{\tilde\rho_1^J(x)\tilde\rho_1^J(y)}{|x-y|}\,dxdy + \iint\frac{(x-y)\cdot x}{|x-y|^3}\tilde\rho_1^J(x)\tilde\rho_2^J(y)\,dxdy.
\end{aligned}$$

Multiplying the first equation of (2.2.5) by $\tilde\rho_1^J$, integrating and combining with (4.0.2), we next get

$$\begin{aligned}
&m_1C_1^J\\
&= -\gamma\int A(\tilde\rho_1^J)\,dx + \iint\frac{\tilde\rho_1^J(x)\tilde\rho_1^J(y)}{|x-y|}\,dxdy + \frac{J^2}{2I(\tilde\rho^J)^2}\int |Px|^2\tilde\rho_1^J\,dx + \iint\frac{\tilde\rho_1^J(x)\tilde\rho_2^J(y)}{|x-y|}\,dxdy\\
&= (5\gamma-6)\int A(\tilde\rho_1^J)\,dx + \frac{5J^2}{2I(\tilde\rho^J)^2}\int |Px|^2\tilde\rho_1^J\,dx + \iint\left(\frac{1}{|x-y|}-\frac{2(x-y)\cdot x}{|x-y|^3}\right)\tilde\rho_1^J(x)\tilde\rho_2^J(y)\,dxdy
\end{aligned}$$

and using (4.0.2) again, one also has

$$\begin{aligned}
&\int A(\tilde\rho_1^J)\,dx\\
&= \frac{1}{4-3\gamma}E^\infty(\tilde\rho_1^J) - \frac{1}{4-3\gamma}\iint\frac{(x-y)\cdot x}{|x-y|^3}\tilde\rho_1^J(x)\tilde\rho_2^J(y)\,dxdy + \frac{J^2}{(4-3\gamma)I(\tilde\rho^J)^2}\int |Px|^2\tilde\rho_1^J\,dx.
\end{aligned}$$

Therefore, we finally see that $m_1C_1^J = F(\tilde\rho_1^J,\tilde\rho_2^J)$. By changing the role of $\tilde\rho_1^J$ and $\tilde\rho_2^J$, it is straightforward to check $m_2C_2^J = F(\tilde\rho_2^J,\tilde\rho_1^J)$.

Remark 4.0.2 *The integral representation of C_i^J in Lemma 4.0.1 plays a crucial role in its quantitative C^2 convergence result to the limiting potential $C_{m_i}^\infty$ (cf Lemma 4.2.1). This convergence result will be importantly used to prove the uniqueness result (cf. Theorem 4.0.4). Indeed, a solution to the variational problem* (2.2.4) *satisfies the self-consistent equations* (2.2.5) *which involve the chemical potential C_i^J on each connected component, and thus the quantitative control of the convergence of C_i^J as in Lemma 4.2.1 will be useful to analyze the perturbed equations around the asymptotic Lane-Emden profiles.*

Lemma 4.0.3 *Let* $\{\tilde\rho^J\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ *be a normalized minimizers to the variational problem* (2.2.4). *Then it solves*

$$\begin{cases} \rho_i^J(x) = B_\gamma \left(\frac{1}{2}\frac{J^2}{I(\rho^J)^2}|Px|^2 + (\frac{1}{|\cdot|} * \rho^J)(x) - \frac{1}{m_i}F(\rho_i^J, \rho_j^J)\right)_+^{\frac{1}{\gamma-1}} \quad \forall x \in B(J^2\mathbf{x}_i, J^2 r_i),\ i = 1,2,\ j \neq i, \\ m_i \bar{x}_1(\rho_i^J) = \frac{I(\rho^J)^2}{J^2} \iint (-1)^{i+1} \frac{x_1 - y_1}{|x-y|^3}\rho_1^J(x)\rho_2^J(y)\, dxdy, \quad \bar{x}_3(\rho_i^J) = 0, \quad i = 1,2, \end{cases} \tag{4.0.3}$$

where $B_\gamma = (\frac{\gamma-1}{K\gamma})^{\frac{1}{\gamma-1}}$ *and* F *is given in* (4.0.1).

Proof By Lemma 3.3.5 and Lemma 4.0.1, it only remains to show

$$m_i \bar{x}_1(\rho_i^J) = \frac{I(\rho^J)^2}{J^2} \iint (-1)^{i+1}\frac{x_1 - y_1}{|x-y|^3}\rho_1^J(x)\rho_2^J(y)\, dxdy, \quad i = 1,2.$$

From a direct computation, we see that for $j \neq i$,

$$\begin{aligned} &\frac{m_i\bar{x}_1(\rho_i^J)J^2}{I(\rho^J)^2} - \iint (-1)^{i+1}\frac{x_1-y_1}{|x-y|^3}\rho_1^J(x)\rho_2^J(y)\, dxdy \\ &= \int \frac{J^2}{I(\rho^J)^2}x_1\rho_i^J\, dx - \int \partial_{x_1}\Phi_{\rho_j^J}\rho_i^J\, dx, \\ &= \int \partial_{x_1}(\frac{J^2}{2I(\rho^J)^2}x_1^2 - \Phi_{\rho_j^J})\rho_i^J\, dx \\ &= \int \partial_{x_1}(A'(\rho_i^J) - \frac{J^2}{2I(\rho^J)^2}x_2^2 + \Phi_{\rho_i^J} + C_i^J)\rho_i^J\, dx \\ &= \int \partial_{x_1}\Phi_{\rho_i^J}\rho_i^J\, dx = \int \partial_{x_1}\Phi_{\rho_i^J}\Delta\Phi_{\rho_i^J}\, dx = 0. \end{aligned}$$

This ends the proof.

Now we state the following uniqueness theorem which implies (*ii*) of Theorem 2.3.2.

Theorem 4.0.4 *Let* $\{\rho^J\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ *be a family of normalized solutions to* (4.0.3). *Suppose that there exists a constant* $R > 0$ *independent of* J *such that*

$$spt(\rho_1^J) \subset B(\bar{x}(\rho_1^J), R), \quad spt(\rho_2^J) \subset B(\bar{x}(\rho_2^J), R)$$

and there hold the following convergences:

$$\begin{cases} \lim_{J\to\infty} \|\rho_1^J(\cdot + \bar{x}(\rho_1^J)) - \tilde\rho^\infty_{m_1}\|_{L^\infty} + \|\rho_2^J(\cdot + \bar{x}(\rho_2^J)) - \tilde\rho^\infty_{m_2}\|_{L^\infty} = 0, \\ \lim_{J\to\infty}\frac{\bar{x}(\rho_1^J)}{J^2} = (\frac{1}{m_1^2 m_2}, 0, 0), \quad \lim_{J\to\infty}\frac{\bar{x}(\rho_2^J)}{J^2} = -(\frac{1}{m_1 m_2^2}, 0, 0), \\ \lim_{J\to\infty}\frac{F(\rho_1^J, \rho_2^J)}{m_1} = C^\infty_{m_1}, \quad \lim_{J\to\infty}\frac{F(\rho_2^J,\rho_1^J)}{m_2} = C^\infty_{m_2}. \end{cases}$$

Then such a family $\{\rho^J\}$ *is unique for sufficiently large* $J > 0$.

The proof of Theorem 4.0.4 involves several steps and technical details. We first give a brief overview of the proof in Section 4.1 and the full proof in Section 4.2.

4.1 Overview of the proof

The purpose of this subsection is to deliver fundamental ideas and abstract settings employed in the proof of Theorem 4.0.4. We first encode the system of equations (4.0.3) into a functional equation

$$u = N^J(u) \tag{4.1.1}$$

for some nonlinear map $N^J : X \to Y$, where X and Y are Banach spaces such that X is continuously embedded in Y. The second step of the proof (Lemma 4.2.1-Lemma 4.2.5) is to obtain the various regularity estimates for N^J including the equicontinuity of N^J and ∇N^J and their uniform convergence to a limit map N^∞. Then our task will be to show that for sufficiently large J, a family of solutions $\{u_J\}$ to (4.1.1) is unique whenever it converges in a suitable sense to a limit $u_\infty \in X$ which is a solution of the limit equation $u = N^\infty(u)$.

The main idea of the proof is as follows. Denoting $u_J = v_J + u_\infty$ and using Taylor expansion, one can see that the equation (4.1.1) is equivalent to

$$v_J - \nabla N^J(u_\infty)[v_J] = \mathcal{R}^J(v_J) + N^J(u_\infty) - N^\infty(u_\infty), \tag{4.1.2}$$

where $\mathcal{R}^J(v_J)$ is the super-linear remainder term in Taylor expansion. The key ingredient of our argument is the following non-degenerate estimate (Lemma 4.2.6)

$$\liminf_{J\to\infty} \inf_{\|\phi\|_X=1} \|\phi - \nabla N^J(u_\infty)[\phi]\|_Y > 0. \tag{4.1.3}$$

The proof of the estimate (4.1.3) strongly relies on the aforementioned regularity estimates on N^J and non-degeneracy of the limit map N^∞ (Theorem 3.1.1.(v) and Proposition 3.2.1.(ii)). With this estimate in hand, suppose that (4.1.2) admits two different families of solutions $\{v_J^1\}$ and $\{v_J^2\}$. Then it follows that

$$(v_J^1 - v_J^2) - \nabla N^J(u_\infty)[v_J^1 - v_J^2] = \mathcal{R}^J(v_J^1) - \mathcal{R}^J(v_J^2).$$

The non-degenerate estimate implies that for large J, there exists a universal constant $c > 0$ such that

$$c\|v_J^1 - v_J^2\|_X \le \|(v_J^1 - v_J^2) - \nabla N^J(u_\infty)[v_J^1 - v_J^2]\|_Y$$

while it is possible to obtain the estimate (Lemma 4.2.7)

$$\|\mathcal{R}^J(v_J^1) - \mathcal{R}^J(v_J^2)\|_Y \le \frac{c}{2}\|v_J^1 - v_J^2\|_X$$

due to the convergence $\|v_J^1 - v_J^2\|_X \to 0$ and the super-linearity of the term $\mathcal{R}^J$. Then we get a contradiction and the uniqueness is proved.

4.2 Proof of Theorem 4.0.4

Let us denote $B_R := B(0,R)$. Define the function space

$$L_0^2(B_R) := \left\{\rho \in L^2(B_R) \mid \bar{x}(\rho) = 0\right\}$$

and intervals

$$I_1 = (\frac{1}{m_1^2 m_2} - r_1, \frac{1}{m_1^2 m_2} + r_1), \qquad I_2 = (-\frac{1}{m_1 m_2^2} - r_2, -\frac{1}{m_1 m_2^2} + r_2).$$

Let us denote $\zeta_i = \bar{x}_1(\rho_i)/J^2$ and redefine as $\rho_i(x) \mapsto \rho_i(x - J^2\zeta_i\vec{e}_{x_1})$. Then for any normalized solution $\rho \in \overline{\mathcal{W}}^J_{m_1,m_2}$ of (4.0.3), we may write

$$\rho(x) = \rho_1(x - J^2\zeta_1\vec{e}_{x_1}) + \rho_2(x - J^2\zeta_2\vec{e}_{x_1}),$$

where $\zeta_1 \in I_1$, $\zeta_2 \in I_2$ and $\rho_1, \rho_2 \in L_0^2(B_R)$. With this transformation, the equations (4.0.3) transforms into

$$(\rho_1, \rho_2, \zeta_1, \zeta_2)^T = N^J(\rho_1, \rho_2, \zeta_1, \zeta_2),$$

where $(\rho_1, \rho_2, \zeta_1, \zeta_2) \in L_0^2(B_R) \times L_0^2(B_R) \times I_1 \times I_2$ and $N^J := (N_1^J, N_2^J, N_3^J, N_4^J)^T$ such that

$$\begin{cases} N_1^J(\rho_1,\rho_2,\zeta_1,\zeta_2) \\ := B_\gamma\left(\dfrac{J^2|P(x+J^2\zeta_1\vec{e}_{x_1})|^2}{2\left(I(\rho_1)+I(\rho_2)+\frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2}\right)^2} + \left(\dfrac{1}{|\cdot|} * \rho_1\right)(x) + \left(\dfrac{1}{|\cdot|} * \rho_2\right)(x + J^2(\zeta_1-\zeta_2)\vec{e}_{x_1}) - \dfrac{1}{m_1}F_1^J\right)_+^{\frac{1}{\gamma-1}}, \\ N_2^J(\rho_1,\rho_2,\zeta_1,\zeta_2) \\ := B_\gamma\left(\dfrac{J^2|P(x+J^2\zeta_2\vec{e}_{x_1})|^2}{2\left(I(\rho_1)+I(\rho_2)+\frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2}\right)^2} + \left(\dfrac{1}{|\cdot|} * \rho_2\right)(x) + \left(\dfrac{1}{|\cdot|} * \rho_1\right)(x + J^2(\zeta_2-\zeta_1)\vec{e}_{x_1}) - \dfrac{1}{m_2}F_2^J\right)_+^{\frac{1}{\gamma-1}}, \\ N_3^J(\rho_1,\rho_2,\zeta_1,\zeta_2) \\ := \dfrac{(I(\rho_1)+I(\rho_2)+\frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2})^2}{m_1 J^4} \displaystyle\iint \dfrac{x_1 - y_1 + J^2(\zeta_1-\zeta_2)}{|x-y+J^2(\zeta_1-\zeta_2)\vec{e}_{x_1}|^3}\rho_1(x)\rho_2(y)\,dxdy, \\ N_4^J(\rho_1,\rho_2,\zeta_1,\zeta_2) \\ := -\dfrac{(I(\rho_1)+I(\rho_2)+\frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2})^2}{m_2 J^4} \displaystyle\iint \dfrac{x_1 - y_1 + J^2(\zeta_1-\zeta_2)}{|x-y+J^2(\zeta_1-\zeta_2)\vec{e}_{x_1}|^3}\rho_1(x)\rho_2(y)\,dxdy. \end{cases}$$

Here F_1^J and F_2^J are given by

$$\begin{aligned} F_1^J(\rho_1,\rho_2,\zeta_1,\zeta_2) &:= F(\rho_1(x - J^2\zeta_1\vec{e}_{x_1}), \rho_2(x - J^2\zeta_2\vec{e}_{x_1})), \\ F_2^J(\rho_1,\rho_2,\zeta_1,\zeta_2) &:= F(\rho_2(x - J^2\zeta_2\vec{e}_{x_1}), \rho_1(x - J^2\zeta_1\vec{e}_{x_1})). \end{aligned} \tag{4.2.1}$$

Let us define

$$\mathfrak{X}_0 := L_0^2(B_R) \times L_0^2(B_R) \times \mathbb{R} \times \mathbb{R}, \quad \mathfrak{X} := L^2(B_R) \times L^2(B_R) \times \mathbb{R} \times \mathbb{R}$$

so that the standard inclusion map $I : \mathfrak{X}_0 \to \mathfrak{X}$ is well defined. For $\xi = (\rho_1, \rho_2, \zeta_1, \zeta_2) \in \mathfrak{X}$, we denote

$$\|\xi\| = \max\Big\{ \|\rho_1\|_{L^2},\ \|\rho_2\|_{L^2},\ |\zeta_1|,\ |\zeta_2| \Big\}.$$

For Banach spaces X, Y and a linear operator A from X to Y, we denote by $\|A\|_{X\to Y}$, the operator norm of A. If A is from $\mathfrak{X}_0$ to $\mathfrak{X}$, then we just denote $\|A\|_{\mathfrak{X}_0\to\mathfrak{X}}$ by $\|A\|$.

Lemma 4.2.1 *For $i = 1, 2$, the functional F_i^J given in (4.2.1) is C^1 on $L_0^2(B_R)\times L_0^2(B_R)\times I_1\times I_2$. Moreover, if we define*

$$S_i^J(\xi) := F_i^J(\xi) - \frac{5\gamma-6}{4-3\gamma}E^\infty(\rho_i), \quad \xi = (\rho_1, \rho_2, \zeta_1, \zeta_2),$$

then S_i^J is C^2 and there exists a constant C depending only on R, r_1, r_2, m_1, m_2 and γ such that

$$|S_i^J(\xi)| + \|\nabla S_i^J(\xi)\|_{\mathfrak{X}_0\to\mathbb{R}} + \|\nabla^2 S_i^J(\xi)\|_{\mathfrak{X}_0\times\mathfrak{X}_0\to\mathbb{R}} \le \frac{C}{J^2}(1+\|\xi\|^2). \tag{4.2.2}$$

In particular, $F_i^J(\xi)$ converges to $\frac{5\gamma-6}{4-3\gamma}E^\infty(\rho_i)$ in C^2 topology.

Proof By the symmetry, we may assume $i = 1$. It is clear that $E^\infty(\rho_1)$ is C^1. We note that S_1^J is given by

$$\begin{aligned} S_1^J =& \frac{8-5\gamma}{2(4-3\gamma)} \frac{J^2\left(I(\rho_1) + J^4\zeta_1^2\int\rho_1\,dx\right)}{\left(I(\rho_1)+I(\rho_2)+\frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2}\right)^2} \\ &+ \frac{2-\gamma}{4-3\gamma}\iint \frac{(x-y+J^2(\zeta_1-\zeta_2)\vec{e}_{x_1})\cdot(x+J^2\zeta_1\vec{e}_{x_1})}{|x-y+J^2(\zeta_1-\zeta_2)\vec{e}_{x_1}|^3}\rho_1(x)\rho_2(y)\,dxdy \\ &+ \iint \frac{\rho_1(x)\rho_2(y)}{|x-y+J^2(\zeta_1-\zeta_2)\vec{e}_{x_1}|}\,dxdy. \end{aligned}$$

It is easy to see that for any $\rho \in L^2(B_R)$, there exists a constant $C(R)$ depending only on R such that

$$\left(\frac{1}{|\cdot|} * \rho\right)(x) \le \frac{C(R)}{1+|x|}\|\rho\|_{L^2} \quad \text{for all } x \in \mathbb{R}^3. \tag{4.2.3}$$

Since $\mathrm{spt}(\rho_i) \in B_R$, $|\zeta_1| \le \frac{1}{m_1^2 m_2} + r_1$, $|\zeta_2| \le \frac{1}{m_1 m_2^2} + r_2$ and $|\zeta_1 - \zeta_2| \ge \frac{m_1+m_2}{m_1^2 m_2^2} - r_1 - r_2 > 0$, we see from (4.2.3) that the estimate (4.2.2) holds true for $k = 0$. To obtain the estimates (4.2.2) for $k = 1, 2$, we need to show

$$\left|\nabla S_1^J(\rho_1, \rho_2, \zeta_1, \zeta_2)(\eta_1, \eta_2, z_1, z_2)\right| \le \frac{C}{J^2}(1+\|\rho_1\|_{L^2}^2 + \|\rho_2\|_{L^2}^2)\|(\eta_1, \eta_2, z_1, z_2)\|,$$

$$\left|\left\langle\nabla^2 S_1^J(\rho_1, \rho_2, \zeta_1, \zeta^2)(\eta_1, \eta_2, z_1, z_2),\ (\eta_1, \eta_2, z_1, z_2)\right\rangle\right| \le \frac{C}{J^2}(1+\|\rho_1\|_{L^2}^2 + \|\rho_2\|_{L^2}^2)\|(\eta_1, \eta_2, z_1, z_2)\|^2.$$

This can be done by computing

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} S_1^J(\rho_1 + t\eta_1, \rho_2 + t\eta_2, \zeta_1 + tz_1, \zeta_2 + tz_2), \\ &\frac{d^2}{dt^2}\Big|_{t=0} S_1^J(\rho_1 + t\eta_1, \rho_2 + t\eta_2, \zeta_1 + tz_1, \zeta_2 + tz_2) \end{aligned}$$

and applying the similar arguments. Since the computations are similar and tedious, we omit them.

We define for $i = 1, 2$,

$$X_i^J(\rho_1, \rho_2, \zeta_1, \zeta_2) :=$$
$$\frac{J^2|P(x + J^2\zeta_i\vec{e}_{x_1})|^2}{2\left(I(\rho_1) + I(\rho_2) + \frac{J^4 m_1 m_2(\zeta_1-\zeta_2)^2}{m_1+m_2}\right)^2} + \left(\frac{1}{|\cdot|} * \rho_i\right)(x) + \left(\frac{1}{|\cdot|} * \rho_j\right)(x + J^2(\zeta_i - \zeta_j)\vec{e}_{x_1}), \quad j \neq i.$$

Arguing similarly with the previous lemma, we can also see there hold the following two lemmas, the proofs of which are omitted.

Lemma 4.2.2 *The map X_i^J is C^2 from $L_0^2(B_R) \times L_0^2(B_R) \times I_1 \times I_2$ to $L^\infty(B_R)$. Moreover, if we denote*

$$X_i^J(\xi) = \left(\frac{1}{|\cdot|} * \rho_i\right) + Q_i^J(\xi), \quad \xi = (\rho_1, \rho_2, \zeta_1, \zeta_2),$$

then there exists a constant $C > 0$ depending only on R, r_1, r_2, m_1, m_2 and γ such that

$$\|Q_i^J(\xi)\|_{L^\infty(B_R)} + \|\nabla Q_i^J(\xi)\|_{\mathfrak{X}_0 \to L^\infty(B_R)} + \|\nabla^2 Q_i^J(\xi)\|_{\mathfrak{X}_0 \times \mathfrak{X}_0 \to L^\infty(B_R)} \leq \frac{C}{J^2}(1 + \|\xi\|).$$

In particular, $X_i^J(\xi)$ converges to $\left(\frac{1}{|\cdot|} * \rho_i\right)$ in C^2 topology.

Remark 4.2.3 *From Lemma 4.2.1–4.2.2, we see that as $J \to \infty$, the limit map of N_i^J is*

$$N_i^\infty(\xi) := B_\gamma \left(\frac{1}{|\cdot|} * \rho_i - \frac{1}{m_i}\frac{5\gamma - 6}{4 - 3\gamma} E^\infty(\rho_i)\right)_+^{\frac{1}{\gamma-1}}, \quad i = 1, 2.$$

Lemma 4.2.4 *For $i = 3, 4$, the map N_i^J is C^2 from $L_0^2(B_R) \times L_0^2(B_R) \times I_1 \times I_2$ to $\mathbb{R}$. Moreover, if we denote*

$$N_i^J(\xi) = \frac{m_1^2 m_2^2}{(m_{i-2})(m_1 + m_2)^2}(\zeta_1 - \zeta_2)^2 \int \rho_1\, dx \int \rho_2\, dx + V_i^J(\xi), \quad \xi = (\rho_1, \rho_2, \zeta_1, \zeta_2),$$

then there exists a constant $C > 0$ depending only on R, r_1, r_2, m_1, m_2 and γ such that for $i = 3, 4$,

$$|V_i^J(\xi)| + \|\nabla V_i^J(\xi)\|_{\mathfrak{X}_0 \to \mathbb{R}} + \|\nabla^2 V_i^J(\xi)\|_{\mathfrak{X}_0 \times \mathfrak{X}_0 \to \mathbb{R}} \leq \frac{C}{J^2}(1 + \|\xi\|^2).$$

In particular, $N_i^J(\xi)$ converges to the limit map

$$N_i^\infty(\xi) := \frac{m_1^2 m_2^2}{(m_{i-2})(m_1 + m_2)^2}(\zeta_1 - \zeta_2)^2 \int \rho_1\, dx \int \rho_2\, dx, \quad i = 3, 4$$

in C^2 topology.

Now, we are ready to prove the uniform C^1 estimate for N^J.

Lemma 4.2.5 *The map N^J is C^1 from $L_0^2(B_R) \times L_0^2(B_R) \times I_1 \times I_2$ to $\mathfrak{X}$. Moreover, there exists a constant C independent of $\xi, \tilde{\xi}$ such that*

$$\begin{cases} \limsup_{J \to \infty} \left(\left\|N^J(\xi)\right\| + \left\|\nabla N^J(\xi)\right\|\right) \leq C\left(1 + \|\xi\|^2\right)^{\frac{2-\gamma}{\gamma-1}+1}, \\ \limsup_{J \to \infty} \left\|\nabla N^J(\xi) - \nabla N^J(\tilde{\xi})\right\| \leq C\left(1 + \|\xi\|^2 + \|\tilde{\xi}\|^2\right)^{\frac{2-\gamma}{\gamma-1}+1}\left(\|\xi - \tilde{\xi}\| + \|\xi - \tilde{\xi}\|^{\frac{2-\gamma}{\gamma-1}}\right) \end{cases}$$

for $\xi, \tilde{\xi} \in L_0^2(B_R) \times L_0^2(B_R) \times I_1 \times I_2$.

Proof We only prove the second estimate because the first one is easier to obtain. By denoting $f(t) = B_\gamma t_+^{1/(\gamma-1)}$, one has for $\phi \in \mathfrak{X}_0$

$$\nabla N_i^J(\xi)[\phi] = f'((X_i^J - F_i^J)(\xi))(\nabla(X_i^J - F_i^J)(\xi)[\phi]), \quad i = 1, 2.$$

Since $|f'(t) - f'(s)| \le C|t-s|^{\frac{2-\gamma}{\gamma-1}}$, we see from Lemmas 4.2.1 and 4.2.2 combined with the mean value theorem that for $i = 1, 2$,

$$\begin{aligned}
&\|\nabla N_i^J(\xi)[\phi] - \nabla N_i^J(\tilde{\xi})[\phi]\| \\
&\le \|f'((X_i^J - F_i^J)(\xi)) - f'((X_i^J - F_i^J)(\tilde{\xi}))\|_{L^\infty(B_R)}\|\nabla(X_i^J - F_i^J)(\xi)[\phi]\| \\
&\quad + \|f'((X_i^J - F_i^J)(\tilde{\xi}))\|_{L^\infty(B_R)}\|\nabla(X_i^J - F_i^J)(\xi)[\phi] - \nabla(X_i^J - F_i^J)(\tilde{\xi})[\phi]\| \\
&\le C(1 + \|\xi\|^2 + \|\tilde{\xi}\|^2)^{\frac{2-\gamma}{\gamma-1}+1}\|\xi - \tilde{\xi}\|^{\frac{2-\gamma}{\gamma-1}}\|\phi\| + C(1 + \|\xi\|^2 + \|\tilde{\xi}\|^2)^{\frac{2-\gamma}{\gamma-1}+1}\|\xi - \tilde{\xi}\|\,\|\phi\|.
\end{aligned}$$

Thus this implies

$$\|\nabla N_i^J(\xi) - \nabla N_i^J(\tilde{\xi})\|_{\mathfrak{X}_0 \to L^2(B_R)} \le C\left(1 + \|\xi\|^2 + \|\tilde{\xi}\|^2\right)^{\frac{2-\gamma}{\gamma-1}+1}\left(\|\xi - \tilde{\xi}\| + \|\xi - \tilde{\xi}\|^{\frac{2-\gamma}{\gamma-1}}\right) \quad i = 1, 2.$$

Finally, as for ∇N_3^J and ∇N_4^J, we invoke Lemma 4.2.4 and the mean value theorem again to get

$$\|\nabla N_i^J(\xi) - \nabla N_i^J(\tilde{\xi})\|_{\mathfrak{X}_0 \to \mathbb{R}} \le C\left(1 + \|\xi\|^2 + \|\tilde{\xi}\|^2\right)\|\xi - \tilde{\xi}\| \quad i = 3, 4.$$

This completes the proof.

To prove Theorem 4.0.4, we need to show that a solution of the equation

$$(\rho_1, \rho_2, \zeta_1, \zeta_2)^T = N^J(\rho_1, \rho_2, \zeta_1, \zeta_2), \qquad \int \rho_1 = m_1, \int \rho_2 = m_2 \tag{4.2.4}$$

near $(\rho_{m_1}^\infty, \rho_{m_2}^\infty, \frac{1}{m_1^2 m_2}, -\frac{1}{m_1 m_2^2})$ is unique for sufficiently large $J > 0$. We note that the limit map $N^\infty = (N_1^\infty, N_2^\infty, N_3^\infty, N_4^\infty)$ given in Remark 4.2.3 and Lemma 4.2.4 has $(\rho_{m_1}^\infty, \rho_{m_2}^\infty, \frac{1}{m_1^2 m_2}, -\frac{1}{m_1 m_2^2})$ as a solution. Then, by denoting

$$\xi^\infty := (\rho_{m_1}^\infty, \rho_{m_2}^\infty, \frac{1}{m_1^2 m_2}, -\frac{1}{m_1 m_2^2}), \qquad (\rho_1, \rho_2, \zeta_1, \zeta_2) = \xi^\infty + (\eta_1, \eta_2, z_1, z_2)$$

and

$$\begin{aligned}
A^J &:= \nabla N^J(\xi^\infty) \\
&= \begin{pmatrix} \partial_{\rho_1} N_1^J(\xi^\infty) & \cdots & \partial_{\zeta_2} N_1^J(\xi^\infty) \\ \vdots & \ddots & \vdots \\ \partial_{\rho_1} N_4^J(\xi^\infty) & \cdots & \partial_{\zeta_2} N_4^J(\xi^\infty) \end{pmatrix},
\end{aligned}$$

the equation (4.2.4) is equivalent to

$$(I - A^J)(\eta_1, \eta_2, z_1, z_2)^T = N^J(\xi^\infty) - N^\infty(\xi^\infty) + \mathcal{R}^J(\eta_1, \eta_2, z_1, z_2), \qquad \int \eta_1 = \int \eta_2 = 0,$$

where

$$\mathcal{R}^J(\eta_1, \eta_2, z_1, z_2) := N^J(\rho_1, \rho_2, \zeta_1, \zeta_2) - N^J(\xi^\infty) - A^J(\eta_1, \eta_2, z_1, z_2)^T.$$

Lemma 4.2.6 *There holds*

$$\liminf_{J\to\infty}\inf_{\phi}\left\{\|(I-A^J)\phi\| \,\middle|\, \phi=(\eta_1,\eta_2,z_1,z_2)\in\mathfrak{X}_0,\ \|\phi\|=1,\ \int\eta_1=\int\eta_2=0\right\}>0.$$

Proof We see from Lemma 4.2.5, the equation (3.1.2) satisfied by $\rho^\infty_{m_i}$ and the fact $\frac{5\gamma-6}{4-3\gamma}\tilde{E}^\infty_m = mC^\infty_m$ (Theorem 3.1.1) that

$$\lim_{J\to\infty}\|A^J - A^\infty\| = 0,$$

where

$$A^\infty(\eta_1,\eta_2,z_1,z_2) :=$$
$$\begin{pmatrix} f'\left((\frac{1}{|\cdot|}*\rho^\infty_{m_1})-C^\infty_{m_1}\right)\left(\frac{1}{|\cdot|}*\eta_1+C^\infty_{m_1}\int\eta_1\right) & 0 & 0 & 0\\ 0 & f'\left((\frac{1}{|\cdot|}*\rho^\infty_{m_2})-C^\infty_{m_2}\right)\left(\frac{1}{|\cdot|}*\eta_2+C^\infty_{m_2}\int\eta_2\right) & 0 & 0\\ \frac{1}{m_1^3m_2}\int\eta_1 & \frac{1}{m_1^2m_2^2}\int\eta_2 & \frac{2m_2z_1}{m_1+m_2} & -\frac{2m_2z_2}{m_1+m_2}\\ -\frac{1}{m_1^2m_2^2}\int\eta_1 & -\frac{1}{m_1m_2^3}\int\eta_2 & -\frac{2m_1z_1}{m_1+m_2} & \frac{2m_1z_2}{m_1+m_2}\end{pmatrix}.$$

It is easy to see that A^∞ is a compact operator. Moreover, observe that for an arbitrary kernel element $(\eta_1,\eta_2,z_1,z_2)\in\mathfrak{X}_0$ of $I-A^\infty$ with $\int\eta_1=\int\eta_2=0$, one has

$$\begin{cases} \eta_1 = f'(\frac{1}{|\cdot|}*\rho^\infty_{m_1}-C^\infty_{m_1})\frac{1}{|\cdot|}*\eta_1,\\ \eta_2 = f'(\frac{1}{|\cdot|}*\rho^\infty_{m_2}-C^\infty_{m_2})\frac{1}{|\cdot|}*\eta_2,\\ z_1 = \frac{1}{m_1^3m_2}\int\eta_1\,dx+\frac{1}{m_1^2m_2^2}\int\eta_2\,dx+\frac{2m_2}{m_1+m_2}z_1-\frac{2m_2}{m_1+m_2}z_2,\\ z_2 = -\frac{1}{m_1^2m_2^2}\int\eta_1\,dx-\frac{1}{m_1m_2^3}\int\eta_2\,dx-\frac{2m_1}{m_1+m_2}z_1+\frac{2m_1}{m_1+m_2}z_2, \end{cases}$$

so that $\eta_1=\eta_2=0$ by Theorem 3.1.1.(v) and consequently $z_1=z_2=0$.

Now, we are ready to end the proof. Arguing indirectly, suppose that there exist sequences $\{J_k\}\to\infty$ and $\phi_k=(\eta_{1,k},\eta_{2,k},z_{1,k},z_{2,k})\in\mathfrak{X}_0$ with $\int\eta_{1,k}=\int\eta_{2,k}=0$ such that $\|\phi_k\|=1$ and $(I-A^{J_k})\phi_k=o(1)$ in $\mathfrak{X}$ as $k\to\infty$. Since

$$\phi_k = A^{J_k}\phi_k+o(1) = (A^{J_k}-A^\infty)\phi_k + A^\infty\phi_k+o(1), \tag{4.2.5}$$

we see from the convergence of A^{J_k} to A^∞, compactness of A^∞ and closedness of $\mathfrak{X}_0$ in $\mathfrak{X}$ that ϕ_k converges in $\mathfrak{X}$ to some $\phi_\infty\in\mathfrak{X}_0$ with $\|\phi_\infty\|=1$, up to a subsequence. Then we use (4.2.5) again to see that $\phi_\infty=(\eta_1^\infty,\eta_2^\infty,z_1^\infty,z_2^\infty)\in\mathfrak{X}_0$ is a nontrivial kernel element of $I-A^\infty$ with $\int\eta_1^\infty=\eta_2^\infty=0$, which is a contradiction. This proves the lemma.

Let $\mathcal{B}_\delta$ be the δ-ball in the space $\mathfrak{X}_0$, i.e.,

$$\mathcal{B}_\delta := \left\{\phi=(\eta_1,\eta_2,z_1,z_2)\in\mathfrak{X}_0 \,\middle|\, \|\phi\|\le\delta\right\}.$$

Lemma 4.2.7 *For any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that if $0 < \delta < \delta_0$, then*

$$\limsup_{J\to\infty} \left\| \mathcal{R}^J(\phi) - \mathcal{R}^J(\tilde{\phi}) \right\| \le \varepsilon \left\| \phi - \tilde{\phi} \right\| \quad \text{for any } \phi, \tilde{\phi} \in \mathcal{B}_\delta.$$

Proof Observe that

$$\begin{aligned}
&\mathcal{R}^J(\phi) - \mathcal{R}^J(\tilde{\phi}) \\
&= N^J(\rho^\infty_{m_1} + \eta_1, \rho^\infty_{m_2} + \eta_2, \frac{1}{m_1^2 m_2} + z_1, -\frac{1}{m_1 m_2^2} + z_2) \\
&\quad - N^J(\rho^\infty_{m_1} + \tilde{\eta}_1, \rho^\infty_{m_2} + \tilde{\eta}_2, \frac{1}{m_1^2 m_2} + \tilde{z}_1, -\frac{1}{m_1 m_2^2} + \tilde{z}_2) - A^J(\phi - \tilde{\phi}) \\
&= \int_0^1 \nabla N(\rho^\infty_{m_1} + \eta_1(s), \rho^\infty_{m_2} + \eta_2(s), \frac{1}{m_1^2 m_2} + z_1(s), \frac{-1}{m_1 m_2^2} + z_2(s)) - A^J \, ds \, (\phi - \tilde{\phi}),
\end{aligned}$$

where $\eta_i(s) := s\eta_i + (1-s)\tilde{\eta}_i, z_i(s) = sz_i + (1-s)\tilde{z}_i$, $i = 1, 2$. Thus we see from Lemma 4.2.5 that for any $\varepsilon > 0$, there exist $\delta_1 > 0$ such that if $\|(\eta_1(s), \eta_2(s), z_1(s), z_2(s)\| < \delta_1$,

$$\begin{aligned}
&\limsup_{J\to\infty} \|\mathcal{R}^J(\phi) - \mathcal{R}^J(\tilde{\phi})\| \\
&\le \int_0^1 \|\nabla N^J(\rho^\infty_{m_1} + \eta_1(s), \rho^\infty_{m_2} + \eta_2(s), \frac{1}{m_1^2 m_2} + z_1(s), \frac{-1}{m_1 m_2^2} + z_2(s)) - A^J\| \, ds \, \|\phi - \tilde{\phi}\| \\
&\le \varepsilon \|\phi - \tilde{\phi}\|.
\end{aligned}$$

By taking $\delta_0 = \delta_1/2$, this completes the proof.

Proof (Completion of proof of Theorem 4.0.4) By the convergence,

$$\lim_{J\to\infty} \|\tilde{\rho}_1^J(\cdot + \bar{x}(\tilde{\rho}_1^J)) - \rho^\infty_{m_1}\|_{L^\infty} + \|\tilde{\rho}_2^J(\cdot + \bar{x}(\tilde{\rho}_2^J)) - \rho^\infty_{m_2}\|_{L^\infty} = 0,$$

it is sufficient to show that the equation for $\phi = (\eta_1, \eta_2, \zeta_1, \zeta_2)$

$$(I - A^J)\phi = \mathcal{R}^J(\phi), \qquad \int \eta_1 = \int \eta_2 = 0 \tag{4.2.6}$$

has a unique solution on a small ball $\mathcal{B}_\delta$ when J is large.

Let $\phi, \tilde{\phi} \in \mathcal{B}_\delta$ be two solutions of (4.2.6) for some δ to be chosen later. Then, we see from Lemma 4.2.6 that there exists a constant $c > 0$ independent of large J such that

$$c\|\phi - \tilde{\phi}\| \le \|(I - A^J)(\phi - \tilde{\phi})\|.$$

Invoking Lemma 4.2.7, we take a suitable $\delta > 0$ satisfying

$$\limsup_{J\to\infty} \left\| \mathcal{R}^J(\phi) - \mathcal{R}^J(\tilde{\phi}) \right\| \le \frac{c}{2} \left\| \phi - \tilde{\phi} \right\|.$$

This implies that for any sufficiently large J,

$$c\|\phi - \tilde{\phi}\| \le \frac{c}{2} \|\phi - \tilde{\phi}\|,$$

from which we see that $\phi = \tilde{\phi}$.

5 Symmetry for the equal mass case

In this section, we prove (*iii*) of Theorem 2.3.2. The strategy is to construct a family of symmetric solutions to the system of equations (4.0.3) satisfying the assumptions of Theorem 4.0.4. Then by Theorem 4.0.4, it should coincide with the normalized minimizer of the minimization problem (2.2.4) so that (*iii*) of Theorem 2.3.2 is proved.

Proposition 5.0.1 *Assume* $m_1 = m_2 = m$ *and* $r_1 = r_2 = r$. *Then for sufficiently large* $J > 0$, *there exists a normalized family of solutions* $\rho^J = \rho_1^J + \rho_2^J \in \overline{\mathcal{W}}^J_{m,m}$ *to* (4.0.3) *satisfying*

$$\rho_1^J(x) = \rho_2^J(R_\pi x), \quad x \in B(J^2\mathbf{x}_1, J^2 r)$$

and the following: there exists a constant $R > 0$ *independent of* J *such that*

$$spt(\rho_1^J) \subset B(\bar{x}(\rho_1^J), R), \quad spt(\rho_2^J) \subset B(\bar{x}(\rho_2^J), R)$$

and there hold the following convergences:

$$\begin{cases} \lim_{J\to\infty} \|\rho_1^J(\cdot + \bar{x}(\rho_1^J)) - \tilde{\rho}_m^\infty\|_{L^\infty} + \|\rho_2^J(\cdot + \bar{x}(\rho_2^J)) - \tilde{\rho}_m^\infty\|_{L^\infty} = 0, \\ \lim_{J\to\infty} \dfrac{\bar{x}(\rho_1^J)}{J^2} = (\dfrac{1}{m^3}, 0, 0), \quad \lim_{J\to\infty} \dfrac{\bar{x}(\rho_2^J)}{J^2} = -(\dfrac{1}{m^3}, 0, 0), \\ \lim_{J\to\infty} \dfrac{F(\rho_1^J, \rho_2^J)}{m} = C_m^\infty, \quad \lim_{J\to\infty} \dfrac{F(\rho_2^J, \rho_1^J)}{m} = C_m^\infty. \end{cases} \tag{5.0.1}$$

Proof In order to obtain a solution to (4.0.3) symmetric with respect to π rotation R_π, we consider the following minimization problem

$$\tilde{E}^J_{\text{min, sym}} := \min\left\{ E^J(\rho) \,\middle|\, \rho \in \overline{\mathcal{W}}^J_{m,m},\ \rho_1(x) = \rho_2(R_\pi x) \ \text{ a.e. } x \in B(J^2\mathbf{x}_1, J^2 r) \right\}. \tag{5.0.2}$$

Similarly with the original problem, any minimizing sequence $\{\rho_k\}$ to (5.0.2) admits a subsequence, still denoted by $\{\rho_k\}$ such that $\{\rho_k\}$ weakly converges in $L^\gamma(\mathbb{R}^3)$ to some $\tilde{\rho}^J_{\text{sym}}$ and $\{\Phi_{\rho_k}\}$ strongly converges to $\Phi_{\tilde{\rho}^J_{\text{sym}}}$ in $\dot{H}^1(\mathbb{R}^3)$. In particular, $\{(\rho_k)_1\}$ and $\{(\rho_k)_2\}$ weakly converge respectively to $(\tilde{\rho}^J_{\text{sym}})_1$ and $(\tilde{\rho}^J_{\text{sym}})_2$ in $L^\gamma(B(J^2\mathbf{x}_1, J^2 r)$ and $L^\gamma(B(J^2\mathbf{x}_2, J^2 r)$. This shows that $\tilde{\rho}^J_{\text{sym}} \in \overline{\mathcal{W}}^J_{m,m}$ and $E^J(\tilde{\rho}^J_{\text{sym}}) \le \tilde{E}^J_{\text{min, sym}}$. Moreover, for any test function $\phi \in C_c^\infty(B(J^2\mathbf{x}_1, J^2 r)$, we have

$$\begin{aligned} \int_{B(J^2\mathbf{x}_1, J^2 r)} (\tilde{\rho}^J_{\text{sym}})_2(R_\pi x)\phi(x)\, dx &= \int_{B(J^2\mathbf{x}_2, J^2 r)} (\tilde{\rho}^J_{\text{sym}})_2(x)\phi(R_{-\pi}x)\, dx \\ = \lim_{k\to\infty} \int_{B(J^2\mathbf{x}_2, J^2 r)} (\rho_k)_2(x)\phi(R_{-\pi}x)\, dx &= \lim_{k\to\infty} \int_{B(J^2\mathbf{x}_1, J^2 r)} (\rho_k)_2(R_\pi x)\phi(x)\, dx \\ = \lim_{k\to\infty} \int_{B(J^2\mathbf{x}_1, J^2 r)} (\rho_k)_1(x)\phi(x)\, dx &= \int_{B(J^2\mathbf{x}_1, J^2 r)} (\tilde{\rho}^J_{\text{sym}})_1(x)\phi(x)\, dx, \end{aligned}$$

which implies $(\tilde{\rho}^J_{\text{sym}})_1(x) = (\tilde{\rho}^J_{\text{sym}})_2(R_\pi x)$ a.e. $x \in B(J^2\mathbf{x}_1, J^2 r)$, and therefore a minimizer $\tilde{\rho}^J_{\text{sym}}$ exists.

Then, by following the variational argument by Auchmuty-Beals (proof of Theorem A in [3]), we see that for each $i = 1, 2$, $(\tilde{\rho}^J_{\mathrm{sym}})_i$ is continuous on $B(J^2\mathbf{x}_i, J^2 r_i)$ and there exists some constants $C^J_{\mathrm{sym},i}$ such that

$$-\frac{J^2}{2I(\tilde{\rho}^J_{\mathrm{sym}})^2}|P(x-\bar{x}(\tilde{\rho}^J_{\mathrm{sym}}))|^2 + A'((\tilde{\rho}^J_{\mathrm{sym}})_i) + \Phi_{\tilde{\rho}^J_{\mathrm{sym}}} = -C^J_{\mathrm{sym},i} \quad \text{in } \{(\tilde{\rho}^J_{\mathrm{sym}})_i > 0\}, \tag{5.0.3}$$

$$-\frac{J^2}{2I(\tilde{\rho}^J_{\mathrm{sym}})^2}|P(x-\bar{x}(\tilde{\rho}^J_{\mathrm{sym}}))|^2 + A'((\tilde{\rho}^J_{\mathrm{sym}})_i) + \Phi_{\tilde{\rho}^J_{\mathrm{sym}}} \geq -C^J_{\mathrm{sym},i} \quad \text{in } B(J^2\mathbf{x}_i, J^2 r_i). \tag{5.0.4}$$

By the π rotating symmetry of $\tilde{\rho}^J_{\mathrm{sym}}$, there automatically holds that $\bar{x}_i(\tilde{\rho}^J_{\mathrm{sym}}) = 0$, $i = 1, 2$, i.e., $P\bar{x}(\tilde{\rho}^J_{\mathrm{sym}}) = 0$.

We first claim that

$$\lim_{J\to\infty} E^\infty((\tilde{\rho}^J_{\mathrm{sym}})_i) = \tilde{E}^\infty_m, \quad i = 1, 2. \tag{5.0.5}$$

Indeed, from the inequality

$$E^J(\tilde{\rho}^J_{\mathrm{sym}}) \leq E^J(\rho^\infty_m(\cdot - J^2\mathbf{x}_1) + \rho^\infty_m(\cdot - J^2\mathbf{x}_2)),$$

where ρ^∞_m is the unique minimizer of the limit variational (3.1.1) with $\bar{x}(\rho^\infty_m) = 0$, one has, as $J \to \infty$

$$E^\infty((\tilde{\rho}^J_{\mathrm{sym}})_1) + E^\infty((\tilde{\rho}^J_{\mathrm{sym}})_2) \leq E^\infty(\rho^\infty_m) + E^\infty(\rho^\infty_m) + o(1).$$

Since $E^\infty(\rho^\infty_m) \leq E^\infty((\tilde{\rho}^J_{\mathrm{sym}})_i)$, $i = 1, 2$, we deduce that the claim holds true. In particular, we also note that $\tilde{E}^J_{\mathrm{min,sym}} < 0$ for sufficiently large $J > 0$.

Secondly, we claim that the constants $C^J_{\mathrm{sym},i}$, $i = 1, 2$ are positive for every sufficiently large $J > 0$. To the contrary, suppose the converse, i.e., there exists a sequence $\{J_k\} \to \infty$ such that $C^{J_k}_{\mathrm{sym},i} \leq 0$. The energy convergence (5.0.5) says $(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i$ is a minimizing sequence of the limit variational problem (3.1.1). Then by Theorem 3.1 in [43], there exists a sequence of translation vectors $\{t_k\} \subset \mathbb{R}^3$ such that after choosing a subsequence, $\{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)\}$ weakly converges in L^γ to ρ^∞_m and $\{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)}\}$ strongly converges in $\dot{H}^1$ to $\Phi_{\rho^\infty_m}$ as $k \to \infty$. In particular, $\{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)}\}$ strongly converges in L^6 to $\Phi_{\rho^\infty_m}$ by the Gagliardo-Nirenberg inequality. Then, extracting a subsequence again if necessary, we may assume that $\{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)}\}$ almost everywhere converges to $\Phi_{\rho^\infty_m}$. Moreover, the energy convergence (5.0.5) and $\dot{H}^1$ convergence of $\{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)}\}$ imply that

$$\lim_{k\to\infty} \|(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)\|_{L^\gamma} = \|\rho^\infty_m\|_{L^\gamma}.$$

Since the Banach space L^γ satisfies the Kadec-Klee property, saying that the weak topology coincides with the norm topology on the unit sphere, we can conclude that $(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)$ strongly converges in L^γ and consequently almost everywhere to to ρ^∞_m. We now pick a point $x_0 \in \mathrm{spt}(\rho^\infty_m)$ such that both of $\{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(x_0 - t_k)\}$ and

$\{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i(\cdot - t_k)}(x_0)\}\left(= \{\Phi_{(\tilde{\rho}^{J_k}_{\mathrm{sym}})_i}(x_0 - t_k)\}\right)$ converge. Then taking lim sup to the both sides of the inequality (5.0.4), we obtain

$$0 > C^\infty_m = A'(\rho^\infty_m(x_0)) + \Phi_{\rho^\infty}(x_0) \geq \limsup_{k\to\infty}(-C^{J_k}_{\mathrm{sym},i}) \geq 0,$$

which is a contradiction and the second claim follows.

Then we now can apply the arguments of proof of Lemma 3.3.8 without modification to obtain the bound

$$\limsup_{J\to\infty} \|\tilde{\rho}^J_{\mathrm{sym}}\|_{W^{1,\infty}(\mathbb{R}^3)} < \infty.$$

This let us to follows the arguments for the proof of Proposition 6.6. in [40] (it requires only uniform L^∞ bound for $\tilde{\rho}^J_{\mathrm{sym}}$) to deduce the existence of a constant $R_{\mathrm{sym}} > 0$ independent of J such that for large $J > 0$,

$$\mathrm{spt}((\tilde{\rho}^J_{\mathrm{sym}})_i) \subset B(\bar{x}((\tilde{\rho}^J_{\mathrm{sym}})_i),\, R_{\mathrm{sym}}) \quad i = 1, 2. \tag{5.0.6}$$

We finally consider the normalization of $\tilde{\rho}^J_{\mathrm{sym}}$ still denoted by $\tilde{\rho}^J_{\mathrm{sym}}$. We note that $\tilde{\rho}^J_{\mathrm{sym}}$ is contained in $\overline{W}^J_{m,m}$ by (5.0.6) and still satisfies the π rotation symmetry because it is obtained by a rotation with respect to x_1x_2 plane and a translation with respect to x_3 direction. Thus $\tilde{\rho}^J_{\mathrm{sym}}$ is a normalized minimizer of the minimization problem (5.0.2) so that we may follow the proof of Lemmas 3.3.5, 4.0.1 and 4.0.3 to see $\tilde{\rho}^J_{\mathrm{sym}}$ solves the system of equations (4.0.3). For obtaining the convergences (5.0.1), we may follow the remaining procedures of the proof of Proposition 3.3.1 without modification. This ends the whole proof of Proposition 5.0.1.

6 Orbital stability for binary stars

In this section, we show the orbital stability of binary stars. To this end, we first introduce the notion of a weak solution to Euler-Poisson equations (1.0.1) for perturbations.

Definition 6.0.1 *Let the triple* $(\rho, u, \Phi) : \mathbb{R}^3 \times [0, T] \to \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \times \mathbb{R}$ *where* $\rho, \rho u, \rho u \otimes u, \rho\nabla\Phi \in L^\infty([0, T]; L^1_{loc}(\mathbb{R}^3))$ *be given. Consider the Cauchy problem for* (1.0.1) *with the initial data* $(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x))$*. We say that* $(\rho, u, \Phi) : \mathbb{R}^3 \times [0, T] \to \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \times \mathbb{R}$ *is a weak solution of the Cauchy problem of Euler-Poisson equations (1.0.1) if for each* $t \in [0, T]$ *and for any test functions* $\psi, \Psi = (\psi_1, \psi_2, \psi_3) \in C^\infty_c(\mathbb{R}^3 \times [0, T])$*, the following hold:*

$$\int_0^t \int_{\mathbb{R}^3} (\rho\partial_t\psi + \rho u \cdot \nabla\psi)\, dxdt = \int_{\mathbb{R}^3} \rho(x, t)\psi(x, t)dx - \int_{\mathbb{R}^3} \rho_0(x)\psi(x, 0)dx \tag{6.0.1}$$

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^3} (\rho u \cdot \partial_t\Psi + \rho u \otimes u \cdot \nabla\Psi - \rho\nabla\Phi \cdot \Psi)\, dxdt \\ &= \int_{\mathbb{R}^3} \rho(x, t)u(x, t) \cdot \Psi(x, t)dx - \int_{\mathbb{R}^3} \rho_0(x)u_0(x) \cdot \Psi(x, 0)dx \end{aligned} \tag{6.0.2}$$

Remark 6.0.2 *By taking suitable test functions, one can easily see that the total mass and and the total angular momentum in x_3-direction are preserved for those weak solutions:*

$$\int_{\mathbb{R}^3} \rho(x,t)dx = \int_{\mathbb{R}^3} \rho_0(x)dx, \quad t \geq 0$$

$$\int_{\mathbb{R}^3} (x_1u_2(x,t) - x_2u_1(x,t))\rho(x,t)dx = \int_{\mathbb{R}^3} (x_1u_{02}(x) - x_2u_{01}(x))\rho_0(x)dx, \quad t \geq 0.$$

In the absence of shock waves, the total energy is also preserved [48] but in general it is non-increasing due to the entropy condition from the second law of thermodynamics. We will not address the existence of global weak solutions of the Euler-Poisson, which is out of scope of this article, but we refer to [37,39] for further discussion on weak solutions, entropy weak solutions and stability.

Next we introduce various distances to be used for stability result.

6.1 Notion of distances for nonlinear stability

Definition 6.1.1 *Let $\tilde\rho \in \mathcal{W}^J_{m_1,m_2}$ be a minimizer of* (2.2.4) *and $\rho \in \mathcal{W}^J_{m_1,m_2}$. If there exist $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$ such that $\tilde\rho^{\theta,\nu}$, $\rho^{\theta,\nu} \in \overline{\mathcal{W}}^J_{m_1,m_2}$, then we define a distance function*

$$d_0(\rho,\tilde\rho) := \int_{\mathbb{R}^3} A(\rho) - A(\tilde\rho) - \left(\frac{J^2}{2I(\tilde\rho)^2}|P(x-\bar{x}(\tilde\rho))|^2 - \Phi_{\tilde\rho}\right)(\rho-\tilde\rho)\,dx.$$

Otherwise we define $d_0(\rho,\tilde\rho) = \infty$.

In the following, by using a strict convexity of A, we show that d_0 is positive and is zero if and only if $\rho = \tilde\rho$.

Lemma 6.1.2 *For any $\rho \in \mathcal{W}^J_{m_1,m_2}$ and any minimizer $\tilde\rho \in \mathcal{W}^J_{m_1,m_2}$ of* (2.2.4)*, one has $d_0(\rho,\tilde\rho) \geq 0$. Moreover, $d_0(\rho,\tilde\rho) = 0$ if and only if $\rho = \tilde\rho$.*

Proof We may only consider the case that there exist $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$ such that $\rho^{\theta,\nu}, \tilde\rho^{\theta,\nu} \in \overline{\mathcal{W}}^J_{m_1,m_2}$. We see from Theorem 2.2.3 that

$$\begin{aligned} d_0(\rho,\tilde\rho) &= \sum_{i=1}^2 \int_{(T^{\theta,\nu})^{-1}(B(J^2\mathbf{x}_i,J^2r_i))} A(\rho) - A(\tilde\rho) - \left(\frac{J^2}{2I(\tilde\rho)^2}|P(x-\bar{x}(\tilde\rho))|^2 - \Phi_{\tilde\rho} - C_i\right)(\rho-\tilde\rho)\,dx \\ &\geq \sum_{i=1}^2 \int_{\mathrm{spt}(\tilde\rho_i)} A(\rho) - A(\tilde\rho_i) - A'(\tilde\rho_i)(\rho-\tilde\rho_i)\,dx. \end{aligned}$$

Since $A(\rho)$ is strictly convex w.r.t ρ, $A(\rho) - A(\tilde\rho_i) - A'(\tilde\rho_i)(\rho-\tilde\rho_i) \geq 0$, and $A(\rho) - A(\tilde\rho_i) - A'(\tilde\rho_i)(\rho-\tilde\rho_i) = 0$ if and only if $\rho_i = \tilde\rho_i$. This proves the lemma.

We next define a distance function taking into account the difference of the center of masses.

Definition 6.1.3 *For ρ and $\tilde\rho$ described in Definition 6.1.1, define*

$$d_1(\rho,\tilde\rho) := d_0(\rho,\tilde\rho) + \frac{J^2}{2}\left(\frac{(m_1+m_2)|P(\bar{x}(\rho)-\bar{x}(\tilde\rho))|^2}{I(\tilde\rho)^2} + \frac{(I(\rho)-I(\tilde\rho))^2}{I(\rho)I(\tilde\rho)^2}\right).$$

The next one measures the deviation of velocity fields.

Definition 6.1.4 *For* $(\rho, u) \in \mathcal{S}^J_{m_1,m_2}$*, define*

$$d_2(u, \tilde{u}_\rho) := \int_{\mathbb{R}^3} \frac{1}{2}|u(x) - \tilde{u}_\rho(x)|^2 \rho(x)\, dx,$$

where $\tilde{u}_\rho$ *is the uniform rotating velocity vector with respect to* $\bar{x}(\rho)$ *with angular velocity* $J/I(\rho)$*, i.e.,*

$$\tilde{u}_\rho = \frac{J}{I(\rho)} \vec{e}_z \times (x - \bar{x}(\rho)).$$

We now show that the distance functions $d_1(\rho, \tilde{\rho})$ and $d_2(u, \tilde{u}_\rho)$ naturally appear when measuring the energy deviation of (ρ, u) from the minimizer $(\tilde{\rho}, \tilde{u}_\rho)$.

Lemma 6.1.5 *Let* $\tilde{\rho}$ *be a minimizer of the variational problem* (2.2.4)*. For* $(\rho, u) \in \mathcal{S}^J_{m_1,m_2}$*, there holds*

$$E(\rho, u) - E_J(\tilde{\rho}) = d_1(\rho, \tilde{\rho}) + d_2(u, \tilde{u}_\rho) - \frac{1}{8\pi}\|\nabla\Phi_\rho - \nabla\Phi_{\tilde{\rho}}\|^2_{L^2(\mathbb{R}^3)}$$

whenever there exist $\theta \in \mathbb{R}$*,* $\nu \in \mathbb{R}^3$ *such that* $\tilde{\rho}^{\theta,\nu}$*,* $\rho^{\theta,\nu} \in \overline{\mathcal{W}}^J_{m_1,m_2}$*.*

Proof We decompose as

$$\begin{aligned} E(\rho, u) - E_J(\tilde{\rho}) &= E(\rho, u) - E_J(\tilde{u}_\rho) + E_J(\tilde{u}_\rho) - E_J(\tilde{\rho}) \\ &= \underbrace{\int_{\mathbb{R}^3} \frac{1}{2}|u|^2 \rho\, dx - \frac{J^2}{2I(\rho)}}_{(A)} + \underbrace{\int_{\mathbb{R}^3} A(\rho) - A(\tilde{\rho})\, dx - \frac{J^2}{2I(\rho)} - \frac{J^2}{2I(\tilde{\rho})}}_{(B)} \\ &\quad - \frac{1}{2}\underbrace{\left(\iint \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy - \iint \frac{\tilde{\rho}(x)\tilde{\rho}(y)}{|x-y|}\, dxdy\right)}_{(C)} \end{aligned}$$

Then, direct computations show

$$\begin{aligned} (A) &= \int_{\mathbb{R}^3} |u|^2 \rho\, dx - 2\frac{J}{I(\rho)}\vec{e}_z \cdot \int_{\mathbb{R}^3} ((x - \bar{x}(\rho)) \times u)\rho\, dx + \frac{J^2}{I(\rho)} \\ &= \int_{\mathbb{R}^3} |u|^2 \rho\, dx - 2\int_{\mathbb{R}^3} (u \cdot \tilde{u}_\rho)\rho\, dx + \int_{\mathbb{R}^3} |\tilde{u}_\rho|^2 \rho\, dx \\ &= \int_{\mathbb{R}^3} |u - \tilde{u}_\rho|^2 \rho\, dx = d_2(u, \tilde{u}_\rho), \end{aligned}$$

where we used $\tilde{u}_\rho = \frac{J}{I(\rho)}\vec{e}_z \times (x - \bar{x}(\rho))$.

$$
\begin{aligned}
(B) &= d_0(\rho,\tilde\rho) + \int_{\mathbb{R}^3}\left(\frac{J^2}{2I(\tilde\rho)^2}|P(x-\bar{x}(\tilde\rho))|^2 - \Phi_{\tilde\rho}\right)(\rho-\tilde\rho)\,dx + \frac{J^2}{2I(\rho)} - \frac{J^2}{2I(\tilde\rho)} \\
&= d_0(\rho,\tilde\rho) + \frac{J^2}{2}\left(\frac{1}{I(\tilde\rho)^2}\left(\int(|P(x-\bar{x}(\tilde\rho))|^2 - |P(x-\bar{x}(\rho))|^2)\rho\,dx\right) + \frac{I(\rho)-I(\tilde\rho)}{I(\tilde\rho)^2} + \frac{1}{I(\rho)} - \frac{1}{I(\tilde\rho)}\right) \\
&\qquad - \int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx \\
&= d_0(\rho,\tilde\rho) + \frac{J^2}{2}\left(\frac{(m_1+m_2)|P(\bar{x}(\rho)-\bar{x}(\tilde\rho))|^2}{I(\tilde\rho)^2} + \frac{(I(\rho)-I(\tilde\rho))^2}{I(\rho)I(\tilde\rho)^2}\right) - \int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx \\
&= d_1(\rho,\tilde\rho) - \int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
(C) &= \iint\frac{\rho(x)\rho(y)}{|x-y|}\,dxdy - \iint\frac{\tilde\rho(x)\tilde\rho(y)}{|x-y|}\,dxdy \\
&\qquad + \iint\frac{(\tilde\rho(x)-\rho(x))\tilde\rho(y)}{|x-y|}\,dxdy + \iint\frac{\tilde\rho(x)(\tilde\rho(x)-\rho(y))}{|x-y|}\,dxdy - 2\int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx \\
&= \iint\frac{(\rho(x)-\tilde\rho(x))(\rho(y)-\tilde\rho(y))}{|x-y|}\,dxdy - 2\int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx \\
&= \frac{1}{4\pi}\|\nabla\Phi_\rho - \nabla\Phi_{\tilde\rho}\|_{L^2}^2 - 2\int_{\mathbb{R}^3}\Phi_{\tilde\rho}(\rho-\tilde\rho)\,dx.
\end{aligned}
$$

Combining (A), (B) and (C), we can complete the proof.

Remark 6.1.6 *From Lemma 6.1.5 we see that the positive part of the energy deviation from the minimal energy is quantitatively given by the distance function $d_1 + d_2$ at the expense of the negative potential energy, and thus the minimizing energy alone is not sufficient for stability. Nevertheless, Lemma 6.1.5 indicates that if $E(\rho,u) - E_J(\tilde\rho)$ and $\|\nabla\Phi_\rho - \nabla\Phi_{\tilde\rho}\|^2_{L^2(\mathbb{R}^3)}$ can be made small, the distance function $d_1 + d_2$ stays small. In fact for our orbital stability result (cf. Theorem 6.4.1), we work with $d_1+d_2+\|\nabla\Phi_\rho-\nabla\Phi_{\tilde\rho}\|^2_{L^2(\mathbb{R}^3)}$ as the total measurement and together with Lemma 6.1.5 we resort to the energy conservation or dissipation property of dynamical solutions and the strong convergence of the potential energy which follows from the compactness result on minimizing sequences (cf. Lemma 6.4.3).*

6.2 Dynamical assumptions for nonlinear stability

(D) There exist $T_J \in (0,\infty]$ and a nonempty class of initial data $\mathcal{I}_J \subset \mathcal{S}^J_{m_1,m_2}$ with the following property: for any $(\rho_0, u_0) \in \mathcal{I}_J$, there exists a weak solution (ρ, u) of (EP) with the initial data (ρ_0, u_0) such that

- (i) $(\rho(\cdot,t), u(\cdot,t))$ exists up to the time interval $[0, T_J]$ and

$$\mathrm{spt}(\rho_1(\cdot,t)) \subset B(\bar{x}(\rho_1(\cdot,t)), J^2 r_0), \quad \mathrm{spt}(\rho_2(\cdot,t)) \subset B(\bar{x}(\rho_2(\cdot,t)), J^2 r_0)$$

  for some $0 < r_0 < \min\{r_1, r_2\}/4$ and all $t \in [0, T_J]$;
- (ii) $E(\rho(\cdot,t), u(\cdot,t)) \le E(\rho_0, u_0)$ for any $t \in [0, T_J]$;

6.3 Remarks on instability in Lyapunov sense

Lemma 6.3.1 *Let $(\rho_0, v_0) \in \mathcal{I}_T$ and (ρ, v) be a solution of (EP) with the initial datum (ρ_0, v_0). Then one has*

$$\bar{x}(\rho(\cdot, t)) = \bar{x}(\rho_0) + t \int_{\mathbb{R}^3} v_0 \rho_0 \, dx.$$

Proof We differentiate $M\bar{x}(\rho(\cdot, t))$ twice with respect to t to get

$$\begin{aligned}\frac{d^2}{dt^2} M\bar{x}(\rho(\cdot, t)) &= \frac{d}{dt}\int_{R^3} x\partial_t \rho \, dx = -\frac{d}{dt}\int_{\mathbb{R}^3} x\nabla \cdot (\rho v)\, dx = \int_{\mathbb{R}^3} \partial_t(\rho v)\, dx \\ &= -\int_{\mathbb{R}^3} \nabla \cdot (\rho v) v \, dx - \int_{\mathbb{R}^3} \rho (v \cdot \nabla) v + \nabla (K\rho^\gamma) + \rho \nabla \Phi \, dx \\ &= -\int_{\mathbb{R}^3} \nabla \cdot (v \otimes v) + \nabla(K\rho^\gamma) - 4\pi \Delta \Phi \nabla \Phi \, dx = 0.\end{aligned}$$

Therefore we can see

$$\bar{x}(\rho(\cdot, t)) = \bar{x}(\rho_0) + t \int_{\mathbb{R}^3} v_0 \rho_0 \, dx.$$

Remark 6.3.2 *Let $\bar{x}_i = \bar{x}(\rho_i(\cdot, t))$ be the center of mass for ρ_i, and let Φ_i be the gravitational potential corresponding to ρ_i so that $\Phi = \sum_{i=1}^2 \Phi_i$ and $\Delta \Phi_i = 4\pi\rho_i$. The same calculation using the evolution equations for (ρ_i, v_i) in the proof of Lemma 6.3.1 reveals*

$$\begin{aligned}\frac{d^2}{dt^2} m_i \bar{x}(\rho_i(\cdot, t)) &= -\int_{\mathbb{R}^3} \rho_i \nabla \Phi dx = -\sum_{k \neq i} \int_{\mathbb{R}^3} \rho_i \nabla \Phi_k dx \\ &= -\sum_{k \neq i} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho_i(x, t) \frac{x - y}{|x - y|^3} \rho_k(y, t) dy dx\end{aligned}$$

which shows that the dynamics of the center of mass for one body is determined by the tidal force due to the other body.

Lemma 6.3.1 shows instability can occur by a translation of center of mass.

6.4 Orbital stability for binary stars

We now state the main result of this section.

Theorem 6.4.1 *Let $\tilde{\rho} \in \mathcal{W}^J_{m_1, m_2}$ be a minimizer to (2.2.4). Assume (D). For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if an initial data $(\rho_0, u_0) \in \mathcal{I}_J$ satisfies*

$$d_1(\rho_0, \tilde{\rho}) + d_2(u_0, \tilde{u}_{\rho_0}) + \|\nabla \Phi_{\rho_0} - \nabla \Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} < \delta,$$

then there exist $\theta(t) \in \mathbb{R}$, $\nu(t) \in \mathbb{R}^3$ such that for every $t \in [0, T_J]$, the solution (ρ, u) to Cauchy problem of (EP) with initial data (ρ_0, u_0) satisfies

$$d_1(\rho^{\theta(t), \nu(t)}(\cdot, t), \tilde{\rho}) + d_2(u(\cdot, t), \tilde{u}_{\rho(\cdot, t)}) + \|\nabla \Phi_{\rho^{\theta(t), \nu(t)}(\cdot, t)} - \nabla \Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} < \varepsilon.$$

Remark 6.4.2 *Theorem 6.4.1 shows that under suitable assumptions on admissible perturbed solutions, McCann's binary star is orbitally stable: if initial data measured using d_1, d_2 and the potential energy is close enough to the binary star solution, the deviation with respect to the same distance stays small up to a rigid motion as long as the solutions exist and the support of the solutions stays bounded. The necessary spatial shifts and rotations appearing in the statement reflect the symmetry and invariance of the system.*

We firstly prove two auxiliary lemmas. The first one is the compactness result on minimizing sequences to the minimization problem (2.2.4).

Lemma 6.4.3 *There holds the following:*

(i) *Any normalized minimizing sequence $\{\rho_n\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ of* (2.2.4) *weakly converges to a normalized minimizer $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ of* (2.2.4) *in L^γ and $\nabla\Phi_{\rho_n}$ strongly converges to $\nabla\Phi_{\tilde{\rho}}$ in L^2, after choosing a subsequence.*

(ii) *For any minimizing sequence $\{\rho_n\} \subset \mathcal{W}^J_{m_1,m_2}$, there exists $\{\theta_n\} \subset \mathbb{R}$ and $\{v_n\} \subset \mathbb{R}^3$ such that $\{\rho_n^{\theta_n,v_n}\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ weakly converges to a minimizer $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ of* (2.2.4) *in L^γ and $\nabla\Phi_{\rho_n^{\theta_n,v_n}}$ strongly converges to $\nabla\Phi_{\tilde{\rho}}$ in L^2, after choosing a subsequence.*

Proof We first point out that any minimizing sequence $\{\rho_n\} \subset \overline{\mathcal{W}}^J_{m_1,m_2}$ of (2.2.4) weakly converges to a minimizer $\tilde{\rho} \in \overline{\mathcal{W}}^J_{m_1,m_2}$ of (2.2.4) in L^γ and $\nabla\Phi_{\rho_n}$ strongly converges to $\nabla\Phi_{\tilde{\rho}}$ in L^2, after choosing a subsequence. This is due to the fact that $\mathrm{spt}(\rho_n) \subset B(J^2\mathbf{x}_1, J^2 r_1) \cup B(J^2\mathbf{x}_2, J^2 r_2)$ so that Lemma 3.2 in [43] applies. Then the assertions (i) and (ii) are just corollaries of this. For (i), we note that the center of mass $\bar{x}(\rho_n)$ should converge to $\bar{x}(\tilde{\rho})$ by the weak convergence in L^γ. For (ii), we just use the definition of the admissible $\mathcal{W}^J_{m_1,m_2}$ and invariance of E^J under a rigid motion.

We next show that the solution for the Cauchy problem belongs the admissible class along the evolution.

Lemma 6.4.4 *There exists $\delta_0 > 0$ such that if an initial data $(\rho_0, u_0) \in \mathcal{I}_J$ satisfies*

$$d_1(\rho_0, \tilde{\rho}) + d_2(u_0, \tilde{u}_{\rho_0}) + \|\nabla\Phi_{\rho_0} - \nabla\Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} < \delta_0,$$

then the solution $(\rho(\cdot,t), u(\cdot,t))$ to Cauchy problem of (EP) with initial data (ρ_0, u_0) belongs to $\mathcal{S}^J_{m_1,m_2}$ for every $t \in [0, T_J]$.

Proof We may assume $\tilde{\rho}$ is normalized, i.e., $\bar{x}(\tilde{\rho}) = 0$, $\bar{x}_1(\tilde{\rho}_1) > 0$ and $\bar{x}_2(\tilde{\rho}_1) = \bar{x}_2(\tilde{\rho}_2) = 0$. The other minimizers can be dealt in the analogous way. For ρ, we define $\hat{\rho}(x) := \rho(R_\theta x + \bar{x}(\rho))$, where θ is chosen so that $\hat{\rho}(x)$ is normalized.

To the contrary suppose not. Then there exist $\{t_n\} \subset [0, T_J]$ and a sequence of initial data $\{\rho_{0,n}, u_{0,n}\} \subset \mathcal{I}_T$ such that

$$d_1(\rho_{0,n}, \tilde{\rho}) + d_2(u_{0,n}, \tilde{u}_{\rho_{0,n}}) + \|\nabla\Phi_{\rho_{0,n}} - \nabla\Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} < \frac{1}{n} \tag{6.4.1}$$

and for the solution (ρ_n, u_n) to Cauchy problem of (EP) with initial data (ρ_0, u_0), $\hat{\rho}_n$ firstly leaves the admissible class $\overline{\mathcal{W}}^J_{m_1,m_2}$ at time t_n.

We claim that the leaving time t_n must be positive. Since $(\rho_{0,n}, u_{0,n}) \in \mathcal{I}_J$, one has

$$\mathrm{spt}((\rho_{0,n})_1 \subset B(\bar{x}((\rho_{0,n})_1), J^2 r_0), \quad \mathrm{spt}((\rho_{0,n})_2) \subset B(\bar{x}((\rho_{0,n})_2), J^2 r_0).$$

Also, combining Theorem 2.2.5.(*ii*) and Theorem 2.3.2.(*i*), we see that

$$\mathrm{spt}(\tilde{\rho}_1) \subset B(J^2(\mathbf{x}_1 + o(1)), R), \quad \mathrm{spt}(\tilde{\rho}_2) \subset B(J^2(\mathbf{x}_2 + o(1)), R).$$

Then the contradiction hypothesis (6.4.1) says $\hat{\rho}_{0,n} \in \overline{\mathcal{W}}^J_{m_1,m_2}$. Since the solution ρ_n has the finite propagation speed for the center of masses, this shows by Assumption (D).(i) the first leaving time t_n is strictly positive.

By the definition of t_n, $\mathrm{spt}((\hat{\rho}_n)_1) \subset \overline{B(J^2\mathbf{x}_1, J^2 r_1)}$, $\mathrm{spt}((\hat{\rho}_n)_2) \subset \overline{B(J^2\mathbf{x}_1, J^2 r_2)}$ and either $\mathrm{spt}((\hat{\rho}_n)_1)$ or $\mathrm{spt}((\hat{\rho}_n)_2)$ touches the boundary at t_n. We just may assume $\mathrm{spt}((\hat{\rho}_n)_1)$ touches the boundary. Then Assumption (D).(*i*) implies that

$$\left|\bar{x}((\hat{\rho}_n)_1(\cdot, t_n)) - J^2\mathbf{x}_1\right| > \frac{3}{4} J^2 r_1. \tag{6.4.2}$$

Now, we take a smaller time than t_n, still denote by t_n such that $\mathrm{spt}((\hat{\rho}_n)_1)$ and $\mathrm{spt}((\hat{\rho}_n)_2)$ are contained in interior of their domains so that $\hat{\rho}_n \in \overline{\mathcal{W}}^J_{m_1,m_2}$ but (6.4.2) still holds true.

Lemma 6.1.5 implies $E(\rho_{0,n}, u_{0,n}) \to E_J(\tilde{\rho})$ as $n \to \infty$. One has from Lemma 2.2.2, Assumption (D).(*ii*) and the invariace of E_J under the group of rigid motions $\{T^{\theta,\nu} \mid \theta \in \mathbb{R}, \nu \in \mathbb{R}^3\}$ that

$$E_J(\hat{\rho}_n(\cdot, t_n)) = E_J(\rho_n(\cdot, t_n)) \le E(\rho_n(\cdot, t_n), u_n(\cdot, t_n)) \le E(\rho_{n,0}, u_{n,0}) \to E_J(\tilde{\rho}), \tag{6.4.3}$$

which means that $\hat{\rho}_n(\cdot, t_n)$ is a normalized minimizing sequence for the variational problem (2.2.4).

Then Lemma 6.4.3 says $\{\hat{\rho}_n(\cdot, t_n)\}$ weakly converges to the unique normalized minimizer $\tilde{\rho}$ in L^γ and

$$\lim_{n\to\infty} \|\nabla\Phi_{\hat{\rho}_n(\cdot,t_n)} - \nabla\Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} = 0.$$

This however makes a contradiction because $\mathrm{spt}(\tilde{\rho}) \cap \mathrm{spt}(\hat{\rho}_n) = \emptyset$ for large n by Theorem 2.2.5.(*ii*) and (6.4.2).

We are now ready to finish the proof of Theorem 6.4.1.

Proof (Completion of Proof of Theorem 6.4.1) The proof follows the similar lines with the proof of Lemma 6.4.4. We also may assume a minimizer $\tilde{\rho}$ belongs to $\overline{\mathcal{W}}^J_{m_1,m_2}$. Arguing indirectly, suppose Theorem 6.4.1 does not hold. Then there exist $\varepsilon_0 > 0$, $\{t_n\} \subset [0, T]$ and a sequence of initial data $\{\rho_{0,n}, u_{0,n}\} \subset \mathcal{I}_T$ such that

$$d_1(\rho_{0,n}, \tilde{\rho}) + d_2(u_{0,n}, \tilde{u}_{\rho_{0,n}}) + \|\nabla\Phi_{\rho_{0,n}} - \nabla\Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} < \frac{1}{n}$$

but the solution (ρ_n, u_n) to Cauchy problem of (EP) with initial data (ρ_0, u_0) satisfies

$$d_1(\rho_n^{\theta,\nu}(\cdot, t_n), \tilde{\rho}) + d_2(u_n(\cdot, t_n), \tilde{u}_{\rho_n(\cdot,t_n)}) + \|\nabla\Phi_{\rho_n^{\theta,\nu}(\cdot,t_n)} - \nabla\Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} \ge \varepsilon_0 \tag{6.4.4}$$

for every $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$ and $n \in \mathbb{N}$.

Lemma 6.1.5 implies $E(\rho_{0,n}, u_{0,n}) \to E_J(\tilde{\rho})$ as $n \to \infty$. One has from Lemma 2.2.2 and Assumption $(D).(ii)$ that

$$E_J(\rho_n(\cdot, t_n)) \leq E(\rho_n(\cdot, t_n), u_n(\cdot, t_n)) \leq E(\rho_{n,0}, u_{n,0}) \to E_J(\tilde{\rho}) \tag{6.4.5}$$

so that $\rho_n(\cdot, t_n)$ is a minimizing sequence for the variational problem (2.2.4) by Lemma 6.4.4.

Then Lemma 6.4.3 says there exists $\{(\theta_n, \nu_n)\} \subset \mathbb{R} \times \mathbb{R}^3$ such that after choosing a subsequence, $\{\rho_n^{\theta_n,\nu_n}(\cdot, t_n)\}$ weakly converges to $\tilde{\rho}$ in L^γ and

$$\lim_{n\to\infty} \|\nabla \Phi_{\rho_n^{\theta_n,\nu_n}(\cdot,t_n)} - \nabla \Phi_{\tilde{\rho}}\|_{L^2(\mathbb{R}^3)} = 0. \tag{6.4.6}$$

Let us denote $\tilde{\rho}_n(x) = \tilde{\rho}((T^{\theta_n,\nu_n})^{-1}x)$. It is straightforward to see that $\tilde{\rho}_n$ is a minimizer of (2.2.4) since $\mathcal{W}^J_{m_1,m_2}$ and E_J is invariant under the group of rigid motions $\{T^{\theta,\nu} \mid \theta \in \mathbb{R},\ \nu \in \mathbb{R}^3\}$. Now, we again use Lemma 6.1.5 to get

$$\begin{aligned}
&E(\rho_n(\cdot, t_n), u_n(\cdot, t_n)) \\
&= E_J(\tilde{\rho}_n) + d_1(\rho_n(\cdot, t_n), \tilde{\rho}_n) + d_2(u_n(\cdot, t_n), \tilde{u}_{\rho_n(\cdot,t_n)}) - \frac{1}{8\pi}\|\nabla \Phi_{\rho_n(\cdot,t_n)} - \nabla \Phi_{\tilde{\rho}_n}\|^2_{L^2(\mathbb{R}^3)} \\
&= E_J(\tilde{\rho}) + d_1(\rho_n^{\theta_n,\nu_n}(\cdot, t_n), \tilde{\rho}) + d_2(u_n(\cdot, t_n), \tilde{u}_{\rho_n(\cdot,t_n)}) - \frac{1}{8\pi}\|\nabla \Phi_{\rho_n^{\theta_n,\nu_n}(\cdot,t_n)} - \nabla \Phi_{\tilde{\rho}}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

This yields $d_1(\rho_n^{\theta_n,\nu_n}(\cdot, t_n), \tilde{\rho}) + d_2(u_n(\cdot, t_n), \tilde{u}_{\rho_n(\cdot,t_n)}) \to 0$ as $n \to \infty$ by (6.4.5) and (6.4.6) but this makes a contradiction with (6.4.4).

7 Applications to binary galaxies

We first introduce various quantities.

Hamiltonian for (VP):

$$E_{\mathrm{VP}}(f) = K_{\mathrm{VP}}(f) + G_{\mathrm{VP}}(f) = \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x,v)\, dxdv - \iint_{\mathbb{R}^6} \frac{1}{2}\frac{\rho_f(x)\rho_f(y)}{|x-y|}\, dxdy,$$

where $\rho_f(x) = \int_{\mathbb{R}^3} f(x,v)\, dv$.

Entropy(Casimir) functional:

$$C(f) = \iint_{\mathbb{R}^6} \beta(f(x,v))\, dxdv = \int_{\mathbb{R}^6} \frac{1}{q}\kappa_q^{q-1}(f(x,v))^q\, dxdv,$$

where $q > 5/3$ and $\kappa_q = \int_0^1 4\pi(1-s)^{\frac{1}{q-1}}\sqrt{2s}\, ds$.

Free energy for (VP):

$$\mathcal{F}(f) = E_{\mathrm{VP}}(f) + C(f).$$

Total mass for (VP):

$$M_{\mathrm{VP}}(f) = \int_{\mathbb{R}^3} \rho_f(x)\, dx = \iint_{\mathbb{R}^6} f(x,v)\, dxdv$$

Total angular momentum for (VP):

$$\mathbf{J}_{\mathrm{VP}}(f) := \int_{\mathbb{R}^6} ((x - \bar{x}(\rho_f)) \times v) f(x,v)\, dxdv = \int_{\mathbb{R}^3} ((x - \bar{x}(\rho_f)) \times u_f)) \rho_f(x)\, dx,$$

where $\bar{x}(\rho_f)$ is the center of mass of ρ_f, i.e., $\bar{x}(\rho_f) = \int_{\mathbb{R}^3} x\rho_f(x)\, dx / M_{\mathrm{VP}}(f)$ and u_f is the mean velocity field of f, i.e.,

$$u_f = \frac{\int_{\mathbb{R}^3} v f(x,v)\, dx}{\rho_f(x)}.$$

Moment of inertia:

$$I_{\mathrm{VP}}(f) = \int_{\mathbb{R}^6} |P(x - \bar{x}(\rho_f))|^2 f(x,v)\, dxdv = I(\rho_f).$$

Admissible class:

$$\mathcal{A} := \left\{ f(x,v) \in L^1(\mathbb{R}^6) \,\middle|\, f(x,v) \geq 0, \quad K_{\mathrm{VP}}(f) + C(f) < \infty, \quad \mathrm{spt}(f) \text{ is bdd} \right\}$$

and

$$\mathcal{A}^J_{m_1,m_2} = \left\{ f \in \mathcal{A} \,\middle|\, \rho_f \in \mathcal{W}^J_{m_1,m_2}, \quad \vec{e}_{x_3} \cdot \mathbf{J}_{\mathrm{VP}}(f) = J \right\}.$$

7.1 Reduction from (VP) energy to (EP) energy

Recall

$$E(\rho, u) = \int_{\mathbb{R}^3} A(\rho(x))\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 \rho(x)\, dx - \frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|}\, dxdy,$$

where $A(\rho) = \frac{K}{\gamma-1}\rho^\gamma$. Here we take $K = \frac{\gamma-1}{\gamma}$ so that $A(\rho) = \frac{1}{\gamma}\rho^\gamma$

In the following Proposition, we show that the minimal level of the kinetic energy-Casimir functional $K_{\mathrm{VP}}(f) + C(f)$ for the Vlasov-Poisson system with the given mean density and mean velocity is given by the associated energy for the Euler-Poisson system for the constraints.

Proposition 7.1.1 *For any given $(\rho_0, u_0) \in \mathcal{R}(\mathbb{R}^3) \times \mathcal{V}(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} |u_0|^2 \rho_0\, dx < \infty$, one has*

$$\int_{\mathbb{R}^3} A(\rho_0)\, dx + \int_{\mathbb{R}^3} \frac{1}{2}|u_0|^2 \rho_0\, dx = \min\left\{ K_{\mathrm{VP}}(f) + C(f) \mid f \in \mathcal{A},\ \rho_f = \rho_0,\ u_f = u_0 \right\},$$

where we take $A(\rho) = \frac{3q-1}{5q-3}\rho^{\frac{5q-3}{3q-1}}$. Moreover the minimum level is uniquely attained by

$$f_0(x,v) = (\beta')^{-1}_+ \left(\lambda(\rho_0(x)) - \frac{1}{2}|v - v_0(x)|^2 \right),$$

where $\lambda(\rho)$ is the Lagrange multiplier determined by $\lambda(\rho) = A'(\rho) = \rho^{\frac{2(q-1)}{3q-1}}$, and is also the inverse function of

$$\mu(\lambda) := \int_{\mathbb{R}^3} (\beta')^{-1}_+ (\lambda - \frac{1}{2}|v|^2)\, dv = \lambda^{\frac{3q-1}{2(q-1)}}.$$

Proof From an algebraic manipulation,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{2}|u_f(x)|^2\rho_f(x)\,dx + \iint_{\mathbb{R}^6} \frac{1}{2}|v-u_f(x)|^2 f(x,v)\,dxdv \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x,v)\,dxdv + \int_{\mathbb{R}^3} |u_f(x)|^2\rho_f(x)\,dx - \iint_{\mathbb{R}^6} u_f(x)\cdot v f(x,v)\,dxdv \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x,v)\,dxdv + \int_{\mathbb{R}^3} |u_f(x)|^2\rho_f(x)\,dx - \int_{\mathbb{R}^3} u_f(x)\cdot u_f(x)\rho_f(x)\,dx \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x,v)\,dxdv,
\end{aligned}$$

we deduce that the equivalent variational problem is

$$\int_{\mathbb{R}^3} A(\rho_0)\,dx = \min\left\{ \iint_{\mathbb{R}^6} \frac{1}{2}|v-u_0(x)|^2 f(x,v) + \beta(f(x,v))\,dxdv \;\Big|\; f\in\mathcal{A},\ \rho_f=\rho_0,\ u_f=u_0 \right\}.$$

Note that the above problem is naturally reduced to finding a minimizer of

$$\min\left\{ \int_{\mathbb{R}^3} \frac{1}{2}|v-u_0|^2 g(v) + \beta(g(v))\,dv \;\Big|\; \int_{\mathbb{R}^3} g(v)\,dv = \rho_0,\ \int_{\mathbb{R}^3} v g(v)\,dv = u_0\rho_0 \right\} \tag{7.1.1}$$

for any given constant value $\rho_0 \geq 0$ and vector $u_0 \in \mathbb{R}^3$. We claim that the minimization problem (7.1.1) has a unique minimizer

$$g_0(v) = (\beta')_+^{-1}(\lambda(\rho) - \frac{1}{2}|v-u_0|^2),$$

which satisfies

$$A(\rho) = \int_{\mathbb{R}^3} \frac{1}{2}|v-u_0|^2 g_0(v) + \beta(g_0(v))\,dv.$$

If this is the case, we deduce the proposition from this by integrating with respect to x variable.

To prove the claim, we take a change of variable $v \mapsto v+u_0$ so that the minimization problem (7.1.1) transforms to

$$\min\left\{ \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 g(v) + \beta(g(v))\,dv \;\Big|\; \int_{\mathbb{R}^3} g(v)\,dv = \rho_0,\ \int_{\mathbb{R}^3} v g(v)\,dv = 0 \right\}. \tag{7.1.2}$$

One can see from Section 2.2 in [43] that the minimum value of (7.1.2) without the constraint $\int_{\mathbb{R}^3} v g(v)\,dv = 0$ is $A(\rho)$ which is attained by

$$\tilde{g}_0(v) := g_0(v+u_0) = (\beta')_+^{-1}(\lambda(\rho) - \frac{1}{2}|v|^2).$$

Since $\tilde{g}_0(v)$ satisfies the constraint $\int_{\mathbb{R}^3} v g(v)\,dv = 0$, we can conclude that $\tilde{g}_0(v)$ is also a minimizer to the minimization problem (7.1.2) itself. This implies the claim holds true.

Motivated by Proposition 7.1.1 we introduce the local Gibbs state which gives rise to the minimal energy level:

Definition 7.1.2 *We define the local Gibbs state G_f associated with $f \in \mathcal{A}$ by*

$$G_f(x,v) := (\beta')_+^{-1}\left(\lambda(\rho_f(x)) - \frac{1}{2}|v - u_f(x)|^2\right).$$

The corollary below immediately follows from Proposition 7.1.1.

Corollary 7.1.3 *For any $f \in \mathcal{A}$,*

$$K_{VP}(G_f) + C(G_f) \le K_{VP}(f) + C(f)$$

and the the equality is attained if and only if $f = G(f)$. Moreover, one has

$$\int_{\mathbb{R}^3} A(\rho_f)dx + \int_{\mathbb{R}^3} \frac{1}{2}|u_f|^2\rho_f\, dx = K_{VP}(G_f) + C(G_f).$$

From the above discussion, we finally arrive at the existence of a family of variationally constructed binary galaxy solutions to (VP).

Theorem 7.1.4 (Existence and properties of a minimizer) *Consider a minimization problem*

$$\mathcal{F}_{min} := \inf_{f \in \mathcal{A}^J_{m_1,m_2}} \mathcal{F}(f). \tag{7.1.3}$$

For any $m_1, m_2 > 0$ and any sufficiently large J, there exists a minimizer $\tilde{f} \in \mathcal{A}^J_{m_1,m_2}$ of the problem (7.1.3) *which satisfies the following:*

(i) *(Self-consistent equations): $\tilde{f}$ satisfies:*

$$\beta'(\tilde{f}(x,v)) = \omega(x_1v_2 - x_2v_1) - \frac{1}{2}|v|^2 - \Phi_{\rho_{\tilde{f}}}(x) - C_i, \quad \forall (x,v) \in \{(\rho_{\tilde{f}})_i > 0\} \times \mathbb{R}^3,\ i = 1,2,$$

where $\omega = \frac{J}{I_{VP}(\tilde{f})}$ is the angular velocity and $C_i > 0$ is a cut-off energy level determined by a Lagrange multiplier.

(ii) *(Time dependent solution): $\tilde{f}(R_{-\omega t}x, R_{-\omega t}v - (0,0,\omega)^T \times (R_{-\omega t}x))$ solves* (VP).

(iii) *(Reduction to* (EP)*): $(\rho_{\tilde{f}}, u_{\tilde{f}})$ is a minimizer of the variational problem* (2.2.2) *described in Theorem 2.3.2.*

Proof The assertions (i) and (ii) follow from direct computations. The assertion (iii) is a consequence of Proposition 7.1.1.

7.2 Orbital stability for binary galaxies

In addition to distances d_1 and d_2 defined in Section 6, we need one more notion of distance d_3 measuring the difference between a distribution function f and its local Gibbs state $G(f)$.

Definition 7.2.1 *For $f \in \mathcal{A}^J_{m_1,m_2}$, define*

$$d_3(f, G(f)) = \iint_{\mathbb{R}^6} \beta(f) - \beta(G(f)) + \frac{1}{2}|v - u_f|^2(f - G(f))\, dvdx,$$

where $G(f)$ is the local Gibbs state given by f, i.e.,

$$G(f) = (\beta')_+^{-1}\left(\lambda(\rho_f) - \frac{1}{2}|v - u_f|^2\right).$$

The distance function d_3 measures the difference between f and its Gibbs state G_f.

The following lemma shows that d_3 makes sense as a distance function.

Lemma 7.2.2 *For $f \in \mathcal{A}^J_{m_1,m_2}$, $d_3(f, G(f)) \geq 0$ and $d_3(f, G(f)) = 0$ if and only if $f = G(f)$.*

Proof Observe that as above

$$\begin{aligned}
d_3(f, G(f)) &= \iint_{\mathbb{R}^6} \beta(f) - \beta(G(f)) + \frac{1}{2}|v - u_f|^2 (f - G(f))\, dvdx \\
&= \iint_{\mathbb{R}^6} \beta(f) - \beta(G(f)) - \beta'(G(f))(f - G(f)) + \lambda(\rho_f)(f - G(f))\, dvdx \\
&= \iint_{\mathbb{R}^6} \beta(f) - \beta(G(f)) - \beta'(G(f))(f - G(f))\, dvdx,
\end{aligned}$$

from which we deduce the lemma follows since β is strictly convex.

The following lemma is analogous to Lemma 6.1.5. This is needed for the proof of orbital stability of a minimizer $\tilde{f}$ to (7.1.3).

Lemma 7.2.3 *Let $\tilde{\rho}$ be a minimizer of the variational problem* (2.2.4)*. For $f \in \mathcal{A}^J_{m_1,m_2}$, there holds*

$$E_{VP}(f) - E_J(\tilde{\rho}) = d_1(\rho, \tilde{\rho}) + d_2(u, \tilde{u}_\rho) + d_3(f, G(f)) - \frac{1}{8\pi}\|\nabla\Phi_\rho - \nabla\Phi_{\tilde{\rho}}\|^2_{L^2(\mathbb{R}^3)}$$

whenever there exist $\theta \in \mathbb{R}$, $\nu \in \mathbb{R}^3$ such that $\tilde{\rho}^{\theta,\nu}_f$, $\rho^{\theta,\nu}_f \in \overline{\mathcal{W}}^J_{m_1,m_2}$.

Proof We decompose as

$$E_{\mathrm{VP}}(f) - E_J(\tilde{\rho}) = E_{\mathrm{VP}}(f) - E(\rho_f, u_f) + E(\rho_f, u_f) - E_J(\tilde{\rho}),$$

from which and Lemma 6.1.5, we see that we may only concern the term $E_{\mathrm{VP}}(f) - E(\rho_f, u_f)$. Observe that

$$\begin{aligned}
&\iint_{\mathbb{R}^3} \frac{1}{2}|u_f(x)|^2 \rho_f(x)\, dx + \iint_{\mathbb{R}^6} \frac{1}{2}|v - u_f(x)|^2 f(x, v)\, dxdv \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x, v)\, dxdv + \int_{\mathbb{R}^3} |u_f(x)|^2 \rho_f(x)\, dx - \iint_{\mathbb{R}^6} u_f(x) \cdot v f(x, v)\, dxdv \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x, v)\, dxdv + \int_{\mathbb{R}^3} |u_f(x)|^2 \rho_f(x)\, dx - \iint_{\mathbb{R}^3} u_f(x) \cdot u_f(x) \rho_f(x)\, dx \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x, v)\, dxdv.
\end{aligned}$$

This shows

$$\begin{aligned}
&E_{\mathrm{VP}}(f) - E(\rho_f, u_f) \\
&= \iint_{\mathbb{R}^6} \frac{1}{2}|v|^2 f(x, v) + \beta(f(x, v))\, dxdv - \int_{\mathbb{R}^3} \frac{1}{2}|u_f(x)|^2 \rho_f(x)\, dx - \int_{\mathbb{R}^3} A(\rho_f(x))\, dx \\
&= \iint_{\mathbb{R}^6} \beta(f(x, v)) + \frac{1}{2}|v - u_f(x)|^2 f(x, v)\, dxdv - \iint_{\mathbb{R}^6} \beta(G_f(x, v)) + \frac{1}{2}|v - u_f(x)|^2 G_f(x, v)\, dxdv \\
&= d_3(f, G_f).
\end{aligned}$$

We introduce the notion of a weak solution for the Vlasov-Poisson system.

Definition 7.2.4 *Let $f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ be given. Consider the Cauchy problem for* (1.0.2) *with the initial data $f(x,v,0) = f_0(x,v)$. We say that $f \in C([0,T]; L^1 \cap L^\infty)$ is a weak solution of the Cauchy problem of the Vlasov-Poisson system* (1.0.2) *if for each $t \in [0,T]$ and for any test functions $\psi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T])$, the following holds:*

$$\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,(\partial_t\psi + v\cdot\nabla_x\psi - \nabla_x\Phi\cdot\nabla_v\psi)\,dxdvdt$$
$$= \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v,t)\psi(x,v,t)dxdv - \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)\psi(x,v,0)dxdv$$

Remark 7.2.5 *For any $f_0 \in L^1 \cap L^\infty$ with compact support, there exists a unique weak solution $f(x,v,t)$ for all $t \geq 0$ such that its $(x,v)-$support is bounded for any finite time interval, and moreover, the total energy $E_{VP}(f)$ is preserved [50]. For other notions of global weak solutions satisfying $E_{VP}(f)(t) \leq E_{VP}(f)(0)$ but without uniqueness, we refer to [1,22]. For classical solutions and propagation of moments and regularity, see [36,41,44].*

The dynamical assumption $(D)'$ for (VP) corresponding to (D) is as follows.

$(D)'$ There exist $T_J' \in (0,\infty]$ and a nonempty class of initial data $\mathcal{I}_J' \subset \mathcal{A}^J_{m_1,m_2}$ with the following property: for any $f_0 \in \mathcal{I}_J'$, there exists a weak solution f of (VP) with the initial data f_0 such that

(*i*) $f(\cdot,t)$ exists up to the time interval $[0,T_J']$ and

$$\mathrm{spt}((\rho_f)_1(\cdot,t)) \subset B(\bar{x}((\rho_f)_1(\cdot,t)), J^2 r_0), \quad \mathrm{spt}((\rho_f)_2(\cdot,t)) \subset B(\bar{x}((\rho_f)_2(\cdot,t)), J^2 r_0)$$

for some $0 < r_0 < \min\{r_1, r_2\}$ and all $t \in [0,T_J']$;

(*ii*) $E_{\mathrm{VP}}(f(\cdot,t)) \leq E_{\mathrm{VP}}(f_0)$ for any $t \in [0,T_J']$;

With $(D)'$ and Lemma 7.2.3, we can obtain the following stability result as in Theorem 6.4.1.

Theorem 7.2.6 *Let $\tilde{f}$ be a minimizer to* (7.1.3). *Assume $(D)'$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if an initial data $f_0 \in \mathcal{I}_J'$ satisfies*

$$d_1(\rho_{f_0},\rho_{\tilde{f}}) + d_2(u_{f_0},\tilde{u}_{\rho_{f_0}}) + d_3(f_0,G(f_0)) + \|\nabla\Phi_{\rho_{f_0}} - \nabla\Phi_{\rho_{\tilde{f}}}\|_{L^2(\mathbb{R}^3)} < \delta,$$

then there exist $\theta(t) \in \mathbb{R}$, $\nu(t) \in \mathbb{R}^3$ such that for every $t \in [0,T_J']$, the solution f to Cauchy problem of (VP) with initial data f_0 satisfies

$$d_1(\rho_f^{\theta(t),\nu(t)}(\cdot,t),\rho_{\tilde{f}}) + d_2(u_f(\cdot,t),\tilde{u}_{\rho_f(\cdot,t)}) + d_3(f(\cdot,t),G(f(\cdot,t))) + \|\nabla\Phi_{\rho^{\theta(t),\nu(t)}(\cdot,t)} - \nabla\Phi_{\rho_{\tilde{f}}}\|_{L^2(\mathbb{R}^3)} < \varepsilon.$$

Proof Since Proposition 7.1.4 says $\rho_{\tilde{f}} = \tilde{\rho}$, where $\tilde{\rho}$ is a minimizer to (2.2.4), we may follow each step of the proof of Theorem 6.4.1 without modification. We omit the details.

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Conflict of interest

The authors declare that they have no conflict of interest.

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