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UNIQUENESS OF SOME CALABI–YAU METRICS ON $\mathbf{C}^n$

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Abstract. We consider the Calabi–Yau metrics on $\mathbf{C}^n$ constructed recently by Yang Li, Conlon–Rochon, and the author, that have tangent cone $\mathbf{C} \times A_1$ at infinity for the $(n-1)$-dimensional Stenzel cone A_1. We show that up to scaling and isometry this Calabi–Yau metric on $\mathbf{C}^n$ is unique. We also discuss possible generalizations to other manifolds and tangent cones.

1 Introduction

On a compact Kähler manifold with vanishing first Chern class, Yau's solution of the Calabi conjecture [Yau78] shows that any Kähler class admits a unique Calabi–Yau, i.e. Ricci-flat Kähler, metric. In the non-compact setting there are many constructions of complete Calabi–Yau manifolds with different asymptotic behaviors, for example, Cheng–Yau [CY80], Tian–Yau [TY90, TY91] to name but a few, and even fixing the Kähler class these metrics are typically not unique. To recover uniqueness, in general one needs to put conditions on the asymptotics of the metric. Our goal in this paper is to prove such a uniqueness result for certain Calabi–Yau metrics on $\mathbf{C}^n$.

A coarse measure of the asymptotic behavior of a complete Calabi–Yau metric, or indeed of any complete metric with non-negative Ricci curvature, is provided by the rate of growth of the volumes of balls as a function of the radius. This must be at least linear (see Calabi [Cal75] and Yau [Yau76]), and at most that of Euclidean space. More refined information is given by the tangent cone at infinity which is obtained as the pointed Gromov–Hausdorff limit of a sequence of rescalings of the manifold by factors converging to zero. Note that in general the tangent cone may depend on the sequence of scalings. When the volume growth is maximal, then any tangent cone is a metric cone (see Cheeger–Colding [CC97]), and if in addition the curvature decays quadratically at infinity then the tangent cone has smooth link and is unique (see Colding–Minicozzi [CM14]).

On $\mathbf{C}^2$ a Calabi–Yau metric with maximal volume growth has tangent cone necessarily given by Euclidean space and is therefore the Euclidean metric (see Tian [Tia06, Theorem 5.2]). However, the Taub-NUT metric on $\mathbf{C}^2$ is a non-flat

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Kähler metric with the same volume form as the Euclidean metric (see LeBrun [LeB93]) which does not have maximal volume growth. It turns out that in higher dimensions even under the maximal volume growth condition the Euclidean metric is not the only possibility, contrary to a conjecture of Tian [Tia06, Remark 5.3], and non-Euclidean tangent cones can appear. For instance, for $n \geq 3$, $\mathbf{C}^n$ admits a complete Calabi–Yau metric ω_0 with tangent cone $\mathbf{C} \times A_1$ at infinity. Here A_1 is the $(n-1)$-dimensional A_1 singularity $x_1^2 + \cdots + x_n^2 = 0$ equipped with the Stenzel cone metric (see Li [Li17], Conlon–Rochon [CR17] and the author's work [Sze17]). These metrics all have the same volume form as the Euclidean metric, and in fact there are infinitely many other metrics with the same volume form exhibiting different tangent cones at infinity. Note that the examples of this kind known so far all have tangent cones with a singular link.

It is therefore natural to try to classify Calabi–Yau metrics with a prescribed tangent cone at infinity. Classification results have previously been obtained by Kronheimer [Kro89] in the case of surfaces, and Conlon–Hein [CH14] in higher dimensions in the asymptotically conical setting, i.e. when the metric converges at a polynomial rate to a Ricci-flat Kähler cone with smooth link. For instance in [CH14] the asymptotically conical Calabi–Yau manifolds with tangent cone A_1 are classified. Compared to these the main novelty in our work is that we are able to deal with tangent cones that do not have isolated singularities. Our main result is the following uniqueness statement for the metric ω_0 on $\mathbf{C}^n$.

Theorem 1.1. *Suppose that ω is a complete Calabi–Yau metric on $\mathbf{C}^n$ with tangent cone $\mathbf{C} \times A_1$ at infinity. Then there is a biholomorphism $F : \mathbf{C}^n \to \mathbf{C}^n$ and a constant $a > 0$ such that $\omega = aF^*\omega_0$.*

We emphasize that for a Calabi–Yau manifold with maximal volume growth the tangent cone at infinity has a natural complex structure on the regular set (which extends in general to the singular set by the main results in [DS15, LS]). When we say that the tangent cone is $\mathbf{C} \times A_1$ we are requiring that the complex structures agree as well as the metric structures, since in principle there may be different complex structures on a given metric cone. It is also worth noting that since a Kähler metric on $\mathbf{C}^n$ is necessarily $\partial\bar{\partial}$-exact, the work of Donaldson–Sun [DS15] can be applied (see Section 3.1) to see that the tangent cone is independent of the sequence of scalings, even when it has a singular link.

The proof of Theorem 1.1 can likely be extended to classify Calabi–Yau metrics on $\mathbf{C}^n$ with other tangent cones, as well as $\partial\bar{\partial}$-exact Calabi–Yau metrics on more general manifolds. We will discuss this in Section 5. The proof relies on two main ingredients. On the one hand, given a $\partial\bar{\partial}$-exact Calabi–Yau metric (X, ω) with tangent cone $C(Y)$, the work of Donaldson–Sun [DS15] gives an algebraic description of the ring of polynomial growth holomorphic functions on (X, ω) in terms of the coordinate ring of $C(Y)$. When $C(Y) = \mathbf{C} \times A_1$ and $X \cong \mathbf{C}^n$, then we can use this description to obtain an embedding $X \to \mathbf{C}^{n+1}$ as the hypersurface $z + x_1^2 + \cdots + x_n^2 = 0$, such that the functions z, x_i have degrees $1, \frac{n-1}{n-2}$ respectively. This is the basic input that allows us to compare the unknown metric (X, ω) with the reference metric $(\mathbf{C}^n, \omega_0)$, which is constructed by viewing $\mathbf{C}^n \subset \mathbf{C}^{n+1}$ as the same hypersurface. We discuss this in Section 3.

While we end up proceeding in a different way, heuristically the idea is that using such an embedding we can hope to find a biholomorphism $F : \mathbf{C}^n \to X$ such that $F^*\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ satisfies the Monge–Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \omega_0^n, \tag{1.1}$$

and in addition φ has subquadratic growth in the sense that $r^{-2}\sup_{B(0,r)}|\varphi| \to 0$ as $r \to \infty$. Here $B(0,r) \subset \mathbf{C}^n$ is the geodesic ball with respect to the metric ω_0. In practice we are not able to do this, but if we could, we would then like to show that $F^*\omega = \omega_0$, in analogy with the uniqueness result of Conlon–Hein [CH13, Theorem 3.1] in the setting of asymptotically conical spaces. Instead, in the proof of Theorem 1.1 we can only find a sequence of suitable biholomorphisms F on larger and larger open balls. The technical heart of the proof is then Proposition 4.1, which roughly speaking says that if on some large R-ball we have a solution of (1.1) such that $R^{-2}\varphi$ is small, then for suitable $\lambda < 1$, on the λR-ball we can find an "equivalent" potential φ' such that $(\lambda R)^{-2}\varphi'$ is even smaller. Such decay estimates are fairly standard in settings where the tangent cones that appear have smooth cross section. However, in our setting we need to analyze L^∞-small solutions of the Monge–Ampère equation on spaces that are close in the Gromov–Hausdorff sense to the singular tangent cone $\mathbf{C} \times A_1$. Here we use the results of Donaldson–Sun [DS12] and the author with Liu [LS18] in order to construct suitable barrier functions. Iterating the decay property Proposition 4.1 and letting $R \to \infty$ then leads to Theorem 1.1.

Finally let us mention some related works for minimal hypersurfaces. Regarding the uniqueness of minimal hypersurfaces with prescribed tangent cone at infinity, Simon–Solomon [SS86] and Mazet [Maz17] showed that minimal hypersurfaces in $\mathbf{C}^{n+1}$ that are asymptotic to certain Simons cones are essentially unique. At the same time, the works by Simon [Sim94, Sim93], and more recently Colombo–Edelen–Spolaor [CES17], address the behavior of minimal submanifolds that are near to a cone with non-isolated singularities, which is also a key point in our case. While the details are very different, there are certainly similarities between our approach and theirs.

2 The Reference Metric

In this section we give some preliminary results about the Calabi–Yau metrics on $\mathbf{C}^n$ constructed in [Li17, CR17, Sze17]. We follow the approach from [Sze17]. We suppose that $f(x_1, \ldots, x_n)$ is a polynomial such that $V_0 = f^{-1}(0) \subset \mathbf{C}^n$ has an isolated normal singularity at the origin. We assume that V_0 admits a Calabi–Yau cone metric $\omega_{V_0} = \sqrt{-1}\partial\overline{\partial}r^2$, whose homothetic action is diagonal with weights

$(w_1, \ldots, w_n)$, and f is homogeneous of degree $d > 2$ under this action. The basic example we are concerned with is $f = x_1^2 + \cdots + x_n^2$, in which case $w_i = \frac{n-1}{n-2}$, and we let $r^2 = |x|^{2\frac{n-2}{n-1}}$. In general it follows from Conlon–Hein [CH13] (see also [Sze17, Section 2]) that the smoothing $V \subset \mathbf{C}^n$ given by the equation $1 + f(x) = 0$ admits a Calabi–Yau metric $\omega_{V_1} = \sqrt{-1}\partial\overline{\partial}\varphi(x)$ with tangent cone V_0 at infinity.

We then consider the hypersurface $X \subset \mathbf{C} \times \mathbf{C}^n$ given by $z + f(x) = 0$, which is biholomorphic to $\mathbf{C}^n$. The main result of [Sze17] is that there exists a Calabi–Yau metric ω_0 on $\mathbf{C}^n$ with tangent cone $X_0 = \mathbf{C} \times V_0$ at infinity which is uniformly equivalent to the metric

$$\omega = \sqrt{-1}\partial\overline{\partial}\Big(|z|^2 + \gamma_1(R\rho^{-\alpha})r^2 + \gamma_2(R\rho^{-\alpha})|z|^{2/d}\varphi(z^{-1/d}\cdot x)\Big)$$

outside a compact set. Here $\gamma_i(s)$ are suitable cutoff functions such that $\gamma_1 + \gamma_2 = 1$, γ_1 is supported where $s > 0$ while γ_2 is supported where $s < 2$; the function R is such that $\sqrt{-1}\partial\overline{\partial}R^2$ defines a cone metric on $\mathbf{C}^n$ with the same homothetic action as V_0; the function $\rho^2 := |z|^2 + R^2$; $\alpha \in (1/d, 1)$ and $z^{-1/d} \cdot x$ is defined using the homothetic action, choosing a branch of log. The form ω defines a metric when restricted to X outside of a compact set, and the Calabi–Yau metric ω_0 that is constructed is asymptotic to ω at infinity, in the sense that $|\omega_0 - \omega|_\omega \to 0$. The volume form of ω_0 is $\sqrt{-1}^{n^2}\Omega \wedge \bar{\Omega}$ for

$$\Omega = \frac{dz \wedge dx_2 \wedge \cdots \wedge dx_n}{\partial_{x_1} f}. \tag{2.1}$$

For more details see [Sze17].

From [Sze17, Proposition 9] we have the following. For large D, we can consider a new embedding $X \to \mathbf{C}^{n+1}$ given by the functions $z' = D^{-1}z, x_i' = D^{-w_i}x_i$. The image has equation

$$Dz' + D^d f(x') = 0,$$

i.e. $D^{1-d}z' + f(x') = 0$, recalling that f has degree d under the homothetic action. We equip this hypersurface X' with the scaled down metric $D^{-2}\omega_0$. Here and below, let us denote by $\Psi(\epsilon)$ a function converging to zero as $\epsilon \to 0$. This function may change from line to line. From [Sze17, Proposition 9], we see that there is a constant $\theta < \Psi(D^{-1})$ satisfying the following. Viewing $X_0 \subset \mathbf{C}^{n+1}$ as $f^{-1}(0)$, we define the map $G : B_{X'}(0,1) \to X_0$ using the nearest point projection on the set where $|x'| > \theta$, and projection onto the z-axis where $|x'| \leq \theta$. Then G is a $\Psi(D^{-1})$-Gromov–Hausdorff approximation to $B_{X_0}(0,1)$. One useful consequence of this is that the distance from the origin in (X, ω_0) is uniformly equivalent to the function ρ. We will need the following.

Proposition 2.1. (a) *The holomorphic functions z, x_i on (X, ω_0) have polynomial growth with degrees $d(z) = 1, d(x_i) = w_i$.*

(b) *Consider the special case $f = x_1^2 + \cdots + x_n^2$. Then the vector fields $2z\partial_z + x_i\partial_{x_i}$ and $a_{jk}x_j\partial_{x_k}$ for skew-symmetric (a_{jk}) all have at most linear growth.*

Proof. The statement in (a) is immediate from the fact that the distance function is uniformly equivalent to ρ.

For part (b), we can work using the description of the metric ω in regions I–V in the proof of [Sze17, Proposition 5] (note that ω is uniformly equivalent to ω_0). In each region we choose new coordinates in which we have a good model for the form ω and so we can bound our vector fields. Let us consider regions I, III and V, the others being very similar.

Region I. Here $R > \kappa\rho$ for some fixed small $\kappa > 0$, and we assume $\rho \in (D/2, 2D)$ for D, which will then be uniformly equivalent to the distance from the origin. We change coordinates to $\tilde{z} = D^{-1}z$ and $\tilde{x} = D^{-1} \cdot x$, and we let $\tilde{r} = D^{-1}r$. In these coordinates X has equation

$$D^{1-d}\tilde{z} + f(\tilde{x}) = 0, \tag{2.2}$$

and in the proof of [Sze17, Proposition 5] the scaled down metric $D^{-2}\omega$ on this hypersurface is compared to the product metric

$$\sqrt{-1}\partial\overline{\partial}(|\tilde{z}|^2 + \tilde{r}^2) \tag{2.3}$$

on the hypersurface with equation $f(\tilde{x}) = 0$ (i.e. the product X_0). Since $|\tilde{z}| < 2$ and $\tilde{r} \in (\kappa/2, 4\kappa)$, as well as $d > 1$, as $D \to \infty$ then in these coordinates the hypersurface (2.2) converges smoothly to X_0. Because of this we can compute the norms of our vector fields with respect to the metric (2.3). For this we have

$$z\partial_z = \tilde{z}\partial_{\tilde{z}}, \quad x_j\partial_{x_k} = \tilde{x}_j\partial_{\tilde{x}_k}.$$

The norms of these vector fields are uniformly bounded for the metric in (2.3), which is uniformly equivalent to $D^{-2}\omega$ (under identifying the two hypersurfaces), and so

$$|z\partial_z|_\omega, |x_j\partial_{x_k}|_\omega < CD$$

for a constant C.

Region III. Here $R \in (K/2, 2K)$, and $K \in (\rho^\alpha, 2\rho^\alpha)$. We suppose $\rho \in (D/2, 2D)$, so $|z|$ is comparable to D. We choose a fixed z_0 such that $|z - z_0| < K$, and we change variables as follows:

$$\tilde{z} = K^{-1}(z - z_0), \quad \tilde{x} = K^{-1} \cdot x, \quad \tilde{r} = K^{-1}r.$$

In these coordinates X is given by the equation

$$K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0,$$

and we compare again to the product metric

$$\sqrt{-1}\partial\overline{\partial}(|\tilde{z}|^2 + \tilde{r}^2)$$

on the hypersurface $f(\tilde{x}) = 0$. We have $|\tilde{z}| < 1$, $\tilde{R} \in (1/2, 2)$, and $K^{-d}z_0 \to 0$ as $K \to \infty$, since $K^d \gg D$. This means that as $K, D \to \infty$, we can measure the norms of our vector fields on X_0 with the product metric. We have

$$z\partial_z = (K\tilde{z} + z_0)K^{-1}\partial_{\tilde{z}}, \quad x_j\partial_{x_k} = \tilde{x}_j\partial_{\tilde{x}_k}.$$

It follows that

$$|z\partial_z|_{K^{-2}\omega} < CK^{-1}D, \quad |x_j\partial_{x_k}|_{K^{-2}\omega} < C.$$

Since D is comparable to the distance from the origin, and $K \ll D$, this implies the estimate we want.

Region V. Here $R < 2\kappa^{-1}\rho^{1/d}$, $\rho \in (D/2, 2D)$, so $|z|$ is comparable to D. We choose a fixed point z_0 with $|z - z_0| < D^{1/d}$, so we also have $|z_0| \sim D$. We scale our metric down by a factor of $|z_0|^{1/d}$, and change coordinates by

$$\tilde{z} = z_0^{-1/d}(z - z_0), \quad \tilde{x} = z_0^{-1/d} \cdot x, \quad \tilde{r} = |z_0|^{-1/d} r.$$

We have $|\tilde{z}|, \tilde{r} < C$ for a uniform C. The equation of X is

$$z_0^{1/d-1}\tilde{z} + 1 + f(\tilde{x}) = 0,$$

and as $D \to \infty$, this converges to hypersurface with equation

$$1 + f(\tilde{x}) = 0,$$

and the metric $|z_0|^{-2/d}\omega$ converges to

$$\sqrt{-1}\partial\bar{\partial}\Big(|\tilde{z}|^2 + \varphi(\tilde{x})\Big). \tag{2.4}$$

We have

$$z\partial_z = z_0^{-1/d}(z_0^{1/d}\tilde{z} + z_0)\partial_{\tilde{z}}, \quad x_j\partial_{x_k} = \tilde{x}_j\partial_{\tilde{x}_k}.$$

The norms of these vector fields are uniformly bounded with respect to (2.4), and so scaling back up, we have

$$|z\partial_z|_\omega < |z_0|^{1/d}C|z_0|^{-1/d}|z_0| < CD, \quad |x_j\partial_{x_k}|_\omega < C|z_0|^{1/d} < CD^{1/d},$$

which gives the required bound. □

2.1 Subquadratic harmonic functions on $\mathbf{C} \times A_1$. Let us consider the tangent cone $C(Y) = \mathbf{C} \times A_1$ embedded in $\mathbf{C} \times \mathbf{C}^n$ as the hypersurface $x_1^2 + \cdots + x_n^2 = 0$, and equipped with the Stenzel cone metric $\sqrt{-1}\partial\bar{\partial}(|z|^2 + |x|^{2\frac{n-2}{n-1}})$. We need to understand the harmonic functions on $C(Y)$ with at most quadratic growth. In Hein–Sun [HS16] a general result is given on Calabi–Yau cones with isolated singularities, saying that the strictly subquadratic harmonic functions are all pluriharmonic (this was first used crucially in Conlon–Hein [CH13]), while the space of exactly quadratic growth harmonic functions decomposes as the sum of pluriharmonic functions and harmonic functions that arise from isometries of the link. See also Chiu [Chi19] for results in the case of more singular cones. We have the following.

Lemma 2.2. *The space $\mathcal{H}_{\leq 2}$ of real harmonic functions on $C(Y)$ with at most quadratic growth are given by linear combinations of the following:*

(1) *the real and imaginary parts of $1, z, z^2, x_i$,*

(2) *the function* $(n-1)|z|^2 - |x|^{2\frac{n-2}{n-1}}$,
(3) *the functions* $|x|^{-\frac{2}{n-1}} a_{jk} x_j \bar{x}_k$, *where* $(a_{jk}) \in \sqrt{-1}\mathfrak{o}(n, \mathbf{R})$ *is a purely imaginary complex orthogonal matrix.*

Proof. A general approach to this result is to extend Hein–Sun [HS16, Theorem 2.14] to singular tangent cones. This can be done along the lines of the work in Chiu [Chi19], using cutoff functions to justify the required integration by parts near the singular set.

Alternatively we can follow the approach from [Sze17, Corollary 12] using the Fourier transform in the $\mathbf{C}$-direction to analyze harmonic functions on the product $\mathbf{C} \times A_1$. The conclusion from this approach is that any harmonic function f of at most quadratic growth can be written as

$$f = f_0 + z f_1 + \bar{z} f_{\bar{1}} + z^2 f_2 + \bar{z}^2 f_{\bar{2}} + |z|^2 f_{1\bar{1}}$$

for functions $f_0, f_1, f_{\bar{1}}, f_2, f_{\bar{2}}, f_{1\bar{1}}$ on the cone A_1. We have

$$\Delta f = \Delta' f_0 + z\Delta' f_1 + \bar{z}\Delta' f_{\bar{1}} + z^2 \Delta' f_2 + \bar{z}^2 \Delta' f_{\bar{2}} + |z|^2 \Delta' f_{1\bar{1}} + f_{1\bar{1}},$$

where Δ' is the Laplacian on A_1. It follows from $\Delta f = 0$ that

$$\begin{aligned} \Delta' f_1 = \Delta' f_{\bar{1}} = \Delta' f_2 = \Delta' f_{\bar{2}} = \Delta' f_{1\bar{1}} = 0, \\ f_{1\bar{1}} + \Delta' f_0 = 0. \end{aligned}$$

In addition since f has at most quadratic growth, $f_1, f_{\bar{1}}, f_2, f_{\bar{2}}$ are all subquadratic harmonic functions, so by [HS16, Theorem 2.14] they are pluriharmonic. Since the non-constant holomorphic functions on A_1 have faster than linear growth, these functions must all be constant. The function $f_{1\bar{1}}$ is harmonic, and $|z|^2 f_{1\bar{1}}$ has at most quadratic growth. It follows that $f_{1\bar{1}} = c$ is constant. Then

$$c + \Delta' f_0 = 0,$$

so $f_0' = (n-1)f_0 - c|x|^{2\frac{n-2}{n-1}}$ is harmonic, and has at most quadratic growth. Using [HS16, Theorem 2.14] again we have that f_0' is a linear combination of real and imaginary parts of $1, x_i$, and functions u such that $V = \nabla u$ is a real holomorphic vector field on A_1 commuting with $r\partial_r$ such that $JV(r) = 0$. We then have $V = \mathrm{Re}(a_{jk} x_j \partial_{x_k})$ for a_{jk} a purely imaginary skew-symmetric matrix. Using the identity $\sqrt{-1}\partial\bar{\partial} u = L_{\nabla u}\sqrt{-1}\partial\bar{\partial} r^2 = \sqrt{-1}\partial\bar{\partial} V(r^2)$, up to adding a pluriharmonic function to u, we have

$$u = V(r^2) = V(|x|^{2\frac{n-2}{n-1}}) = \frac{n-2}{n-1}|x|^{-\frac{2}{n-1}} a_{jk} x_j \bar{x}_k.$$

The result follows from this. □

The functions in (1) are all the pluriharmonic functions of at most quadratic growth, while (2) and (3) correspond to automorphisms of $C(Y)$ commuting with the homothetic scaling, which has weights $(1, \frac{n-1}{n-2}, \dots, \frac{n-1}{n-2})$ on $(z, x_1, \dots, x_n)$. As in the proof the functions $|x|^{-\frac{2}{n-1}} a_{jk} x_j \bar{x}_k$ correspond to the vector fields $V_{\mathbf{a}} = \frac{n-1}{n-2}\mathrm{Re}(a_{jk} x_j \partial_{x_k})$, in the sense that

$$V_{\mathbf{a}}(|z|^2 + |x|^{2\frac{n-2}{n-1}}) = |x|^{-\frac{2}{n-1}} a_{jk} x_j \bar{x}_k.$$

These vector fields preserve the hypersurfaces $cz + x_1^2 + \dots + x_n^2 = 0$ for all c as well as the volume form Ω.

Similarly, the function $(n-1)|z|^2 - |x|^{2\frac{n-2}{n-1}}$ corresponds to the real holomorphic vector field $W = \mathrm{Re}((n-1)z\partial_z - \frac{n-1}{n-2} x_i \partial_{x_i})$, i.e.

$$W(|z|^2 + |x|^{2\frac{n-2}{n-1}}) = (n-1)|z|^2 - |x|^{2\frac{n-2}{n-1}}.$$

This vector field W preserves $C(Y) \subset \mathbf{C}^{n+1}$, however it does not preserve the hypersurfaces $cz + x_1^2 + \dots + x_n^2 = 0$. Instead we let

$$V = \mathrm{Re}\left(z\partial_z + \frac{1}{2} x_i \partial_{x_i}\right)$$

which does preserve all of these hypersurfaces. The vector field V satisfies

$$L_V \Omega = \frac{n}{2}\Omega,$$

and

$$V(|z|^2 + |x|^{2\frac{n-2}{n-1}}) - \frac{1}{2}(|z|^2 + |x|^{2\frac{n-2}{n-1}}) = \frac{1}{2}|z|^2 - \frac{1}{2(n-1)}|x|^{2\frac{n-2}{n-1}},$$

which is a scalar multiple of the function in (2). We conclude the following.

Lemma 2.3. *Suppose that h is a harmonic function on $C(Y)$ with at most quadratic growth, and write $h = h_{ph} + h_{aut}$, where h_{ph} is in the span of the type (1) functions in Lemma 2.2 and is pluriharmonic, while h_{aut} is in the span of the type (2) and (3) functions.*

We can find a real holomorphic vector field V preserving the hypersurfaces $cz + x_1^2 + \dots + x_n^2 = 0$, and a constant β such that $L_V\Omega = n\beta\Omega$, and

$$V(|z|^2 + |x|^{2\frac{n-2}{n-1}}) - \beta(|z|^2 + |x|^{2\frac{n-2}{n-1}}) = h_{aut}.$$

In addition we have $|\beta| \leq C\|h\|$ and $V = a_0 z\partial_z + a_{jk} x_j \partial_{x_k}$ with $|a_0|, |a_{jk}| \leq C\|h\|$ for a constant C, where $\|h\|$ denotes the L^2 norm on $B(0,1) \subset C(Y)$.

3 Special Embeddings

In this section (X, η) is a complete Calabi–Yau manifold such that X is biholomorphic to $\mathbf{C}^n$, and X has tangent cone $C(Y) = \mathbf{C} \times A_1$ at infinity. Let us fix a basepoint $p \in X$ and denote by B_i the ball $B(p, 2^i)$ with the metric $2^{-2i}\eta$, so that B_i is a unit ball in the scaled down metric. By assumption, the sequence B_i converges to the unit ball $B(0,1) \subset C(Y)$ in the Gromov–Hausdorff sense. We will view $C(Y) \subset \mathbf{C}^{n+1}$ as defined by the equation $x_1^2 + \cdots + x_n^2 = 0$ in terms of the coordinates z, x_i on $\mathbf{C}^{n+1}$. The cone $C(Y)$ is equipped with the Ricci-flat Stenzel metric given by

$$\sqrt{-1}\partial\overline{\partial}(|z|^2 + |x|^{2\frac{n-2}{n-1}}),$$

which has volume form $\sqrt{-1}^{n^2}\Omega \wedge \bar{\Omega}$ in terms of the Ω from (2.1). The main result of this section is the following.

Proposition 3.1. *There is a sequence of holomorphic embeddings*

$$F_i : X \to \mathbf{C}^{n+1}$$

with the following properties:

(1) *the image $F_i(X)$ is given by the equation*

$$a_i z + x_1^2 + \cdots + x_n^2 = 0$$

for some $a_i > 0$,

(2) *the volume form η^n satisfies*

$$2^{-2ni}\eta^n = F_i^*(\sqrt{-1}^{n^2}\Omega \wedge \bar{\Omega}),$$

(3) *$a_i/a_{i+1} \to 2^{n/(n-2)}$ as $i \to \infty$,*

(4) *on the ball B_i the map F_i gives a $\Psi(i^{-1})$-Gromov–Hausdorff approximation to the embedding $B(0,1) \to \mathbf{C}^{n+1}$. More precisely, we have a $\Psi(i^{-1})$-Gromov–Hausdorff approximation $g : B_i \to B(0,1)$ such that $|F_i - g| < \Psi(i^{-1})$ on B_i. Recall that here $B(0,1) \subset \mathbf{C}^{n+1}$ is the unit ball of $C(Y)$ under our embedding, and $\Psi(i^{-1})$ denotes a function converging to zero as $i \to \infty$.*

The main input for this result is the work of Donaldson–Sun [DS15] on the algebro-geometric study of tangent cones, and we first review the results that we use.

3.1 Donaldson–Sun theory. In [DS12, DS15], Donaldson–Sun consider non-collapsed Gromov–Hausdorff limits of compact polarized Kähler manifolds with bounded Ricci curvature. We observe that for many of the arguments compactness is not required (see also Liu [Liu17, Liu16] for related work in the non-compact setting). More precisely, suppose that $(M_i, L_i, \omega_i, p_i)$ is a sequence of complete pointed

n-dimensional Kähler manifolds with line bundles $L_i \to M_i$ equipped with Hermitian metrics with curvature $-\sqrt{-1}\omega_i$. In addition suppose that we have the Einstein condition $\mathrm{Ric}(\omega_i) = \lambda_i \omega_i$ with $|\lambda_i| \leq 1$, and the non-collapsing condition $\mathrm{Vol}(B(p_i, 1)) > \kappa > 0$ for all i, for a fixed $\kappa > 0$. If in addition we were to assume that the manifolds were compact, then the sequence would be in the class $\mathcal{K}(n, \kappa)$ considered in [DS15].

Let us suppose that (Z, p) is the pointed Gromov–Hausdorff limit of the sequence (M_i, ω_i, p_i). Note that, up to choosing a subsequence, such a limit exists by Gromov compactness. Then [DS15, Theorems 1.1, 1.2, 1.3] hold, i.e. Z has the structure of a normal complex analytic space, and tangent cones to Z have the structure of affine varieties and are unique. To see this note that the basic construction in [DS12] that is used in the arguments is to "graft" a holomorphic function from a tangent cone to Z onto M using cutoff functions for sufficiently large i, and then use the Hörmander L^2-estimate to perturb the resulting approximately holomorphic section of (a power of) L_i to a holomorphic section s. The grafting is a local construction, and the Hörmander estimate holds on complete Kähler manifolds (see e.g. [Dem, Theorem 4.5]). Finally one uses Moser iteration and Bochner–Weitzenbock type formulas to bound the L^∞-norms of s and ∇s in terms of the L^2-norm of s (see [DS12, Proposition 2.1]). Since under the non-collapsing condition and Ricci curvature lower bound we can control the Sobolev constant on geodesic balls (see e.g. Anderson [And92, Theorem 4.1]), the same estimates hold in our setting.

We need to use the results in [DS15, Section 3.4] where the assumptions are that the limit space (Z, p) is a scaled limit of a sequence in $\mathcal{K}(n, \kappa)$ with scaling factors tending to infinity. We claim, however, that the same results hold true for a complete, exact Calabi–Yau manifold (M, ω) with maximal volume growth. In other words, assuming that $\omega = \sqrt{-1}\partial\bar{\partial}\psi$ for a global Kähler potential ψ and $\mathrm{Vol}(B(p, r)) > \kappa r^{2n}$ for some $p \in M$, $\kappa > 0$ and all $r > 0$. The basic reason is that in this case the tangent cone at infinity is still the Gromov–Hausdorff limit of a sequence of polarized Kähler manifolds as above: for any sequence $\lambda_i \to 0$, we can consider the sequence $(M_i, \omega_i, L_i, p_i)$, where $M_i = M, \omega_i = \lambda_i^2 \omega, p_i = p$, and L_i is the trivial bundle equipped with the metric $e^{-\lambda_i^2 \psi}$. Up to choosing a subsequence, this sequence converges in the Gromov–Hausdorff sense to a tangent cone at infinity $C(Y)$ of (M, ω). Using this, the arguments in [DS15, Section 2.2] can be applied to the limit space $C(Y)$ (instead of (Z, p) in the statements of the propositions there) without any changes. In particular [DS15, Proposition 2.9] holds, showing that holomorphic functions on a ball in $C(Y)$ can be approximated by holomorphic functions on suitable balls in M_i. This is a crucial ingredient in [DS15, Proposition 3.26], which leads to the algebro-geometric description of the tangent cone $C(Y)$ in terms of the ring of polynomial growth holomorphic functions on (M, ω).

Let us briefly recall the results that we need from [DS15, Section 3.4], where for us M plays the role of Z there. If $R(M)$ denotes the ring of polynomial growth holomorphic functions on M, then $R(M)$ has a filtration

$$\mathbf{C} = I_0 \subset I_1 \subset \cdots \subset R(M).$$

Here I_k is the space of polynomial growth holomorphic functions on M with degree at most d_k, where $0 = d_0 < d_1 < \cdots$ are the possible growth rates. For a holomorphic function f on X, the growth rate $d(f)$ is defined by

$$d(f) = \lim_{r\to\infty} (\log r)^{-1} \sup_{B(p,r)} \log |f|,$$

and f has polynomial growth if $d(f) < \infty$. It follows from [DS15, Proposition 3.26] that these growth rates are the same as the possible growth rates on the tangent cone $C(Y)$ and the dimensions $\dim I_k$ are equal to the corresponding dimensions on $C(Y)$. Let us write R_{d_k} for the functions of degree d_k on $C(Y)$, and $\mu_k = \dim R_{d_k}$.

By [DS15, Proposition 3.26] we can find decompositions $I_k = I_{k-1} \oplus J_k$, where $\dim J_k = \mu_k$ and J_k admits an adapted sequence of bases. This means that, for fixed k, we have a sequence of bases $\{G^i_1, \ldots, G^i_{\mu_k}\}$ for J_k satisfying

(1) $\|G^i_a\|_{B_i} = 1$ for all a, and for $a \neq b$ we have $\lim_{i\to\infty} \int_{B_i} G^i_a \overline{G^i_b} = 0$. Here $\|\cdot\|_{B_i}$ denotes the L^2-norm on B_i, and as above, B_i is the ball $B(p, 2^i)$ scaled down to unit size.
(2) $G^{i+1}_a = \mu_{ia} G^i_a + p^i_a$ for scalars μ_{ia}, and $p^i_a \in \mathbf{C}\langle G^i_1, \ldots G^i_{a-1}\rangle$ with $\|p^i_a\|_{B_i} \to 0$ as $i \to \infty$.
(3) $\mu_{ia} \to 2^{d_k}$ as $i \to \infty$.

Suppose now that the coordinate ring $R(C(Y))$ is generated by $\bigoplus_{k\leq k_0} R_{d_k}$. It follows then that $R(M)$ is generated by $\bigoplus_{k\leq k_0} J_k$ and the adapted bases of J_k for $k \leq k_0$ define embeddings $F_i : M \to \mathbf{C}^N$. Furthermore, under the Gromov–Hausdorff convergence $B_i \to B(0,1) \subset C(Y)$, the components of the maps F_i converge to an L^2-orthonormal basis of $R_{d_0} \oplus \cdots \oplus R_{d_{k_0}}$ on $B(0,1)$ and these define an embedding $B(0,1) \to \mathbf{C}^N$.

3.2 Proof of Proposition 3.1. We now specialize to the setting of Proposition 3.1. It will be helpful to write down the homogeneous holomorphic functions of low degree on $C(Y) = \mathbf{C} \times A_1$. Note that they are all spanned by polynomials in z, x_i, and $d(z) = 1, d(x_i) = \frac{n-1}{n-2}$. We treat three cases separately:

- $n = 3$. In this case we have

$$\begin{aligned} R_0 &= \langle 1\rangle, \\ R_1 &= \langle z\rangle, \\ R_2 &= \langle z^2, x_i\rangle, \\ R_3 &= \langle z^3, zx_i\rangle, \\ R_4 &= \langle z^4, z^2x_i, x_ix_j\rangle, \end{aligned} \tag{3.1}$$

  where in R_4 one term is redundant because of the equation $x_1^2 + \cdots + x_n^2 = 0$.

- $n = 4$. Here we have

$$\begin{aligned} R_0 &= \langle 1 \rangle, \\ R_1 &= \langle z \rangle, \\ R_{3/2} &= \langle x_i \rangle, \\ R_2 &= \langle z^2 \rangle, \\ R_{5/2} &= \langle z x_i \rangle, \\ R_3 &= \langle z^3, x_i x_j \rangle, \end{aligned}$$

where again one term in R_3 is redundant.

- $n > 4$.

$$\begin{aligned} R_0 &= \langle 1 \rangle, \\ R_1 &= \langle z \rangle, \\ R_{\frac{n-1}{n-2}} &= \langle x_i \rangle, \\ R_2 &= \langle z^2 \rangle, \\ R_{\frac{2n-3}{n-2}} &= \langle z x_i \rangle, \\ R_{\frac{2n-2}{n-2}} &= \langle x_i x_j \rangle, \end{aligned}$$

where again one term in $R_{\frac{2n-2}{n-2}}$ is redundant.

For simpicity we focus on the case $n = 3$. The discussion in the other cases is completely analogous. The ring $R(C(Y))$ is generated by $R_1 \oplus R_2$, and so by the results of Donaldson–Sun [DS15] discussed above, $R(X)$ is generated by $I_2 = J_0 \oplus J_1 \oplus J_2$. This space of holomorphic functions on X with at most quadratic growth has $\dim I_2 = 6$ and admits a sequence of adapted bases $\{G_1^i, \ldots, G_6^i\}$. The growth rates of the functions are $d(G_1^i) = 0, d(G_2^i) = 1$, and $d(G_j^i) = 2$ for $j = 3, 4, 5, 6$, and for each i we obtain an embedding

$$F_i : X \to \mathbf{C}^6$$

with components G_j^i. On the balls B_i these maps converge in the Gromov–Hausdorff sense to an embedding $F_\infty : B(0,1) \to \mathbf{C}^6$. The map F_∞ is given by functions on $C(Y)$ with the degrees specified above that are orthonormal on $B(0,1)$. Up to a unitary transformation commuting with the homothetic scaling (which has degrees $(0,1,2,2,2,2)$) we can assume that $F_\infty = (1, z, z^2, x_1, x_2, x_3)$. We can modify our sequence of adapted bases by the same unitary transformation, so that we still have $F_i \to F_\infty$ as $i \to \infty$.

Since I_0 consists of just the constants, the first component of F_i is constant. In addition, we have $\|(G_2^i)^2 - G_3^i\|_{B_i} \to 0$ by the convergence of F_i to F_∞, while also $(G_2^i)^2 - G_3^i \in I_2$. It follows that $G_3^i = (G_2^i)^2 + \sum_a q_{ia} G_a^i$, where $|q_{iq}| < \Psi(i^{-1})$. Therefore dropping the first and third components of F_i, we still obtain embeddings

$F_i' : X \to \mathbf{C}^4$. Let us now fix i, and abusing notation, let us denote by z, x_1, x_2, x_3 the pullbacks under F_i' to X of the coordinate functions on $\mathbf{C}^4$. By construction we have $d(z) = 1, d(x_i) = 2$. The 20 functions

$$1, z, z^2, x_i, z^3, zx_i, z^4, z^2x_j, x_jx_k$$

are all in the space I_4 of at most quartic growth functions on X, but by the earlier discussion $\dim I_4 = 19$. Therefore we have one linear dependency between them, which determines the equation f_i in $\mathbf{C}^4$ defining $F_i'(X)$. Because of the Gromov–Hausdorff convergence of the F_i' to the embedding of $B(0,1)$ satisfying $x_1^2+x_2^2+x_3^2 = 0$, the equation $f_i(z, x_1, x_2, x_3) = 0$ has to be a perturbation of this equation. We can apply a linear transformation in x_1, x_2, x_3 that is $\Psi(i^{-1})$-close to the identity to transform the quadratic expression in the x_j that appears in f_i to the quadric $x_1^2 + x_2^2 + x_3^2$. Next we can complete the square in the x_j, applying changes of coordinates of the form $x_j \mapsto x_j + a_jz^2 + b_jz + c_j$ with small a_j, b_j, c_j, to eliminate all terms of the form x_j, zx_j, z^2x_j in f_i. We have now reduced our equation to one of the form

$$\tilde{f}_i(z) + x_1^2 + x_2^2 + x_3^2 = 0, \tag{3.2}$$

where $\tilde{f}_i(z)$ is a quartic polynomial in z with coefficients of order $\Psi(i^{-1})$. Since X is biholomorphic to $\mathbf{C}^3$, $\tilde{f}_i$ must actually be linear, so $f_i(z) = d_iz + e_i$ for $|d_i|, |e_i| < \Psi(i^{-1})$ with $d_i \neq 0$. We can assume that $d_i > 0$ by multiplying z by a unit complex number.

So far we have only performed coordinate changes $\Psi(i^{-1})$-close to the identity, and so the new maps $F_i'' : X \to \mathbf{C}^4$ that we obtain still converge on B_i, in the Gromov–Hausdorff sense, to our standard embedding $B(0,1) \to \mathbf{C}^4$. The image $F_i''(X)$ satisfies the equation

$$e_i + d_iz + x_1^2 + x_2^2 + x_3^2 = 0.$$

We want to perform a further coordinate change $z \mapsto z + d_i^{-1}e_i$ so that the equation reduces to $d_iz + x_1^2 + x_2^2 + x_3^2 = 0$. However, this may not be a small coordinate change since we do not yet have information about the relative sizes of d_i, e_i. For this we need the following.

LEMMA 3.2. *Suppose that we have polynomial growth holomorphic functions w, y_1, y_2, y_3 on X such that $d(w) = 1$ and $d(y_j) = 2$, and*

$$cw + y_1^2 + y_2^2 + y_3^2 = 0$$

for some $c \neq 0$. Then there exists a point $o \in X$ such that $w(o) = y_j(o) = 0$, and moreover this point is independent of the choice of functions w, y_j as above.

Proof. In the above discussion we have already constructed functions z, x_j satisfying

$$c'z + x_1^2 + x_2^2 + x_3^2 = 0.$$

Let us denote by $o \in X$ the point where $z(o) = x_i(o) = 0$. In terms of these functions, because of the degree restrictions, we must have

$$\begin{aligned} w &= az + b, \\ \mathbf{y} &= A\mathbf{x} + \mathbf{c}z^2 + \mathbf{d}z + \mathbf{e}. \end{aligned}$$

Here $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{c}, \mathbf{d}, \mathbf{e}$ are vectors, and A is a matrix. By scaling we can assume that $c = c' = 1$ to simplify notation. We have

$$0 = w + \mathbf{y}^T\mathbf{y} = az + b + (A\mathbf{x} + \mathbf{c}z^2 + \mathbf{d}z + \mathbf{e})^T(A\mathbf{x} + \mathbf{c}z^2 + \mathbf{d}z + \mathbf{e}). \tag{3.3}$$

By examining the dimensions of the spaces R_{d_k} in (3.1) we see that, up to scalar multiples, there is only one linear equation satisfied by the functions

$$1, z, z^2, x_j, z^3, zx_j, z^4, z^2x_j, x_jx_k,$$

namely $z + \mathbf{x}^T\mathbf{x} = 0$. Therefore if we also have $w + \mathbf{y}^T\mathbf{y} = 0$, then from (3.3) we first find that A^TA is a multiple of the identity, and in particular A is invertible. It then follows that we have $\mathbf{c} = \mathbf{d} = \mathbf{e} = 0$, and in turn $b = 0$. Therefore $w(o) = y_j(o) = 0$, and conversely o is the unique point where w, y_j all vanish. □

Let us return to the proof of Proposition 3.1 and the map $F_i'' : X \to \mathbf{C}^4$ whose image satisfies the equation

$$d_i(z + d_i^{-1}e_i) + x_1^2 + x_2^2 + x_3^2 = 0.$$

By Lemma 3.2, we have $z(o) = -d_i^{-1}e_i$. However, since F_i converges to the embedding $B(0, 1) \to \mathbf{C}^4$, we also have $F_i(o) \to 0$. For this, note that under the rescaled metric in the ball B_i (centered at p) the point o is contained in a ball of radius $C2^{-i}$ around p, where $C = \mathrm{dist}_\eta(p, o)$. In particular we have $d_i^{-1}e_i \to 0$, and so we can perform a final small coordinate change to reduce to the case $e_i = 0$.

Next we consider the volume forms. The pullback $(F_i'')^*(\Omega)$ defines a polynomial growth, nowhere vanishing holomorphic volume form on X. By the Calabi–Yau condition on η,

$$F_i''^*(\sqrt{-1}\Omega \wedge \bar{\Omega}) = |g_i|^2\eta^3,$$

where g_i is a nowhere vanishing polynomial growth holomorphic function on X. This means that g_i can be written as a polynomial in x_1, x_2, x_3, and therefore it is constant since it has no zeros. By the volume convergence under Gromov–Hausdorff convergence [Col97], remembering that we are using the scaled down metric $2^{-2i}\eta$ in the convergence, we find that $|g_i|^2 2^{6i} \to 1$ as $i \to \infty$. Scaling z by a factor $\Psi(i^{-1})$-close to 1, we can arrange that $|g_i|^2 = 2^{-6i}$.

Finally we address the claim that $a_i/a_{i+1} \to 8$. Let us denote the components of our maps F_i'' by $z_i, x_{i,1}, x_{i,2}, x_{i,3}$, so that

$$a_iz_i + x_{i,1}^2 + x_{i,2}^2 + x_{i,3}^2 = 0$$

with $a_i > 0$. From the proof of Lemma 3.2 we know that

$$\begin{aligned} z_{i+1} &= c_i z_i, \\ \mathbf{x}_{i+1} &= \sqrt{\frac{a_{i+1}c_i}{a_i}} A_i \mathbf{x}_i, \end{aligned}$$

for a constant c_i and a complex orthogonal matrix A_i. Using that $d(z_i) = 1, d(x_{i,j}) = 2$, and that the $z_i, x_{i,j}$ have normalized L^2-norm approximately equal to 1 on B_i we have $c_i \to 2^{-1}$ and $\sqrt{a_i^{-1}a_{i+1}c_i} \to 2^{-2}$ as $i \to \infty$. It follows that $a_i^{-1}a_{i+1} \to 2^{-3}$.

The cases $n = 4$ and $n > 4$ can be treated in an almost identical way, except that in both cases we have $\dim I_2 = 5$, so we do not have to worry about the function z^2. In addition in the equation analogous to (3.2), the degree of $\tilde{f}_i$ is at most cubic when $n = 4$, and at most quadratic when $n > 4$. We leave the details to the reader.

4 Decay of the Kähler Potential

In this section, we let $\omega = c^2\omega_0$ be a scaled down copy of the reference Calabi–Yau metric on $\mathbf{C}^n$, and we denote by $o \in (\mathbf{C}^n, \omega)$ the origin. For a given $\epsilon > 0$ we will assume that $c < 1$ is sufficiently small so that $d_{GH}(B(o, \epsilon^{-1}), B(0, \epsilon^{-1})) < \epsilon$, where $0 \in C(Y)$ is the origin in the cone $C(Y) = \mathbf{C} \times A_1$. Since $C(Y)$ is the tangent cone at infinity, this condition is equivalent to the Gromov–Hausdorff distance $d_{GH}(B(o, 1), B(0, 1))$ being sufficiently small. As in Section 2 we have an embedding $F_c : \mathbf{C}^n \to \mathbf{C}^{n+1}$ using the functions $cz, c^{\frac{n-1}{n-2}}x_i$, with image given by the equation

$$c^{\frac{n}{n-2}}z + x_1^2 + \cdots + x_n^2 = 0.$$

In addition the maps F_c on $B(o, 1)$ converge in the Gromov–Hausdorff sense to the standard embedding $B(0, 1) \to \mathbf{C}^{n+1}$ of the unit ball in $C(Y)$ as $c \to 0$.

The main technical result of the paper is the following.

Proposition 4.1. *There are λ_0, α such that if $\lambda < \lambda_0$ and ϵ is sufficiently small (depending on λ), then we have the following. Suppose that c above is small enough so that $d_{GH}(B(o, \epsilon^{-1}), B(0, \epsilon^{-1})) < \epsilon$, and we have a smooth function u on $B(o, 1)$ satisfying $\sup_{B(o,1)} |u| < \epsilon$ and*

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = \omega^n.$$

Then we can find a constant β, an automorphism g of $\mathbf{C}^n$ fixing the origin o, and a smooth function u' on $B(o, 1/2)$ satisfying

(1) $\beta g^*(\omega + \sqrt{-1}\partial\overline{\partial}u) = \omega + \sqrt{-1}\partial\overline{\partial}u'$,
(2) $(\beta g^*\omega)^n = \omega^n$,
(3) $\sup_{B(o,\lambda)} |u'| \leq \lambda^{2+\alpha} \sup_{B(o,1)} |u|$.

In this work in property (3) the constant λ^2 would suffice, however we expect that the result has other applications that need the better constant $\lambda^{2+\alpha}$. The strategy of the proof is to show that once ϵ is sufficiently small, the function u is close to a harmonic function on $C(Y)$. We then modify u by subtracting a pluriharmonic function and applying an automorphism to get a function u' with faster than quadratic decay using Lemma 2.3. The main difficulty is to control the behavior of u near the singular set of $C(Y)$. We will achieve this by using the maximum principle with a suitable barrier function in order to show that if u concentrates near the singular set, then u decays rapidly when passing from $B(o,1)$ to $B(o,1/2)$. Note that much of the argument works in more general settings than what we are considering. The one place where we will use the fairly explicit form of the reference metric ω_0 on $\mathbf{C}^n$ is when we need to control the action of the automorphisms on the Kähler potential of ω near the singular set using Proposition 2.1.

To begin, we note that on any compact set away from the singular set of $C(Y)$ we can apply the small perturbation result of Savin [Sav07] to get regularity of $|u|$ once ϵ is sufficiently small. For $\theta > 0$ we define the set N_θ to be the θ-neighborhood of the singular set under the Gromov–Hausdorff approximation:

$$N_\theta = \{(z,\mathbf{x})\,|\,2^{1/2}|x|^{\frac{n-2}{n-1}} < \theta\},$$

where we note that on $C(Y)$ the function $2|x|^{2\frac{n-2}{n-1}}$ is the distance squared from the singular set with respect to the Ricci-flat cone metric.

Lemma 4.2. *Let $\theta > 0$. There exist $C_k > 0$ depending on θ with the following property. If, in the setting of Proposition 4.1, ϵ is sufficiently small (depending on θ), then on $B(o,1-\theta)\backslash N_\theta$ we have*

$$|u|_{C^k(B(o,1-\theta)\backslash N_\theta)} < C_k \sup_{B(o,1)} |u|.$$

Proof. Note that by the Cheeger–Colding theory [CC97] and Anderson's epsilon regularity result [And90], we can bound the harmonic radius of ω on $B(o,1-\theta)\backslash N_\theta$ once ϵ is sufficiently small. Then for any $\delta > 0$, if $\sup_{B(o,1)}|u|$ is sufficiently small, we can apply Savin's result [Sav07] in harmonic coordinates to obtain $|u|_{C^3(B(o,1-\theta/2))} < \delta$. We have the equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = \omega^n,$$

which we can write as

$$\left[n\omega^{n-1} + \binom{n}{2}\omega^{n-2}\wedge\sqrt{-1}\partial\overline{\partial}u + \cdots + (\sqrt{-1}\partial\overline{\partial}u)^{n-1}\right]\wedge\sqrt{-1}\partial\overline{\partial}u = 0. \quad (4.1)$$

If δ is sufficiently small (so that $\sqrt{-1}\partial\overline{\partial}u$ is small), then we can view this as a uniformly elliptic homogeneous equation for u with C^α coefficients (the expression in square brackets determining the coefficients of the equation). The Schauder estimates imply that on $B(o,1-3\theta/4)$ we have the bound $|u|_{C^{2,\alpha}} \le C_2 \sup_{B(o,1)}|u|$. We can bootstrap this estimate to obtain the required higher order bounds. □

We will apply this result in the following form several times.

LEMMA 4.3. *Suppose that we have a sequence of functions u_i as in Proposition 4.1 such that $\epsilon_i \to 0$. Let $v_i = (\sup_{B(o_i,1)} |u_i|)^{-1} u_i$. Then there is a (possibly vanishing) bounded harmonic function h on $B(0,1) \subset C(Y)$ such that under the Gromov–Hausdorff convergence, for a subsequence, we have $v_i \to h$ in C^∞ uniformly on $B(o_i, 1-\theta)\backslash N_\theta$ for any $\theta > 0$.*

Proof. By Lemma 4.2, for any given $\theta > 0$ the functions v_i satisfy uniform C^∞ bounds on $B(o_i, 1-\theta)\backslash N_\theta$ for sufficiently large i, and they are all bounded by 1 on $B(o_i, 1)$. By a diagonal argument we can find a subsequence such that $v_i \to h$ for a function h on $B(0,1) \subset C(Y)$, with the convergence taking place in C^∞ on any $B(o_i, 1-\theta)\backslash N_\theta$.

To see that h is harmonic, note that just like the equation (4.1), v_i satisfies

$$\left[n\omega_i^{n-1} + \binom{n}{2} \omega_i^{n-2} \wedge \sqrt{-1}\partial\overline{\partial} u_i + \cdots + (\sqrt{-1}\partial\overline{\partial} u_i)^{n-2} \right] \wedge \sqrt{-1}\partial\overline{\partial} v_i = 0.$$

By Lemma 4.2, on $B(o_i, 1-\theta)\backslash N_\theta$ we have $u_i \to 0$ in C^∞, and so passing to the limit in these equations we get that h is harmonic on $B(0, 1-\theta)\backslash N_\theta$. This holds for any θ, and so h is harmonic on the regular part of $B(0,1)$. In addition h is bounded, so it is harmonic in a weak sense across the singular set too. □

4.1 Construction of a barrier function. We suppose that we are in the setting of Proposition 4.1. The following provides the barrier function used in the maximum principle argument below.

PROPOSITION 4.4. *There is a constant $D > 0$ with the following property. Let $\theta > 0$. There is a constant $C_\theta > 0$ such that if ϵ is sufficiently small (depending on θ), then there is a smooth real-valued function v on $B(o,1)$ satisfying the following properties:*

1. *$|\partial\bar{\partial} v|_\omega < C_\theta$ on $B(o, 1-\theta/2)$.*
2. *$v(q) > D^{-1}\theta^{-1/2}$ whenever $q \in N_\theta \cap \partial B(o, 1-\theta)$.*
3. *$v > D^{-1}$ on $B(o,1)$, and $v < D$ on $B(o, 1/2)$.*
4. *On $B(o, 1-\theta/2)$ the function v satisfies the differential inequality*

$$\sum_i \mu_i + \mu_{max} < 0,$$

where the μ_i are the eigenvalues of $\sqrt{-1}\partial\overline{\partial} v$ relative to ω, and μ_{max} is the largest eigenvalue.

Proof. Let $(z, 0)$ be a point in the singular set of $C(Y)$ with $|z| = 1$, and let $q \in B(o, 2)$ be within ϵ of $(z,0)$ under the Gromov–Hausdorff approximation. Using that $C(Y)$ is a cone also when centered at $(z,0)$, once ϵ is sufficiently small we can apply [LS18, Proposition 3.1] to find a good Kähler potential φ for ω on $B(q,3)$ in the sense that we have $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$, while also

$$|\varphi - d(q,\cdot)^2/2| < \Psi(\epsilon) \tag{4.2}$$

on $B(q,3)$. By adding a constant of order $\Psi(\epsilon)$ we can assume that $\varphi > 0$. We have $\Delta\varphi = n$ (using the complex Laplacian), and from the Cheeger–Colding estimate [CC96] together with the Cheng–Yau gradient estimate [CY75] we have

$$\int_{B(q,2)} \left| |\nabla^{1,0}\varphi|^2 - \varphi \right|^2 < \Psi(\epsilon),$$

where we emphasize that we are taking the $(1,0)$-part of the derivative of φ. At the same time from the Bochner formula (computing in normal coordinates),

$$\Delta|\nabla^{1,0}\varphi|^2 = (\varphi_i\varphi_{\bar{i}})_{j\bar{j}} = \varphi_{ij}\overline{\varphi_{ij}} + \varphi_{i\bar{j}}\overline{\varphi_{i\bar{j}}} \geq n,$$

since $\varphi_{i\bar{j}} = \omega_{i\bar{j}}$. So

$$\Delta\Big(|\nabla^{1,0}\varphi|^2 - \varphi\Big) \geq 0,$$

and the mean value inequality implies that

$$|\nabla^{1,0}\varphi|^2 \leq \varphi + \Psi(\epsilon).$$

Let us now consider the function $\varphi^{-3/4}$. We have

$$\sqrt{-1}\partial\bar{\partial}\varphi^{-3/4} = -\frac{3}{4}\varphi^{-7/4}\sqrt{-1}\partial\bar{\partial}\varphi + \frac{21}{16}\varphi^{-11/4}\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi.$$

Fix $y \in B(o,1)$. We can choose orthonormal coordinates for ω at y such that $\partial\varphi = \varphi_1 dz^1$ and $\sqrt{-1}\partial\bar{\partial}\varphi$ is the identity matrix. By the estimate above, we have $|\varphi_1|^2 \leq \varphi + \Psi(\epsilon)$. The eigenvalues of $\sqrt{-1}\partial\bar{\partial}\varphi^{-3/4}$ therefore satisfy

$$\begin{aligned} -\frac{3}{4}\varphi^{-7/4} \leq \mu_1 &\leq -\frac{3}{4}\varphi^{-7/4} + \frac{21}{16}\varphi^{-11/4}(\varphi + \Psi(\epsilon)), \\ \mu_2, \ldots, \mu_n &= -\frac{3}{4}\varphi^{-7/4}. \end{aligned}$$

The maximum eigenvalue is necessarily μ_1 and so

$$\begin{aligned} \sum \mu_i + \mu_{max} &= 2\mu_1 + (n-1)\mu_2 \leq 2\mu_1 + 2\mu_2 \\ &\leq -3\varphi^{-7/4} + \frac{21}{8}\varphi^{-11/4}(\varphi + \Psi(\epsilon)) \\ &= \varphi^{-11/4}\left(-\frac{3}{8}\varphi + \frac{21}{8}\Psi(\epsilon)\right). \end{aligned}$$

As long as $\varphi > 7\Psi(\epsilon)$ we obtain

$$\sum \mu_i + \mu_{max} < 0. \tag{4.3}$$

In particular by (4.2) this holds if $y \in B(o, 1-\theta)$ and ϵ is sufficiently small (depending on θ).

We also have $\varphi > \theta^2/4$ on $B(o, 1-\theta)$ once ϵ is sufficiently small, and so from the bounds for the eigenvalues we have $|\sqrt{-1}\partial\bar{\partial}\varphi^{-3/4}| < C_\theta$ on $B(o, 1-\theta)$ (where C_θ is of order $\theta^{-7/2}$, although we do not need this). On $B(o,1/2)$ we have $\varphi > 1/10$ once ϵ is sufficiently small, and this leads to an upper bound $\varphi^{-3/4} < 10^{3/4}$ on $B(o,1/2)$. At the same time $\varphi < 1$ on $B(o,1)$ for sufficiently small ϵ, and so $\varphi^{-3/4} > 1$ on $B(o,1)$.

Note that once ϵ is sufficiently small, we have $\varphi < 16\theta^2$ on $B(q,4\theta)$, and so

$$\varphi^{-3/4} > 16^{-3/4}\theta^{-3/2} \text{ on } B(q,4\theta).$$

This means $\varphi^{-3/4}$ is large near q, but we want a function that is large at all points in $N_\theta \cap \partial B(o,1-\theta)$. To achieve this, we define v to be an average of functions $\varphi^{-3/4}$ constructed for different points in $S^1 \times \{0\} \in \mathbf{C} \times A_1$. For a given $\theta > 0$, we pick $z_1, \dots, z_K$ on the unit circle, and $q_i \in B(o,2)$ which are ϵ-close to $(z_i, 0) \in C(Y)$ under our Gromov–Hausdorff approximation, so that the 4θ-balls $B(q_i, 4\theta)$ cover $N_\theta \cap \partial B(o, 1-\theta)$. For sufficiently small ϵ we can achieve this with $K < c\theta^{-1}$ for a uniform c. We consider the functions $\varphi_i^{-3/4}$ constructed as above, based at the points q_i, and define

$$v = \frac{1}{K}\sum_{i=1}^{K} \varphi_i^{-3/4}.$$

Then v satisfies the required properties:

(1) $|\sqrt{-1}\partial\overline{\partial}v|_\omega < C_\theta$, since we are taking an average of functions that satisfy this estimate.

(2) If $q \in N_\theta \cap B(o, 1-\theta)$, then by assumption, there is a q_i such that $q \in B(q_i, 4\theta)$. It follows that

$$v(q) > \frac{1}{K}\varphi_i^{-3/4}(q) > c^{-1}\theta \cdot 16^{-3/4}\theta^{-3/2} = c^{-1}16^{-3/4}\theta^{-1/2},$$

which gives the required lower bound.

(3) We have $\varphi_i^{-3/4} > 1$ on $B(o,1)$ for all i, so v satisfies the same estimate. Similarly, $\varphi_i^{-3/4} < 10^{3/4}$ on $B(o,1/2)$, and so v satisfies the same.

(4) Each $\varphi_i^{-3/4}$ satisfies the required differential inequality (4.3) on $B(o, 1-\theta/2)$, and the expression $\sum_i \mu_i + \mu_{max}$ is convex on the space of Hermitian matrices. Therefore v satisfies the same differential inequality. $\square$

Note that this result can easily be generalized to other cones of the form $C(Y) = \mathbf{C}^k \times C(Y')$, where Y' has an isolated singularity, but we have crucially used that all singular points in $C(Y)$ can be taken to be a vertex of $C(Y)$. We still expect that with some additional work a similar barrier function can be constructed for more general cones.

We now use the maximum principle to obtain the following important decay property.

PROPOSITION 4.5. *There is a constant $C > 0$ with the following property. Let $A > 10$. There exists $\theta > 0$ depending on A such that, if in the setting of Proposition 4.1 ϵ is sufficiently small (depending on A, θ) and*

$$\sup_{B(o,1)\setminus N_\theta} |u| \le \tau < \epsilon,$$
$$\sup_{B(o,1)} |u| \le A\tau,$$

then we have

$$\sup_{B(o,1/2)} |u| \le C\tau.$$

Note that C does not depend on A, θ.

We first need the following lemma.

LEMMA 4.6. *Suppose that $\mu_i > -1$ are constants for $i = 1, \dots, n$ such that*

$$\prod_{i=1}^{n} (1 + \mu_i) = 1.$$

Then there is a $\delta_0 > 0$ depending only on n such that if $\mu_i < \delta_0$ for all i, then

$$\sum_{i=1}^{n} \mu_i + \mu_{max} \ge 0,$$

while if $\mu_i > -\delta_0$ for all i, then

$$\sum_{i=1}^{n} \mu_i + \mu_{min} \le 0.$$

Here μ_{max}, μ_{min} are the largest and smallest of the μ_i, respectively.

Note that the stronger inequality $\sum_i \mu_i \ge 0$ follows directly from the inequality between arithmetic and geometric means without assuming any upper bound on the μ_i, but we prefer to state the above in a way that the lower and upper bounds are symmetric.

Proof. Suppose that $\mu_{max} = \delta < \delta_0$. For all j we have

$$1 + \mu_j = \frac{1}{\prod_{i \ne j} (1 + \mu_i)} \ge (1 + \delta)^{-(n-1)}.$$

We can choose δ_0 so that if $0 < \delta < \delta_0$, then

$$(1 + \delta)^{-(n-1)} \ge 1 - n\delta,$$

so we find that $\mu_j \geq -n\delta$ for all j. Then if $\delta < 1$ we have

$$1 = \prod_{i=1}^{n}(1+\mu_i) \leq 1 + \sum_{i=1}^{n} \mu_i + C_n\delta^2$$

for a constant C_n depending only on n, since the remaining terms are all at least quadratic in the μ_i. It follows that

$$\sum_{i=1}^{n} \mu_i + \mu_{max} \geq -C_n\delta^2 + \delta.$$

Finally, if δ is sufficiently small, then $C_n\delta^2 \leq \delta$. The argument for the second statement is completely analogous. □

Proof of Proposition 4.5. Let us choose $\theta = A^{-2}$ and assume ϵ is sufficiently small to apply Proposition 4.4. Let $\Lambda > 0$ satisfy

$$\tau = \Lambda^{-1}D^{-1},$$

so that

$$\begin{aligned} \sup_{B(o,1)\setminus N_\theta} |u| &\leq \Lambda^{-1}D^{-1}, \\ \sup_{B(o,1)} |u| &\leq A\tau = \Lambda^{-1}D^{-1}\theta^{-1/2}. \end{aligned}$$

In terms of v given by Proposition 4.4, set $\tilde{v} = \Lambda^{-1}v$, so that $\tilde{v} > u$ on $\partial B(o, 1-\theta)$ by properties (2) and (3). We assume in addition that ϵ is sufficiently small (and so τ is sufficiently small) so that $\Lambda^{-1}C_\theta < \delta_0$ for the δ_0 from Lemma 4.6. □

Claim. $\tilde{v} > u$ on $B(o, 1-\theta)$.

Proof of Claim. If this were not the case, then setting

$$t_0 = \inf\{t > 0 \,:\, \tilde{v} + t > u \text{ on } B(o, 1-\theta)\},$$

the graph of $\tilde{v} + t_0$ will lie above the graph of u, and the two graphs will touch at a point $q \in B(o, 1-\theta)$. At q we must have $\sqrt{-1}\partial\overline{\partial}u(q) \leq \sqrt{-1}\partial\overline{\partial}\tilde{v}(q)$. In orthonormal coordinates at q we have $\sqrt{-1}\partial\overline{\partial}\tilde{v}(q) \leq \Lambda^{-1}C_\theta\mathrm{Id}$ by property (1) in Proposition 4.4, and so the eigenvalues μ_i of $\sqrt{-1}\partial\overline{\partial}u(q)$ are bounded above by $\Lambda^{-1}C_\theta$. Since $\Lambda^{-1}C_\theta < \delta_0$, Lemma 4.6 implies that $\sum \mu_i + \mu_{max} \geq 0$, but this contradicts property (4) in Proposition 4.4, since $\sqrt{-1}\partial\overline{\partial}u(q) \leq \sqrt{-1}\partial\overline{\partial}\tilde{v}(q)$. This proves the claim.

Using the claim, it follows from property (3) that

$$\sup_{B(o,1/2)} u \leq \Lambda^{-1}D = D^2\tau,$$

which gives the required upper bound for u.

The lower bound for u is proved similarly, the only difference being that we compare u to the function $-\Lambda^{-1}v$ instead and we use the second statement in Lemma 4.6. □

4.2 Proof of Proposition 4.1. We prove Proposition 4.1 by contradiction. Let us suppose that we have a sequence u_i as in the statement of the proposition on balls $B(o_i,1)$ with corresponding constants $\epsilon_i \to 0$ such that the conclusion of the proposition fails. Here $o_i \in (\mathbf{C}^n, \omega_i)$ is the origin. We will show that along a subsequence, for sufficiently large i we can find β_i, g_i, u_i' satisfying the required properties, giving a contradiction.

Step 1. Let us write $\kappa_i = \sup_{B(o_i,1)} |u_i| \to 0$. By Lemma 4.3 we have a harmonic function h on $B(0,1) \in C(Y)$ such that, after choosing a subsequence, $\kappa_i^{-1} u_i \to h$ in C^∞ uniformly on $B(o_i, 1-\theta)\setminus N_\theta$ for any $\theta > 0$. Note that we have $|h| \leq 1$, and it is possible that $h = 0$. Let us write $h = h^{\leq 2} + h^{>2}$ for the decomposition of h into pieces with at most quadratic and faster than quadratic growth. In addition we decompose $h^{\leq 2} = h_{ph} + h_{aut}$, where h_{ph} is pluriharmonic, and h_{aut} is in the span of the functions of type (2) and (3) in Lemma 2.2. From Lemma 2.3 we obtain a real holomorphic vector field V on $\mathbf{C}^{n+1}$ preserving the hypersurfaces

$$az + x_1^2 + \cdots + x_n^2 = 0,$$

and a constant β such that $L_V\Omega = n\beta\Omega$, and at the same time on $C(Y) \subset \mathbf{C}^4$ we have

$$V(|z|^2 + |x|^{2\frac{n-1}{n-2}}) - \beta(|z|^2 + |x|^{2\frac{n-1}{n-2}}) = h_{aut}.$$

We define the automorphism $g_i = \exp(\kappa_i V)$ on $\mathbf{C}^{n+1}$ and the constants $\beta_i = \kappa_i\beta$.

For any $\theta > 0$, the spaces $B(o_i,1)\setminus N_\theta$ converge smoothly to $B(0,1)\setminus N_\theta$ inside $\mathbf{C}^{n+1}$, and for sufficiently large i we can use the nearest point projection to identify them. On $B(o_i,1)$ we have Kähler potentials φ_i for ω_i which converge smoothly to $|z|^2 + |x|^{2\frac{n-1}{n-2}}$ as $i \to \infty$, uniformly on $B(o_i,1)\setminus N_\theta$ for any $\theta > 0$. It follows from this that

$$\sup_{B(o_i,1)\setminus N_\theta} \left| \kappa_i V\varphi_i - \beta_i\varphi_i - \kappa_i h_{aut} \right| \leq \kappa_i \Psi(i^{-1}\,|\,\theta),$$

and so we also have

$$\sup_{B(o_i,1)\setminus N_\theta} \left| e^{-\beta_i} g_i^*\varphi_i - \varphi_i - \kappa_i h_{aut} \right| \leq \kappa_i \Psi(i^{-1}\,|\,\theta).$$

Here $\Psi(i^{-1}\,|\,\theta)$ denotes a function which for fixed θ converges to zero as $i \to \infty$. At the same time, using Proposition 2.1, we have a uniform bound $|V|_{\omega_i} < C$ on $B(o_i,1)$, since this ball is a ball centered at the origin in our reference metric $(\mathbf{C}^n, \omega_0)$ scaled down to unit size. Together with the uniform gradient bound for φ_i (since $\Delta_{\omega_i}\varphi_i = n$), this implies

$$\sup_{B(o_i,1)} \left| e^{-\beta_i} g_i^*\varphi_i - \varphi_i \right| \leq C\kappa_i.$$

To deal with h_{ph} note that h_{ph} is in the span of the real and imaginary parts of $1, z, z^2, x_i$, and so h_{ph} also defines a pluriharmonic function $h_{ph,i}$ on $B(o_i,1)$ for all

i under the embeddings into $\mathbf{C}^{n+1}$. For any $\theta > 0$ we have

$$\sup_{B(o_i,1)\setminus N_\theta} \left| h_{ph,i} - h_{ph} \right| < \Psi(i^{-1} \,|\, \theta),$$

where as above, we are using the nearest point projection on $B(o_i, 1)\setminus N_\theta$ for sufficiently large i to view h_{ph} as a function on $B(o_i, 1)$. We also clearly have

$$\sup_{B(o_i,1)} |h_{ph,i}| \leq C.$$

We can now define

$$u_i' = u_i - \kappa_i h_{ph,i} - (e^{-\beta_i} g_i^* \varphi_i - \varphi_i).$$

By the construction we have

$$\omega_i + \sqrt{-1}\partial\overline{\partial} u_i = e^{-\beta_i} g_i^* \omega_i + \sqrt{-1}\partial\overline{\partial} u_i', \tag{4.4}$$

and $e^{-\beta_i} g_i^* \omega_i$ has the same volume form as ω_i. By the estimates above, we have

$$\sup_{B(o_i,1)\setminus N_\theta} |u_i' - \kappa_i h^{>2}| \leq \Psi(i^{-1} \,|\, \theta)\kappa_i,$$

and also

$$\sup_{B(o_i,1)} |u_i'| \leq C\kappa_i.$$

Letting $\theta \to 0$, we find that

$$\|u_i' - \kappa_i h^{>2}\|_{L^2(B(o_i,1))} \leq \Psi(i^{-1})\kappa_i,$$

where near the singular set we use a Gromov–Hausdorff approximation to view $h^{>2}$ as a function on $B(o_i, 1)$.

Since $h^{>2}$ has faster than quadratic growth, there exists an $\alpha > 0$ (depending only on the cone $C(Y)$) such that

$$\|h^{>2}\|_{B(0,\lambda)} \leq \lambda^{2+2\alpha} \|h^{>2}\|_{B(0,1)}.$$

Here and below, for any ball B we define

$$\|f\|_B = \left(\mathrm{Vol}(B)^{-1} \int_B |f|^2 \right)^{1/2}$$

to be the L^2-norm normalized by the volume of the ball B. We therefore have

$$\|\kappa_i h^{>2}\|_{B(0,\lambda)} \leq C\lambda^{2+2\alpha}\kappa_i,$$

while also

$$\begin{aligned} \|\kappa_i h^{>2}\|_{B(0,\lambda)} &\geq \|u_i'\|_{B(o_i,\lambda)} - \|\kappa_i h^{>2} - u_i'\|_{B(o_i,\lambda)} \\ &\geq \|u_i'\|_{B(o_i,\lambda)} - C_\lambda \|\kappa_i h^{>2} - u_i'\|_{B(o_i,1)} \\ &\geq \|u_i'\|_{B(o_i,\lambda)} - C_\lambda \Psi(i^{-1})\kappa_i \end{aligned}$$

for a constant C_λ depending on λ. Combining these, we get

$$\|u_i'\|_{B(o_i,\lambda)} \le (C\lambda^{2+2\alpha} + C_\lambda \Psi(i^{-1}))\kappa_i.$$

Once i is sufficiently large (depending on λ), we get

$$\|u_i'\|_{B(o_i,\lambda)} \le 2C\lambda^{2+2\alpha}\kappa_i.$$

Step 2. Let us apply the construction in Step 1 to 8λ instead of λ, and let us scale the balls $B(o_i, 8\lambda)$ up to unit size. We denote the origins of the scaled up balls by o_i', and also let $\omega_i' = (8\lambda)^{-2}e^{-\beta_i}g_i^*\omega_i$. Letting $U_i' = (8\lambda)^{-2}u_i'$, on $B(o_i', 1)$ we have

$$(\omega_i' + \sqrt{-1}\partial\overline{\partial}U_i')^n = \omega_i'^n,$$

and for fixed λ, as $i \to \infty$, the metrics ω_i' on $\mathbf{C}^n$ satisfy the same assumptions as ω in the statement of Proposition 4.1 for arbitrarily small ϵ. By Step 1, (replacing C by a larger constant if necessary), we have

$$\begin{aligned} \sup_{B(o_i',1)} |U_i'| &\le C\lambda^{-2}\kappa_i, \\ \|U_i'\|_{B(o_i',1)} &\le C\lambda^{2\alpha}\kappa_i. \end{aligned} \tag{4.5}$$

By Lemma 4.3 we have a harmonic function H on $B(0,1) \subset C(Y)$ such that, after choosing a subsequence,

$$\frac{U_i'}{\sup_{B(o_i',1)} |U_i'|} \to H,$$

the convergence being in C^∞ uniformly on $B(o_i', 3/4)\setminus N_\theta$ for any $\theta > 0$. There are two cases to consider:

- Suppose that $H \neq 0$. Then by the L^∞-bound for harmonic functions on $C(Y)$ we have

  $$\sup_{B(0,1/2)} |H| \le C\|H\|_{B(0,1)}.$$

  It follows that for any $\theta > 0$, once i is sufficiently large, we have

  $$\begin{aligned} \frac{\sup_{B(o_i',1/2)\setminus N_\theta} |U_i'|}{\sup_{B(o_i',1)} |U_i'|} &\le 2 \sup_{B(0,1/2)} |H| \\ &\le 2C\|H\|_{B(0,1)} \\ &\le 4C\frac{\|U_i'\|_{B(o_i',1)}}{\sup_{B(o_i',1)} |U_i'|}, \end{aligned}$$

  and so using (4.5) we have

  $$\sup_{B(o_i',1/2)\setminus N_\theta} |U_i'| \le 4C^2\lambda^{2\alpha}\kappa_i.$$

- Suppose that $H = 0$. Then for any $\theta > 0$ we have

$$\sup_{B(o_i',1/2)\setminus N_\theta} |U_i'| \leq \lambda^{2+2\alpha} \sup_{B(o_i',1)} |U_i'| \leq C\lambda^{2\alpha}\kappa_i,$$

once i is sufficiently large.

In either case, applying Proposition 4.5 to $B(o_i', 1/2)$, we can choose $\theta > 0$ depending on λ such that for sufficiently large i we have

$$\sup_{B(o_i',1/4)} |U_i'| \leq C'\lambda^{2\alpha}\kappa_i$$

for a constant C' depending only on $C(Y)$. After rescaling, we get the bound $\sup_{B(o_i,2\lambda)} |u_i'| \leq 64C'\lambda^{2+2\alpha}\kappa_i$, and from (4.4) we have

$$e^{\beta_i}(g_i^{-1})^*(\omega_i + \sqrt{-1}\partial\overline{\partial}u_i) = \omega_i + \sqrt{-1}\partial\overline{\partial}\left(e^{\beta_i}(g_i^{-1})^*u_i'\right).$$

Letting $u_i'' = e^{\beta_i}(g_i^{-1})^*u_i'$ we have

$$\sup_{B(o_i,\lambda)} |u_i''| \leq 100C'\lambda^{2+2\alpha}\kappa_i$$

as long as $g_i^{-1}(B(o_i,\lambda)) \subset B(o_i, 2\lambda)$, and $e^{\beta_i} < 3/2$. Both of these estimates will hold once i is sufficiently large (depending on λ). Finally we just need to ensure that λ is sufficiently small so that $100C'\lambda^{2+2\alpha} < \lambda^{2+\alpha}$. □

4.3 Proof of Theorem 1.1. We now prove the main result, Theorem 1.1. Suppose that we have a Calabi–Yau metric (X', η) with X' biholomorphic to $\mathbf{C}^n$ and with tangent cone $\mathbf{C} \times A_1$ at infinity. Let us write (X, ω_0) for our reference metric on $\mathbf{C}^n$ discussed in Section 2. We have the origin $o \in X$, and using Lemma 3.2 we also have a distinguished basepoint $o' \in X'$. By the discussion in Section 2 we have embeddings $F_i : X \to \mathbf{C}^{n+1}$ such that $F_i(o) = 0$, and the image $F_i(X)$ has the equation

$$a_i z + x_1^2 + \cdots + x_n^2 = 0,$$

where $a_i = 2^{-in/(n-2)}$. At the same time using Proposition 3.1 we have embeddings $F_i' : X' \to \mathbf{C}^{n+1}$ such that $F_i'(o') = 0$, and the image $F_i'(X')$ satisfies the equation

$$a_i' z + x_1^2 + \cdots + x_n^2 = 0$$

with $a_i' > 0$. In addition the unit balls for the scaled down metrics $2^{-2i}\omega_0, 2^{-2i}\eta$ converge in the Gromov–Hausdorff sense to the unit ball $B(0,1) \subset C(Y)$, and the maps F_i, F_i' converge to the standard embedding $B(0,1) \subset \mathbf{C}^{n+1}$ as $i \to \infty$.

We need to find suitable $j(i)$ such that after a further scaling by a bounded factor, the images $F_{j(i)}'(X')$ satisfy the same equations as $F_i(X)$. Since $a_i/a_{i+1} = 2^{n/(n-2)}$ and $a_i'/a_{i+1}' \to 2^{n/(n-2)}$, for all sufficiently large i we can find $j(i)$ such that

$$C_n^{-1}a_{j(i)}' < a_i < C_n a_{j(i)}'$$

for a dimensional constant C_n, and $j(i) \to \infty$ as $i \to \infty$. Let us compose $F'_{j(i)}$ with an automorphism g_i of $\mathbf{C}^{n+1}$ of the form $(z, \mathbf{x}) \mapsto (c_i z, c_i^{\frac{n-1}{n-2}} \mathbf{x})$ for $c_i > 0$, where $\mathbf{x} = (x_1, \dots, x_n)$. Then $g_i \circ F'_{j(i)}(X')$ satisfies the equation

$$c_i^{\frac{n}{n-2}} a'_{j(i)} z + x_1^2 + \cdots + x_n^2 = 0,$$

so we can choose $c_i \in (C_n^{-1}, C_n)$ so that $g_i \circ F'_{j(i)}(X')$ satisfies the same equation as $F_i(X)$. The balls $B_\eta(o', c_i^{-1} 2^{-j(i)})$ with the scaled metrics $\eta_i = c_i^{-2} 2^{-2j(i)} \eta$ still converge to the unit ball $B(0,1) \subset C(Y)$, and on these balls the maps $g_i \circ F'_{j(i)}(X')$ still converge to the standard embedding of $B(0,1)$ into $\mathbf{C}^{n+1}$ as $i \to \infty$. Moreover the volume form of η_i is the pullback of $\sqrt{-1}^{n^2} \Omega \wedge \bar{\Omega}$ under $g_i \circ F'_{j(i)}$.

Let us write $\omega_i = 2^{-2i} \omega_0$ and B_i for the unit ball around o with the metric ω_i. Similarly let B'_i be the unit ball around o' with the metric η_i. We have biholomorphisms $\Phi_i : X \to X'$ defined by

$$\Phi_i = (g_i \circ F'_{j(i)})^{-1} \circ F_i$$

which satisfy $\Phi_i(o) = o'$ and $\Phi_i^*(\eta_i^n) = \omega_i^n$. We claim that on the ball B_i we have

$$|\Phi_i^* d_{\eta_i}(o', \cdot) - d_{\omega_i}(o, \cdot)| < \Psi(i^{-1}). \tag{4.6}$$

For this let $x \in B_i$ and $x' = \Phi_i(x)$. By the construction and Proposition 3.1 we have $\Psi(i^{-1})$-Gromov–Hausdorff approximations $G : B_i \to B(0,1)$ and $G' : B'_i \to B(0,1)$ satisfying $G(o) = 0, G'(o') = 0$ such that viewing $B(0,1) \subset \mathbf{C}^{n+1}$ under the standard embedding, we have $|G(x) - F_i(x)| < \Psi(i^{-1})$ and $|G'(x') - g_i \circ F'_{j(i)}(x')| < \Psi(i^{-1})$. Note that $F_i(x) = g_i \circ F'_{j(i)}(x')$ by assumption, and so $|G(x) - G'(x')| < \Psi(i^{-1})$. At the same time, under the cone metric on $B(0,1)$, the distance from 0 is Hölder continuous with respect to the Euclidean distance (it is given up to a factor by $|x|^{\frac{n-2}{n-1}}$), and so this means

$$|d_{B(0,1)}(0, G(x)) - d_{B(0,1)}(0, G'(x'))| < \Psi(i^{-1}).$$

Since G, G' are Gromov–Hausdorff approximations, we get

$$|d_{\omega_i}(o, x) - d_{\eta_i}(o', x')| < \Psi(i^{-1})$$

as claimed.

The balls B_i, B'_i are both $\Psi(i^{-1})$-Gromov–Hausdorff close to the unit ball in $C(Y)$. For sufficiently large i we can then use [LS18, Proposition 3.1] to find Kähler potentials φ_i, φ'_i for ω_i, η_i respectively, such that

$$|\varphi_i - d_{\omega_i}(o, \cdot)^2/2| < \Psi(i^{-1}),$$
$$|\varphi'_i - d_{\eta_i}(o', \cdot)^2/2| < \Psi(i^{-1}).$$

Using (4.6) this means that on $B_{\omega_i}(o,1)$ we can write

$$\Phi_i^*(\eta_i) = \omega_i + \sqrt{-1}\partial\overline{\partial}u_i,$$

with $\sup_{B_{\omega_i}(o,1)} |u_i| < \Psi(i^{-1})$.

We will next apply Proposition 4.1. We can assume that the λ in the proposition is of the form $\lambda = 2^{-m}$ for an integer m. We can also choose $i_0 > 0$ such that the assumptions of Proposition 4.1 hold for ω_i and u_i on $B_{\omega_i}(o,1)$ for all $i \geq i_0$. Let us fix a large $k > 0$ and apply the proposition for $i = i_0 + km$. We have

$$\sup_{B_{\omega_{i_0+km}}(o,1)} |u_{i_0+km}| < \epsilon_k,$$

where $\lim_{k\to\infty} \epsilon_k = 0$. We find a β_k, g_k and u'_k such that

$$\beta_k g_k^* \Phi_k^* \eta_{i_0+km} = \omega_{i_0+km} + \sqrt{-1}\partial\overline{\partial}u'_k \tag{4.7}$$

on $B_{\omega_{i_0+km}}(o,1/2)$ together with the estimate

$$\sup_{B_{\omega_{i_0+km}}(o,\lambda)} |u'_k| \leq \lambda^{2+\alpha}\epsilon_k.$$

Note that $B_{\omega_{i_0+km}}(o,\lambda) = B_{\omega_{i_0+(k-1)m}}(o,1)$. Scaling (4.7) up by a factor of λ^{-2}, we have

$$\lambda^{-2}\beta_k g_k^* \Phi_k^* \eta_{i_0+km} = \omega_{i_0+(k-1)m} + \sqrt{-1}\partial\overline{\partial}\lambda^{-2}u'_k,$$

and $\sup_{B_{\omega_{i_0+(k-1)m}}(o,1)} |\lambda^{-2}u'_k| \leq \lambda^\alpha \epsilon_k \leq \epsilon_k$, where we dropped the λ^α factor. Note that by construction the volume forms of $\lambda^{-2}\beta_k g_k^*\Phi_k^*\eta_{i_0+km}$ and $\omega_{i_0+(k-1)m}$ are equal, and so we can apply Proposition 4.1 again, iterating the above argument. After k steps we obtain a constant Λ_k, a biholomorphism $G_k : X \to X'$ satisfying $G_k(o) = o'$, and a function U_k such that

$$G_k^*(\Lambda_k \eta) = \omega_{i_0} + \sqrt{-1}\partial\overline{\partial}U_k \tag{4.8}$$

on $B_{\omega_{i_0}}(o,1)$, together with the estimate

$$\sup_{B_{\omega_{i_0}}(o,1)} |U_k| \leq \epsilon_k,$$

such that

$$(\omega_{i_0} + \sqrt{-1}\partial\overline{\partial}U_k)^n = \omega_{i_0}^n.$$

Here we have absorbed the additional scaling between η and η_k into the constant Λ_k. Note that fixing i_0 we can let $k \to \infty$, and once ϵ_k is sufficiently small, we can apply Savin's small perturbation result [Sav07] to find that on $B_{\omega_{i_0}}(o,1/2)$ we have $U_k \to 0$ in C^∞. Since $G_k(o) = o'$, we find, using (4.8), that for sufficiently large k,

$$B_{\Lambda_k\eta}(o',1/4) \subset G_k(B_{\omega_{i_0}}(o,1/2)) \subset B_{\Lambda_k\eta}(o',1),$$

and moreover $G_k^*(\Lambda_k\eta) \to \omega_{i_0}$ in C^∞ on $B_{\omega_{i_0}}(o, 1/2)$. If $\Lambda_k \to 0$, then this is a contradiction since η is not flat, and so the curvature of $B_{\Lambda_k\eta}(o', 1/4)$ blows up as $\Lambda_k \to 0$. Similarly $\Lambda_k \to \infty$ leads to a contradiction since ω_0 is not flat. Choosing a subsequence we can assume $\Lambda_k \to \Lambda_\infty > 0$. It follows that we can then take a limit $G_k \to G_\infty$ on $B_{\omega_{i_0}(o,1/4)}$ which gives a holomorphic isometry

$$G_\infty : B_{\omega_{i_0}}(o, 1/4) \to B_{\Lambda_\infty\eta}(o', 1/4).$$

We can repeat the same argument for any $i > i_0$ and extract a global holomorphic isometry between (X, ω_0) and $(X', \Lambda\eta)$ for a suitable Λ. □

5 Further Directions

The approach that we used to prove Theorem 1.1 can be applied in more general situations. One natural generalization would be to study the uniqueness of all of the metrics constructed in the author's work [Sze17], or by Conlon–Rochon [CR17], given their tangent cones. The places where we used the specific choice $\mathbf{C} \times A_1$ for the tangent cone were in Lemma 2.3 in order to understand the quadratic growth harmonic functions, and in Proposition 3.1 which allowed us to construct embeddings of a given Calabi–Yau space as a specific hypersurface. When we consider more general tangent cones, then these results need to be suitably modified. We expect that in general the Calabi–Yau metric with a given tangent cone is not unique, however we hope that our methods can be used to describe the moduli space of such metrics.

To illustrate this, let us consider the next simplest example, namely the metric ω_0 on $\mathbf{C}^3$ with tangent cone $\mathbf{C} \times A_2$ at infinity, constructed by viewing $\mathbf{C}^3 \subset \mathbf{C}^4$ as the hypersurface

$$z + x_1^2 + x_2^2 + y^3 = 0. \tag{5.1}$$

This metric has the property that we have holomorphic functions z, x_1, x_2, y whose degrees satisfy $d(z) = 1, d(x_1) = 3, d(x_2) = 3, d(y) = 2$, and which satisfy (5.1). Suppose now that (X, η) is another Calabi–Yau metric with the same tangent cone, and we try to argue as in Proposition 3.1. The same arguments show that we can embed X into $\mathbf{C}^4$ as a hypersurface given by a linear equation in monomials of total degree at most 6. Moreover this equation is a small perturbation of the equation $x_1^2 + x_2^2 + y^3 = 0$ defining the tangent cone. As in Proposition 3.1 we can perform simplifications, but we cannot always reduce to the equation (5.1). Instead we may end up with an equation of the form

$$az + by + x_1^2 + x_2^2 + y^3 = 0$$

for small constants a, b, with $a \neq 0$. Note that this defines a hypersurface biholomorphic to $\mathbf{C}^3$. When $b \neq 0$, then we cannot make the change of coordinate $z' = z + a^{-1}by$ to reduce to an equation of the form (5.1), since y has faster growth than z. Indeed, we expect that one can construct a one-parameter family of inequivalent Calabi–Yau metrics on $\mathbf{C}^3$ with tangent cone $\mathbf{C}\times A_2$ using the methods from [Li17, CR17, Sze17], viewing $\mathbf{C}^3 \subset \mathbf{C}^4$ as the hypersurface

$$z + by + x_1^2 + x_2^2 + y^3 = 0.$$

More precisely we expect the following.

Conjecture 5.1. *Up to scaling and isometry there is a one parameter family of Calabi–Yau metrics on $\mathbf{C}^3$ with tangent cone $\mathbf{C} \times A_2$ at infinity.*

In view of the gluing construction by Li [Li18] of collapsing Calabi–Yau metrics on threefolds (see the discussion in [Li18, Section 4.2]), such metrics could arise as a suitable blowup limit of a collapsing family of CY metrics on a threefold that has a fibration locally of the form $(x_1, x_2, y) \mapsto x_1^2 + x_2^2 + y^3$.

A different generalization would be to consider Calabi–Yau metrics on more general spaces than $\mathbf{C}^n$. The crucial prerequisite for applying the Donaldson–Sun theory in Section 3, as well as [LS18, Proposition 3.1], was that the metric ω_0 is $\partial\bar\partial$-exact, and we expect that our methods can be extended to classifying such exact Calabi–Yau metrics. For instance the smoothing $Q^n \subset \mathbf{C}^n$ of the n-dimensional A_1 singularity,

$$1 + x_1^2 + \cdots x_{n+1}^2 = 0,$$

is expected to admit a Calabi–Yau metric with tangent cone $\mathbf{C} \times A_1$ in terms of the $(n-1)$-dimensional A_1 singularity (see e.g. [Li18, Section 4.2]), and the methods used in Theorem 1.1 could lead to a uniqueness result for this metric.

More generally, from the argument in Proposition 3.1 we can read off which manifolds can admit a $\partial\bar\partial$-exact Calabi–Yau metric with a given tangent cone. For instance the following is a natural conjecture to make.

Conjecture 5.2. *Let $n > 4$. The only $\partial\bar\partial$-exact Calabi–Yau manifolds of dimension n with tangent cone $\mathbf{C} \times A_1$ are $\mathbf{C} \times Q^{n-1}$, $\mathbf{C}^n$ and Q^n. Moreover up to scaling and isometry each of these manifolds admits a unique such Calabi–Yau metric.*

The three cases correspond to the function $\tilde{f}_i$ in the equation analogous to (3.2) having degree 0, 1 or 2. When $n \leq 4$ then there would be more possibilities. For both Conjectures 5.1 and 5.2 we expect that the proof of Theorem 1.1 can be extended to prove the classification results, once the corresponding existence results are shown using the techniques of [Li17, CR17, Sze17].

Note that some of the results of Donaldson–Sun [DS15] can also be extended to the case when the metric is not exact, under the assumption that the tangent cone is smooth away from the vertex (see Liu [Liu17]). This is closer to the setting of asymptotically conical Calabi–Yau metrics considered by Conlon–Hein [CH14] who also obtained classification results for Calabi–Yau metrics with prescribed tangent cone. At the moment there is little that we can say in this direction about general Calabi–Yau manifolds with tangent cones that have non-isolated singularities, beyond the result in [LS] that each tangent cone is a normal affine variety.

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