Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #55, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Fermionic diagonal coinvariants and exterior Lefschetz elements

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Abstract. Let W be a complex reflection group acting irreducibly on its reflection representation V. The group W acts diagonally on the exterior algebra $\wedge(V \oplus V^*)$ over the direct sum of V with its dual space V^*. We study the *W-fermionic diagonal coinvariant ring FDR_W* obtained by quotienting $\wedge(V \oplus V^*)$ by the ideal generated by W-invariants with vanishing constant term.

Keywords: reflection group, fermions, diagonal coinvariants

1 The W-Fermionic Diagonal Coinvariant Ring

Let $X_n = (x_1, \ldots, x_n)$ and $Y_n = (y_1, \ldots, y_n)$ be two lists of n variables and let $\mathbb{C}[X_n, Y_n]$ be the polynomial ring in these $2n$ variables over the complex numbers. The symmetric group acts diagonally on $\mathbb{C}[X_n, Y_n]$, viz.

$$w \cdot x_i := x_{w(i)} \qquad w \cdot y_i := y_{w(i)} \qquad \text{for all } w \in S_n \text{ and } 1 \leq i \leq n.$$

Let $\mathbb{C}[X_n, Y_n]^{S_n} := \{f \in \mathbb{C}[X_n, Y_n] \,:\, w \cdot f = f \text{ for all } w \in S_n\}$ be the corresponding invariant subring and let $\mathbb{C}[X_n, Y_n]^{S_n}_+$ be the space of S_n-invariants with vanishing constant term. The *diagonal coinvariant ring* is the quotient

$$DR_n := \mathbb{C}[X_n, Y_n] / \langle \mathbb{C}[X_n, Y_n]^{S_n}_+ \rangle. \tag{1.1}$$

This is a doubly graded S_n-module, with one grading coming from X_n and another coming from Y_n. Haiman [6] proved that DR_n carries (up to sign twist) the permutation action of S_n on size n parking functions and gave an expression for the bigraded S_n-isomorphism type of DR_n in terms of the ∇ operator on symmetric functions.

Over the last couple years, many researchers in algebraic combinatorics [2, 3, 4, 5, 10, 11, 14] have considered variations on the diagonal coinvariant ring DR_n. Some of these

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modules [2, 3, 10, 11, 14] have involved the use of not only commuting variables, but also anticommuting variables. This was motivated in part by the *superspace* ring

$$\Omega_n = \mathbb{C}[x_1, \ldots, x_n] \otimes \wedge\{\theta_1, \ldots, \theta_n\}$$

given by the tensor product of a rank n polynomial ring with a rank n exterior algebra. The ring Ω_n has a long history in physics; the variables x_i are called *bosonic* (with a power x_i^2 corresponding to two indsintinguishable bosons in state i) and the variables θ_i are called *fermionic* (with the relation $\theta_i^2 = 0$ corresponding to the *Pauli Exclusion Principle*: two fermions cannot occupy the same state at the same time). The 'super' in superspace comes from 'supersymmetry' between bosons and fermions; in **Section 4** we recall a beautiful conjecture of F. Bergeron [2] on supersymmetry in S_n-coinvariant theory and explain (Remark 4.3) how it relates to our work.

In this extended abstract we define and study a variant on DR_n in which all variables are fermionic (i.e. anticommuting). Our construction extends simply and uniformly from the symmetric group S_n to any complex reflection group W. Before defining our ring, we recall the relevant reflection group terminology.

Let $V = \mathbb{C}^n$ be an n-dimensional complex vector space. An element $t \in \mathrm{GL}(V)$ is called a *reflection* if it has finite order and if the fixed space $V^t := \{v \in V : t \cdot v = v\}$ has codimension one in V. A *complex reflection group* is a finite subgroup $W \subseteq \mathrm{GL}(V)$ which is generated by reflections. The group W acts naturally on V; we call W *irreducible* if V is an irreducible W-module. If W is irreducible, we say that W has *rank* $n = \dim V$ and call V the *reflection representation* of W.

Let W be a complex reflection group of rank n acting irreducibly on its reflection representation V. We may construct several W-modules from V, including

- the n-dimensional dual space V^* of all linear functionals $V \to \mathbb{C}$,
- the $2n$-dimensional direct sum $V \oplus V^*$ of V with its dual space, and finally
- the 2^{2n}-dimensional exterior algebra $\wedge(V \oplus V^*)$ over $V \oplus V^*$.

This last space $\wedge(V \oplus V^*)$ may be regarded as a bigraded W-module by placing V in bidegree $(1, 0)$ and V^* in bidegree $(0, 1)$. The (i, j)-graded piece $\wedge(V \oplus V^*)_{i,j}$ is then given by $\wedge^i V \otimes \wedge^j V^*$.

It will sometimes be convenient to coordinatize our spaces. Let $\Theta_n = (\theta_1, \ldots, \theta_n)$ be a basis of V and let $\Xi_n = (\xi_1, \ldots, \xi_n)$ be the corresponding dual basis of V^* characterized by $\xi_i(\theta_j) = \delta_{i,j}$ (Kronecker delta). Write

$$\wedge\{\Theta_n, \Xi_n\} := \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\}$$

for the exterior algebra (over $\mathbb{C}$) generated by these $2n$ symbols. We have a natural identification of bigraded W-modules $\wedge\{\Theta_n, \Xi_n\} \cong \wedge(V \oplus V^*)$. The following quotient ring is our object of study.

Definition 1.1. *Let W be an irreducible complex reflection group acting on its reflection representation $V = \mathbb{C}^n$. Let $\wedge(V \oplus V^*)^W_+ \subseteq \wedge(V \oplus V^*)$ be the subspace of W-invariants with vanishing constant term and let $\langle \wedge(V \oplus V^*)^W_+ \rangle$ be the ideal generated by this subspace. The* W-fermionic diagonal coinvariant ring *is the quotient*

$$FDR_W := \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)^W_+ \rangle. \tag{1.2}$$

The ring FDR_W is a bigraded W-module, with one grading coming from V and another from V^*. We denote its bigraded decomposition as

$$FDR_W = \bigoplus_{i,j=0}^{n} (FDR_W)_{i,j}. \tag{1.3}$$

In our coordinate model, we may write

$$FDR_W = \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}^W_+ \rangle. \tag{1.4}$$

2 Bigraded Structure of FDR_W

As with any representation, the first and most basic question one can ask about FDR_W is its vector space dimension and the dimension of its bigraded pieces $(FDR_W)_{i,j}$. Here we get a combinatorial surprise: an appearance of the *Catalan* and *Narayana numbers*:

$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} \qquad \mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \tag{2.1}$$

Theorem 2.1. *Let W be a rank n irreducible reflection group and let $FDR_W = \bigoplus_{i,j=0}^{n}(FDR_W)_{i,j}$ be the fermionic diagonal coinvariant ring.*

1. *We have* $\dim FDR_W = \binom{2n+1}{n}$.
2. *The bigraded component $(FDR_W)_{i,j}$ is zero unless $i + j \leq n$.*
3. *When $i + j \leq n$ we have* $\dim(FDR_W)_{i,j} = \binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j-1}$.
4. *For any $0 \leq k \leq n$ we have* $\dim(FDR_W)_{k,n-k} = \mathrm{Nar}(n,k)$*, the Narayana number.*
5. *We have* $\sum_{k=0}^{n} \dim(FDR_W)_{k,n-k} = \mathrm{Cat}(n)$*, the Catalan number.*

Theorem 2.1 says that the ring FDR_W lives in the 'triangle' of bidegrees defined by $i, j \geq 0$ and $i + j \leq n$. The 'extreme' bidegree components $i + j = n$ give a Catalan-dimensional W-module with a natural direct sum decomposition into Narayana-dimensional submodules. The natural GL_2-action on the $2 \times n$ matrix of variables $\left(\begin{smallmatrix} \theta_1 & \cdots & \theta_n \\ \xi_1 & \cdots & \xi_n \end{smallmatrix}\right)$

induces actions on $\wedge\{\Theta_n, \Xi_n\}$, its quotient FDR_W, and on its Catalan-dimensional submodule $\bigoplus_{i+j=n}(FDR_W)_{i,j}$. The representation theory of GL_2 combines with point (4.) of Theorem 2.1 to explain algebraically the palindromicity and unimodality of the sequence $(\mathrm{Nar}(n,k))_{k=1}^n$ of Narayana numbers.

Remark 2.2. *While there exist Catalan numbers* $\mathrm{Cat}(W)$ *and Narayana numbers* $\mathrm{Nar}(W,k)$ *defined for any reflection group W in terms of invariant degrees, only their type A versions show up in Theorem 2.1. In contrast to the bosonic case, results in fermionic diagonal coinvariant theory tend to depend on only the rank of the group W, rather than W itself.*

Our next result enhances Theorem 2.1 by describing the bigraded W-isomorphism type of FDR_W. We state our answer in terms of the *Grothendieck ring* of W. This is the $\mathbb{Z}$-algebra generated by isomorphism classes $[U]$ of W-modules subject to relations $[U] = [U'] + [U'']$ for any short exact sequence $0 \to U' \to U \to U'' \to 0$. Multiplication in the Grothendieck ring corresponds to tensor product, viz. $[U] \cdot [U'] := [U \otimes U']$ where W acts diagonally on $U \otimes U'$.

Theorem 2.3. *Let W be a rank n irreducible reflection group. For any* $i + j \leq n$, *in the Grothendieck ring of W there holds the equation*

$$[(FDR_W)_{i,j}] = [\wedge^i V] \cdot [\wedge^j V^*] - [\wedge^{i-1} V] \cdot [\wedge^{j-1} V^*] \tag{2.2}$$

where we interpret $\wedge^{-1} V = \wedge^{-1} V^* = 0$.

Our next result gives a basis of FDR_W. To do this, we use our coordinate model $FDR_W = \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^W \rangle$ and interpret exterior monomials in $\wedge\{\Theta_n, \Xi_n\}$ in terms of certain lattice paths.

We denote by $\Pi(n)$ the family of n-step lattice paths in $\mathbb{Z}^2$ which

- start at the origin $(0,0)$ and
- consist of up-steps $(1,1)$, down-steps $(1,-1)$, and horizontal steps $(1,0)$, where
- every horizontal step has one of the decorations θ or ξ.

We define $\Pi(n)_{\geq 0} \subseteq \Pi(n)$ to be the subfamily of paths which remain weakly above the x-axis.

Paths in $\Pi(n)$ encode monomials in $\wedge\{\Theta_n, \Xi_n\}$. To see how, let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Pi(n)$ be a path with steps $\sigma_1, \ldots, \sigma_n$. For $1 \leq i \leq n$, we assign the step σ_i the following weight:

$$\mathrm{wt}(\sigma_i) = \begin{cases} 1 & \text{if } \sigma_i \text{ is an up-step,} \\ \theta_i & \text{if } \sigma_i \text{ is a horizontal step decorated } \theta, \\ \xi_i & \text{if } \sigma_i \text{ is a horizontal step decorated } \xi, \\ \theta_i \xi_i & \text{if } \sigma_i \text{ is a down-step.} \end{cases} \tag{2.3}$$

The path σ itself is weighted by the product of its step weights:

$$\mathrm{wt}(\sigma) = \mathrm{wt}(\sigma_1) \cdots \mathrm{wt}(\sigma_n). \tag{2.4}$$

The family $\{\mathrm{wt}(\sigma) \,:\, \sigma \in \Pi(n)\}$ of all path weights is precisely the family of monomials in $\wedge\{\Theta_n, \Xi_n\}$ (up to sign). By considering paths which do not sink below the x-axis, we get a monomial basis of FDR_W.

Theorem 2.4. *Let W be a rank n irreducible complex reflection group. The set*

$$\{\mathrm{wt}(\sigma) \,:\, \sigma \in \Pi(n)_{\geq 0}\} \tag{2.5}$$

descends to a monomial basis of FDR_W.

In fact, the basis of Theorem 2.4 is a *standard monomial basis* in the sense of (exterior) Gröbner theory. We close this section by remarking that the conclusions Theorems 2.1, 2.3, and 2.4 hold in greater generality than their hypotheses.

Remark 2.5. *The results of this section apply not just to a complex reflection group $W \subseteq \mathrm{GL}(V)$, but to any subgroup $G \subseteq \mathrm{GL}(V)$ for which the modules $\wedge^0 V, \wedge^1 V, \ldots, \wedge^{\dim V} V$ are inequivalent G-irreducibles. One example of such a group G is the general linear group $\mathrm{GL}(V)$ itself.*

3 The case of the symmetric group

In this section we specialize to the case $W = S_n$ of the symmetric group. In this setting, it is traditional to consider the n-dimensional permutation representation $U = \mathbb{C}^n$ rather than its $(n-1)$-dimensional irreducible reflection submodule V. In the coinvariant context, this has only minor effects. Indeed, we have $U = V \oplus U^{S_n}$ and $U^* = V^* \oplus (U^*)^{S_n}$ so that

$$\begin{aligned} \wedge\,(U \oplus U^*) &\cong \wedge[(V \oplus U^{S_n}) \oplus (V^* \oplus (U^*)^{S_n})] \\ &\cong \wedge[(V \oplus V^*) \oplus (U^{S_n} \oplus (U^*)^{S_n}] \cong [\wedge(V \oplus V^*)] \otimes [\wedge(U^{S_n} \oplus (U^*)^{S_n})] \end{aligned}$$

and modding out by the ideals generated by S_n-invariants with vanishing constant term gives

$$\wedge(U \oplus U^*)/\langle \wedge(U \oplus U^*)_+^{S_n} \rangle \cong \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)_+^{S_n} \rangle. \tag{3.1}$$

We may therefore harmlessly adopt the convenient realization of the S_n-structure of

$$FDR_n := FDR_{S_n} \cong \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{S_n} \rangle \tag{3.2}$$

by subscript permutation.

To state Theorems 2.1 and 2.3 for the S_n-module FDR_n, we recall some ideas from symmetric group representation theory. Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ be the ring of symmetric functions in an infinite variable set $X = (x_1, x_2, \dots)$ with Λ_n denoting its n^{th} graded piece. Bases of Λ_n are indexed by partitions $\lambda \vdash n$. Given a partition λ, we let $s_\lambda(X)$ be the associated *Schur function*.

Irreducible representations of S_n are indexed by partitions of n; if $\lambda \vdash n$ is a partition we let V^λ be the corresponding S_n-irreducible. Any finite-dimensional S_n-module V has a unique decomposition $V = \bigoplus_{\lambda \vdash n} c_\lambda V^\lambda$ for some multiplicities $c_\lambda \geq 0$. The *Frobenius image* of V is the symmetric function $\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda$ obtained by replacing the irreducible V^λ with the Schur function s_λ.

As an example of what the FDR_n modules look like, the Frobenius images of the bigraded pieces of FDR_4 are displayed in matrix format below, where the rows correspond to increasing θ-degree and the columns correspond to increasing ξ-degree.

$$FDR_4 \leftrightarrow \begin{pmatrix} s_4 & s_{31} & s_{211} & s_{1111} \\ s_{31} & s_{211} + s_{22} + s_{31} & s_{1111} + s_{211} + s_{22} & \\ s_{211} & s_{1111} + s_{211} + s_{22} & & \\ s_{1111} & & & \end{pmatrix} \tag{3.3}$$

The symmetry of (3.3) across the main diagonal reflects the symmetry between the Θ_n and Ξ_n in the definition of FDR_n.

The natural external product on the graded ring $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ corresponds to induction product of symmetric group modules. Far more mysterious is the *Kronecker product* $*$ defined on each graded piece Λ_n by $s_\lambda * s_\mu = \sum_{\gamma \vdash n} g_{\lambda,\mu,\gamma} \cdot s_\gamma$ where the *Kroencker coefficient* $g_{\lambda,\mu,\gamma} \geq 0$ is given by $V^\lambda \otimes V^\mu \cong_{S_n} \bigoplus_{\gamma \vdash n} g_{\lambda,\mu,\gamma} \cdot V^\gamma$. No general combinatorial formula is known for the numbers $g_{\lambda,\mu,\gamma}$.

Theorem 3.1. *The following facts hold concerning the bigraded S_n-module*

$$FDR_n = \bigoplus_{i,j=0}^{n} (FDR_n)_{i,j}.$$

1. *We have* $\dim FDR_n = \binom{2n-1}{n}$.

2. *We have $(FDR_n)_{i,j} = 0$ unless $i + j \leq n - 1$.*

3. *When $i + j \leq n - 1$ we have*

$$\mathrm{Frob}(FDR_n)_{i,j} = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})} \tag{3.4}$$

   where we interpret the second term to be zero when $i = 0$ or $j = 0$.

4. *For any $0 \leq k \leq n - 1$ we have* $\dim(FDR)_{k,n-k-1} = \mathrm{Nar}(n-1, k)$.

5. *We have* $\sum_{k=0}^{n-1} \dim(FDR)_{k,n-k-1} = \mathrm{Cat}(n-1)$.

Theorem 3.1 (1) proves a conjecture of Mike Zabrocki [15]. The Kronecker products of hook-shaped Schur functions appearing in Theorem 3.1 (2) may be evaluated using work of Rosas [12]. The discrepancies between Theorem 3.1 and Theorems 2.1 and 2.3 come from the reducibility of the permutation representation of S_n.

A glance at the matrix (3.3) shows some patterns. There is a unique copy of the trivial representation s_4 in bidegree $(0,0)$ and a single copy of the sign representation s_{1111} in the extreme bidegrees (i,j) with $i+j=3$. These patterns, along with an expression for the bigraded multiplicities of hook representations, generalize as follows. Let $\langle -,- \rangle$ be the *Hall inner product* on Λ_n obtained by declaring the Schur basis $\{s_\lambda \,:\, \lambda \vdash n\}$ to be orthonormal. We also define the *q,t-number* $[m]_{q,t} := q^{m-1} + q^{m-2}t + \cdots + qt^{m-2} + t^{m-1}$.

Theorem 3.2. *For any partition* $\lambda \vdash n$, *let* $c_\lambda(q,t) := \sum_{i,j \geq 0} \langle \mathrm{Frob}(FDR_n)_{i,j}, s_\lambda \rangle \cdot q^i t^j$ *be the bigraded multiplicity of* V^λ *in* FDR_n.

1. *We have* $c_\lambda(q,t) = 0$ *unless* $\lambda_3 < 3$.

2. *We have* $c_{(n)}(q,t) = 1$.

3. *We have* $c_{(1^n)}(q,t) = [n]_{q,t}$.

4. *We have* $c_{(n-k,1^k)}(q,t) = [k+1]_{q,t} + (qt) \cdot [k]_{q,t}$ *for any* $0 < k < n-1$.

4 F. Bergeron's Combinatorial Supersymmetry Conjecture

The diagonal coinvariants DR_n (two sets of bosonic variables and no fermionic variables) and the ring FDR_n (no bosonic variables and two sets of fermionic variables) motivate the study of a multigraded coinvariant S_n-quotient involving r sets of bosonic variables and s sets of fermionic variables. To formalize this, for $n,r,s \geq 0$ let $X_{r\times n}$ be an $r \times n$ matrix of bosonic variables and $\Theta_{s \times n}$ be an $s \times n$ matrix of fermionic variables and let

$$S(n;r,s) := \mathbb{C}[X_{r\times n}] \otimes \wedge\{\Theta_{s\times n}\} \tag{4.1}$$

be the tensor product of the polynomial ring over $X_{r\times n}$ with the exterior algebra over $\Theta_{s\times n}$.

The symmetric group S_n acts in a 'multidiagonal' way on $S(n;r,s)$ by simultaneously permuting the columns of $X_{r\times n}$ and $\Theta_{s\times n}$. Let $I(n;r,s) \subseteq S(n;r,s)$ be the ideal generated by the S_n-invariants with vanishing constant term and consider the *boson-fermion coinvariant ring*

$$R(n;r,s) := S(n;r,s)/I(n;r,s). \tag{4.2}$$

The ring $R(n;r,s)$ is an S_n-module which carries an r-fold grading in the bosonic variable sets and an s-fold grading in the fermionic variable sets. Explicit generators for the defining ideal $I(n;r,s)$ of $R(n;r,s)$ were found by Orellana and Zabrocki [10]. Special cases of the ring $R(n;r,s)$ are as follows.

- When $r = s = 0$, the ring $R(n;0,0)$ is the ground field $\mathbb{C}$.
- When $r = 1$ and $s = 0$, the ring $R(n;1,0)$ is the classical coinvariant algebra $\mathbb{C}[X_n]/\langle \mathbb{C}[X_n]_+^{S_n} \rangle$ of the symmetric group. This S_n-module carries the regular representation $\mathbb{C}[S_n]$ and presents the cohomology of the variety $\mathcal{F}\ell_n$ of complete flags in $\mathbb{C}^n$.
- When $r = 2$ and $s = 0$, the ring $R(n;2,0)$ is the diagonal coinvariant ring DR_n.
- When $r = 0$ and $s = 1$, it is not difficult to see that $R(n;0,1) = \wedge\{\Theta_n\}/\langle \wedge\{\Theta_n\}_+^{S_n} \rangle$ has dimension 2^{n-1}.
- When $r = s = 1$, the ring $R(n;1,1)$ is the *superspace coinvariant ring* $\Omega_n/\langle (\Omega_n)_+^{S_n} \rangle$ studied in [11, 14]. It is conjectured that the dimension of this quotient counts ordered set partitions of $\{1,\ldots,n\}$.
- When $r = 2$ and $s = 1$, Zabrocki conjectured [14] that $R(n;2,1)$ gives a representation-theoretic model for the symmetric function $\Delta'_{e_{k-1}} e_n$ as k ranges from 1 to n, where Δ' is a *delta operator*; see [14] for details.
- When $r = 0$ and $s = 2$, the ring $R(n;0,2) = FDR_n$ is the quotient ring studied in this extended abstract.

F. Bergeron proposed [2] the following unified approach to studying $R(n;r,s)$ as r and s vary. The general linear group GL_r acts on the matrix $X_{r\times n}$ of bosonic variables by left multiplication. Similarly, the matrix $\Theta_{s\times n}$ of fermionic variables carries an action of GL_s. These actions combine to give an action of $\mathcal{G} := \mathrm{GL}_r \times \mathrm{GL}_s \times S_n$ on the ring $S(n;r,s)$ and its quotient $R(n;r,s)$. We describe how to record the $\mathcal{G}$-character of $R(n;r,s)$ as a formal power series. For the remainder of this section, we assume knowledge of the power series operation of *plethysm*.

Let W be a $\mathcal{G}$-module. To capture the isomorphism type of W in a polynomial, we need three alphabets of variables: an r-letter alphabet $Q_r = (q_1,\ldots,q_r)$ tracking GL_r, an s-letter alphabet $Z_s = (z_1,\ldots,z_s)$ tracking GL_s, and an infinite alphabet $X = (x_1,x_2,\ldots)$ tracking S_n. For any partition $\lambda \vdash n$, we let W^λ be the λ-isotypic component of W as an S_n-module; the space W^λ carries a $\mathrm{GL}_r \times \mathrm{GL}_s$-action. We define the *$\mathcal{G}$-character* $\mathrm{ch}_{\mathcal{G}}(W)$ to be the formal power series

$$\mathrm{ch}_{\mathcal{G}}(W)[Q_r + Z_s; X] := \sum_{\lambda\vdash n} \mathrm{trace}_{W^\lambda}(\mathrm{diag}(q_1,\ldots,q_r)\times \mathrm{diag}(z_1,\ldots,z_s))\cdot s_\lambda[X] \tag{4.3}$$

where we take $q_1, \ldots, q_r$ and $z_1, \ldots, z_s$ to be nonzero complex numbers so that the product $\mathrm{diag}(q_1, \ldots, q_r) \times \mathrm{diag}(z_1, \ldots, z_s)$ of diagonal matrices is an element of $\mathrm{GL}_r \times \mathrm{GL}_s$ acting on W^λ. The formal power series $\mathrm{ch}_{\mathcal{G}}(W)[Q_r + Z_s; X]$ may be written uniquely as

$$\mathrm{ch}_{\mathcal{G}}(W)[Q_r + Z_s; X] = \sum_{\ell(\mu) \leq r} \sum_{\ell(\nu) \leq s} \sum_{\lambda \vdash n} d_{\mu,\nu,\lambda} \cdot s_\mu[Q_r] \cdot s_\nu[Z_s] \cdot s_\lambda[X] \tag{4.4}$$

for some integers $d_{\mu,\nu,\lambda} \geq 0$.

Remark 4.1. *Although the indices r and s are suppressed in our product group notation $\mathcal{G} = \mathrm{GL}_r \times \mathrm{GL}_s \times S_n$, they reappear in the q-alphabet and z-alphabet sizes in $\mathrm{ch}_{\mathcal{G}}(W)[Q_r + Z_s; X]$. The notation $\mathrm{ch}_{\mathcal{G}}(W)[Q_r + Z_s; X]$ is therefore unambiguous.*

Let $Q = (q_1, q_2, \ldots)$ be an infinite list of q-variables. F. Bergeron showed [1] that the limit of the 'purely bosonic' $\mathcal{G}$-characters of $R(n; r, 0)$:

$$\mathcal{E}_n[Q; X] := \lim_{r \to \infty} \mathrm{ch}_{\mathcal{G}}(R(n; r, 0))[Q_r + Z_0; X] \tag{4.5}$$

is a well-defined formal power series in the variables Q and X. The following conjecture states that that the $\mathcal{G}$-characters of **all** of the rings $R(n; r, s)$ may be determined from $\mathcal{E}_n[Q; X]$ (which only has *a priori* knowledge of the $R(n; r, 0)$).

Conjecture 4.2 (F. Bergeron [2])**.** "Combinatorial Supersymmetry Conjecture": *For any integers $n, r, s \geq 0$ the $\mathcal{G}$-character of $R(n; r, s)$ is given by*

$$\mathrm{ch}_{\mathcal{G}}(R(n; r, s))[Q_r + Z_s; X] = \mathcal{E}_n[Q - \epsilon Z; X] \mid_{\substack{Q \to Q_r \\ Z \to Z_s}} . \tag{4.6}$$

Here ϵ is the plethystic epsilon and the subscript $Q \to Q_r, Z \to Z_r$ on the left-hand-side means to evaluate $q_i \to 0$ whenever $i > r$ and $z_j \to 0$ whenever $j > s$.

Conjecture 4.2 implies that knowledge of all of the 'purely bosonic' quotients $R(n; r, 0)$ for $r \geq 0$ would determine all of the the boson-fermion rings $R(n; r, s)$. This conjecture also implies that knowledge of all of the 'purely fermionic' quotients $R(n; 0, s)$ for $s \geq 0$ would determine all of the rings $R(n; r, s)$. Conjecture 4.2 therefore suggests that studying $R(n; 0, s)$ will become very difficult as s grows.

Remark 4.3. *The bigraded hook-shaped multiplicities $c_\lambda(q, t)$ in Theorem 3.2 are in accordance with the prediction of Conjecture 4.2, giving evidence for combinatorial supersymmetry.*

5 Proof idea: Exterior Lefschetz theory

In this final section, we describe the main ideas of the proofs of Theorems 2.1 and 2.3; for complete proofs see that full version [9] of this work. The idea is to develop a (nontraditional) Lefschetz theory for the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. We start by recalling classical Lefschetz theory.

Let $A = \bigoplus_{d=0}^n A_d$ be a finite-dimensional graded commutative $\mathbb{C}$-algebra. The algebra A satisfies *Poincaré Duality (PD)* if $A_n \cong \mathbb{C}$ and if multiplication $A_d \otimes A_{n-d} \to A_n \cong \mathbb{C}$ is a perfect pairing for each $0 \leq d \leq n$. This implies that $\dim A_d = \dim A_{n-d}$. Assuming A satisfies PD, an element $\ell \in A_1$ is a *(strong) Lefschetz element* if the map $\ell^{n-2d} \times (-) : A_d \to A_{n-d}$ is bijective for each $d < n/2$. If A has a Lefschetz element, it is said to satisfy the *Hard Lefschetz (HL)* property.

Algebras A which satisfy PD and HL arise naturally in geometry. If X is a compact complex manifold, the cohomology ring $A = H^\bullet(X;\mathbb{C})$ satisfies PD and HL where we take $A_d := H^{2d}(X;\mathbb{C})$.

For some spaces X, PD and HL for $H^\bullet(X;\mathbb{C})$ may be understood combinatorially. As an example, recall that the *Boolean poset* $B(n)$ consists of subsets of $\{1,\ldots,n\}$ ordered by containment and that the i^{th} rank $B(n)_i$ of $B(n)$ consists of i-element subsets of $\{1,\ldots,n\}$.

Theorem 5.1. *For any $1 \leq i \leq j \leq n$, define an $\binom{n}{i} \times \binom{n}{j}$ matrix $M(n;i,j)$ with rows indexed by $B(n)_i$ and columns indexed by $B(n)_j$ by*

$$M(n;i,j)_{S,T} = \begin{cases} 1 & S \subseteq T \\ 0 & else. \end{cases}$$

For all $0 \leq i \leq n/2$, the matrix $M(n;i,n-i)$ is invertible.

For example, when $n = 4$ and $i = 1$, Theorem 5.1 asserts that the matrix

$$M(4;1,3) = \begin{array}{c} \\ \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \end{array} \begin{array}{c} \begin{array}{cccc} \{1\} & \{2\} & \{3\} & \{4\} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

is invertible.

The origins of Theorem 5.1 are difficult to trace. Stanley [13] gave a proof of Theorem 5.1 in the context of his theory of *differential posets*. Hara and Watanabe [7] gave another proof of Theorem 5.1 in which they calculated the (nonzero) determinant of $M(n;i,n-i)$. Hara and Watanabe interpreted Theorem 5.1 in a geometric context by showing that the cohomology ring

$$H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1;\mathbb{C}) = \mathbb{C}[x_1,\ldots,x_n]/\langle x_1^2,\ldots,x_n^2\rangle \tag{5.1}$$

of the n-fold product of the Riemann sphere $\mathbb{P}^1$ with itself satisfies PD and HL with $x_1 + \cdots + x_n$ serving as a Lefschetz element.

The exterior algebra $\wedge\{\Theta_n, \Xi_n\}$ has a natural bigrading

$$\wedge\{\Theta_n, \Xi_n\} = \bigoplus_{i,j=0}^{n} \wedge\{\Theta_n, \Xi_n\}_{i,j}.$$

We have $\wedge\{\Theta_n, \Xi_n\}_{n,n} \cong \mathbb{C}$ and the multiplication map

$$\wedge\{\Theta_n, \Xi_n\}_{i,j} \otimes \wedge\{\Theta_n, \Xi_n\}_{n-i,n-j} \to \wedge\{\Theta_n, \Xi_n\}_{n,n} \cong \mathbb{C}$$

is a perfect pairing for all $0 \leq i, j \leq n$. In this way, the algebra $\wedge\{\Theta_n, \Xi_n\}$ satisfies a bigraded kind of PD. The following result describes a kind of HL satisfied by $\wedge\{\Theta_n, \Xi_n\}$.

Theorem 5.2. *Consider the element $\delta \in \wedge\{\Theta_n, \Xi_n\}_{1,1}$ given by*

$$\delta := \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n. \tag{5.2}$$

Whenever $i + j \leq n$, the linear map

$$\delta^{n-i-j} \times (-) : \wedge\{\Theta_n, \Xi_n\}_{i,j} \longrightarrow \wedge\{\Theta_n, \Xi_n\}_{n-j,n-i} \tag{5.3}$$

is a bijection.

Informally, Theorem 5.2 says that δ_n is a Lefschetz-like element for the exterior algebra $\wedge\{\Theta_n, \Xi_n\}_{1,1}$. Theorem 5.2 is proven using Theorem 5.1.

Using our realization of Θ_n as a basis of the reflection representation V on which our reflection group W acts and Ξ_n as the dual basis of V^*, one sees that δ is fixed by the action of W (and, in fact, by the action of the full general linear group $\mathrm{GL}(V)$). We close by describing how Theorems 2.1 and 2.3 are proven.

Proof. (of Theorems 2.1 and 2.3, sketch) A result of Steinberg (see [8, Thm. A, §24-3, p.250]) states that the exterior powers $\wedge^0 V, \wedge^1 V, \ldots, \wedge^n V$ are inequivalent irreducible W-modules. From this one deduces that the defining ideal $\langle \wedge(V \oplus V^*)^W_+ \rangle$ of FDR_W is principal and generated by the element δ of Theorem 5.2, so that $FDR_W = \wedge(V \oplus V^*)/\langle\delta\rangle$.

Whenever a composition $f \circ g$ of two maps is a bijection, the map f is surjective and the map g is injective. Theorem 5.2 therefore implies that the map

$$\delta \times (-) : \wedge(V \oplus V^*)_{i,j} \longrightarrow \wedge(V \oplus V^*)_{i+1,j+1} \tag{5.4}$$

is surjective whenever $i + j \geq n$ and injective whenever $i + j < n$. In terms of the quotient $FDR_W = \wedge(V \oplus V^*)/\langle\delta\rangle$, the W-equivariance of δ means that $(FDR_W)_{i,j} = 0$ if $i + j > n$ and $[(FDR_W)]_{i,j} = [\wedge(V \oplus V^*)_{i,j}] - [\wedge(V \oplus V^*)_{i-1,j-1}]$ otherwise. $\square$

Theorem 5.2 is also used in the proof of Theorem 2.4 (giving a bigraded basis of FDR_W). Theorem 5.2 shows that the bigraded pieces of FDR_W have the appropriate dimensions.

Acknowledgements

The authors are grateful to François Bergeron, Jim Haglund, Eugene Gorsky, Vic Reiner, Richard Stanley, and Mike Zabrocki for many helpful conversations.

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