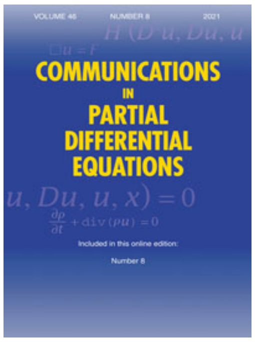

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Weak Harnack inequalities for eigenvalues and constant rank theorems

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ABSTRACT
We consider convex solutions of nonlinear elliptic equations which satisfy the structure condition of Bian and Guan. We prove a weak Harnack inequality for the eigenvalues of the Hessian of these solutions. This can be viewed as a quantitative version of the constant rank theorem.

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1. Introduction

Constant rank theorems in PDE have a long history, starting with work of Caffarelli and Friedman [1], Yau (see [2]) and then developed further by Korevaar and Lewis [3], Caffarelli et al. [4], Bian and Guan [5, 6] and others [7–13]. These results assert that a convex solution u of a certain class of elliptic or parabolic equations has Hessian D^2u of constant rank.

Constant rank theorems, also known as the "microscopic convexity principle", have been used to establish "macroscopic" convexity properties of solutions to PDEs on convex domains, now a vast area of research (see [1, 14–34] and the references therein).

One method for establishing a constant rank theorem is to compute with an expression involving the elementary symmetric polynomials σ_k of the eigenvalues

$$\lambda_1 \leq \cdots \leq \lambda_n$$

of the Hessian D^2u. Bian and Guan [5] considered solutions of nonlinear elliptic equations

$$F(D^2u, Du, u, x) = 0,$$

under a convexity condition for F (see (1.4) below) and proved a constant rank theorem using a differential inequality for the quantity $\sigma_{\ell+1} + \frac{\sigma_{\ell+2}}{\sigma_{\ell+1}}$. The authors [13] gave a new proof of the Bian and Guan result using the simple linear expression

$$\lambda_\ell + 2\lambda_{\ell-1} + \cdots + \ell\lambda_1, \tag{1.1}$$

and a method of induction. This approach exploited the concavity of the sums of the lowest eigenvalues.

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In this paper we will also assume the Bian and Guan structure conditions. Building on the method of [13] and making use again of the expression (1.1) we directly prove a weak Harnack inequality for each of the eigenvalues λ_i. This states that the L^q norm for some $q > 0$ is bounded above by the infimum. We view this Harnack inequality as a *quantitative* version of the constant rank theorem of Bian and Guan, which follows as an immediate consequence.

Another difference between the current paper and [13] is that we compute differential inequalities directly, holding almost everywhere, for sums of the eigenvalues λ_i. In particular we avoid here the device of approximation by polynomials.

We now state our results precisely. Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and let F be a real-valued function

$$F = F(A, p, u, x) \in C^2(\mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \times B),$$

where $\mathrm{Sym}_n(\mathbb{R})$ is the vector space of symmetric $n \times n$ matrices with real entries. We assume that F satisfies the condition of 00-Guan [5] that for each $p \in \mathbb{R}^n$,

$$(A, u, x) \in \mathrm{Sym}_n^+(\mathbb{R}) \times \mathbb{R} \times B \mapsto F(A^{-1}, p, u, x) \quad \text{is locally convex,} \tag{1.2}$$

where $\mathrm{Sym}_n^+(\mathbb{R})$ is the subset of $\mathrm{Sym}_n(\mathbb{R})$ that are strictly positive definite. Suppose that $u \in C^3(B)$ is a convex solution of

$$F(D^2u, Du, u, x) = 0, \tag{1.3}$$

subject to the ellipticity condition that for all $\xi \in \mathbb{R}^n$,

$$\Lambda^{-1}|\xi|^2 \leq F^{ij}(D^2u, Du, u, x)\xi^i\xi^j \leq \Lambda|\xi|^2, \quad \text{on } B, \tag{1.4}$$

for a positive constant $\Lambda > 0$, where F^{ij} is the derivative of F with respect to the (i, j)th entry A_{ij} of A. Our main result is as follows.

Theorem 1.1. *Let u be as above and let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of D^2u. Then there exist positive constants C_0, q depending only on n, Λ, $||u||_{C^3(B)}$ and $||F||_{C^2}$ such that for each $\ell = 1, ..., n$,*

$$||\lambda_\ell||_{L^q(B_{1/2})} \leq C_0 \inf_{B_{1/2}} \lambda_\ell,$$

where $B_{1/2} = B_{1/2}(0)$ is the ball in $\mathbb{R}^n$ centered at 0 of radius 1/2.

This implies in particular the constant rank theorem of Bian and Guan [5]:

Corollary 1.1. *The Hessian D^2u has constant rank in B.*

Indeed, applying Theorem 1.1 on appropriately scaled balls, the sets

$$\{x \in B | \mathrm{rank}(D^2u(x)) \leq k\}, \quad k = 0, 1, 2, ..., n,$$

are open in B. On the other hand the sets $\{x \in B | \mathrm{rank}(D^2u(x)) \geq k\}$ are open in B by continuity of the eigenvalues of D^2u. A consequence is that the sets

$$\{x \in B | \mathrm{rank}(D^2u(x)) = k\}$$

are open and closed in B, giving the corollary.

We now give an outline of the paper. In Section 2 we recall some definitions and known results about semi-concave functions and in particular we provide a proof of the semi-concavity of the sum of the first k eigenvalues of D^2u for u in C^4. We also give a version of the weak Harnack inequality for subsolutions of elliptic equations.

In Section 3, under the assumption that u is in C^4, we prove that the key differential inequality

$$F^{ab}Q_{ab} \leq CQ + \sum_{i=1}^{k} b^{i,j}(\lambda_i)_j, \quad \text{for } Q = Q_k = \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1, \tag{1.5}$$

holds almost everywhere, where C and $b^{i,j}$ are bounded. This improves on the analogous result in [13] where the inequality is proved for approximating polynomials. We note that the method of Section 3 includes proofs of a first variation formula for λ_i (Lemma 3.2) and a second variation inequality (Lemma 3.3), which hold almost everywhere.

In Section 4 we complete the proof of Theorem 1.1. We cannot directly apply the weak Harnack inequality to Q satisfying (1.5) because its right hand side includes derivatives of $\lambda_1, \ldots, \lambda_k$. We get around this difficulty by considering a new quantity

$$R = \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{1/2}, \quad \text{for } \varepsilon > 0.$$

By exploiting the concavity of the square root function, R is shown to satisfy the differential inequality

$$F^{ab}R_{ab} \leq CR,$$

almost everywhere. We apply the weak Harnack inequality to R and then let $\varepsilon \to 0$ to obtain Theorem 1.1.

2. Preliminaries

In this section we collect some elementary and well-known results which we will need in the sequel.

2.1. Semi-concave functions

Let U be a bounded convex subset of $\mathbb{R}^n$. A real-valued function f on U is *semi-concave* if there exists a constant M such that $g = f - M|x|^2$ is concave. We call M the semi-concavity constant for f on U. Observe that every function in $C^2(\bar{U})$ is automatically semi-concave.

Equivalently, a continuous function f is semi-concave if for some M'

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \leq M'|x-y|^2, \quad \text{for all } x, y \in U. \tag{2.1}$$

It is a classical result that a concave function f is Lipschitz continuous and hence differentiable almost everywhere (we write its derivative as $Df(x)$ if it exists at x). Moreover, by a theorem of Alexandrov, the second derivative of f exists almost everywhere in the sense that there is a second order Taylor expansion at almost every x (see for example [35, Theorem 2.6.4]). This holds too then for semi-concave functions f on

U. More explicitly, at almost every $x \in U$, the derivative $Df(x) = (f_1(x), ..., f_n(x))$ exists and there is a symmetric matrix which we write as $D^2 f(x) = (f_{ij}(x))$ such that

$$f(y) = f(x) + f_i(x)(y-x)_i + \frac{1}{2} f_{ij}(x)(y-x)_i(y-x)_j + o(|y-x|^2), \quad \text{as } |y-x| \to 0, \tag{2.2}$$

where as usual we are summing repeated indices from 1 to n.

The following proposition is well-known (see for example [36]), but we include a brief proof for the reader's convenience.

Proposition 2.1. *Let $u \in C^4(\bar{U})$, for a bounded convex set $U \subset \mathbb{R}^n$. Denote by $\lambda_1(x) \le \cdots \le \lambda_n(x)$ the eigenvalues of the Hessian $D^2 u(x)$. Then for each $k = 1, ..., n$ the map $\bar{U} \to \mathbb{R}$ given by*

$$x \mapsto \lambda_1(x) + \cdots + \lambda_k(x)$$

is semi-concave.

Proof. We claim the following: the map $\sigma : \mathrm{Sym}_n(\mathbb{R}) \to \mathbb{R}$ given by

$$\sigma(A) = \lambda_1(A) + \cdots + \lambda_k(A)$$

is increasing and concave on $\mathrm{Sym}_n(\mathbb{R})$, and is Lipschitz continuous with Lipschitz constant depending only on n and k. To see the claim, note that given fixed unit vectors $V_1, ..., V_k \in \mathbb{R}^n$, the function

$$A \mapsto \sum_{\alpha=1}^{k} A_{ij} V_\alpha^i V_\alpha^j,$$

is linear, increasing and has bounded Lipschitz constant depending only on n and k. Here we are writing $V_\alpha = (V_\alpha^1, ..., V_\alpha^n)$ and $A = (A_{ij})_{i,j=1}^n$. But we can define

$$\sigma(A) = \inf\left\{ \sum_{\alpha=1}^{k} A_{ij} V_\alpha^i V_\alpha^j | V_1, ..., V_k \text{ are orthonormal} \right\}.$$

The map σ is clearly increasing, and it is concave since the infimum of concave functions is concave. Moreover, it is an elementary fact that for any normed vector space $(X, ||\cdot||)$, if $f_s : X \to \mathbb{R}$, for $s \in S$, is a family of functions which are uniformly Lipschitz continuous:

$$|f_s(x) - f_s(y)| \le C||x-y||, \quad x, y \in X$$

then $f := \min_{s\in S} f_s$, assuming the minimum is attained at each point and is finite, is also Lipschitz continuous with the same constant C. The claim follows.

For $x, y \in \bar{U}$, using concavity of σ,

$$\begin{aligned} &\frac{\sigma(D^2u(x)) + \sigma(D^2u(y))}{2} - \sigma\left(D^2u\left(\frac{x+y}{2}\right)\right) \\ &\quad \le \sigma\left(\frac{D^2u(x) + D^2u(y)}{2}\right) - \sigma\left(D^2u\left(\frac{x+y}{2}\right)\right). \end{aligned}$$

But then since u is in $C^4(\bar{U})$ we have that $D_{ij}u$ is semi-concave for every i, j and hence

$$\frac{D_{ij}u(x)+D_{ij}u(y)}{2}-D_{ij}u\left(\frac{x+y}{2}\right)\leq M|x-y|^2.$$

Then, increasing M if necessary, we have the following inequality of symmetric matrices:

$$\frac{D^2u(x)+D^2u(y)}{2}-D^2u\left(\frac{x+y}{2}\right)\leq M|x-y|^2\mathrm{Id}.$$

Since σ is increasing and Lipschitz continuous,

$$\begin{aligned}&\frac{\sigma(D^2u(x))+\sigma(D^2u(y))}{2}-\sigma\left(D^2u\left(\frac{x+y}{2}\right)\right)\\&\leq\sigma\left(D^2u\left(\frac{x+y}{2}\right)+M|x-y|^2\mathrm{Id}\right)-\sigma\left(D^2u\left(\frac{x+y}{2}\right)\right)\\&\leq C|x-y|^2,\end{aligned}$$

completing the proof of the proposition. □

We end the subsection with an elementary proposition.

Proposition 2.2. *Let U be a bounded convex set in $\mathbb{R}^n$ and V an open interval in $\mathbb{R}$. Suppose that $f:U\to V$ is semiconcave and $h:V\to\mathbb{R}$ is increasing, Lipschitz continuous and concave. Then $h\circ f:U\to\mathbb{R}$ is semiconcave.*

Proof. We have for $x,y\in U$,

$$\begin{aligned}&\frac{h(f(x))+h(f(y))}{2}-h\left(f\left(\frac{x+y}{2}\right)\right)\\&\leq h\left(\frac{f(x)+f(y)}{2}\right)-h\left(f\left(\frac{x+y}{2}\right)\right)\qquad(h\ \text{concave})\\&\leq h\left(f\left(\frac{x+y}{2}\right)+M|x-y|^2\right)-h\left(f\left(\frac{x+y}{2}\right)\right)\qquad(f\ \text{semiconcave},\ h\ \text{increasing})\\&\leq C|x-y|^2,\qquad(h\ \text{Lipschitz})\end{aligned}$$

as required. □

2.2. The weak Harnack inequality

We state a weak Harnack inequality for semi-concave functions. It follows by approximation from the classical weak Harnack inequality for functions in $W^{2,n}$ (see [37, Theorem 9.22]). Similar statements, with slightly different hypotheses, can be found in [38] and [13].

Proposition 2.3. *Consider the operator $Lv=a^{ij}D_{ij}v+b^iD_iv+cv$ with bounded coefficients on the unit ball $B\subset\mathbb{R}^n$ and with a^{ij} satisfying the uniform ellipticity condition $\Lambda^{-1}|\xi|^2\leq a^{ij}\xi_i\xi_j\leq\Lambda|\xi|^2$ for all $\xi\in\mathbb{R}^n$ for $\Lambda>0$. Let v be a semi-concave nonnegative function on B satisfying $Lv\leq f$ almost everywhere in B for $f\in L^n(B)$. Then on the half size ball B',*

$$\left(\frac{1}{|B'|}\int_{B'} v^q\right)^{1/q} \leq C\left(\inf_{B'} v + ||f||_{L^n(B)}\right), \tag{2.3}$$

for positive constants C *and* q *depending only on* n, Λ, *bounds for* b^i *and* c *and the radius of the ball* B.

Proof. We include a brief proof for the sake of completeness. Let $\tilde{B}$ be a ball such that $B' \subset\subset \tilde{B} \subset\subset B$. Since v is semi-concave, it is Lipschitz continuous and twice differentiable almost everywhere. For $\varepsilon > 0$ let v_ε be a standard mollification of v. Then $v_\varepsilon \to v$ uniformly and $D^k v_\varepsilon \to D^k v$ for $k = 1, 2$ almost everywhere on $\tilde{B}$ as $\varepsilon \to 0$ (the second assertion is a direct consequence of the expansion (2.2)). Let $\delta > 0$ be given. Then by Egorov's Theorem there exists a set $K_\delta \subset \tilde{B}$ such that $|\tilde{B} \setminus K_\delta| \leq \delta$ and $D^k v_\varepsilon \to D^k v$ uniformly on K_δ for $k = 1, 2$. Then we may choose $\varepsilon > 0$ sufficiently small so that

$$Lv_\varepsilon = Lv + L(v_\varepsilon - v) \leq f + \delta, \qquad \text{almost everywhere on } K_\delta.$$

On the other hand, since v is semi-concave we have an upper bound for $D^2 v_\varepsilon$ and hence on $\tilde{B} \setminus K_\delta$

$$Lv_\varepsilon \leq M,$$

for a constant $M \geq 1$ depending on the C^0 norm, semi-concavity constant and Lipschitz bound of v as well as bounds on the coefficients of L. It follows that almost everywhere on $\tilde{B}$ we have

$$Lv_\varepsilon \leq f + g,$$

for a function $g \in L^\infty(\tilde{B})$ with $||g||_{L^n(\tilde{B})} \leq CM\delta^{1/n}$. Then we apply [37, Theorem 9.22] to the smooth nonnegative function v_ε to obtain

$$\left(\int_{B'} v_\varepsilon^q\right)^{1/q} \leq C \inf_{B'} v_\varepsilon + C||f + g||_{L^n(\tilde{B})} \leq C\left(\inf_{B'} v_\varepsilon + ||f||_{L^n(B)} + M\delta^{1/n}\right),$$

for uniform positive constants C, q. Then let $\delta \to 0$, so that in addition $\varepsilon \to 0$ and we obtain (2.3). □

We emphasize that the constants C, q in the above proposition are independent of the semi-concavity constant of v.

3. A Differential inequality

As in the introduction, let u solve the equation (1.3) subject to the conditions (1.2) and (1.4). In this section, we make the following:

Additional assumption: $u \in C^4(B)$.

Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of $D^2 u$. By Proposition 2.1, the map $x \mapsto \lambda_1 + \cdots + \lambda_k$ is semi-concave on compact convex subsets of B. It follows that $\lambda_1, \ldots, \lambda_n$ are twice differentiable almost everywhere on B. The goal of this section is the following differential inequality.

Lemma 3.1. *For $1 \leq k \leq n$, define*

$$Q := Q_k := \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1.$$

Then

$$F^{ab}Q_{ab} \leq CQ + \sum_{i=1}^{k} b^{i,j}(\lambda_i)_j, \quad \textit{almost everywhere on } B, \tag{3.1}$$

where C is a uniform constant and the $b^{i,j}$ are uniformly bounded functions on B.

In the above, "uniform" means that the constants depend only on n, Λ, $||u||_{C^3(B)}$ and $||F||_{C^2}$. In particular, the constants do not depend on a C^4 bound for u.

To establish the lemma, we prove two rather general results about the first and second derivatives of the eigenvalues λ_i, which will hold almost everywhere. In the case when the eigenvalues λ_i are all distinct, there are well-known formulae for their first and second derivatives (see for example [39]). To deal with eigenvalues with multiplicity we adapt an approach of Brendle-Choi-Daskalopoulos [40, Lemma 5] where similar statements to Lemmas 3.2 and 3.3 below are proved for λ_1. Crucially, these lemmas only hold at a point where the eigenvalues are twice differentiable.

Fix an x_0 at which the λ_i are twice differentiable, and choose coordinates at x_0 such that D^2u is diagonal with entries $u_{ii} = \lambda_i$. Moreover, we suppose that N of the λ_i's are distinct at x_0, and define $1 \leq \mu_1 < \cdots < \mu_N = n$ by

$$\lambda_1 = \cdots = \lambda_{\mu_1} < \lambda_{1+\mu_1} = \cdots = \lambda_{\mu_2} < \lambda_{1+\mu_2} = \cdots\cdots = \lambda_{\mu_N} = \lambda_n.$$

We also define $\mu_0 = 0$ so that the multiplicities of the eigenvalues of D^2u at x_0 are $\mu_1 - \mu_0, \mu_2 - \mu_1, \ldots, \mu_N - \mu_{N-1}$.

Lemma 3.2. *For each $j = 1, 2, \ldots, N$ we have at x_0,*

$$u_{k\ell i} = (\lambda_{1+\mu_{j-1}})_i \delta_{k\ell}, \qquad \textit{for } 1 + \mu_{j-1} \leq k, \ell \leq \mu_j. \tag{3.2}$$

for $i = 1, 2, \ldots, n$.

Proof. We prove this by induction on j. We first prove the case $j = 1$, which states that

$$u_{k\ell i} = (\lambda_1)_i \delta_{k\ell}, \quad \text{for } 1 \leq k, \ell \leq \mu_1, \quad 1 \leq i \leq n. \tag{3.3}$$

Let $V = (V^1, \ldots, V^n)$ be a constant unit vector field defined in a neighborhood of x_0. Then by definition of λ_1, the function h defined by

$$h := u_{k\ell}V^kV^\ell - \lambda_1,$$

has $h(x) \geq 0$ for x near x_0. Choose V with $V^k(x_0) = 0$ for $k > \mu_1$ so that we have $h(x_0) = 0$ and h has a local minimum at x_0. Moreover, h is twice differentiable at x_0.

Then at x_0,

$$0 = h_i = \sum_{k,\ell \leq \mu_1} u_{k\ell i}V^kV^\ell - \sum_{k,\ell \leq \mu_1} (\lambda_1)_i \delta_{k\ell}V^kV^\ell,$$

using the fact that V is a unit vector. Then (3.3) follows since we are free to choose $V^k(x_0)$ for $k \leq \mu_1$.

For the inductive step, assume (3.2) holds for $1 \leq j \leq p$. Let $V_1, \ldots, V_{\mu_p}$ be the constant unit vector fields in the $\partial/\partial x_1, \ldots, \partial/\partial x_{\mu_p}$ directions. That is, writing V_α in component form as $V_\alpha = (V^1_\alpha, ..., V^n_\alpha)$, we have $V^q_\alpha = \delta_{q\alpha}$. Next let $W = V_{1+\mu_p}$ be an arbitrary constant unit vector field in the span of the directions $\partial/\partial x_{1+\mu_p}, ..., \partial/\partial x_{\mu_{p+1}}$.

Consider the quantity

$$\begin{aligned} h &= \sum_{\alpha=1}^{1+\mu_p} u_{k\ell} V^k_\alpha V^\ell_\alpha - \sum_{j=1}^{p} (\mu_j - \mu_{j-1}) \lambda_{1+\mu_{j-1}} - \lambda_{1+\mu_p} \\ &= \sum_{\alpha=1}^{1+\mu_p} u_{k\ell} V^k_\alpha V^\ell_\alpha - \sum_{j=1}^{p} \sum_{1+\mu_{j-1}\le\alpha\le\mu_j} \lambda_{1+\mu_{j-1}} \delta_{k\ell} V^k_\alpha V^\ell_\alpha - \lambda_{1+\mu_p} \\ &= \sum_{j=1}^{p} \sum_{1+\mu_{j-1}\le\alpha\le\mu_j} (u_{k\ell} - \lambda_{1+\mu_{j-1}} \delta_{k\ell}) V^k_\alpha V^\ell_\alpha + u_{k\ell} W^k W^\ell - \lambda_{1+\mu_p}. \end{aligned}$$

Now note that $h(x_0) = 0$ and $h(x) \ge 0$ for x near x_0. Since h achieves a minimum at x_0 we have

$$\begin{aligned} 0 = h_i(x_0) &= \sum_{j=1}^{p} \sum_{1+\mu_{j-1}\le\alpha\le\mu_j} (u_{k\ell i} - (\lambda_{1+\mu_{j-1}})_i \delta_{k\ell}) V^k_\alpha V^\ell_\alpha + u_{k\ell i} W^k W^\ell - (\lambda_{1+\mu_p})_i \\ &= (u_{k\ell i} - \delta_{k\ell} (\lambda_{1+\mu_p})_i) W^k W^\ell, \end{aligned}$$

where for the last line we used the inductive hypothesis and the fact that W is a unit vector. Since W is an arbitrary unit vector in the span of $\partial/\partial x_{1+\mu_p}, ..., \partial/\partial x_{\mu_{p+1}}$, we have proved (3.2) for $j = p + 1$ and the result follows by induction. □

As above, pick coordinates at x_0 such that $D^2 u$ is diagonal with entries $u_{ii} = \lambda_i$. Fix m between 1 and n. Define $\rho \in \{m, m+1, ..., n\}$ to be the largest integer such that $\lambda_\rho = \lambda_m$ at x_0, so that

$$0 \le \lambda_1 \le \cdots \le \lambda_m = \lambda_{m+1} = \cdots = \lambda_\rho < \lambda_{\rho+1} \le \cdots \le \lambda_n.$$

Then we have the following lemma on the second derivatives of the λ_α. We emphasize that this only holds at the point x_0 where the λ_i are twice differentiable.

Lemma 3.3. *As symmetric $n \times n$ matrices we have at x_0,*

$$\sum_{\alpha=1}^{m} (\lambda_\alpha)_{ab} \le \sum_{\alpha=1}^{m} u_{\alpha\alpha ab} + 2 \sum_{\alpha=1}^{m} \sum_{q>\rho} \frac{u_{q\alpha a} u_{q\alpha b}}{\lambda_\alpha - \lambda_q}. \tag{3.4}$$

Proof. Let $V_1, ..., V_m$ be smooth mutually orthogonal unit vector fields defined in a neighborhood of x_0 with $V_\alpha(x_0)$ the unit vector in the $\partial/\partial x_\alpha$ direction. In particular, writing $V_\alpha = (V^1_\alpha, ..., V^n_\alpha)$ we have $V^q_\alpha = \delta_{q\alpha}$ at x_0.

We consider the quantity

$$h(x) = \sum_{\alpha=1}^{m} u_{k\ell} V^k_\alpha V^\ell_\alpha - \sum_{\alpha=1}^{m} \lambda_\alpha,$$

which has $h(x_0) = 0$ and $h(x) \ge 0$ for x near x_0. In particular h achieves a minimum at x_0, and moreover, h is twice differentiable at x_0.

We prescribe the first and second derivatives of the V_α at x_0 as follows. For $1 \le \alpha \le m$, and $1 \le a \le n$,

$$\partial_a V_\alpha^q = \begin{cases} 0, & q \le \rho \\ \dfrac{u_{\alpha q a}}{\lambda_\alpha - \lambda_q}, & q > \rho. \end{cases}$$

For $1 \le \alpha, \beta \le m$ and $1 \le a, b \le n$,

$$\partial_a \partial_b V_\beta^\alpha = -\frac{1}{2} \sum_{q>\rho} \frac{u_{\alpha q a} u_{\beta q b}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)} - \frac{1}{2} \sum_{q>\rho} \frac{u_{\alpha q b} u_{\beta q a}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)}$$

noting that when $\alpha = \beta$ we have

$$\partial_a \partial_b V_\alpha^\alpha = -\sum_{q>\rho} \frac{u_{\alpha q a} u_{\alpha q b}}{(\lambda_\alpha - \lambda_q)^2}.$$

We take $\partial_a \partial_b V_\alpha^q = 0$ with $q > m$.

We first check these definitions are consistent with the V_α being orthonormal vectors. Compute at x_0, for $\alpha, \beta = 1, \ldots, m$,

$$\partial_a \left(\sum_q V_\alpha^q V_\beta^q \right) = \sum_q (\partial_a V_\alpha^q) V_\beta^q + \sum_q V_\alpha^q \partial_a V_\beta^q = 0,$$

since $\partial_a V_\alpha^q$ and $\partial_a V_\beta^q$ vanish when $q \le m \le \rho$. And

$$\begin{aligned} \partial_a \partial_b \left(\sum_q V_\alpha^q V_\beta^q \right) &= \sum_{q>\rho} (\partial_a V_\alpha^q)(\partial_b V_\beta^q) + \sum_{q>\rho} (\partial_b V_\alpha^q)(\partial_a V_\beta^q) + \partial_a \partial_b V_\alpha^\beta + \partial_a \partial_b V_\beta^\alpha \\ &= \sum_{q>\rho} \frac{u_{\alpha q a} u_{\beta q b}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)} + \sum_{q>\rho} \frac{u_{\alpha q b} u_{\beta q a}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)} \\ &\quad - \sum_{q>\rho} \frac{u_{\alpha q a} u_{\beta q b}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)} - \sum_{q>\rho} \frac{u_{\alpha q b} u_{\beta q a}}{(\lambda_\alpha - \lambda_q)(\lambda_\beta - \lambda_q)} \\ &= 0, \end{aligned}$$

as required.

Since h has a minimum at x_0 we have the inequality of matrices:

$$\begin{aligned} 0 \le h_{ab} &= \sum_{\alpha=1}^m \Big\{ u_{\alpha\alpha ab} - (\lambda_\alpha)_{ab} + 2u_{k\ell a} (\partial_b V_\alpha^k) V_\alpha^\ell + 2u_{k\ell b} (\partial_a V_\alpha^k) V_\alpha^\ell \\ &\quad + 2u_{k\ell} (\partial_a V_\alpha^k)(\partial_b V_\alpha^\ell) + 2u_{k\ell} (\partial_a \partial_b V_\alpha^k) V_\alpha^\ell \Big\} \\ &= \sum_{\alpha=1}^m \left\{ u_{\alpha\alpha ab} - (\lambda_\alpha)_{ab} + 4 \sum_{q>\rho} \frac{u_{q\alpha a} u_{\alpha q b}}{\lambda_\alpha - \lambda_q} + 2 \sum_{q>\rho} \lambda_q \frac{u_{\alpha q a} u_{\alpha q b}}{(\lambda_\alpha - \lambda_q)^2} - 2\lambda_\alpha \sum_{q>\rho} \frac{u_{\alpha q a} u_{\alpha q b}}{(\lambda_\alpha - \lambda_q)^2} \right\} \\ &= \sum_{\alpha=1}^m \left\{ u_{\alpha\alpha ab} - (\lambda_\alpha)_{ab} + 2 \sum_{q>\rho} \frac{u_{q\alpha a} u_{\alpha q b}}{\lambda_\alpha - \lambda_q} \right\}, \end{aligned}$$

giving (3.4). □

We can now complete the proof of Lemma 3.1, making crucial use of the convexity condition. Following [5], we observe that the convexity condition (1.2) can be written as: for every symmetric matrix $(X_{ab}) \in \mathrm{Sym}_n(\mathbb{R})$, vector $(Z_a) \in \mathbb{R}^n$ and $Y \in \mathbb{R}$,

$$\begin{aligned} 0 \leq\ & F^{ab,rs} X_{ab} X_{rs} + 2F^{ar} A^{bs} X_{ab} X_{rs} + F^{x_a,x_b} Z_a Z_b \\ & - 2F^{ab,u} X_{ab} Y - 2F^{ab,x_r} X_{ab} Z_r + 2F^{u,x_a} Y Z_a + F^{u,u} Y^2, \end{aligned} \tag{3.5}$$

where we are evaluating the derivatives of F at (A, p, u, x) for a positive definite matrix A. We are writing A^{ij} for the (i, j)th entry of the inverse matrix of A. We remark that we do not require the condition (1.2) to hold for the entire space $\mathrm{Sym}_n^+(\mathbb{R}) \times \mathbb{R} \times B$, but only for a certain subset which depends on the solution u.

Proof of Lemma 3.1. Observe that

$$Q = \sum_{m=1}^{k} \sum_{\alpha=1}^{m} \lambda_\alpha.$$

We will compute at a point x_0 at which the λ_i are twice differentiable and D^2u is diagonal with entries $u_{ii} = \lambda_i$. From Lemma 3.3 we have

$$\begin{aligned} F^{ab} Q_{ab} &= \sum_{m=1}^{k} \sum_{\alpha=1}^{m} F^{ab} (\lambda_\alpha)_{ab} \\ &\leq \sum_{m=1}^{k} \sum_{\alpha=1}^{m} F^{ab} u_{\alpha\alpha ab} + 2 \sum_{m=1}^{k} \sum_{\alpha=1}^{m} \sum_{q>\rho_m} F^{ab} \frac{u_{q\alpha a} u_{q\alpha b}}{\lambda_\alpha - \lambda_q}, \end{aligned}$$

where we are writing ρ_m for the largest integer $\rho_m \in \{m, m+1, \ldots, n\}$ with the property that $\lambda_{\rho_m} = \lambda_m$ at x_0. Differentiating the equation

$$F(D^2u, Du, u, x) = 0,$$

twice in the x_α direction gives

$$\begin{aligned} 0 =\ & F^{ab} u_{ab\alpha\alpha} + F^{p_a} u_{a\alpha\alpha} + F^u u_{\alpha\alpha} \\ & + F^{ab,rs} u_{ab\alpha} u_{rs\alpha} + F^{p_a,p_b} u_{a\alpha} u_{b\alpha} + F^{u,u} u_\alpha^2 + F^{x_\alpha,x_\alpha} \\ & + 2F^{ab,p_r} u_{ab\alpha} u_{r\alpha} + 2F^{ab,u} u_{ab\alpha} u_\alpha + 2F^{ab,x_\alpha} u_{ab\alpha} \\ & + 2F^{p_a,u} u_{a\alpha} u_\alpha + 2F^{p_a,x_\alpha} u_{a\alpha} + 2F^{u,x_\alpha} u_\alpha. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned} F^{ab} Q_{ab} \leq\ & \sum_{m=1}^{k} \sum_{\alpha=1}^{m} \Bigg\{ 2 \sum_{q>\rho_m} F^{ab} \frac{u_{q\alpha a} u_{q\alpha b}}{\lambda_\alpha - \lambda_q} - \sum_{a,b,r,s>\rho_k} F^{ab,rs} u_{ab\alpha} u_{rs\alpha} - F^{u,u} u_\alpha^2 - F^{x_\alpha,x_\alpha} \\ & - 2 \sum_{a,b>\rho_k} F^{ab,u} u_{ab\alpha} u_\alpha - 2 \sum_{a,b>\rho_k} F^{ab,x_\alpha} u_{ab\alpha} - 2F^{u,x_\alpha} u_\alpha \Bigg\} \\ & + CQ + \sum_{i=1}^{k} b^{i,j} (\lambda_i)_j + \sum_{1 \leq \alpha < \beta \leq \rho_k} c^{j,\alpha,\beta} u_{\alpha\beta j}, \end{aligned}$$

for uniformly bounded $b^{i,j}, c^{j,\alpha,\beta}$. Here we are using Lemma 3.2 which implies that if $1 \leq \alpha \leq \rho_k$ then at x_0,

$$u_{\alpha\alpha j} = (\lambda_i)_j,$$

for some $1 \le i \le k$ with $\lambda_i = \lambda_\alpha$. But

$$\begin{aligned}\sum_{1\le\alpha<\beta\le\rho_k} c^{j,\alpha,\beta} u_{\alpha\beta j} &\le C \sum_{1\le\alpha<\beta\le\rho_k} (\lambda_\beta - \lambda_\alpha) + 2 \sum_{1\le\alpha<\beta\le\rho_k,\, \lambda_\alpha\ne\lambda_\beta} F^{ab} \frac{u_{\alpha\beta a} u_{\alpha\beta b}}{\lambda_\beta - \lambda_\alpha} \\ &\le CQ + 2\sum_{\alpha=1}^{k-1} \sum_{\rho_\alpha<q\le\rho_k} \frac{F^{ab} u_{\alpha q a} u_{\alpha q b}}{\lambda_q - \lambda_\alpha} \\ &\le CQ + 2\sum_{m=1}^{k-1}\sum_{\alpha=1}^{m} \sum_{\rho_m<q\le\rho_k} F^{ab} \frac{u_{q\alpha a} u_{q\alpha b}}{\lambda_q - \lambda_\alpha},\end{aligned} \tag{3.7}$$

where for the first line we note that if $1 \le \alpha < \beta \le \rho_k$ with $\lambda_\alpha = \lambda_\beta$ then by Lemma 3.2 we have $u_{\alpha\beta j} = 0$.

We now apply the convexity assumption (3.5), taking for each fixed α,

$$X_{ab} = \begin{cases} -u_{ab\alpha} & \text{if } a, b > \rho_k \\ 0 & \text{otherwise,} \end{cases} \quad Z_a = \begin{cases} 1 & \text{if } a = \alpha \\ 0 & \text{otherwise,} \end{cases} \quad Y = u_\alpha. \tag{3.8}$$

This gives for each $\alpha = 1, \ldots, m$,

$$\begin{aligned} 0 \le &\sum_{a,b,r,s>\rho_k} F^{ab,rs} u_{ab\alpha} u_{rs\alpha} + 2 \sum_{a,b,q>\rho_k} F^{ab} \frac{u_{qa\alpha} u_{qb\alpha}}{\lambda_q} + F^{x_\alpha, x_\alpha} \\ &+2 \sum_{a,b>\rho_k} F^{ab,u} u_{ab\alpha} u_\alpha + 2 \sum_{a,b>\rho_k} F^{ab,x_\alpha} u_{ab\alpha} + 2F^{u,x_\alpha} u_\alpha + F^{u,u} u_\alpha^2. \end{aligned} \tag{3.9}$$

Observe that

$$2\sum_{m=1}^{k}\sum_{\alpha=1}^{m} \sum_{a,b,q>\rho_k} F^{ab} \frac{u_{qa\alpha} u_{qb\alpha}}{\lambda_q} \le 2\sum_{m=1}^{k}\sum_{\alpha=1}^{m}\sum_{q>\rho_k} F^{ab} \frac{u_{q\alpha a} u_{q\alpha b}}{\lambda_q - \lambda_\alpha}. \tag{3.10}$$

Combining all of the above gives

$$F^{ab} Q_{ab} \le CQ + \sum_{i=1}^{k} b^{i,j} (\lambda_i)_j, \tag{3.11}$$

as required. □

4. Proof of Theorem 1.1

We will first assume that $u \in C^4(B)$ and then remove this assumption by approximation. The quantity $Q_k = \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1$ is semi-concave and from Lemma 3.1 we have almost everywhere

$$F^{ab} (Q_k)_{ab} \le CQ_k + \sum_{i=1}^{k} b^{i,j} (\lambda_i)_j,$$

for $b^{i,j}$ bounded. For $\varepsilon > 0$ and a fixed $\ell \in \{1, 2, \ldots, n\}$, consider

$$R = \sum_{k=1}^{\ell} (Q_k + \varepsilon)^{1/2}.$$

Then by Proposition 2.2, since the map $x \mapsto (x+\varepsilon)^{1/2}$ is increasing, Lipschitz continuous and concave for $x \geq 0$, we see that R is semiconcave (but note that its constant of semi-concavity will depend on ε). R is twice differentiable almost everywhere. At such a point, we compute

$$\begin{aligned} F^{ab}R_{ab} &= \frac{1}{2}\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}F^{ab}(Q_k)_{ab} - \frac{1}{4}\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-3/2}F^{ab}(Q_k)_a(Q_k)_b \\ &\leq \frac{1}{2}\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}\left(CQ_k + \sum_{i=1}^{k} b^{i,j}(\lambda_i)_j\right) - c\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-3/2}|DQ_k|^2 \\ &\leq CR + C\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}\sum_{i=1}^{k}|D\lambda_i| - c\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-3/2}|DQ_k|^2, \end{aligned}$$

for uniform constants $C, c > 0$ (the constant C may differ from line to line). We claim that

$$\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}\sum_{i=1}^{k}|D\lambda_i| \leq C\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}|DQ_k|, \tag{4.1}$$

for a universal constant C. We do this by induction on ℓ. The case $\ell = 1$ is trivial. Assume it holds for $\ell - 1$. We need to show that

$$(Q_\ell+\varepsilon)^{-1/2}\sum_{i=1}^{\ell}|D\lambda_i| \leq C\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}|DQ_k|.$$

But using $Q_{\ell-1} \leq Q_\ell$, the inductive hypothesis and the definition of Q_ℓ we have

$$\begin{aligned} (Q_\ell+\varepsilon)^{-1/2}\sum_{i=1}^{\ell}|D\lambda_i| &\leq (Q_{\ell-1}+\varepsilon)^{-1/2}\sum_{i=1}^{\ell-1}|D\lambda_i| + (Q_\ell+\varepsilon)^{-1/2}|D\lambda_\ell| \\ &\leq C\sum_{k=1}^{\ell-1}(Q_k+\varepsilon)^{-1/2}|DQ_k| + (Q_\ell+\varepsilon)^{-1/2}\left(|DQ_\ell| + C\sum_{i=1}^{\ell-1}|D\lambda_i|\right) \\ &\leq C\sum_{k=1}^{\ell-1}(Q_k+\varepsilon)^{-1/2}|DQ_k| + (Q_\ell+\varepsilon)^{-1/2}|DQ_\ell| \\ &\quad + (Q_{\ell-1}+\varepsilon)^{-1/2}C\sum_{i=1}^{\ell-1}|D\lambda_i| \\ &\leq C\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}|DQ_k|. \end{aligned}$$

This completes the proof of (4.1).

It follows that, for constants $C, c > 0$ independent of ε,

$$F^{ab}R_{ab} \leq CR + C\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-1/2}|DQ_k| - c\sum_{k=1}^{\ell}(Q_k+\varepsilon)^{-3/2}|DQ_k|^2$$
$$\leq CR,$$

using the bound

$$(Q_k+\varepsilon)^{-1/2}|DQ_k| \leq \frac{\delta}{2}(Q_k+\varepsilon)^{-3/2}|DQ_k|^2 + \frac{1}{2\delta}(Q_k+\varepsilon)^{1/2},$$

which holds for any $\delta > 0$.

Since R is semi-concave, the weak Harnack inequality (Proposition 2.3 above) implies that for a uniform $q > 0$ and C,

$$||R||_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} R. \tag{4.2}$$

In particular, the constant C is independent of ε. Hence we can let $\varepsilon \to 0$ to obtain the same estimate for $\sum_{k=1}^{\ell} Q_k^{1/2}$. Write

$$S = \left(\sum_{k=1}^{\ell} Q_k^{1/2}\right)^2.$$

Now observe that for a C depending only on n we have

$$\frac{1}{C}\lambda_\ell \leq S \leq C\lambda_\ell.$$

We have

$$||S||^{1/2}_{L^{q/2}(B_{1/2})} \leq C \inf_{B_{1/2}} S^{1/2},$$

and squaring both sides gives

$$||S||_{L^{q/2}(B_{1/2})} \leq C \inf_{B_{1/2}} S$$

and hence

$$||\lambda_\ell||_{L^{q/2}(B_{1/2})} \leq C \inf_{B_{1/2}} \lambda_\ell.$$

This completes the proof of the theorem in the case that $u \in C^4(B)$.

For $u \in C^3(B)$ as in the statement of the theorem, the elliptic equation satisfied by u implies that $u \in W^{4,p}_{\text{loc}}(B)$ for all p. Fix $p > n$ and a ball $\tilde{B}$ with $B_{1/2} \subset\subset \tilde{B} \subset\subset B$. Then we can find a sequence of smooth convex functions $u^{(s)}$ in a neighborhood of $\tilde{B}$ such that $u^{(s)} \to u$ in $W^{4,p}(\tilde{B})$ as $s \to \infty$. This also implies that $u^{(s)} \to u$ in $C^3(\tilde{B})$. We wish to apply Lemma 3.1 to each $u^{(s)}$. Although $u^{(s)}$ does not solve $F(D^2u, Du, u, x) = 0$ we see from (3.6) and $F \in C^2$ that $\tilde{u} := u^{(s)}$ satisfies

$$
\begin{aligned}
0 = {} & \tilde{F}^{ab}\tilde{u}_{ab\alpha\alpha} + \tilde{F}^{p_a}\tilde{u}_{a\alpha\alpha} + \tilde{F}^{u}\tilde{u}_{\alpha\alpha} \\
& + \tilde{F}^{ab,rs}\tilde{u}_{ab\alpha}\tilde{u}_{rs\alpha} + \tilde{F}^{p_a,p_b}\tilde{u}_{a\alpha}\tilde{u}_{b\alpha} + \tilde{F}^{u,u}\tilde{u}_\alpha^2 + \tilde{F}^{x_\alpha,x_\alpha} \\
& + 2\tilde{F}^{ab,p_r}\tilde{u}_{ab\alpha}\tilde{u}_{r\alpha} + 2\tilde{F}^{ab,u}\tilde{u}_{ab\alpha}\tilde{u}_\alpha + 2\tilde{F}^{ab,x_\alpha}\tilde{u}_{ab\alpha} \\
& + 2\tilde{F}^{p_a,u}\tilde{u}_{a\alpha}\tilde{u}_\alpha + 2\tilde{F}^{p_a,x_\alpha}\tilde{u}_{a\alpha} + 2\tilde{F}^{u,x_\alpha}\tilde{u}_\alpha - f^{(s)},
\end{aligned}
\tag{4.3}
$$

where $f^{(s)} \to 0$ in $L^p(\tilde{B})$ as $s \to \infty$. Here we are using $\tilde{F}$ to indicate that we are evaluating the derivatives of F at $\tilde{u}$. Carrying out the rest of the proof of Lemma 3.1 with this extra term $f^{(s)}$ gives almost everywhere on $\tilde{B}$,

$$
\tilde{F}^{ab}\tilde{Q}_{ab} \le C\tilde{Q} + \sum_{i=1}^{k} b^{i,j}(\tilde{\lambda}_i)_j + f^{(s)},
$$

modifying $f^{(s)}$ if necessary, and evaluating $\tilde{\lambda}_i, \tilde{Q}$ etc. with respect to $\tilde{u}$. Then $\tilde{R}$ satisfies almost everywhere on $\tilde{B}$,

$$
\tilde{F}^{ab}\tilde{R}_{ab} \le C\tilde{R} + \frac{1}{2}\varepsilon^{-1/2}f^{(s)}.
$$

Applying Proposition 2.3 we obtain

$$
||\tilde{R}||_{L^q(B_{1/2})} \le C \inf_{B_{1/2}} \tilde{R} + C\varepsilon^{-1/2}||f^{(s)}||_{L^n(\tilde{B})}
$$

and letting $s \to \infty$ gives

$$
||R||_{L^q(B_{1/2})} \le C \inf_{B_{1/2}} R,
$$

for a constant C independent of ε. The rest of the proof goes through as above, and this completes the proof of Theorem 1.1.

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