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Harmonic bases for generalized coinvariant algebras

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Abstract. Let $k \leq n$ be nonnegative integers and let λ be a partition of k. S. Griffin recently introduced a quotient $R_{n,\lambda}$ of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n variables which simultaneously generalizes the Delta Conjecture coinvariant rings of Haglund–Rhoades–Shimozono and the cohomology rings of Springer fibers studied by Tanisaki and Garsia–Procesi. We describe the space $V_{n,\lambda}$ of harmonics attached to $R_{n,\lambda}$ and produce a harmonic basis of $R_{n,\lambda}$ indexed by certain ordered set partitions $\mathcal{OP}_{n,\lambda}$. Our description of $V_{n,\lambda}$ involves injective tableaux and Vandermonde determinants and combinatorics of our harmonic basis is governed by a new extension of the *Lehmer code* of a permutation to $\mathcal{OP}_{n,\lambda}$.

Keywords: coinvariant ring, harmonic space, Vandermonde

1 Introduction: Quotient Rings and Harmonic Spaces

In his Ph.D. thesis [4], Sean Griffin introduced the following remarkable family of quotients of the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ in n variables. Given a subset $S \subseteq [n] := \{1, 2, \ldots, n\}$ and $d \geq 0$, let $e_d(S)$ be the degree d elementary symmetric polynomial in the variable set $\{x_i : i \in S\}$. For example, we have

$$e_2(1457) = x_1x_4 + x_1x_5 + x_1x_7 + x_4x_5 + x_4x_7 + x_5x_7.$$

By convention, we set $e_d(S) = 0$ whenever $|S| < d$.

For $k \geq 0$, in this paper we use the term *partition of* k to mean a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ of **nonnegative** integers with $\lambda_1 + \cdots + \lambda_s = k$. We write $|\lambda| = k$ or $\lambda \vdash k$ to mean that λ is a partition of k and call s the number of *parts* of $\lambda = (\lambda_1, \ldots, \lambda_s)$. For example, if $\lambda = (4, 2, 2, 0, 0)$ we have $\lambda \vdash 8$ and the partition λ has 5 parts.

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Definition 1.1 (Griffin [4]). *Let $k \leq n$ be nonnegative integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s \geq 0)$ be a partition of k with s parts. Write $\lambda' = (\lambda'_1 \geq \cdots \geq \lambda'_n \geq 0)$ for the conjugate partition of λ, padded with trailing zeros to have n parts.*

Let $I_{n,\lambda} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal

$$I_{n,\lambda} = \langle x_1^s, \ldots, x_n^s \rangle + \langle e_d(S) \, : \, S \subseteq [n] \text{ and } d > |S| - \lambda'_n - \lambda'_{n-1} - \cdots - \lambda'_{n-|S|+1} \rangle \quad (1.1)$$

and let $R_{n,\lambda} := \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$ be the associated quotient ring

As an example, suppose $n = 9$ and $\lambda = (3,2,2,0)$ so that $k = 7$ and $s = 4$. The conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_9)$ is given by $(3,3,1,0,0,0,0,0,0)$. The ideal $I_{9,\lambda} \subseteq \mathbb{Q}[\mathbf{x}_9]$ is generated by $x_1^4, \ldots, x_9^4$ together with the polynomials

$$\begin{array}{llll} e_i(S) & S \subseteq [9] & |S| = 9 & i = 9,8,7,6,5,4,3, \\ e_j(T) & T \subseteq [9] & |T| = 8 & j = 8,7,6,5, \\ e_d(U) & U \subseteq [9] & |U| = 7 & d = 7. \end{array}$$

Griffin's rings $R_{n,\lambda}$ generalize several important classes of quotient rings in algebraic combinatorics.

- When $k = s = n$ and $\lambda = (1^n)$, the ideal $I_{n,\lambda}$ is generated by the n elementary symmetric polynomials $e_1(\mathbf{x}_n), e_2(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)$ in the full variable set $\{x_1, \ldots, x_n\}$ and $R_{n,\lambda}$ is the classical *coinvariant ring* $R_n := \mathbb{Q}[\mathbf{x}_n]/\langle e_1(\mathbf{x}_n), e_2(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n) \rangle$ attached to the symmetric group $\mathfrak{S}_n$. The ring $R_{n,\lambda}$ presents the cohomology of the complete flag variety of type A_{n-1}.

- When $k = n$ and $\lambda \vdash n$ is arbitrary, the ring $R_{n,\lambda}$ is the *Tanisaki quotient* studied by Tanisaki [10] and Garsia–Procesi [3] which presents the cohomology of the *Springer fiber* $\mathcal{B}_\lambda$ attached to the partition λ.

- When $\lambda = (1^k, 0^{s-k})$ has all parts less or equal to 1, the rings $R_{n,\lambda}$ were introduced by Haglund, Rhoades, and Shimozono [7] to give a representation-theoretic model for the Haglund–Remmel–Wilson *Delta Conjecture* [6]. Pawlowski–Rhoades proved that these rings present the cohomology of a certain moduli space $X_{n,k,s}$ of line configurations [8].

The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Q}[\mathbf{x}_n]$ by subscript permutation. The ideals $I_{n,\lambda}$ are graded and $\mathfrak{S}_n$-stable, so $R_{n,\lambda}$ is a graded $\mathfrak{S}_n$-module. Generalizing results from [3, 7], Griffin calculated [4] the graded $\mathfrak{S}_n$-isomorphism type of $R_{n,\lambda}$.

In this extended abstract we study the rings $R_{n,\lambda}$ as graded $\mathbb{Q}$-vector spaces. In the special case $k = s = n$ and $\lambda = (1^n)$, the classical coinvariant ring R_n has a number of interesting bases which are important for different reasons. Perhaps the simplest of these was discovered by E. Artin [1], who used Galois Theory to prove that the family of 'sub-staircase monomials'

$$\{x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \,:\, 0 \leq c_i \leq n-i\} \tag{1.2}$$

descends to a basis for R_n. Extending earlier results of [3, 7], Griffin discovered the appropriate generalization of 'sub-staircase' to obtain a monomial basis of $R_{n,\lambda}$; his result is quoted in Theorem 2.3 below.

Our main goal is to describe the *harmonic space* of the quotient ring $R_{n,\lambda}$ and so derive a *harmonic basis* of this quotient ring. In order to motivate harmonic spaces and bases, we recall some technical issues that arise in the study of quotient rings.

Let $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ be any homogeneous ideal with quotient ring $R = \mathbb{Q}[\mathbf{x}_n]/I$. In algebraic combinatorics, one is often interested in calculating algebraic invariants of R such as its dimension or Hilbert series. A frequent impediment to computing these invariants is that, given $f \in \mathbb{Q}[\mathbf{x}_n]$, it can be difficult to decide whether $f + I = 0$ in R. Harmonic spaces can be used to replace quotients with subspaces, circumventing this problem.

For $f = f(x_1, \ldots, x_n) \in \mathbb{Q}[\mathbf{x}_n]$, let $\partial f := f(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ be the differential operator on $\mathbb{Q}[\mathbf{x}_n]$ obtained by replacing each x_i appearing in f with the partial derivative $\partial/\partial x_i$. The ring $\mathbb{Q}[\mathbf{x}_n]$ acts on itself by

$$f \odot g := (\partial f)(g) \quad \text{for all } f, g \in \mathbb{Q}[\mathbf{x}_n]. \tag{1.3}$$

That is, the polynomial $f \odot g$ is obtained by first turning f into a differential operator ∂f, and then applying ∂f to g.

For $f, g \in \mathbb{Q}[\mathbf{x}_n]$, we define a number $\langle f, g \rangle \in \mathbb{Q}$ by

$$\langle f, g \rangle := \text{constant term of } f \odot g. \tag{1.4}$$

Then $\langle -, - \rangle$ is an inner product on $\mathbb{Q}[\mathbf{x}_n]$ for which the set $\{x_1^{a_1} \cdots x_n^{a_n} \,:\, a_i \geq 0\}$ of monomials forms an orthogonal basis.

For a homogeneous ideal $I \subseteq \mathbb{Q}[\mathbf{x}_n]$, the *harmonic space* V of I is the graded subspace of $\mathbb{Q}[\mathbf{x}_n]$ given by the orthogonal complement

$$V = I^{\perp} = \{g \in \mathbb{Q}[\mathbf{x}_n] \,:\, \langle f, g \rangle = 0 \text{ for all } f \in I\}. \tag{1.5}$$

Writing $R = \mathbb{Q}[\mathbf{x}_n]/I$, standard results of linear algebra imply that $\mathbb{Q}[\mathbf{x}_n] = V \oplus I$ so that any vector space basis for V projects onto a basis of R. Any basis of V (and its image basis in R) is called a *harmonic basis*. If the ideal I is $\mathfrak{S}_n$-invariant, the $\mathfrak{S}_n$-invariance of the inner product $\langle -, - \rangle$ furnishes an isomorphism of graded $\mathfrak{S}_n$-modules $R \cong V$. Harmonic spaces are important for the following two reasons.

1. As alluded to above, elements of the quotient ring R are cosets whereas elements of the harmonic space V are honest polynomials. Expensive calls to polynomial division inherent to Buchberger-like algorithms involved in the study of R are avoided when one uses the harmonic model V.

2. It is often of interest to extend a quotient involving $\mathbf{x}$-variables $\{x_1, \ldots, x_n\}$ to an additional set of $\mathbf{y}$-variables $\{y_1, \ldots, y_n\}$. A key example is the extension of the classical coinvariant ring R_n to the diagonal coinvariant ring DR_n. While it can be unclear how to modify the ideal I defining the above quotient ring R to incorporate $\mathbf{y}$-variables in a natural way, the harmonic space V can sometimes by so extended by closing under the 'polarization' operators $y_1(\partial/\partial x_1) + \cdots + y_n(\partial/\partial x_n)$ together with $\partial/\partial y_1, \ldots, \partial/\partial y_n$.

Definition 1.2. *Let $V_{n,\lambda} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the harmonic space of $I_{n,\lambda}$.*

In the classical case $k = s = n$ and $\lambda = (1, \ldots, 1)$ so that $R_{n,\lambda} = R_n$, the harmonic space has the following description. Recall that the *Vandermonde determinant* $\delta_n \in \mathbb{Q}[\mathbf{x}_n]$ is the polynomial

$$\delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{1.6}$$

The harmonic space $V_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ corresponding to R_n is generated by δ_n as a $\mathbb{Q}[\mathbf{x}_n]$-module. Equivalently, the space V_n is the smallest subspace of $\mathbb{Q}[\mathbf{x}_n]$ containing δ_n which is closed under the partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$. A harmonic basis of R_n is given by applying sub-staircase monomials (as differential operators) to δ_n:

$$\{(x_1^{c_1} \cdots x_n^{c_n}) \odot \delta_n \,:\, 0 \leq c_i \leq n - i\}. \tag{1.7}$$

In the Springer fiber case $k = n$ with λ arbitrary, the harmonic space $V_{n,\lambda}$ was described by N. Bergeron and Garsia [2] using 'partial Vandermonde' polynomials.

The main results of this extended abstract are as follows.

- We show that $V_{n,\lambda}$ is generated as a $\mathbb{Q}[\mathbf{x}_n]$-module by an explicit family δ_T of polynomials indexed by *injective tableaux T* of shape λ with entries $\leq n$ (Theorem 3.1).
- We describe a harmonic basis for $V_{n,\lambda}$ indexed by certain *ordered set partitions* (Theorem 3.2).
- In order to describe the harmonic basis of Theorem 3.2, we introduce a *coinversion statistic* coinv and an associated *coinversion code* `code` on ordered set partitions which generalize the classical Lehmer code on permutations (Section 2). The definition of our coinversion code is reminiscent of the diagonal inversion statistic dinv in the theory of Macdonald polynomials [5]. The coinversion statistic may be used to calculate the Hilbert series of $R_{n,\lambda}$ (Corollary 3.3).

2 Coinversion codes for ordered set partitions

2.1 Coinversion codes

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s) \vdash k$ and $n \geq k$, we let $\mathcal{OP}_{n,\lambda}$ be the family of sequences $\sigma = (B_1, \ldots, B_s)$ of subsets of $[n] := \{1, \ldots, n\}$ such that $[n] = B_1 \sqcup \cdots \sqcup B_s$ (disjoint union) and $|B_i| \geq \lambda_i$ for all i. We call elements of $\mathcal{OP}_{n,\lambda}$ *ordered set partitions*, despite the fact that when $\lambda_i = 0$ we allow the possibility that $B_i = \varnothing$. For example, if $n = 16$ and $\lambda = (3,3,2,2,0,0)$ (so that $s = 6$) we have

$$(\{1,3,5,9\}, \{6,7,8,10,14\}, \{2,12,15\}, \{4,13\}, \varnothing, \{11,16\}) \in \mathcal{OP}_{n,\lambda}. \tag{2.1}$$

When $\lambda = (1^n) \vdash n$, we have a natural identification of ordered set partitions in $\mathcal{OP}_{n,\lambda}$ with permutations in $\mathfrak{S}_n$. The *coinversion number* $\mathrm{coinv}(w)$ of a permutation $w = w(1) \cdots w(n)$ in $\mathfrak{S}_n$ counts the number of noninversions in w, i.e.

$$\mathrm{coinv}(w) := |\{1 \leq i < j \leq n \,:\, w(i) < w(j)\}|. \tag{2.2}$$

The *coinversion code* of w is the sequence $(c_1, \ldots, c_n)$ where

$$c_j := |\{1 \leq i < j \,:\, w(i) < w(j)\}|. \tag{2.3}$$

This is a variant on the *Lehmer code* of the permutation w. The coinversion code $(c_1, \ldots, c_n)$ refines the coinversion number in the sense that $c_1 + \cdots + c_n = \mathrm{coinv}(w)$. The map $w \mapsto (c_1, \ldots, c_n)$ sending w to its coinversion code gives a bijection from $\mathfrak{S}_n$ to the set $[n-1] \times [n-2] \times \cdots \times [0]$ of sequences which fit below a staircase. These substaircase sequences also index the Artin basis $\{x_1^{c_1} \cdots x_1^{c_n} \,:\, 0 \leq c_i \leq n - i\}$ of the classical coinvariant ring R_n.

Our first results extend the notion of coinversion number and coinversion code from $\mathfrak{S}_n$ to $\mathcal{OP}_{n,\lambda}$. To define these statistics, we adopt a tableau-like representation of ordered set partitions in $\mathcal{OP}_{n,\lambda}$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ is a partition and $\sigma = (B_1, \ldots, B_s) \in \mathcal{OP}_{n,\lambda}$, the *container diagram* of σ is obtained by

- first placing from left to right λ_i top-justified boxes in column i (these boxes are called the *container*), and then
- filling column i with the elements of B_i in increasing order from bottom to top, where we will have entries which 'float' above the boxes whenever $|B_i| > \lambda_i$.

For example, when $n = 16$ and $\lambda = (3,3,2,2,0,0)$, the container diagram of the ordered set partition σ of (2.1) is shown below.

$$\begin{array}{cccccc} & 14 & & & & 16 \\ 9 & 10 & 15 & & \varnothing & 11 \\ \hline \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & & \\ \cline{1-4} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & & \\ \cline{1-4} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & & & \\ \cline{1-2} \end{array} \tag{2.4}$$

Empty blocks in ordered set partitions give rise to empty columns in container diagrams. The condition $|B_i| \geq \lambda_i$ corresponds to every box in the container being filled with a number. Numbers which do not lie in the container are called *floating*.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition and let $\sigma \in \mathcal{OP}_{n,\lambda}$ be an ordered set partition, thought of in terms of its container diagram. For $1 \leq i < j \leq n$, we say that the pair (i,j) is a *coinversion* of σ when one of the following three conditions hold:

- i is not floating, j is to the right of i in σ, i and j are in the same row of σ, and $i < j$,
- i is not floating, j is to the left of i in σ, j is one row below i in σ, and $i < j$, or
- i is floating, j is to the right of i in σ, j is at the top of its container, and $i < j$.

The first two conditions may be depicted schematically as

$$\boxed{i} \quad \cdots \quad \boxed{j} \quad \text{and} \quad \begin{array}{ccc} \boxed{} & \cdots & \boxed{i} \\ \boxed{j} & & \end{array}$$

where i and j are in adjacent rows in the diagram on the right. The last condition may be depicted

$$\begin{array}{ccc} i & & \\ & \ddots & \\ \boxed{} & & \boxed{j} \end{array}$$

where j is at the top of the container and there is an arbitrary positive number of rows separating i and j.

Remark 2.1. *The conditions defining coinversions for non-floating indices are the same as those used to define the statistic* dinv *which arises in the Haglund–Haiman–Loehr monomial expansion of the modified Macdonald polynomials [5].*

For $1 \leq i \leq n$ we define a number $c_i \geq 0$ by

$$c_i := \begin{cases} |\{i < j \,:\, (i,j) \text{ is a coinversion of } \sigma\}| & \text{if } i \text{ is not floating} \\ |\{i < j \,:\, (i,j) \text{ is a coinversion of } \sigma\}| + (p-1) & \text{if } i \text{ is floating in the } p^{\text{th}} \text{ block of } \sigma. \end{cases} \tag{2.5}$$

The *coinversion code* of σ is given by

$$\texttt{code}(\sigma) := (c_1, \ldots, c_n) \tag{2.6}$$

and the *coinversion number* of σ is given by

$$\text{coinv}(\sigma) := c_1 + \cdots + c_n. \tag{2.7}$$

Rhoades and Wilson [9] defined $\texttt{code}(\sigma)$ in the special case where $\lambda = (1^k)$.

As an example of these concepts, consider the ordered set partition $\sigma \in \mathcal{OP}_{16,(3,3,2,2,0,0)}$ appearing in (2.4). Let $(c_1, \ldots, c_{16})$ be the sequence $\texttt{code}(\sigma)$. The entry 2 forms coinversions with 4 and 6, so that $c_2 = 2$. The entry 10 is floating in column 2, and forms coinversions with 12 and 13 so that $c_{10} = 2 + (2-1) = 3$. We have

$$\texttt{code}(\sigma) = (c_1, \ldots, c_{16}) = (1,2,2,1,3,0,0,2,2,3,5,1,0,1,2,5).$$

Adding this sequence yields $\text{coinv}(\sigma) = 30$.

2.2 Shuffles and the code map

The code map $\sigma \mapsto \texttt{code}(\sigma)$ sends ordered set partitions in $\mathcal{OP}_{n,\lambda}$ to length n sequences of nonnegative integers. It turns out that this map is injective; to describe its image we need some definitions.

Recall that a *shuffle* of two sequences $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ is an interleaving $(c_1, \ldots, c_{r+s})$ of these sequences which preserves the relative order of the a's and b's. Analogously (or by induction), one can define a shuffle of any number of sequences.

Let $k \leq n$ be nonnegative integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition of k with s nonnegative parts. Write its conjugate as $\lambda' = (\lambda_1' \geq \cdots \geq \lambda_k')$. We define $\mathcal{C}_{n,\lambda}$ to be the family of length n sequences $(c_1, \ldots, c_n)$ of nonnegative integers which are componentwise $\leq$ some shuffle of the $k+1$ (possibly empty) sequences

$$(\lambda_1' - 1, \lambda_1' - 2, \ldots, 1, 0), \ldots, (\lambda_k' - 1, \lambda_k' - 2, \ldots, 1, 0), \text{ and } (s-1, s-1, \ldots, s-1)$$

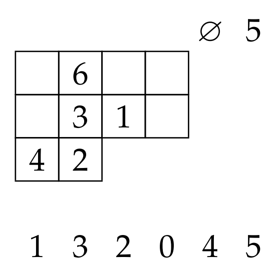
where the final sequence has $n-k$ copies of $s-1$.

Theorem 2.2. *Let $k \leq n$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s) \vdash k$ be a partition of k. The map $\texttt{code} : \mathcal{OP}_{n,\lambda} \to \mathcal{C}_{n,\lambda}$ is a bijection.*

Proof. (Sketch) We describe the inverse map $\iota : \mathcal{C}_{n,\lambda} \to \mathcal{OP}_{n,\lambda}$ in terms of an insertion algorithm. For any ordered set partition $(B_1, \ldots, B_s)$, we place the blocks $B_1, \ldots, B_s$ in the container diagram corresponding to λ, from left to right. For example, if $\lambda = (3,3,2,2,0,0)$ and

$$(B_1, \cdots, B_s) = (\{4\}, \{2,3,6\}, \{1\}, \varnothing, \varnothing, \{5\})$$

we obtain the diagram

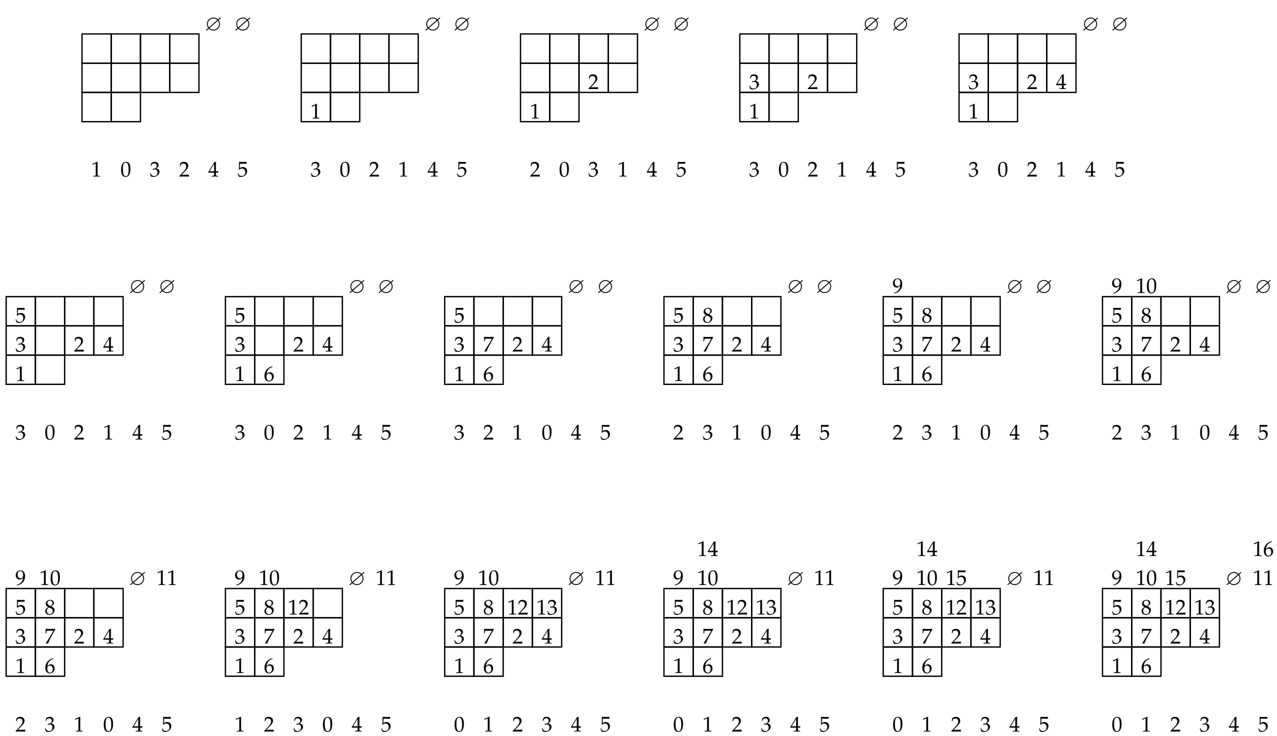


where the first, third, and fourth columns from the left remain unfilled. We label the columns of the diagram of $(B_1, \ldots, B_s)$ with the s distinct *coinversion labels* as follows. We first label the a columns which contain no empty boxes (including those columns corresponding to empty parts of λ) from right to left with $s-1, s-2, \ldots, s-a$, then label the b columns which contain a single empty box from left to right with $s-a-1, s-a-2, \ldots, s-a-b$, then label the c columns which contain two empty boxes from left to right with $s-a-b-1, s-a-b-2, \ldots, s-a-b-c$, and so on. Note that coinversion labels are assigned 'in the opposite direction' when labeled filled and unfilled columns. The coinversion labels are displayed below the columns of the container diagram.

Suppose we have a sequence $(c_1, \ldots, c_n) \in \mathcal{OP}_{n,\lambda}$. We define $\iota(c_1, \ldots, c_n)$ by starting with the 'empty' diagram corresponding to $(\varnothing, \ldots, \varnothing)$ and, for each $i = 1, 2, \ldots, n$, inserting i into the unique column with coinversion label c_i (updating coinversion labels as we go). For example if $\lambda = (3,3,2,2,0,0)$ (so that $s = 6$) and $n = 16$, the sequence

$$(c_1, \ldots, c_{16}) = (1,2,2,1,3,0,0,2,2,3,5,1,0,1,2,5) \in \mathcal{C}_{16,\lambda}$$

inserts to



One verifies that ι is well-defined and that `code` and ι are mutually inverse. $\square$

Theorem 2.2 is proven by constructing an insertion-style inverse map $\mathcal{C}_{n,\lambda} \to \mathcal{OP}_{n,\lambda}$. The family of sequences $\mathcal{C}_{n,\lambda}$ arises algebraically in the study of $R_{n,\lambda}$. Griffin proved that $R_{n,\lambda}$ has the following Artin-like monomial basis [4].

Theorem 2.3 (Griffin [4]). *Let $k \leq n$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s) \vdash k$. The set of monomials*

$$\{x_1^{c_1} \cdots x_n^{c_n} \,:\, (c_1, \ldots, c_n) \in \mathcal{C}_{n,\lambda}\} \tag{2.8}$$

descends to a vector space basis of $R_{n,\lambda}$.

Theorem 2.2 gives a direct combinatorial link between the monomial basis of Theorem 2.3 and ordered set partitions. Theorem 2.2 will also be useful in study of harmonic bases.

3 The harmonic space $V_{n,\lambda}$

3.1 Injective tableaux and harmonic generators

We regard $\mathbb{Q}[\mathbf{x}_n]$ as a module over itself by the differential operator action $f \odot g := \partial f(g)$. The harmonic space $V_{n,\lambda}$ of $R_{n,\lambda}$ forms a $\mathbb{Q}[\mathbf{x}_n]$-submodule of $\mathbb{Q}[\mathbf{x}_n]$. The submodule $V_{n,\lambda}$ is not principal, but we may describe a generating set combinatorially.

Let λ be a partition. An *tableau* of shape λ is a filling $T : \lambda \to \{1, 2, \ldots\}$ of the boxes of λ with positive integers. The tableau T is *injective* if its entries are distinct and *column strict* if its entries increase strictly going down columns. We let $\mathrm{Inj}(\lambda; \leq n)$ be the family of injective and column strict tableaux of shape λ with entries $\leq n$. A sample element of $\mathrm{Inj}((4, 2, 1, 0, 0); \leq 9)$ appears below; observe that the entries 7 and 8 do not appear in this tableau.

$$\begin{array}{|c|c|c|c|} \hline 2 & 1 & 3 & 9 \\ \hline 5 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \tag{3.1}$$

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition with s parts. We attach a polynomial $\delta_T \in \mathbb{Q}[\mathbf{x}_n]$ to any tableau $T \in \mathrm{Inj}(\lambda; \leq n)$ as follows. For any subset $S \subseteq [n]$, we let $\delta_S := \prod_{\substack{i,j \in S \\ i<j}} (x_i - x_j)$ be the Vandermonde determinant in the family of variables indexed by S. If the tableau T has columns $C_1, \ldots, C_r$ we set

$$\delta_T := \delta_{C_1} \cdots \delta_{C_r} \times \prod_{i \notin T} x_i^{s-1} \tag{3.2}$$

where the final product is over all indices $1 \leq i \leq n$ which do not appear in the tableau T. For example, if $\lambda = (4, 2, 1, 0, 0)$ (so that $s = 5$) and T is as in (3.1) then

$$\begin{aligned} \delta_T &= \delta_{\{2,5,6\}} \times \delta_{\{1,4\}} \times \delta_{\{3\}} \times \delta_{\{9\}} \times x_7^4 x_8^4 \\ &= (x_2 - x_5)(x_2 - x_6)(x_5 - x_6) \times (x_1 - x_4) \times 1 \times 1 \times x_7^4 x_8^4. \end{aligned}$$

The polynomials δ_T indexed by injective tableaux generate the harmonic space $V_{n,\lambda}$.

Theorem 3.1. *Let $k \leq n$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition of k. The harmonic space $V_{n,\lambda}$ is the smallest subspace of $\mathbb{Q}[\mathbf{x}_n]$ which*

- *contains the polynomial δ_T for every tableau $T \in \mathrm{Inj}(\lambda, \leq n)$, and*
- *is closed under the partial derivative operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$.*

Equivalently, the set $\{\delta_T : T \in \mathrm{Inj}(\lambda, \leq n)\}$ generates $V_{n,\lambda}$ as a $\mathbb{Q}[\mathbf{x}_n]$-module.

Theorem 3.1 gives an effective way to realize the $\mathfrak{S}_n$-module $R_{n,\lambda}$ on a computer. The use of harmonic polynomials in $V_{n,\lambda}$ avoids the computationally expensive calls to the multivariate polynomial long division involved in Gröbner-theoretic algorithms operating on the quotient $R_{n,\lambda} = \mathbb{Q}[\mathbf{x}_n]/I_{n,\lambda}$. The description of $V_{n,\lambda}$ in terms of injective tableaux is also arguably simpler than the quotient ring formulation of $R_{n,\lambda}$ in Definition 1.1.

3.2 Harmonic bases

Theorem 3.1 furnishes a natural algorithm spanning set of the quotient ring $R_{n,\lambda}$: starting with the family of polynomials $\{\delta_T : T \in \mathrm{Inj}(\lambda, \leq n)\}$, apply all possible partial derivatives to obtain a set of polynomials

$$\{(\partial/\partial x_1)^{b_1} \cdots (\partial/\partial x_n)^{b_n} \delta_T : T \in \mathrm{Inj}(\lambda; \leq n),\ b_i \geq 0\} \tag{3.3}$$

which spans $R_{n,\lambda}$. Since there are finitely many injective tableaux, the set (3.3) is finite.

It is natural to ask for a basis of $R_{n,\lambda}$ extracted from the spanning set (3.3). Such a basis may be obtained using container diagrams and coinversion codes. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of n with s parts, we think of an ordered set partition $\sigma \in \mathcal{OP}_{n,\lambda}$ in terms of its container diagram. We define an injective tableau $T(\sigma) \in \mathrm{Inj}(\lambda; \leq n)$ by letting the i^{th} column of T be the entries in the i^{th} row of the container of σ. We also define a sequence $\texttt{maxcode}(\sigma) = (a_1, \ldots, a_n)$ of nonnegative integers by letting

$$a_i := \begin{cases} b & \text{if } i \text{ appears in } T(\sigma) \text{ with } b \text{ boxes directly below it,} \\ s-1 & \text{if } i \text{ does not appear in } T(\sigma). \end{cases} \tag{3.4}$$

We define the polynomial $\delta_\sigma \in \mathcal{OP}_{n,\lambda}$ by the rule

$$\delta_\sigma := (x_1^{a_1-c_1} \cdots x_n^{a_n-c_n}) \odot \delta_{T(\sigma)} \tag{3.5}$$

where $\texttt{code}(\sigma) = (c_1, \ldots, c_n)$ is the coinversion code of σ.

As an example of these concepts, if $n = 16$ and $\lambda = (3, 3, 2, 2, 0, 0)$ (so that $s = 6$) and σ is the ordered set partition of (2.4) (rewritten below)

$$\sigma = \begin{array}{cccccc}
 & 14 & & & & 16 \\
9 & 10 & 15 & & \varnothing & 11 \\
\hline
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{13} & & \\
\hline
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & & \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} & & & & \\
\cline{1-2}
\end{array}$$

which gives the injective tableau $T(\sigma) \in \mathrm{Inj}(\lambda; \leq n)$

$$T(\sigma) = \begin{array}{|c|c|c|}
\hline
5 & 2 & 1 \\
\hline
8 & 3 & 6 \\
\hline
12 & 4 & \multicolumn{1}{c}{} \\
\cline{1-2}
13 & 7 & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array}$$

and the corresponding `maxcode` sequence

$$\texttt{maxcode}(\sigma) = (a_1, \ldots, a_{16}) = (1, 3, 2, 1, 3, 0, 0, 2, 5, 5, 5, 1, 0, 5, 5, 5).$$

The coinversion code of σ is

$$\texttt{code}(\sigma) = (c_1, \ldots, c_{16}) = (1, 2, 2, 1, 3, 0, 0, 2, 2, 3, 5, 1, 0, 1, 2, 5)$$

and we have $\texttt{maxcode}(\sigma) - \texttt{code}(\sigma) = (0, 1, 0, 0, 0, 0, 0, 0, 3, 2, 0, 0, 0, 4, 3, 0)$ so that

$$\delta_\sigma = (x_1^0 x_2^1 x_3^0 x_4^0 x_5^0 x_6^0 x_7^0 x_8^0 x_9^3 x_{10}^2 x_{11}^0 x_{12}^0 x_{13}^0 x_{14}^4 x_{15}^3 x_{16}^0) \odot \delta_{T(\sigma)}.$$

Theorem 3.2. *Let $k \leq n$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition of k. The set*

$$\{\delta_\sigma \,:\, \sigma \in \mathcal{OP}_{n,\lambda}\} \tag{3.6}$$

is a harmonic basis of $R_{n,\lambda}$. The lexicographical leading term of δ_σ has exponent sequence $\texttt{code}(\sigma)$.

It follows from the definitions that the degree of the polynomial δ_σ is the coinversion number $\mathrm{coinv}(\sigma)$. Theorem 3.2 therefore gives a combinatorial formula for the Hilbert series of $R_{n,\lambda}$.

Corollary 3.3. *Let $k \leq n$ be positive integers and let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ be a partition of k. We have*

$$\mathrm{Hilb}(R_{n,\lambda}; q) = \sum_{\sigma \in \mathcal{OP}_{n,\lambda}} q^{\mathrm{coinv}(\sigma)}. \tag{3.7}$$

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References

[1] E. Artin. *Galois Theory, Second edition*. Notre Dame Math Lectures, no. 2. Notre Dame: University of Notre Dame, 1944.

[2] N. Bergeron and A. Garsia. "On certain spaces of harmonic polynomials". In: *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, 51–86, Contemp. Math. **138**, *Amer. Math. Soc., Providence, RI*, 1992.

[3] A. M. Garsia and C. Procesi. "On certain graded S_n-modules and the q-Kostka polynomials". *Adv. Math.* **94 (1)** (1992), pp. 82–138.

[4] S. Griffin. "Ordered set partitions, Garsia-Procesi modules, and rank varieties". To appear, *Trans. Amer. Math. Soc.* 2020. arXiv:2004.00788.

[5] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for the Macdonald polynomials". *J. Amer. Math. Soc.* **18** (2005), pp. 735–761.

[6] J. Haglund, J. Remmel, and A. T. Wilson. "The Delta Conjecture". *Trans. Amer. Math. Soc.* **370** (2018), pp. 4029–4057.

[7] J. Haglund, B. Rhoades, and M. Shimozono. "Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture". *Adv. Math.* **329** (2018), pp. 851–915.

[8] B. Pawlowski and B. Rhoades. "A flag variety for the Delta Conjecture". *Trans. Amer. Math. Soc.* **372** (2019), pp. 8195–8248.

[9] B. Rhoades and A. T. Wilson. "Line configurations and r-Stirling partitions". *J. Comb.* **10 (3)** (2019), pp. 411–431.

[10] T. Tanisaki. "Defining ideals of the closures of conjugacy classes and representations of the Weyl groups". *Tohoku Math. J.* **33 (4)** (1982), pp. 575–585.