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Dyson gas on a curved contour

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Abstract

We introduce and study a model of a logarithmic gas with inverse temperature β on an arbitrary smooth closed contour in the plane. This model generalizes Dyson's gas (the β-ensemble) on the unit circle. We compute the non-vanishing terms of the large N expansion of the free energy (N is the number of particles) by iterating the 'loop equation' that is the Ward identity with respect to reparametrizations and dilatation of the contour. We show that the main contribution to the free energy is expressed through the spectral determinant of the Neumann jump operator associated with the contour, or equivalently through the Fredholm determinant of the Neumann–Poincare (double layer) operator. This result connects the statistical mechanics of the Dyson gas to the spectral geometry of the interior and exterior domains of the supporting contour.

Keywords: Dyson, logarithmic gas, loop equation, free energy

1. Introduction

The logarithmic gases were introduced by F Dyson in the seminal papers [1], where it was shown that eigenvalues of random matrices can be represented as a statistical ensemble of charged particles with 2D Coulomb (logarithmic) interaction at inverse temperature $\beta > 0$. Particles can be located either on a circle or on an interval of a straight line. The partition function of the circular Dyson's ensemble is

$$Z_N = \oint_{|z_1|=1} \dots \oint_{|z_N|=1} |\Delta_N(\{z_i\})|^{2\beta} |\mathrm{d}z_1| \dots |\mathrm{d}z_N|, \qquad z_j = \mathrm{e}^{\mathrm{i}\theta_j},$$
$$|\mathrm{d}z_j| = \mathrm{d}\theta_j. \tag{1.1}$$

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Here

$$\Delta_N(\{z_i\}) = \prod_{j<k}^{N} (z_j - z_k) \tag{1.2}$$

is the Vandermonde determinant. A comprehensive description of logarithmic gases can be found in [2].

The values $\beta = \frac{1}{2}, 1, 2$ are special. In these cases, the logarithmic gases (on various supports) can be represented either through free fermions or through some random matrix ensembles. Furthermore, for these special values of β the log-gases possess an integrable structure. In particular, the $\beta = 1$ partition function is the tau-function of the 2D Toda lattice hierarchy (see (4.44) and (4.45) below). Positive integer values of β are also special. In particular, the two-point correlation function then permits a β-dimensional integral representation valid for any N. Other values of β do not seem to possess special properties. Nevertheless, a few particular integrals where computed explicitly for any values of β. In particular, the integral (1.1) equals

$$Z_N = (2\pi)^N \frac{\Gamma(1 + N\beta)}{\Gamma^N(1 + \beta)}$$

(see [1]). This is not the case for correlation functions, though.

In this paper, we introduce a natural generalization of the Dyson ensemble (1.1): the logarithmic gas on an arbitrary closed contour (a deformed circle) and study how the partition function and various correlation functions depend on the geometry of the contour. Namely, we consider the β-ensemble of N particles on a simple closed contour Γ on a plane. An early reference on this problem is [3] (section 7). For further convenience, we place the gas into N-independent external potential W. The partition function reads

$$Z_N = \oint_\Gamma \dots \oint_\Gamma \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{k=1}^{N} \mathrm{e}^{W(z_k)} |\mathrm{d}z_k|. \tag{1.3}$$

Clearly, the dimension of Z_N is $[\text{length}]^{\beta N^2 + (1-\beta)N}$. The real-valued function W is defined in a strip-like neighborhood of the contour and depends on both z and $\bar{z}$ (we write $W = W(z)$ just for simplicity of the notation). The meaning of the potential W may be understood as a metric on the contour, $\mathrm{d}s = \mathrm{e}^W |\mathrm{d}z|$. As a functional of W, the partition function is the generating functional of correlation functions of densities. A similar problem having support on the complex plane, the 2D Dyson gas, had been considered by us earlier (see [4–7]).

We are mostly interested in the large N limit, $N \to \infty$, and in the large N expansion of the free energy. Let us introduce the small parameter

$$\hbar = \frac{1}{N}, \tag{1.4}$$

then in the large N limit the behavior of the partition function is of the form

$$Z_N = N!\, N^{(\beta-1)N}\, \mathrm{e}^F,$$

where

$$F = \frac{F_0}{\hbar^2} + \frac{F_1}{\hbar} + F_2 + O(\hbar), \quad \hbar \to 0. \tag{1.5}$$

The factors $\mathrm{e}^{F_0/\hbar^2}$ and $\mathrm{e}^{F_1/\hbar}$ carry the dimensions $[\text{length}]^{\beta N^2}$ and $[\text{length}]^{(1-\beta)N}$ respectively. We are interested in the contributions that do not vanish as $\hbar \to 0$, i.e., in the first three terms F_0, F_1 and F_2 of the large N expansion of the free energy F.

We obtain explicit expressions for F_0, F_1 and F_2 in terms of the conformal map $w_{\text{ext}}(z)$ of the exterior domain bounded by the contour Γ to the exterior of the unit circle (normalized as $w_{\text{ext}}(\infty) = \infty$, $w'_{\text{ext}}(\infty) > 0$) and the Neumann jump operator $\hat{\mathcal{N}}$ (which is defined below, see (2.9)). The leading contribution, F_0, equals the electrostatic energy of a charged 2D conductor of the shape Γ (neutralized by a charge at infinity). The next-to-leading contribution, F_1, reflects the entropy and the excluded volume (the effect that the place occupied by a particle is no longer accessible for others). The most interesting quantity is F_2. It is shown to be closely related to the spectral theory of the interior and exterior domains. Setting $W = 0$, we have:

$$\begin{aligned} F_0 &= \beta \log r, \\ F_1 &= -(\beta - 1) \log \frac{er}{2\pi\beta} + \log \frac{2\pi}{\Gamma(\beta)}, \\ F_2 &= \frac{(\beta-1)^2}{8\pi\beta} \oint_\Gamma \log \rho_0 \hat{\mathcal{N}} \log \rho_0 \mathrm{d}s - \frac{1}{2} \log \frac{P}{2\beta} + \frac{1}{2} \log \det{}' \hat{\mathcal{N}}. \end{aligned} \tag{1.6}$$

Here $r = \left(w'_{\text{ext}}(\infty)\right)^{-1}$ is the external conformal radius of the domain D bounded by the contour (we also call it the conformal radius of the exterior domain $\mathbb{C}\backslash\mathsf{D}$), $\rho_0(z) = \frac{1}{2\pi}|w'_{\text{ext}}(z)|$ is the mean density of particles in the leading order, P is the perimeter of the contour Γ, $\mathrm{d}s = |\mathrm{d}z|$ is the line element along the contour and $\det' \hat{\mathcal{N}}$ is the regularized spectral determinant of the Neumann jump operator with the zero mode removed. The result for F_2 can be naturally divided into 'classical' part having an electrostatic nature (the first term) and the 'quantum' part which origin is similar to the gravitational anomaly of the free bosonic field localized on the contour. The quantum part can be also represented as the logarithm of the Fredholm determinant associated to the domain surrounded by the contour Γ. It is explicitly given by equation (4.36). We also calculate one- and two-point correlation functions of densities of the particles in the large N limit.

Our main technical tool is the so-called loop equation, which is an exact relation connecting one-and two-point correlation functions. It follows from the invariance of the partition function under re-parameterizations of the contour (admissible changes of integration variables). The term 'loop equation' goes back to the terminology used in the theory of random matrices but long before it is known as Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy in statistical mechanics or as Ward identity w.r.t. dilatations in quantum field theory. Although this relation is to some extent tautological, it becomes really meaningful when combined with the large N expansion. We show that the general form of the loop equation for contours Γ of arbitrary shape is a condition for the boundary value of a holomorphic quantity $T(z)$ closely related to the stress-energy tensor of a Gaussian field in 2D often utilized in conformal field theory (CFT). Its mean value $\langle T(z)\rangle$ is a holomorphic function both in the interior and exterior domains with a jump across the contour Γ. The boundary condition for the mean value is equivalent to the loop equation. It states that the jump of the off-diagonal tangential-normal component $\left[\langle T_{sn}(z)\rangle\right]_\Gamma = \left\langle T^{(+)}_{sn}(z)\right\rangle - \left\langle T^{(-)}_{sn}(z)\right\rangle$ across the contour ($z \in \Gamma$, $T^{(+)}$ is the value from inside and $T^{(-)}$ from outside) vanishes. Below we will use the notation $[\ldots]_\Gamma$ for a jump across the contour.

Another technical tool is the rule of variation of the partition function under small deformations of the contour. We show that this variation is expressed through the jump $\left[\langle T_{nn}(z)\rangle\right]_\Gamma$

of the normal–normal component of $\langle T(z)\rangle$ across the contour:

$$\hbar^2\delta\,\log\,Z_N = -\frac{1}{\pi\beta}\oint_{\Gamma}\left[\langle T_{nn}(z)\rangle\right]_{\Gamma}\delta n(z)|\mathrm{d}z|. \tag{1.7}$$

Here $\delta n(z)$ is the infinitesimal normal displacement of the contour at the point $z\in\Gamma$. This formula, combined with the large N expansion of the loop equation, allows one to find variations of F_0, F_1 and F_2 and then, having at hand the variations, to restore them up to constant terms which do not depend on the shape of the contour. In principle, using the iterative procedure, one can also find the higher corrections to the free energy but the calculations become rather involved. This procedure is in parallel to the one which we used for the 2D Dyson gas in [6].

The organization of the paper is as follows. In section 2 we introduce the distribution functions and correlation functions of the main observables (density of charges, the potential created by them and an analog of the stress-energy tensor). In section 3 we obtain general relations valid at a finite N and express the loop equation and the variation of the partition function in terms of boundary values of $T(z)$. Section 4 is devoted to the large N expansion. There we develop a systematic procedure of the large N expansion and obtain explicit formulas for the first three orders of the large N expansion. In section 5 we calculate correlation functions of the analogs of vertex operators. There are also six appendices. In appendix A we list a few simple but useful formulas for differential geometry of a contour. In appendices B and C we review the Green functions of Laplace operator and discuss the Neumann jump operator. In appendix D we present some facts about the double layer potentials and Fredholm eigenvalues associated with the contour Γ. Appendix E contains the technique of variation of contour integrals we used in the text. In appendix F we discuss functional determinants of the Laplace operators in interior and exterior domains.

2. Some definitions and general relations

The partition function of our model is defined by the integral (1.3). We start with some definitions and general relations.

2.1. Distribution functions

Integrating in the rhs of (1.3) over a part of the variables keeping the others fixed, one can define a set of quantities which are proportional to the probability distributions to find particles at given points. For example,

$$R(a) = \frac{1}{Z_N}\,\mathrm{e}^{W(a)}\oint_{\Gamma}|\Delta_{N-1}(z_i)|^{2\beta}\prod_{j=1}^{N-1}|a-z_j|^{2\beta}\,\mathrm{e}^{W(z_j)}\,|\mathrm{d}z_j| \tag{2.1}$$

is the mean density of particles while

$$\begin{aligned} R(a,b) = {} & \frac{\hbar(N-1)}{Z_N}|a-b|^{2\beta}\,\mathrm{e}^{W(a)+W(b)} \\ & \times\oint_{\Gamma}|\Delta_{N-2}(z_i)|^{2\beta}\prod_{j=1}^{N-2}|a-z_j|^{2\beta}|b-z_j|^{2\beta}\,\mathrm{e}^{W(z_j)}\,|\mathrm{d}z_j| \end{aligned} \tag{2.2}$$

is the pair distribution function. Obviously, $R(a,a)=0$ which means that two particles cannot be found in one and the same point. The normalization is chosen so that the functions $R(a)$, $R(a,b)$ are dimensionless and

$$\oint_\Gamma R(z)|\mathrm{d}z| = 1, \qquad \oint_\Gamma R(a,z)|\mathrm{d}z| = (N-1)\hbar\, R(a).$$

Let $A(z_1,\ldots,z_N)$ be an arbitrary symmetric function of the variables z_i; its mean value is defined as

$$\langle A\rangle = \frac{1}{Z_N}\oint_\Gamma\ldots\oint_\Gamma A(z_1,\ldots,z_N)\prod_{i<j}|z_i-z_j|^{2\beta}\prod_{k=1}^{N}\mathrm{e}^{W(z_k)}|\mathrm{d}z_k|. \tag{2.3}$$

Clearly, the mean values $\langle\sum_i f(z_i)\rangle$ and $\left\langle\sum_{i\neq j}f(z_i,z_j)\right\rangle$, etc are expressed through the distribution functions as follows:

$$\begin{aligned}\left\langle\sum_i f(z_i)\right\rangle &= \frac{1}{\hbar}\oint_\Gamma R(z)f(z)|\mathrm{d}z|,\\ \left\langle\sum_{i\neq j}f(z_i,z_j)\right\rangle &= \frac{1}{\hbar^2}\oint_\Gamma\oint_\Gamma R(z,\zeta)f(z,\zeta)|\mathrm{d}z||\mathrm{d}\zeta|\,.\end{aligned} \tag{2.4}$$

2.2. *Density, potential and their correlation functions*

The most important observables in the model are density of particles and 2D Coulomb field created by them. The density is defined as

$$\rho(z) = \hbar\sum_{i=1}^{N}\delta_\Gamma(z,z_i), \tag{2.5}$$

where $\delta_\Gamma(z,z')$ is the δ-function on Γ defined by the property $\oint_\Gamma f(z)\delta_\Gamma(z,a)|\mathrm{d}z| = f(a)$ for $a\in\Gamma$ and a continuous function f. The field

$$\varphi(z) = -\beta\hbar\sum_i \log|z-z_i|^2 = -2\beta\oint_\Gamma \log|z-\xi|\,\rho(\xi)|\mathrm{d}\xi| \tag{2.6}$$

is the 2D Coulomb potential created by the particles. Equation (2.6) says that $\varphi(z)$ is the potential of a simple layer with density ρ. The simple layer potential $\varphi(z)$ is harmonic everywhere in the plane except for the contour Γ, where it is continuous across the boundary but has a jump of the normal derivative

$$[\partial_n\varphi(z)]_\Gamma = \partial_n^+\varphi(z) - \partial_n^-\varphi(z) = 4\pi\beta\rho(z), \quad z\in\Gamma, \tag{2.7}$$

where ∂_n^+ and ∂_n^- are normal derivatives inside and outside the contour Γ respectively (with the unit normal vector looking outward in both cases). At infinity the field φ behaves as

$$\varphi(z) = -2\beta\,\log|z| + O(|z|^{-1}). \tag{2.8}$$

Note the absence of the $O(1)$ term.

Equation (2.7) allows one to invert formula (2.6) in terms of the Neumann jump operator $\hat{\mathcal{N}}$, the operator that takes a function f on Γ to the jump of the normal derivative of its harmonic continuations to the domains inside and outside Γ (denoted by f_H and f^H respectively):

$$\mathcal{N} f = \partial_n^+ f_H - \partial_n^- f^H. \tag{2.9}$$

See appendix C for details. We will also use the Dirichlet-to-Neumann operators $\hat{\mathcal{N}}^{\pm}$ which send a function f on Γ to normal derivatives of its harmonic continuations to the interior (exterior) domains:

$$\hat{\mathcal{N}}^+ f = \partial_n^+ f_H, \qquad \hat{\mathcal{N}}^- f = \partial_n^- f^H, \tag{2.10}$$

so that $\hat{\mathcal{N}} = \hat{\mathcal{N}}^+ - \hat{\mathcal{N}}^-$. In what follows, we denote the interior domain by D, then the exterior one is $\mathbb{C}\backslash\mathsf{D}$, $\Gamma = \partial\mathsf{D}$. Using the Neumann jump operator, we write equation (2.7) as

$$\rho(z) = \frac{1}{2\pi}\, |w'_{\mathrm{ext}}(z)| + \frac{1}{4\pi\beta}\, \hat{\mathcal{N}}\varphi(z), \tag{2.11}$$

where w_{ext} is the conformal map from $\mathbb{C}\backslash\mathsf{D}$ onto the exterior of the unit circle such that $w_{\mathrm{ext}}(z) = z/r + O(1)$ as $z \to \infty$ with a real positive r which is called the conformal radius of the exterior domain. Note that $|w'_{\mathrm{ext}}(z)| = \partial_n^-\, \log|w_{\mathrm{ext}}(z)|$, $z \in \Gamma$.

Correlation functions of densities can be obtained as variational derivatives of $\log Z_N$ w.r.t. the potential. For example,

$$\langle \rho(z)\rangle = \hbar\, \frac{\delta\, \log\, Z_N}{\delta W(z)}, \qquad \langle \rho(z_1)\rho(z_2)\rangle_{\mathrm{c}} = \hbar\, \frac{\delta\, \langle \rho(z_1)\rangle}{\delta W(z_2)} = \hbar^2 \frac{\delta^2\, \log\, Z_N}{\delta W(z_1)\delta W(z_2)}. \tag{2.12}$$

Here $\langle \rho(z_1)\rho(z_2)\rangle_{\mathrm{c}} = \langle \rho(z_1)\rho(z_2)\rangle - \langle \rho(z_1)\rangle\, \langle \rho(z_2)\rangle$ is the connected part of the correlation function. These formulas follow from the fact that variation of the partition function w.r.t. W inserts $\sum_i \delta_\Gamma(z, z_i)$ into the integral. Clearly,

$$\langle \rho(a)\rangle = R(a), \qquad \langle \rho(a)\rho(b)\rangle = R(a, b) + \hbar\, \langle \rho(a)\rangle\, \delta_\Gamma(a, b). \tag{2.13}$$

Correlation functions of φ's can be obtained from those of ρ's using equation (2.6). An equivalent method is to consider linear response of the system to a small change of the potential $\delta_a W(z) = \varepsilon \log|z - a|^2$ which means inserting a point-like charge $\varepsilon \to 0$ at the point a. We have the exact relations

$$\delta_a\, \log\, Z_N = -\frac{\varepsilon}{\beta\hbar}\, \langle \varphi(a)\rangle \tag{2.14}$$

and

$$\delta_a \langle X\rangle = -\frac{\varepsilon}{\beta\hbar} \langle X\varphi(a)\rangle_{\mathrm{c}} \tag{2.15}$$

for any X. In particular,

$$\delta_a \langle \varphi(z)\rangle = -\frac{\varepsilon}{\beta\hbar} \langle \varphi(z)\varphi(a)\rangle_{\mathrm{c}}. \tag{2.16}$$

2.3. Stress tensor

A holomorphic traceless tensor

$$T(z) = \beta^2\hbar^2 \sum_{j\neq k} \frac{1}{(z-z_j)(z-z_k)} + \beta\hbar^2 \sum_j \frac{1}{(z-z_j)^2} + 2\beta\hbar^2 \sum_j \frac{\partial W(z_j)}{z-z_j} \quad (2.17)$$

plays an important role. We refer to it as stress tensor and utilize it in the next section. Here we express it in terms of the fields φ and ρ:

$$T(z) = (\partial\varphi(z))^2 + (1-\beta)\hbar\,\partial^2\varphi(z) + 2\beta\hbar \oint_\Gamma \frac{\partial W(\xi)\rho(\xi)}{z-\xi}\,|\mathrm{d}\xi|. \quad (2.18)$$

The mean value $\langle T(z)\rangle$ is a holomorphic function for all $z \notin \Gamma$ given by

$$\langle T(z)\rangle = 2\beta^2 \oint_\Gamma \frac{|\mathrm{d}\xi|}{z-\xi} \oint_\Gamma \frac{R(\xi,\zeta)|\mathrm{d}\zeta|}{\xi-\zeta} + \beta\hbar \oint_\Gamma \frac{R(\xi)|\mathrm{d}\xi|}{(z-\xi)^2} + 2\beta\hbar \oint_\Gamma \frac{\partial W(\xi)R(\xi)}{z-\xi}\,|\mathrm{d}\xi|, \quad (2.19)$$

where the identities (2.4) are used. On the contour Γ it has a jump which follows from the Sokhotsky–Plemelj formula

$$\oint_\Gamma \frac{g(\xi)|\mathrm{d}\xi|}{\xi - z_\pm} = \fint_\Gamma \frac{g(\xi)|\mathrm{d}\xi|}{\xi - z} \pm \pi\,\overline{\nu(z)}g(z),$$

where $\nu(z) = -i\,\mathrm{d}z/|\mathrm{d}z|$ is the outward unit normal vector to Γ at the point z and $z_\pm$ means that z approaches the contour from the interior/exterior. The second integral in (2.19) can be transformed as follows:

$$\oint_\Gamma \frac{R(\xi)|\mathrm{d}\xi|}{(z-\xi)^2} = i\oint_\Gamma R(\xi)\overline{\nu(\xi)}\,\mathrm{d}\frac{1}{\xi-z} = -i\oint_\Gamma \frac{\mathrm{d}(R(\xi)\overline{\nu(\xi)})}{\xi-z}$$
$$= -\oint_\Gamma \frac{[i\partial_s R(\xi) + \kappa(\xi)R(\xi)]\,\overline{\nu(\xi)}\,|\mathrm{d}\xi|}{\xi-z}.$$

Here ∂_s is the tangential derivative along Γ and $\kappa(z) = i\nu(z)\partial_s\overline{\nu(z)}$ is the curvature of the contour at the point z. Using the notation

$$[h(z)]_\Gamma = h(z_+) - h(z_-), \quad z \in \Gamma \quad (2.20)$$

introduced in section 1 for the jump across Γ, we have:

$$\left[\langle T(z)\rangle\right]_\Gamma = 2\pi\beta\,\overline{\nu(z)}\left(2\beta\oint_\Gamma \frac{R(z,\zeta)|\mathrm{d}\zeta|}{\zeta-z} - 2\hbar\partial W(z)R(z) - \hbar\,\overline{\nu(z)}\,(\kappa(z)+i\partial_s)\,R(z)\right). \quad (2.21)$$

The first integral is convergent at $\zeta = z$ because $R(z,\zeta) \sim |z-\zeta|^{2\beta}$ as $\zeta \to z$.

In what follows it will be instructive to divide the mean value of T in two parts, 'classical' and 'quantum': $\langle T\rangle = \langle T\rangle^{(\mathrm{cl})} + \langle T\rangle^{(\mathrm{q})}$. The 'classical' part is obtained from (2.18) by disregarding the connected part of the pair correlation function:

$$\langle T(z)\rangle^{(\mathrm{cl})} = (\langle\partial\varphi(z)\rangle)^2 + (1-\beta)\hbar\,\partial^2\langle\varphi(z)\rangle + 2\beta\hbar\oint_\Gamma \frac{\partial W(\xi)\,\langle\rho(\xi)\rangle}{z-\xi}\,|\mathrm{d}\xi|. \quad (2.22)$$

The 'quantum' contribution

$$\langle T(z)\rangle^{(\mathrm{q})} = \left\langle (\partial\varphi(z))^2\right\rangle_{\mathrm{c}} \tag{2.23}$$

is the connected part.

3. Exact relations at finite *N*

There are two basic relations: one is an identity for correlation functions which follow from the invariance of the partition function under re-parametrizations of the contour (the 'loop equation'), another describes the variation of the partition function under small deformations of the contour. As we shall see, both can be expressed in terms of the jump of $\langle T(z)\rangle$ across the contour.

3.1. Loop equation

Let us start from the obvious identity

$$\sum_j \oint_\Gamma \ldots \oint_\Gamma \partial_{s_j}\left(\epsilon(z_j)\prod_{i<k}|z_i - z_k|^{2\beta}\prod_m \mathrm{e}^{W(z_m)}\right)\prod_{l=1}^N |\mathrm{d}z_l| = 0, \tag{3.1}$$

which expresses invariance of the partition function under re-parametrizations of the contour (changes of the integration variables). Here $\epsilon(z)$ is an arbitrary differentiable function on Γ. In terms of the mean value (2.3) the identity takes the form

$$\left\langle \sum_j \left(\partial_{s_j}\epsilon(z_j) + \epsilon(z_j)\partial_{s_j}W(z_j)\right) + \beta\sum_{j\neq k}\epsilon(z_j)\partial_{s_j}\,\log|z_j - z_k|^2\right\rangle = 0,$$

which, according to (2.4), can be rewritten through the distribution functions:

$$\oint_\Gamma |\mathrm{d}z|\epsilon(z)\left(\hbar\partial_s W(z)R(z) - \hbar\partial_s R(z) + \beta\oint_\Gamma R(z,\xi)\partial_s\,\log|z-\xi|^2|\mathrm{d}\xi|\right) = 0. \tag{3.2}$$

Since this identity holds for any $\epsilon(z)$, it implies that

$$\beta\oint_\Gamma R(z,\xi)\partial_s\,\log|z-\xi|^2\,|\mathrm{d}\xi| + \hbar\partial_s W(z)R(z) - \hbar\partial_s R(z) = 0. \tag{3.3}$$

In fact this exact relation is the first equation of an infinite system connecting multi-point distribution functions, called the BBGKY hierarchy (after Bogoliubov, Born, Green, Kirkwood and Yvone). Note also that equation (3.3) can be obtained in a simpler way by differentiating the definition of $R(z)$ (2.1) with respect to z along the contour Γ (cf [8]).

It remains to notice that equation (3.3) is equivalent to the following condition for the jump of $\langle T(z)\rangle$ given by (2.21):

$$\mathcal{I}m\left(\nu^2(z)[\langle T(z)\rangle]_\Gamma\right) = 0, \quad z\in\Gamma. \tag{3.4}$$

In the frame with axes normal and tangential to the contour the components of the 'stress tensor' T read

$$\begin{aligned}4T_{nn} &= -4T_{ss} = \nu^2 T + \bar\nu^2\bar T,\\ 4T_{sn} &= i\nu^2 T - i\bar\nu^2\bar T.\end{aligned} \tag{3.5}$$

Then the loop equation states that the tangential-normal component is continuous across the boundary:

$$[\langle T_{sn}(z)\rangle]_\Gamma = 0, \quad z \in \Gamma. \tag{3.6}$$

We comment that the boundary condition is different from that used in the boundary CFT. There $\langle T_{sn}(z)\rangle = 0$ when the argument approaches the boundary from one side.

If Γ is the unit circle, then $\nu(z) = z$. Equation (3.3) acquires the form

$$\beta \oint_{|\xi|=1} R(z,\xi)\frac{z+\xi}{z-\xi}\frac{\mathrm{d}\xi}{\xi} + \hbar\partial_s W(z)R(z) - \hbar\partial_s R(z) = 0.$$

It is equivalent to the more customary form of the loop equation

$$\sum_j \left\langle \beta\sum_{k\neq j}\frac{z}{(z-z_j)(z-z_k)} + \frac{z}{(z-z_j)^2} + \frac{z_j\partial W(z_j) - \bar{z}_j\bar{\partial}W(z_j)}{z-z_j} - \frac{\beta N - \beta + 1}{z-z_j}\right\rangle = 0$$

which follows from the obvious identity

$$\sum_j \oint_\Gamma \cdots \oint_\Gamma \partial_{s_j}\left(\frac{1}{z-z_j}\prod_{i<k}|z_i - z_k|^{2\beta}\prod_m \mathrm{e}^{W(z_m)}\right)\prod_{l=1}^N |\mathrm{d}z_l| = 0$$

valid for any $z \in \mathbb{C}$ (outside of the unit circle).

Using the expression (2.18) for the stress tensor, we rewrite (3.6) in terms of correlation functions of the fields φ and ρ. We have:

$$\mathcal{I}m\left\{\nu^2(z)\left(\left[\left\langle(\partial\varphi(z))^2\right\rangle\right]_\Gamma + \hbar(1-\beta)\left[\left\langle\partial^2\varphi(z)\right\rangle\right]_\Gamma + 2\beta\hbar\left[\oint_\Gamma \frac{\partial W(\xi)\,\langle\rho(\xi)\rangle}{z-\xi}\,|\mathrm{d}\xi|\right]_\Gamma\right)\right\} = 0.$$

Let us consider the three terms in the lhs separately. In the first term, we separate the connected part and use equation (A6) from appendix A to write the other two:

$$2\mathcal{I}m\left\{\nu^2\left[\left\langle(\partial\varphi)^2\right\rangle\right]_\Gamma\right\} = -\left[\partial_n\langle\varphi\rangle\right]_\Gamma\partial_s\langle\varphi\rangle + 2\mathcal{I}m\left\{\nu^2\left[\left\langle(\partial\varphi)^2\right\rangle_{\mathrm{c}}\right]_\Gamma\right\}.$$

Here and below in similar calculations we take into account that the function $\langle\varphi\rangle$ is continuous across the contour, so the jumps like $[\partial_s\langle\varphi\rangle]_\Gamma$ vanish. In the second term, we use equation (A8) from appendix A to get

$$2\mathcal{I}m\left\{\nu^2\left[\left\langle\partial^2\varphi\right\rangle\right]_\Gamma\right\} = -\partial_s\left[\partial_n\langle\varphi\rangle\right]_\Gamma = -4\pi\beta\partial_s\langle\rho\rangle.$$

In the third term we apply the Sokhotsky–Plemelj formula:

$$2\mathcal{I}m\left\{\nu^2(z)\left[\oint_\Gamma \frac{\partial W(\xi)\,\langle\rho(\xi)\rangle}{z-\xi}\,|\mathrm{d}\xi|\right]_\Gamma\right\} = 2\pi\partial_s W(z)\langle\rho(z)\rangle.$$

Combining the three terms, we bring the loop equation to the form

$$\langle\rho(z)\rangle\left(\partial_s\langle\varphi(z)\rangle - \hbar\partial_s W(z)\right) - (\beta-1)\hbar\,\partial_s\langle\rho(z)\rangle = \frac{1}{2\pi\beta}\mathcal{I}m\left(\nu^2(z)\left[\left\langle(\partial\varphi(z))^2\right\rangle_{\mathrm{c}}\right]_\Gamma\right) \tag{3.7}$$

which will be used for the large N expansion. The lhs comes from the classical part of the stress-energy tensor while the rhs is due to the 'quantum' fluctuations. Note that the three terms of the loop equation (3.7) have orders $O(\hbar^0)$, $O(\hbar^1)$ and $O(\hbar^2)$ respectively.

A comment on the relation to CFT is in order. The boundary condition there is $\langle T_{sn}\rangle = 0$, not $[\langle T_{sn}\rangle]_\Gamma = 0$. With this boundary condition, under a change of the geometry (a deformation of the contour) correlation functions transform according to the CFT rules. This is not the case in our model, and so the similarity with CFT is not complete.

3.2. Deformations of contour

Let us describe small deformations of the contour Γ by a normal displacement $\delta n(z)$ of the point $z \in \Gamma$ (positive for displacements in the outward direction). Below we use the rule for variation of the contour integral $I = \oint_\Gamma f(z)|\mathrm{d}z|$ given in [4] (see also appendix E). Let f be a function defined in a strip-like neighborhood of the contour, then

$$\delta I = \oint_\Gamma \delta n(z)(\partial_n + \kappa(z)) f(z)|\mathrm{d}z| + \oint_\Gamma \delta f(z)|\mathrm{d}z|. \tag{3.8}$$

Here $\kappa(z)$ is the curvature of Γ at the point z. The first term is due to the deformation of the contour. The second contribution is due to the variation of the function f itself (in the case if it depends on the shape of the contour).

According to (3.8), we have:

$$\delta Z_N = \sum_j \oint_\Gamma \dots \oint_\Gamma \delta n(z_j)\left(\partial_{n_j} + \kappa(z_j)\right)\left(\prod_{i<k}|z_i - z_k|^{2\beta}\prod_m \mathrm{e}^{W(z_m)}\right)\prod_{l=1}^N |\mathrm{d}z_l|.$$

Similarly to the derivation of equation (3.2) from identity (3.1), we write:

$$\delta \log Z_N = \left\langle \sum_j \left(\kappa(z_j) + \partial_{n_j} W(z_j) + \beta \sum_{k\neq j} \partial_{n_j} \log|z_j - z_k|^2\right)\delta n(z_j)\right\rangle$$

or, in terms of the mean density and the pair distribution function,

$$\hbar^2 \delta \log Z_N = \oint_\Gamma |\mathrm{d}z| \delta n(z)\left(\hbar\partial_n W(z) R(z) + \hbar\kappa(z) R(z) \right.$$
$$\left. + \beta \oint_\Gamma R(z,\xi)\partial_n \log|z-\xi|^2 |\mathrm{d}\xi|\right).$$

Comparing with (2.21), one can rewrite this relation in terms of the jump of $\langle T(z)\rangle$:

$$\hbar^2 \delta \log Z_N = -\frac{1}{2\pi\beta}\oint_\Gamma \mathcal{R}e\left(\nu^2(z)\left[\langle T(z)\rangle\right]_\Gamma\right)\delta n(z)\,|\mathrm{d}z|. \tag{3.9}$$

It can be written in some other forms. In terms of the normal–normal component of the stress-energy tensor we have

$$\hbar^2 \delta \log Z_N = -\frac{1}{\pi\beta}\oint_\Gamma \left[\langle T_{nn}(z)\rangle\right]_\Gamma \delta n(z)\,|\mathrm{d}z|. \tag{3.10}$$

Combining (3.9) with the loop equation which says that the quantity $\left\langle \nu^2(z)[T(z)]_\Gamma\right\rangle$ is real, we can also write

$$\hbar^2 \delta \log Z_N = -\frac{1}{2\pi\beta}\oint_\Gamma \nu^2(z)\;\langle [T(z)]_\Gamma\rangle\;\delta n(z)|\mathrm{d}z|. \tag{3.11}$$

For some particular deformations equation (3.9) gets simplified. For example, set $\delta n(z) = \delta_a n(z)$, where

$$\delta_a n(z) = 2\varepsilon \,\mathcal{R}e \left(\frac{\overline{\nu(z)}}{z-a} \right)$$

then a simple calculation of residues yields

$$\beta\hbar^2 \delta \,\log Z_N = -2\varepsilon \,\mathcal{R}e \,\langle T(a)\rangle \,. \tag{3.12}$$

For the variation $\delta_W \log Z_N$ caused by a small change of the potential, we obtain:

$$\hbar\delta_W \,\log Z_N = \left\langle \hbar \sum_j \delta W(z_j) \right\rangle = \oint_\Gamma \langle \rho(z)\rangle \,\delta W(z)|\mathrm{d}z|. \tag{3.13}$$

The total variation is $\delta \log Z_N + \delta_W \log Z_N$.

Let us represent the jump of $\langle T_{nn}\rangle$ in the form similar to (3.7). We have:

$$4\big[\langle T_{nn}(z)\rangle\big]_\Gamma = 2\mathcal{R}e \left\{ \nu^2(z) \left(\big[\big\langle(\partial\varphi(z))^2\big\rangle\big]_\Gamma + \hbar(1-\beta)\big[\big\langle\partial^2\varphi(z)\big\rangle\big]_\Gamma \right.\right.$$

$$\left.\left. + 2\beta\hbar \left[\oint_\Gamma \frac{\partial W(\xi)\,\langle\rho(\xi)\rangle}{z-\xi}\,|\mathrm{d}\xi| \right]_\Gamma \right) \right\}.$$

In the first term, we separate the connected part and use equation (A5) from appendix A in the rest:

$$2\mathcal{R}e \left\{ \nu^2 \big[\big\langle(\partial\varphi)^2\big\rangle\big]_\Gamma \right\} = \frac{1}{2}\big[\partial_n\langle\varphi\rangle\big]_\Gamma \big(\partial_n^+\langle\varphi\rangle + \partial_n^-\langle\varphi\rangle\big)$$
$$+ 2\mathcal{R}e \left\{ \nu^2 \big[\big\langle(\partial\varphi)^2\big\rangle_{\mathrm{c}}\big]_\Gamma \right\}.$$

In the second term, we use equation (A7) from appendix A to get

$$2\mathcal{R}e \left\{ \nu^2 \big[\big\langle\partial^2\varphi\big\rangle\big]_\Gamma \right\} = -\kappa\big[\partial_n\langle\varphi\rangle\big]_\Gamma = -4\pi\beta\kappa\,\langle\rho\rangle\,.$$

In the third term we apply the Sokhotsky–Plemelj formula:

$$2\mathcal{R}e \left\{ \nu^2(z) \left[\oint_\Gamma \frac{\partial W(\xi)\,\langle\rho(\xi)\rangle}{z-\xi}\,|\mathrm{d}\xi| \right]_\Gamma \right\} = -2\pi\partial_n W(z)\langle\rho(z)\rangle\,.$$

Combining the three terms, we obtain:

$$\begin{aligned} 4\big[\langle T_{nn}(z)\rangle\big]_\Gamma = {} & 2\pi\beta\,\langle\rho(z)\rangle\,\big\{\partial_n^+\langle\varphi(z)\rangle + \partial_n^-\langle\varphi(z)\rangle - 2\hbar\partial_n W(z) \\ & + 2(\beta-1)\hbar\kappa(z)\big\} \\ & + 2\mathcal{R}e \left\{ \nu^2(z)\big[\big\langle(\partial\varphi(z))^2\big\rangle_{\mathrm{c}}\big]_\Gamma \right\}. \end{aligned} \tag{3.14}$$

Similarly to (3.7), the first line comes from the classical part of the stress tensor while the second one is due to the quantum contribution.

4. Large *N* expansion

The large N (small $\hbar$) behavior of $\log Z_N$ is of the form (1.5). Our task is to obtain explicit expressions for the contributions to the free energy F_0, F_1 and F_2.

We assume that $\langle\rho(z)\rangle$ can be expanded in integer powers of $\hbar$:

$$\langle\rho(z)\rangle = \rho_0(z) + \hbar\rho_1(z) + \hbar^2\rho_2(z) + \cdots \tag{4.1}$$

Then a similar expansion holds for $\langle\varphi(z)\rangle$: $\langle\varphi\rangle = \varphi_0 + \hbar\varphi_1 + \hbar^2\varphi_2 + \cdots$. Their coefficients are related as $4\pi\beta\rho_j = [\partial_n\varphi_j]_\Gamma$.

4.1. The leading order

In the leading order, the problem is equivalent to a 2D electrostatic problem of finding the equilibrium distribution of a charged conducting fluid. It can be easily solved by requiring the total electrostatic potential created by the charges and by the background to be constant along the contour. This will be the leading order of the regular iterative procedure based on the loop equation we develop below.

The connected correlation function in the right-hand side of the loop equation (3.7) is of order $O(\hbar^2)$. Therefore, at the 0th step, the loop equation states that

$$\rho_0\,\partial_s\varphi_0 = 0$$

which means that φ_0 is constant along Γ. Then

$$\rho_0(z) = \frac{1}{2\pi}\,|w'_{\mathrm{ext}}(z)|, \quad z \in \Gamma. \tag{4.2}$$

This is the solution to the electrostatic problem of continuous charge distribution over the contour. In this approximation discreetness of the particles is neglected. The charges form a simple layer whose electric potential $\varphi_0(z) = -2\beta\oint_\Gamma \log|z-\xi|\rho_0(\xi)|\mathrm{d}\xi|$ reads

$$\varphi_0(z) = \begin{cases} -2\beta\,\log r & \text{for } z \in \mathsf{D} \\ -2\beta\,\log r - 2\beta\,\log|w_{\mathrm{ext}}(z)| & \text{for } z \in \mathbb{C}\backslash\mathsf{D}. \end{cases} \tag{4.3}$$

Next, we use (3.14) to find that

$$4\left[\langle T_{nn}\rangle\right]_\Gamma \cong 2\pi\beta\rho_0\left(\partial_n^+\varphi_0 + \partial_n^-\varphi_0\right). \tag{4.4}$$

Here and below $\cong$ means equality in the leading order (up to higher powers of $\hbar$). This should be substituted into (3.10). After some simple transformations, one can find δF_0 with F_0 given by

$$F_0 = \beta\,\log r. \tag{4.5}$$

This is the electrostatic energy of the charge distribution with density (4.2).