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Chiral anomaly in Euler fluid and Beltrami flow

P.B. Wiegmann^a and A.G. Abanov^b

^a*Kadanoff Center for Theoretical Physics, University of Chicago,
5640 South Ellis Ave, Chicago, IL 60637, U.S.A.*

^b*Department of Physics and Astronomy and Simons Center for Geometry and Physics,
Stony Brook University,
Stony Brook, NY 11794, U.S.A.*

E-mail: wiegmann@uchicago.edu, alexandre.abanov@stonybrook.edu

ABSTRACT: We show that barotropic flows of a perfect, charged, classical fluid exhibit an anomaly analogous to the chiral anomaly known in quantum field theories with Dirac fermions. A prominent effect of the chiral anomaly is the transport electric current in the fluid at equilibrium with the chiral reservoir. We find that it is also a property of celebrated Beltrami flows — stationary solutions of the Euler equation with an extensive helicity.

KEYWORDS: Anomalies in Field and String Theories, Chern-Simons Theories, Field Theory Hydrodynamics, Topological States of Matter

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1 Introduction

Interacting quantum fermionic systems can behave as perfect fluids. Prominent examples are superconductors, the superfluid ^3He , the fractional quantum Hall effect, and one-dimensional electronic systems, e.g., the Luttinger liquid. Recently the possibility that a quark-gluon plasma arising in collisions of heavy ions may turn into a fluid came to light. The fundamental particles in quark-gluon plasma, ^3He and some other examples are the Dirac fermions. A characteristic feature of quantum field theories (QFT) with Dirac fermions is quantum anomalies that are mainly insensitive to interactions. Therefore, if the interaction turns Dirac fermions into a fluid, anomalies must also be an inherent part of the hydrodynamics. This argument set off a search for relativistic hydrodynamics models consistent with the anomalies (see, refs. [1–3], and references therein). In a recent paper, [4], the effect of the axial-current anomaly had been identified in the foundational description of fluid dynamics — the conventional classical Euler equation for barotropic flow (relativistic and non-relativistic alike).

If the Dirac fermions retain the chiral imbalance (the numbers of left and right moving particles are not equal), a fermionic system exhibits a significant anomaly-driven physical effect: an electric current runs across the system at the thermodynamic equilibrium. We show that this and other anomaly-based properties are readily apparent in Beltrami flows of the perfect classical fluid. We recall that the Beltrami flows are stationary flows with an

extensive helicity (i.e., proportional to the volume of the system). Superfluid $^3\text{He-A}$ may serve as a platform for the experimental realization of the effects discussed in this work, where there is experimental evidence of the axial current anomaly [5], and of the realization of the Beltrami flow [6].

Chiral anomaly in Dirac fermions. We briefly review the chiral anomaly in the fermionic quantum field theories before turning to fluid dynamics. Dirac fermions, whose Hamiltonian density is

$$H = \psi_L^\dagger \boldsymbol{\sigma} (i \boldsymbol{\nabla} + \mathbf{A}) \psi_L - \psi_R^\dagger \boldsymbol{\sigma} (i \boldsymbol{\nabla} + \mathbf{A}) \psi_R \quad (1.1)$$

in a time-independent magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ possess two conserved charges. They are the electric charge Q and the axial charge referred to as the chirality Q_A

$$Q = \int \rho d\mathbf{x} = N_L + N_R, \quad Q_A = \int \rho_A d\mathbf{x} = N_L - N_R. \quad (1.2)$$

The charge densities are the sum $\rho = \psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R$ and the difference $\rho_A = \psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R$ of the left and the right occupation numbers of chiral (Weyl) components of the Dirac multiplet (ψ_L, ψ_R) . They are temporal components of the 4-vector current $j = (j^0, \mathbf{j})$ and 4-axial current $j_A = (j_A^0, \mathbf{j}_A)$.

As a system with two conserved charges, Dirac fermions could be brought in contact with a reservoir which maintains the chemical potential μ coupled to the particle density ρ and the *chiral chemical potential* μ^A coupled to the chirality density ρ_A . Such a system may reside in thermodynamic equilibrium with the reservoir. Then microstates are described by a grand canonical ensemble with the Gibbs distribution $e^{-\mathcal{H}/T}$ with

$$\mathcal{H} = H - \int (\mu \rho + \mu^A \rho_A) d\mathbf{x}, \quad (1.3)$$

where H is the energy of the isolated system. At zero temperature $T = 0$, the case we consider, the equilibrium state is the ground state of $\mathcal{H}$ where the chemical potentials μ and μ^A determine the charges Q and Q_A . The chemical potential μ may be treated as a temporal component of the external electromagnetic 4-vector potential $A = (\mu, \mathbf{A})$ and the chiral chemical potential μ^A as a temporal component of the external axial 4-vector potential $A^A = (\mu^A, \mathbf{A}^A)$ commonly considered in the Dirac QFT. In the main part of the paper we focus on the case $\mathbf{A}^A = 0$. The general case is briefly considered in section 9.

In the familiar case, when the chemical potential is coupled to the particle number, the equilibrium state assumes no flow. The situation is fundamentally different when the reservoir maintains the chiral chemical potential μ^A , which in its turn causes a chiral imbalance. In this case, the equilibrium state carries an electric transport current proportional to and directed along the magnetic field

$$\mathbf{j}_{\text{trans}} = k \mu^A \mathbf{B}. \quad (1.4)$$

This result had a tangled history and met interpretational difficulties. Physics applications started perhaps from the work of Vilenkin [7]. Recently (1.4) had been explored in

heavy-ion collisions [8] among other fields. A clear interpretation of eq. (1.4) as a ballistic transport at equilibrium is presented in ref. [9] (see, also [10]). In one spatial dimension the analog of eq. (1.4) is the Landauer formula $j_{\text{trans}} = k\mu^A$ [11, 12] for the experimentally observed Landauer-Sharvin effect of the resistance of ballistic transport. There μ^A is a voltage drop across the bulk and the constant k is the conductance quantized in fundamental units e^2/h . The effect of the transport current at equilibrium also discussed in connection with non-centro-symmetric semiconductors and cosmology. For some references and applications of the effects of the chiral imbalance in various domains of physics, see refs. [13–15].

The effect of the flux at equilibrium has roots in the axial-current anomaly obtained by Adler [16] and Bell and Jackiw [17] in 1969 (see also ref. [18]). That is, under a physics requirement that the vector current is conserved in an isolated system, the divergence of the axial current is given by the formula¹

$$\partial \cdot j_A = k \mathbf{E} \cdot \mathbf{B}, \quad k = 2. \quad (1.5)$$

The coefficient k is the value of the triangle diagram with one axial and two vector vertices. In units of the magnetic flux quantum $\Phi_0 = hc/e$, the coefficient k is equal 2, the number of Weyl components in the Dirac multiplet. Then, if the system is coupled to the reservoirs, the same triangular diagram which entered (1.5), determines that a gradient of the chiral chemical potential causes the vector current to diverge

$$\partial \cdot j = k \mathbf{E}^A \cdot \mathbf{B}, \quad \mathbf{E}^A := \nabla \mu^A. \quad (1.6)$$

Eq. (1.4) is a particular stationary solution of the chiral anomaly equation (1.6). We comment that eq. (1.4) is valid for $\mu^A = \text{const}$ even though the r.h.s. of (1.6) vanishes.

Helicity and Beltrami flow of a perfect fluid. Now we turn to the electrically charged barotropic flows of a perfect fluid. The fluid’s Hamiltonian is

$$H = \int \left(\frac{1}{2} m \rho v^2 + \varepsilon[\rho] \right) d\mathbf{x}. \quad (1.7)$$

Here $\rho, \mathbf{v}$ are the fluid density and velocity, m is the mass of fluid particles. In barotropic flow the energy density ε is a function of ρ .

Like Dirac fermions, barotropic flows in three dimensions possess two conserved charges. One is the particle number $Q = \int \rho d\mathbf{x}$. The second conserved charge is the *fluid helicity*

$$\mathcal{H} = h^{-2} \int m \mathbf{v} \cdot \boldsymbol{\omega} d\mathbf{x}. \quad (1.8)$$

Here, $\boldsymbol{\omega} = \nabla \times m \mathbf{v}$ is the fluid vorticity, and h is a normalization constant of the dimension of energy $\times$ time which makes helicity dimensionless. The root of the helicity conservation can be traced to its topological nature: helicity as a linkage of vortex lines [19]. Another related root is the degeneracy of the Poisson structure of the Euler hydrodynamics. The

¹Throughout the paper we use a dot to denote the contraction of adjacent indices, i.e., $\partial \cdot j$ denotes $\partial_\alpha j^\alpha = \dot{\rho} + \nabla \cdot \mathbf{j}$.

latter gives helicity the meaning of the Casimir invariant (see, e.g., [20]). It is conserved regardless of the choice of the Hamiltonian.

For the reasons described above, the fluid could be coupled with two reservoirs as in (1.3). One maintains the particle density (the electric charge), the second maintains the helicity

$$\mathcal{H} = \int \left[\frac{1}{2} m \rho \mathbf{v}^2 + \varepsilon[\rho] - \mu \rho - \mu^A m \mathbf{v} \cdot \boldsymbol{\omega} \right] d\mathbf{x}. \quad (1.9)$$

In the next section we explain how to add the electromagnetic field to the chiral coupling. Then we will show that the formulas for anomalies (1.5), (1.6), (1.4) hold to the Euler fluid. Furthermore, if the normalization constant h in (1.8) is identified with the Planck constant, the formulas become identical. With this in mind, we will use the units of h and Φ_0 , setting $h = \Phi_0 = 1$ in (1.9) and below.

We focus on the electric transport current flown in the equilibrium state (1.4). From the fluid perspectives the Gibbs equilibrium state is a stationary (or steady) flow. We will argue that the chiral reservoir yields a stationary flow broadly known as a *Beltrami flow* [21]. We recall that the (generalized) Beltrami flow is defined by the condition that velocity and vorticity are collinear at every point

$$\text{curl } \mathbf{v} = \kappa \mathbf{v}. \quad (1.10)$$

Beltrami flows carry an extensive helicity described by the parameter κ , which, as we will show, is determined by μ^A . Then we show that the Beltrami flow conducts the electric current given by the formulas identical to the formulas (1.4), (1.6) of the fermionic QFT.

Axial current anomaly in fluid dynamics and chiral coupling. In magnetic field the definition of helicity amounts to replacing the momentum $m\mathbf{v}$ in (1.8) by the canonical momentum

$$m\mathbf{v} \rightarrow \boldsymbol{\pi} := m\mathbf{v} + \mathbf{A}, \quad (1.11)$$

where $\mathbf{A}$ is the vector potential so that $\mathbf{B} = \nabla \times \mathbf{A}$. Under this definition the helicity

$$\mathcal{H} = \int \boldsymbol{\pi} \cdot \nabla \times \boldsymbol{\pi} d\mathbf{x} \quad (1.12)$$

remains conserved [22, 23] (see, also the section VI.2.C of [21]). However, the helicity $\mathcal{H}$ is no longer an additive quantity, as its density $\boldsymbol{\pi} \cdot \nabla \times \boldsymbol{\pi}$ is not a local Eulerian field (at a given fluid momentum $m\mathbf{v}$ helicity density explicitly depends on the vector potential $\mathbf{A}$). This creates the difficulty of coupling helicity to the reservoir. The problem had been addressed in [4]. There we argued that in magnetic field the analog of the axial charge is the fluid chirality $Q_A = \int \rho_A d\mathbf{x}$, not the helicity. The chirality density ρ_A is a local Eulerian field

$$\rho_A = m\mathbf{v} \cdot (\boldsymbol{\omega} + 2\mathbf{B}), \quad (1.13)$$

and, therefore is an additive quantity which at $\mathbf{B} = 0$ is equal to the helicity density. The magnetic field modifies the angular velocity of a charged parcel by the Larmor frequency $\mathbf{B}/m$. This effectively changes vorticity as $\boldsymbol{\omega} \rightarrow \boldsymbol{\omega} + 2\mathbf{B}$. It is not difficult to find the

relation between chirality and helicity. Their difference depends only on external fields and is equal twice magnetic helicity

$$Q_A = \mathcal{H} - 2 \int \mathbf{A} \cdot \mathbf{B} d\mathbf{x} . \quad (1.14)$$

Based on this arguments we conclude that the chirality density ρ_A given by (1.13), not the helicity density (1.12) enters to the augmented energy (1.3). However, there is a caveat. There are certain space-time configurations of the electromagnetic field which may change the magnetic helicity, and therefore change the fluid chirality (1.14). It follows from (1.14) that

$$\frac{d}{dt} Q_A = 2 \int \mathbf{E} \cdot \mathbf{B} d\mathbf{x} . \quad (1.15)$$

The local form of this equation

$$\partial_t \rho_A + \nabla \cdot \mathbf{j}_A = 2 \mathbf{E} \cdot \mathbf{B} \quad (1.16)$$

is the axial-current anomaly (1.5) [see (6.2) below for the explicit formula for the axial flux $\mathbf{j}_A$]. Like in the case of fermions the chiral reservoir is coupled to a charge whose conservation is subject to the anomaly. This is the source of the effects we study.

Before we come to the main part of the paper, a few comments are in order.

Perhaps the most interesting and relevant case for applications occurs at $\mu, \mu^A = \text{const.}$ In this case, $\mathcal{H} = H - \mu Q - \mu^A Q_A$ and one may think that the reservoir contacts the system at its boundaries. Still, we assume that the chemical potentials may vary in space as that clarifies computational and structural aspects of hydrodynamics and the anomalies. In this setting, the system is in contact with the reservoir at any point in the bulk.

Similarities between anomalies in QFT and the fluid dynamics also hold in neutral rotating systems. In this case, the angular velocity is identified with the Larmor frequency $\mathbf{B}/m$ and the electric field with the gradient of an electrochemical potential or external forces.

From the perspective of fluid dynamics, fluid motion subject to the chiral reservoir is an interesting problem. To the best of our knowledge, the hydrodynamics in such a setting has not been addressed so far. We develop it below within the Hamiltonian description of the fluid dynamics. We choose to describe a non-relativistic fluid. As in the case of Dirac fermions, the chemical potential μ^A is the temporal components of a general axial potential $A^A = (\mu^A, \mathbf{A}^A)$. In the main part of this paper, we set $\mathbf{A}^A = 0$. In the last section, we extend our results for a general axial potential $A^A = (\mu^A, \mathbf{A}^A)$ and present the formulas in a covariant form readily applicable to relativistic fluid dynamics.

2 The Hamiltonian and the Poisson algebra

First, we express the fluid chirality density (1.13) in terms of the canonical momentum $\boldsymbol{\pi}$ (1.11)

$$\rho_A = (\boldsymbol{\pi} - \mathbf{A}) \cdot (\nabla \times \boldsymbol{\pi} + \mathbf{B}) \quad (2.1)$$

and use it in the augmented Hamiltonian (1.3)

$$\mathcal{H}[\rho, \boldsymbol{\pi}] = \int \left[\frac{\rho}{2m} (\boldsymbol{\pi} - \mathbf{A})^2 + \epsilon[\rho] - \mu\rho - \mu^\Lambda \rho_\Lambda \right] d\mathbf{x}. \quad (2.2)$$

The canonical momentum and the fluid density generate the Lie-Poisson algebra

$$\begin{aligned} \{\rho(\mathbf{x}), \rho(\mathbf{x}')\} &= 0, \\ \{\rho(\mathbf{x}), \boldsymbol{\pi}(\mathbf{x}')\} &= \boldsymbol{\nabla}_{\mathbf{x}'} \delta(\mathbf{x} - \mathbf{x}'), \\ \{\boldsymbol{\pi}(\mathbf{x}) \times \boldsymbol{\pi}(\mathbf{x}')\} &= (2/\rho)(\boldsymbol{\nabla} \times \boldsymbol{\pi}) \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (2.3)$$

Here in the last line the left hand side in components reads $\{\boldsymbol{\pi}(\mathbf{x}) \times \boldsymbol{\pi}(\mathbf{x}')\}^i = \epsilon^{ijk} \{\pi_j(\mathbf{x}) \times \pi_k(\mathbf{x}')\}$. Throughout the paper we routinely drop the argument $\mathbf{x}$ when it does not compromise the meaning. The brackets (2.3) yield the bracket for the chirality density

$$\{\rho_\Lambda(\mathbf{x}), \rho(\mathbf{x}')\} = (\boldsymbol{\nabla} \times \boldsymbol{\pi} + \mathbf{B}) \cdot \boldsymbol{\nabla}_{\mathbf{x}'} \delta(\mathbf{x} - \mathbf{x}'). \quad (2.4)$$

For completeness we write down the remaining brackets also following from (2.3)

$$\{\boldsymbol{\pi}(\mathbf{x}), \rho_\Lambda(\mathbf{x}')\} = -\rho^{-1}(\boldsymbol{\nabla} \times \boldsymbol{\pi}) \times (m\mathbf{v} \times \boldsymbol{\nabla}_{\mathbf{x}'} \delta(\mathbf{x} - \mathbf{x}')), \quad (2.5)$$

$$\{\rho_\Lambda(\mathbf{x}), \rho_\Lambda(\mathbf{x}')\} = -\left[\boldsymbol{\nabla} \times \left(\rho^{-1}(\boldsymbol{\nabla} \times \boldsymbol{\pi}) \cdot m\mathbf{v} \right) m\mathbf{v} \right] \cdot \boldsymbol{\nabla}_{\mathbf{x}'} \delta(\mathbf{x} - \mathbf{x}'). \quad (2.6)$$

The form of the brackets (2.4), (2.5), (2.6) reflects that the fluid chirality $Q_\Lambda = \int \rho_\Lambda d\mathbf{x}$ is the Casimir function of the Poisson algebra, i.e., Poisson-commute with all hydrodynamic fields.

Magnetic field as a central extension. We notice that the magnetic field prevents the r.h.s. of (2.4) from being expressed solely in terms of the canonical momentum $\boldsymbol{\pi}$. The magnetic field acts as a *central extension* of the Poisson algebra. In section 7 we show that the central extension in the Poisson structure is essentially equivalent to the formulation of the effect of the anomaly in terms of the divergence of the currents (1.5), (1.6). Now we comment on how the axial anomaly in the form of (1.15) follows from the bracket (2.4). First we write the Hamiltonian equation $\dot{Q}_\Lambda = \{Q_\Lambda, \mathcal{H}\}$ and notice that the only contribution comes from the term $-\int \mu \rho d\mathbf{x}$ in (2.2). Other terms are divergence and are eliminated by the integration. Then with the help of (2.4) we compute

$$\dot{Q}_\Lambda = -\{Q_\Lambda, \int \mu \rho d\mathbf{x}\} = 2 \int \boldsymbol{\nabla} \mu \cdot \mathbf{B} d\mathbf{x}. \quad (2.7)$$

This yields (1.15) after identifying $\boldsymbol{\nabla} \mu$ with the electric field $\mathbf{E}$.

We remark that if magnetic field possess monopoles with a charge $q_{\text{mon}} = -\int \boldsymbol{\nabla} \cdot \mathbf{B} d\mathbf{x}$, then (2.7) yields the central extension of the Poisson algebra of the charges

$$\{Q, Q_\Lambda\} = 2q_{\text{mon}}. \quad (2.8)$$

The same formula holds for Dirac fermions (see, e.g., refs. [18, 24]). It follows from the current algebra of Faddeev [25] (see, also [9]), where the magnetic field also yields the central extension of the algebra of fermionic currents.

Conjugate fields. We will use the fields conjugate to the density, the momentum, and to the external electromagnetic vector potential. The field conjugate to the density reads

$$\pi_0 := -\delta\mathcal{H}/\delta\rho|_{\pi} = -\Phi + \mu, \quad (2.9)$$

where the Bernoulli function Φ and the enthalpy w are given by

$$\Phi = \frac{1}{2}m\mathbf{v}^2 + w[\rho], \quad w = d\varepsilon/d\rho. \quad (2.10)$$

The field conjugate to the momentum is the *particle current*

$$\mathcal{J} = \delta\mathcal{H}/\delta\boldsymbol{\pi}|_{\rho}, \quad (2.11)$$

whose meaning we will discuss shortly.

Our primary interest is the electric (or vector) current which we define as a linear response to the electromagnetic vector potential $\mathbf{A}$ (at a fixed fluid momentum $\boldsymbol{\pi}$) evaluated at zero electromagnetic field²

$$\mathbf{j} = -(\delta\mathcal{H}/\delta\mathbf{A}|_{\boldsymbol{\pi}})|_{\mathbf{A}=0}. \quad (2.12)$$

We further clarify the definition of the electric current in section 6 where we discuss the Lorentz force.

We remark that in an isolated Galilean invariant fluid, the fluid momentum per particle, the electric current and the vector flow field $\mathbf{v}$ are proportional to the fluid velocity $\mathbf{v}$. The chiral reservoir makes these three vector fields (1.11), (2.12), (2.11) different.

3 Flow vector field and the continuity equation

Frozen-in fields are fluid substances whose Lie derivatives along a *flow vector field* vanish. We denote the flow vector field as $\mathcal{U}$. The physical meaning of $\mathcal{U}$ is the true fluid velocity, which is not necessarily equals to $\mathbf{v}$ serving as a notation for the fluid momentum $\mathbf{p} = m\mathbf{v}$. The particle current $\mathcal{J} = \rho\mathcal{U}$, defined by eq. (2.11), is the subject of the continuity equation. Indeed, consider the Hamiltonian equation

$$\dot{\rho}(\mathbf{x}) = \{\rho(\mathbf{x}), \mathcal{H}\} = \int \{\rho(\mathbf{x}), \boldsymbol{\pi}\} \frac{\delta\mathcal{H}}{\delta\boldsymbol{\pi}}, \quad (3.1)$$

where the integration goes over the omitted argument. Then with the help of Poisson algebra and (2.11) we obtain that the particle 4-current $\mathcal{J} = (\rho, \mathcal{J})$ is divergence-free

$$\partial \cdot \mathcal{J} \equiv \dot{\rho} + \boldsymbol{\nabla} \cdot \mathcal{J} = 0. \quad (3.2)$$

We see that the flow along the vector field $\mathcal{U}$ advects the fluid mass. Later we will see that vorticity is also advected along $\mathcal{U}$.

²We comment that the electric current in [9] is defined as $\mathcal{J}$.

The explicit form of the particle current follows from eqs. (1.3), (1.7), (1.11) and its definition (2.11)

$$\mathcal{J} = \rho \mathbf{v} - 2\mu^A(\boldsymbol{\omega} + \mathbf{B}) - \mathbf{E}_A \times m\mathbf{v} \quad (3.3)$$

$$= \rho \mathbf{v} - \mu^A(\boldsymbol{\omega} + 2\mathbf{B}) - \nabla \times (\mu^A m\mathbf{v}), \quad (3.4)$$

where we denote $\mathbf{E}^A = \nabla \mu^A$. In the important case, when μ^A is constant across the fluid, the term $\int \mu^A \rho_A d\mathbf{x} = \mu^A Q_A$ in (1.3) is the Casimir invariant that does not affect the equations of motion. In this case

$$\mathcal{J}|_{\mu^A=\text{const}} = \rho \mathbf{v} - 2\mu^A(\boldsymbol{\omega} + \mathbf{B}), \quad (3.5)$$

so that $\mathcal{J}$ differs from $\rho \mathbf{v}$ by a divergence-free term.

4 Vector current and its anomaly

Applying the definition (2.12) to the Hamiltonian (2.2) we obtain the explicit expression of the vector current in terms of the Eulerian fields $\rho, \mathbf{v}^3$

$$\mathbf{j} = \rho \mathbf{v} + \mathbf{E}^A \times m\mathbf{v}. \quad (4.1)$$

If μ^A is uniform $\mathbf{E}^A = 0$ and the vector current does not deviate from $\rho \mathbf{v}$. Otherwise, the vector current (4.1) receives a component normal to velocity, a feature reminiscent of the Hall effect.

From (4.1) and (3.3) we obtain the relation between the particle current and the vector current

$$\mathcal{J} = \mathbf{j} - 2\nabla \times (\mu^A m\mathbf{v}) - 2\mu^A \mathbf{B}. \quad (4.2)$$

Then, the continuity equation (3.2), written in terms of the vector current yields the chiral anomaly (1.6)

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 2\nabla \mu^A \cdot \mathbf{B}. \quad (4.3)$$

Hence, from the perspective of the fluid dynamics, the chiral anomaly (1.6) is yet another form of the continuity equation emphasizing the difference between the vector current $\mathbf{j}$ and the particle current $\mathcal{J}$ given by (4.2).

The reader should not be surprised that the electric current conducted by the flow is not conserved (4.3). The chiral reservoir represented by a non-uniform chiral chemical potential is able to supply an electric charge locally into the system bulk. The fluid is mere a subsystem of a complete system which includes the reservoir. For example, one could think about the reservoir as a species of charged particles that do not participate in the flow before they enter or are removed from the flow as required by a given chemical potential.

³We assume that the definition (2.12) remains intact even if the system is in the magnetic field. In more general situations, the electric current may contain additional divergence-free micro-currents (aka non-minimal terms) triggered by the magnetic field.

5 Euler equation

We now proceed with the Euler equation. We compute

$$\dot{\pi}(\mathbf{x}) = \{\pi(\mathbf{x}), \mathcal{H}\} = \int \left[\{\pi(\mathbf{x}), \rho\} \frac{\delta \mathcal{H}}{\delta \rho} + \{\pi(\mathbf{x}), \boldsymbol{\pi}\} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\pi}} \right], \quad (5.1)$$

where the integration goes over the omitted argument. With the help of the Poisson brackets (2.3) and the definitions (2.9), (2.11) we obtain the Euler equation

$$\dot{\boldsymbol{\pi}} - \nabla \pi_0 = \boldsymbol{\mathcal{U}} \times (\nabla \times \boldsymbol{\pi}). \quad (5.2)$$

If there is no chiral coupling, the flow vector field $\boldsymbol{\mathcal{U}}$ equals $\mathbf{v}$ and (5.2) becomes the familiar Euler equation for the barotropic flow of a charged fluid $m\dot{\mathbf{v}} + \nabla(\frac{m}{2}\mathbf{v}^2 + w) - \mathbf{v} \times \boldsymbol{\omega} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.

The form (5.2) of the Euler equation is the basis of the geometric interpretation of the Euler hydrodynamics. It is instructive to write the Euler equation in the space-time covariant form. Let us introduce a space-time 4-momentum

$$\pi_\alpha = (\pi_0, \boldsymbol{\pi}) \quad (5.3)$$

and the canonical vorticity 2-form

$$\Omega_{\alpha\beta} = \partial_\alpha \pi_\beta - \partial_\beta \pi_\alpha. \quad (5.4)$$

The flow vector field $\boldsymbol{\mathcal{U}}$ relates the components of the vorticity so that (5.2) can be written as $\Omega_{0i} = \mathcal{U}^j \Omega_{ji}$. This means that the flow advects the vorticity along the vector field $\boldsymbol{\mathcal{U}}$ (the Helmholtz law).

In the space-time notations $\mathcal{J} = (\rho, \boldsymbol{\mathcal{J}})$, the Euler equation receives the compact form

$$\mathcal{J}^\alpha \Omega_{\alpha\beta} = 0. \quad (5.5)$$

known in relativistic hydrodynamics as the Carter-Lichnerowicz equation [26–28].

6 Momentum-energy equations and the Lorentz force

An insight about forces the reservoir exerts on the fluid comes from the equations for the fluid momentum $\rho\mathbf{v}$ and the energy density $\mathcal{E} = \frac{1}{2}m\rho v^2 + \varepsilon$. We obtain them by transforming the Euler equation (5.2) with the help of (3.2), (3.3)

$$\begin{cases} \partial_t(\rho\mathbf{v}) + \partial_i(v_i\rho\mathbf{v}) + \nabla P = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} + \rho_A \mathbf{E}_A, \\ \partial_t\mathcal{E} + \nabla \cdot \mathbf{j}_\mathcal{E} = \mathbf{E} \cdot \mathbf{j} + \mathbf{E}^A \cdot \mathbf{j}_A, \end{cases} \quad (6.1)$$

where $P = \rho w - \varepsilon$, is the pressure and $\mathbf{j}_\mathcal{E} = (w + mv^2/2)\rho\mathbf{v}$ is the energy flux of the fluid and

$$\mathbf{j}_A = \Phi(\boldsymbol{\omega} + 2\mathbf{B}) - m\mathbf{v} \times ((\boldsymbol{\omega} + \mathbf{B}) \times \boldsymbol{\mathcal{U}}) + \mathbf{E} \times m\mathbf{v}. \quad (6.2)$$

In the next section, we show that (6.2) is the axial current of (1.16). The l.h.s. of these equations are the space-time divergence of the momentum and the energy flux of the isolated fluid, the r.h.s. are forces exerted by the reservoirs and electromagnetic field.

The highlights of these equations are (i) $\mathbf{E}^A$ acts on the fluid axial current $(\rho_A, \mathbf{j}_A)$ similarly to the action of the electric field $\mathbf{E}$ acts on the vector current $(\rho, \mathbf{j})$, and (ii) the vector current $\mathbf{j}$ (4.1) defining the Lorentz force is the electric current defined by (2.12).

7 Axial-current and its anomaly

In a similar manner we derive the axial anomaly equation (1.16). With the help of the brackets (2.4), (2.5) we compute $\partial_t \rho_A = \{\rho_A, \mathcal{H}\}$. Proceeding similarly to (5.1) and using the notations (2.9), (2.11) we obtain

$$\partial_t \rho_A(\mathbf{x}) = \int \left(\{\rho_A(\mathbf{x}), \boldsymbol{\pi}\} \cdot \boldsymbol{\mathcal{J}} - \{\rho_A(\mathbf{x}), \rho\} \pi_0 \right), \quad (7.1)$$

[the integration goes over the omitted argument]. Then with the help of (2.4) and (2.5), we obtain the axial anomaly equation (1.16) and the hydrodynamic expression for the axial current. The latter is given by (6.2) and enters the evolution equation (6.1) for the energy density.

Finally, with the help of the Euler equation (5.2) and using the notation

$$p_\alpha := \pi_\alpha - A_\alpha = \left(-w[\rho] - \frac{1}{2} m \mathbf{v}^2, m \mathbf{v} \right), \quad (7.2)$$

we write the chirality 4-current $j_A = (\rho_A, \mathbf{j}_A)$ in a space-time covariant form previously obtained in [4]

$$j_A^\alpha = \epsilon^{\alpha\beta\gamma\delta} p_\beta (\partial_\gamma p_\delta + F_{\gamma\delta}), \quad (7.3)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic field tensor.

8 Beltrami flow

Now we are equipped to discuss stationary flows. We assume the external fields are either time-independent, or depend on time adiabatically. The stationary (or steady) flows give extrema to $\mathcal{H}$. The variation of $\mathcal{H}$ with respect to the fluid density is the Bernoulli function (2.9). The variation with respect to the momentum is the flow vector field (2.11). In the stationary flow, whose density and velocity we denote by $\bar{\rho}, \bar{\mathbf{v}}$, both these variations should be zero. This gives the conditions

$$\begin{cases} \pi_0 = 0 : & w[\bar{\rho}] + \frac{1}{2} m \bar{\mathbf{v}}^2 = \mu, \\ \boldsymbol{\mathcal{J}} = 0 : & \rho \bar{\mathbf{v}} = 2\mu^A (\bar{\boldsymbol{\omega}} + \mathbf{B}) + \mathbf{E}^A \times m \bar{\mathbf{v}}. \end{cases} \quad (8.1)$$

The first line is the Bernoulli condition. The second defines the generalized Beltrami flow in the magnetic field. Together they determine $\bar{\rho}, \bar{\rho}_A$ as functions of μ, μ^A . For example, at $\mathbf{B} = 0$ we have the relation $\mu^A \bar{\rho}_A = (\mu - w[\bar{\rho}]) \bar{\rho}$.

We recall (see eq. 1.10) that in the absence of magnetic field, the velocity of the generalized Beltrami flow is an eigenfunction of the curl operator

$$\text{curl } \bar{\mathbf{v}} = \kappa \bar{\mathbf{v}}. \quad (8.2)$$

In our case $\text{curl } \mathbf{v} = \frac{1}{\sqrt{\mu^A}} \nabla (\sqrt{\mu^A} \times \mathbf{v})$ and $\kappa = \rho/2m\mu^A$. Extension by the magnetic field amounts to the shift by the Larmor frequency $\text{curl } \mathbf{v} \rightarrow \text{curl } \mathbf{v} + \mathbf{B}/m$.

Locally the Beltrami flow describes a helix along a vortex line. The pitch of the helix of the order $1/\kappa$ (see, e.g., [29]). A particularly interesting case is the Beltrami flow proper, when $\kappa = \text{const}$. In this case, the Beltrami flow is incompressible as $\text{div } \bar{\mathbf{v}} = 0$, where $\text{div } \mathbf{v} = \frac{1}{\sqrt{\mu^\Lambda}} \nabla \cdot (\sqrt{\mu^\Lambda} \mathbf{v})$. An important feature of the streamlines of the incompressible Beltrami flow is that they are not integral, and some are known to be chaotic [21]. A similar situation occurs in some compressible (generalized) Beltrami flows [30] (see also [31] for conditions when compressible solutions exist). We do not elaborate on this aspect further but remark that a possibility of chaotic advection of particles (Lagrangian turbulence) in the context of fermionic QFT seems to be an unexplored aspect of anomalies. The Beltrami flow appears as a hydrodynamic realization of the ground state of the Dirac fermions with a chiral imbalance.

Transport current of the Beltrami flow. Let us evaluate the vector and the axial current on the Beltrami flow. This way we obtain the currents at the equilibrium. Combining (4.1), (6.2), (8.1) we obtain

$$\bar{\mathbf{j}} = 2\mu^\Lambda \mathbf{B} + 2\nabla \times (\mu^\Lambda m \bar{\mathbf{v}}), \quad \bar{\mathbf{j}}_A = 2\mu \mathbf{B} + \nabla \times (\mu m \bar{\mathbf{v}}). \quad (8.3)$$

If the chemical potentials are constant the vector and the axial currents are⁴

$$\bar{\mathbf{j}} = 2\mu^\Lambda (\mathbf{B} + \boldsymbol{\omega}), \quad \bar{\mathbf{j}}_A = 2\mu \left(\mathbf{B} + \frac{1}{2} \boldsymbol{\omega} \right), \quad \mu, \mu^\Lambda = \text{const}. \quad (8.4)$$

We notice that both currents (8.3) consist of the transport part and the micro-current as

$$\bar{\mathbf{j}} = \mathbf{j}_{\text{trans}} + \mathbf{j}_{\text{micro}}, \quad \bar{\mathbf{j}}_A = \mathbf{j}_{A \text{ trans}} + \mathbf{j}_{A \text{ micro}} \quad (8.5)$$

The transport currents

$$\mathbf{j}_{\text{trans}} = 2\mu^\Lambda \mathbf{B}, \quad \mathbf{j}_{A \text{ trans}} = 2\mu \mathbf{B}. \quad (8.6)$$

transfer hydrodynamics substances along magnetic lines propagating across the entire system. The microscopic currents are the curl $\mathbf{j}_{\text{micro}} = 2\nabla \times (\mu^\Lambda m \bar{\mathbf{v}})$, $\mathbf{j}_{A \text{ micro}} = \nabla \times (\mu m \bar{\mathbf{v}})$. At constant chemical potentials the micro-currents are

$$\mathbf{j}_{\text{micro}} = 2\mu^\Lambda \boldsymbol{\omega}, \quad \mathbf{j}_{A \text{ micro}} = \mu \boldsymbol{\omega}. \quad (8.7)$$

They describe a stationary movement of electric and axial charges localized on vortex lines.

As we remarked in the section 7 one can trace the origin of the transport current to the central extension of the Poisson algebra (2.4). The expressions (8.3)–(8.4) are the main results of this work. We have shown that the equilibrium state of the fluid coupled to the chiral reservoir (i) is a Beltrami flow and that (ii) the Beltrami flow transports both electric and the chirality charges in a way identical to the anomaly-driven flows of Dirac fermions.

⁴Formulas similar to (8.4) appear in various hydrodynamic models (other than Euler hydrodynamics) aimed to be consistent with the chiral anomaly. There the term proportional to the magnetic field and the term proportional to the vorticity in the vector current is dubbed by the ‘chiral magnetic effect’ and the ‘chiral vortical effect’, respectively (see, e.g., [32] and references therein). Similar terms in the axial current are sometimes referred to as the ‘chiral separation effect’ and the ‘chiral vortical separation effect’, respectively [13]. The coefficients vary across the literature.

9 Flows in external axial potential

We conclude the paper by extending the coupling to the chiral reservoir from μ^A to a full axial 4-potential $\mu^A \rightarrow A_\alpha^A = (\mu^A, \mathbf{A}^A)$. The extended coupling brings various currents to space-time-covariant forms and further clarifies the parallels between the fluid dynamics and Dirac fermions in the axial field. The results can be obtained by means of covariant extensions of the formulas of the main text. We present the main expressions here and publish a detailed derivation separately.

The covariant extension is based on the main property of the Euler hydrodynamics: the fluid momentum $p_\alpha = \pi_\alpha - A_\alpha$, and the particle current $\mathcal{J}^\alpha$, are $\mathbb{R}^4$ -covector and vector, respectively, regardless of the space-time structure, Galilean or Minkowski. For that reason the formulas below are equally valid in the relativistic case.

We start from the currents. The axial current (7.3) does not depend on the axial potential and has already appeared in the space-time covariant form. In turn, the vector current does not depend on the vector potential. The particle 4-current is the covariant extension of (4.2). Together the currents read

$$\begin{cases} j^\alpha = \rho u^\alpha + {}^\star F_A^{\alpha\beta} p_\beta, \\ j_A^\alpha = \epsilon^{\alpha\beta\gamma\delta} p_\beta (\partial_\gamma p_\delta + F_{\gamma\delta}), \\ \mathcal{J}^\alpha = j^\alpha + 2 {}^\star F^{\alpha\beta} A_\beta^A + 2 \epsilon^{\alpha\beta\gamma\delta} \partial_\gamma (A_\beta^A p_\delta). \end{cases} \quad (9.1)$$

In these formulas we denote the 4-velocity $u^\alpha = (1, \mathbf{v})$, or $u^\alpha = (1, \mathbf{v}/c)/\sqrt{1 - \mathbf{v}^2/c^2}$ in the relativistic case. Also, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field tensor, ${}^\star F^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ is the dual field tensor, and we use similar notations for the axial field tensors.

As we mentioned in the section 5, the covariant form of the Euler equation appears as the Carter-Lichnerowicz equation (5.5). We repeat it here

$$\begin{cases} \mathcal{J}^\alpha (\partial_\alpha \pi_\beta - \partial_\beta \pi_\alpha) = 0, \\ \partial_\alpha \mathcal{J}^\alpha = 0. \end{cases}$$

The covariant form of the divergence of the stress tensor extends the equation (6.1) and the divergence of the vector current extends the formula (4.3). Together they are given by

$$\begin{cases} \partial_\beta T_\alpha^\beta = F_{\alpha\beta} j^\beta + F_{\alpha\beta}^A j_A^\beta, \\ \partial \cdot j = F \cdot {}^\star F^A = 2(\mathbf{E}^A \cdot \mathbf{B} + \mathbf{B}^A \cdot \mathbf{E}), \\ \partial \cdot j_A = \frac{1}{2} F \cdot {}^\star F = 2\mathbf{E} \cdot \mathbf{B}. \end{cases} \quad (9.2)$$

The equations (9.1) and (9.2) retain their form for the relativistic fluid. For the non-relativistic fluid the components of the momentum-stress-energy tensor in (9.2) are: the energy density $-T_0^0 = \frac{1}{2} m \rho \mathbf{v}^2 + \varepsilon[\rho]$, the energy flux $-T_0^i = \rho v^i (\frac{1}{2} m v^2 + w)$, the momentum density $T_i^0 = m \rho v_i$, and the momentum flux $T_i^j = m \rho v_i v^j + P \delta_i^j$. For a relativistic fluid $T_\alpha^\beta = \rho p_\alpha u^\beta + P \delta_\alpha^\beta$, and the entries in (9.1) are the relativistic 4-velocity $u^\alpha = (1, \mathbf{v}/c)/\sqrt{1 - \mathbf{v}^2/c^2}$ and the relativistic 4-momentum $p_\alpha = (mc + c^{-1}w)u_\alpha$.

We comment that starting from the eqs. (9.2) and given momentum-stress-energy tensor, one could derive the expressions for the currents (9.1) as a consistency of (9.2). This approach had been explored in [1].⁵

10 Summary

Summing up, we have shown that anomaly-based properties of QFT with Dirac fermions coupled to the axial vector potential are identical to the kinematic properties of the Euler fluid coupled to the reservoir, which maintains the fluid helicity. The result is captured by eqs. (9.1), (9.2). From the perspective of fluid dynamics, the equation for the stress tensor (the first equation in (9.2)) is the complete equation of motion. Once the stress tensor and the currents are expressed through the fluid density and the velocity as in (9.1), the divergence of the currents (the last two equations in (9.2)) follow. Now let us look on (9.2) from the perspective of Dirac fermions. The first equation in (9.2) is valid for the classical Dirac equation, or quantum alike. However, the divergences of the currents (the last two equations) are of a quantum nature. They require a short-distance regularization which preserves the gauge invariance. That regularization yields the anomaly in the divergence of the currents. Once the divergence is determined, the equations must be completed by relations expressing the stress tensor and the axial current in terms of the vector current. We have shown that the Euler fluid provides such relation in the form of eqs. (9.1). Another result of the paper is a close parallel between the Beltrami flow in fluid dynamics and the ground state of the Dirac fermions. The Beltrami flow stands out as a fascinating property of fluid dynamics. In turn, the quantum anomalies might be understood from the properties of the ground state of the Dirac fermions (perturbed by the external fields). A close parallel between these objects of seemingly unrelated fields seems noteworthy. Interpretation and consequences of the chaotic nature of the Beltrami flow in the quantum field theory are especially interesting.

Equations of motions (9.2) and the hydrodynamic expressions for the vector and the axial currents (9.1) are gauge invariant with respect to both the vector $A \rightarrow A + \partial\varphi$ and the axial $A^A \rightarrow A^A + \partial\varphi^A$ gauge transformations. However, the particle current $\mathcal{J}$ given by the third line of (9.1) and the Beltrami flow defined by the condition $\mathcal{J} = 0$ are not invariant under the axial gauge transformation. As a result, the quantities evaluated on the stationary flow (8.1) do not generally possess the axial gauge symmetry. Despite the axial gauge symmetry of equations of motion, the symmetry does not hold at the equilibrium state (or the ground state in QFT). This is the essence of the anomaly: the (axial) symmetry of equations of motions is not a symmetry of their solutions. This property gives rise to the main effect we discussed in the paper. The equilibrium state (the Beltrami flow) possesses a non-vanishing mass flux described in section 5.2. Even if the axial vector potential is a pure gauge, so that the axial field is a gradient $A_\beta^A = (2\pi)^{-1} \partial_\beta \Theta$, and for that reason drops out of the equations of motions (9.2), the particle current $\mathcal{J}$, hence the state of equilibrium

⁵We comment on the formal relation between eqs. (9.1), (9.2) and the formulas of refs. [1, 33]. Equations $\partial_\beta T_\alpha^\beta = F_{\alpha\beta} j_{[1]}^\beta$ and $\partial_\mu j_{[1]}^\mu = C \mathbf{E} \cdot \mathbf{B}$ of [1, 33] follow from (9.2) in a particular background when the vector and the axial potentials are set proportional $A^A = CA/6$. Under this condition $j_{[1]}$ of ref. [1] is a particular linear combination $j + (C/6)j_A$ of the currents given by (9.1).

$\mathcal{J} = 0$ depends on Θ . As a result, the equilibrium state transports the mass/charge with an adiabatic change of the theta-angle as $\dot{j}_{\text{trans}} = (1/\pi)\mathbf{B}\dot{\Theta}$. The amount of charge crossing the surface element $d\sigma$ when Θ changes by 2π is $2Bd\sigma$, i.e., twice the magnetic flux threading the surface.

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