

MODEL-THEORETIC ELEKES-SZABÓ IN THE STRONGLY MINIMAL CASE

ARTEM CHERNIKOV AND SERGEI STARCHENKO

ABSTRACT. We prove a generalizations of the Elekes–Szabó theorem [8] for relations definable in strongly minimal structures that are interpretable in distal structures.

1. INTRODUCTION AND PRELIMINARIES

Our notation is mostly standard. For $n \in \mathbb{N}$ we denote by $[n]$ the set $[n] = \{1, \dots, n\}$. If X is a set and $n \in \mathbb{N}$, then we write $A \subseteq_n X$ to denote that A is a subset of X with $|A| \leq n$. Given a binary relation $E \subseteq X \times Y$ and $a \in X, b \in Y$, we write $E_a = \{b' \in Y : (a, b') \in E\}$ and $E_b = \{a' \in X : (a', b) \in E\}$ to denote the fibers of E at a and b , respectively. As usual, for functions $f, g: \mathbb{N} \rightarrow \mathbb{R}$ we write $f(n) = O(g(n))$ if there is a positive $C \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $f(n) \leq Cg(n)$ for all $n > n_0$. As usual, given $s, t \in \mathbb{N}$ we say that a bipartite graph $E \subseteq X \times Y$ is $K_{s,t}$ -free if it does not contain a copy of the complete bipartite graph $K_{s,t}$ with parts of size s and t , respectively.

We will use freely some standard model-theoretic notions such as saturated models, algebraic closure (by which we will always mean the algebraic closure in $\mathcal{M}^{\text{eq}}$) and Morley rank (see e.g. [13, 22]).

- Definition 1.1.** (1) We say that a subset $F \subseteq X \times Y$ is *cartesian* if $I \times J \subseteq F$ for some infinite $I \subseteq X, J \subseteq Y$.
 (2) We say that a subset $F \subseteq S_1 \times S_2 \times \dots \times S_k$ is *cylindrical* if it is cartesian as a subset of $S_i \times \hat{S}_i$ for some $i \in [k]$, where $\hat{S}_i = \prod_{j \neq i} S_j$.

Let $\mathcal{M}$ be a sufficiently saturated first order structure and let X, Y, Z be strongly minimal sets definable in $\mathcal{M}$. Let $F \subseteq X \times Y \times Z$ be a definable set of Morley rank 2. As usual, we say that $\bar{a} = (a_1, a_2, a_3) \in F$ is *generic in F* over a set of parameters $C \subseteq X \times Y \times Z$ if $\text{RM}(\bar{a}/C) = \text{RM}(F) = 2$.

Definition 1.2. We say that a definable relation F as above is *group-like* if there is a group G of Morley rank 1 and degree 1 (hence, abelian) definable in $\mathcal{M}$ over a small set $C \subseteq X \times Y \times Z$, elements $g_1, g_2, g_3 \in G$,

and $\alpha_1 \in X$, $\alpha_2 \in Y$, $\alpha_3 \in Z$ such that α_i and g_i are inter-algebraic over C for all $i \in [3]$ (i.e. $\alpha_i \in \text{acl}(g_i C)$ and $g_i \in \text{acl}(\alpha_i C)$), $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in F$ is generic in F over C and $g_1 \cdot g_2 \cdot g_3 = 1$ in G .

We can now state our main result.

Theorem 1.3 (Main Theorem). *Let X, Y, Z be strongly minimal sets definable in a sufficiently saturated structure $\mathcal{M}$ and let $F \subseteq X \times Y \times Z$ be a definable set of Morley rank 2. Assume in addition that $\mathcal{M}$ is interpretable in a distal structure (see Section 2.1). Then one of the following holds.*

- (a) *There is some real $\varepsilon > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$ we have*

$$|F \cap A \times B \times C| = O(n^{2-\varepsilon}).$$

- (b) *F is group-like.*

- (c) *F is cylindrical.*

Remark 1.4. Theorem 1.3 can be viewed as a generalization of the Elekes-Szabó Theorem [8] which established it for $\mathcal{M} = \mathbb{C}$ the field of complex numbers (a strongly minimal structure), which is interpretable in the field of reals — a distal structure.

Remark 1.5. Various improvements of the Elekes-Szabó theorem, including explicit bounds on ε , have been obtained [7, 16, 24]. In our general situation we don't optimize the bounds, even though they can be calculated explicitly in terms of the size of the available distal cell decomposition, see Section 2.1.

Remark 1.6. After the completion of this paper, generalizations to definable hypergraphs of arbitrary arity and dimension were obtained in [2] over $\mathbb{C}$, in [18] for a higher arity version of the Elekes-Rónyai theorem over $\mathbb{C}$, and in [4] for hypergraphs of arbitrary arity and dimension definable in arbitrary stable structures with distal expansions, as well as in arbitrary $\mathcal{o}$ -minimal structures.

Proposition 1.7. *If the Morley degree of F is 1, then the three cases in Theorem 1.3 are mutually exclusive.*

Proof. (a) and (c) are incompatible. Assume F is cylindrical, say $F \subseteq (X \times Y) \times Z$ is cartesian. Then for any n there are some $D \subseteq X \times Y, C \subseteq Z$ with $|D| = |C| = n$ such that $D \times C \subseteq F$. Let A, B be the projections of D on X and Y , respectively. Then $|A|, |B| \leq n$ and $|F \cap A \times B \times C| \geq n^2$.

(b) and (c) are incompatible. By assumption F has Morley rank 2 and degree 1, so it contains a unique generic type. We work over some

saturated model. Assume that F is cylindrical, say $F \subseteq (X \times Y) \times Z$ is cartesian. Then there exist $\bar{a} = (a_1, a_2) \in X \times Y$ and $b \in Z$ such that $(a_1, a_2, b) \in F$, $\bar{a} \notin \text{acl}(b)$ and $b \notin \text{acl}(\bar{a})$. Since Z is strongly minimal, it follows that $\bar{a} \perp b$ and $(\bar{a}, b)$ is generic in F . So a generic of F has to be of the form (α, β, γ) with $\text{RM}(\alpha\beta) = \text{RM}(\gamma) = 1$ and $\alpha\beta \perp \gamma$. On the other hand, if F is group-like, then for its generic (α, β, γ) we must have $\text{RM}(\alpha\beta) = 2$ and $\gamma \in \text{acl}(\alpha, \beta)$.

(a) and (b) are incompatible. Let G be a definable abelian group with $\text{RM}(G) = 1$, and let $F_G \subseteq G^3$ be the (non-definable) relation given by

$$(a, b, c) \in F_G : \iff a \perp b \text{ are generic in } G \text{ and } a + b + c = 0.$$

First we establish high edge count for the relation F_G .

Claim 1.8. *There exists some $c = c(G) \in \mathbb{R}_{>0}$ such that for arbitrary large $n \in \mathbb{N}$ there exist some $A, B, C \subseteq_n G$ such that:*

$$|\{F_G \cap (A \times B \times C)\}| \geq cn^2.$$

Proof of Claim 1.8. Assume first that G has a generic type $p(x)$ such that some/any realization of p is of infinite order. Choose some $a, b \models p$ with $a \perp b$, then $nb \in \text{acl}(b)$ and $b \in \text{acl}(nb)$, hence $a \perp nb$ and nb is generic, for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$, and consider the set $A = \{a, a + b, a + 2b, \dots, a + (n-1)b\}$, then $|A| = n$. By forking calculus, the elements of A are all generic in G and pairwise independent: $a + sb \perp a + tb$ for $s \neq t$. Note that for any $s, t < \frac{n}{2}$ we have $(a + sb) + (a + tb) = a + (s+t)b \in A$ as $s+t < n$, hence taking $B := A$ and $C := -A$, we see that

$$|F_G \cap (A \times B \times C)| \geq \left| \left\{ (s, t) : s \neq t \text{ and } s, t < \frac{n}{2} \right\} \right| \geq \binom{n}{2} - \frac{n}{2} \geq \frac{1}{6}n^2,$$

for all sufficiently large n , as wanted.

Otherwise, all generics of G have the same finite order, say k , and as every elements in a stable group is a product of two generics, by commutativity this implies that $kx = 0$ for all $x \in G$. Then, by classification of abelian groups, $px = 0$ holds in G for some prime p , and G is an $\mathbb{F}_p$ -vector space. Fix any $m \in \mathbb{N}$ and choose independent generic elements $a_1, \dots, a_m$ in G . Let $A := \langle a_1, \dots, a_m \rangle$ be the $\mathbb{F}_p$ -linear span of $\{a_1, \dots, a_m\}$, then $|A| = p^m := n$. By forking calculus we have: every $0 \neq a \in A$ is generic in G , and for any $a, b \in A$, $a \perp b$ if and only if a and b are linearly independent over $\mathbb{F}_p$. But then we have

$$|F_G \cap A^3| \geq |\{(a, b) \in A^2 : a \text{ and } b \text{ are } \mathbb{F}_p\text{-linearly independent}\}|$$

$$= (p^m - 1)(p^m - p) \geq \frac{1}{4}n^2$$

for all $n \gg p$. $\square$

Now assume that F is group-like, witnessed by G and α_i, g_i (we suppress the parameter set to simplify the notation). We want to transfer high edge count from F_G to F . As α_i is inter-algebraic with g_i , we can choose some formulas φ_i and $k \in \mathbb{N}_{\geq 1}$ such that

($\dagger$) $\models \varphi_i(\alpha_i, g_i)$, and $|\varphi_i(\alpha, M)|, |\varphi_i(M, g)| \leq k$ for all $i \in \{1, 2, 3\}$ and $\alpha \in X \cup Y \cup Z, g \in G$.

As G has Morley rank and degree 1, there is a unique generic type, hence by stationarity we have the equality of types $g_1 g_2 \equiv g'_1 g'_2$ for any generic $g_1 \perp g_2, g'_1 \perp g'_2$ in G . As $\models F(\alpha_1, \alpha_2, \alpha_3)$ by assumption, taking an automorphism we see that:

($\dagger\dagger$) for every $(g_1, g_2, g_3) \in F_G$ there exist some $\alpha_1 \in \varphi_1(X, g_1), \alpha_2 \in \varphi_2(Y, g_2), \alpha_3 \in \varphi_3(Z, g_3)$ such that $(\alpha_1, \alpha_2, \alpha_3) \in F$.

Now let $c \in \mathbb{R}_{>0}$ be as given by Claim 1.8 for G . Let $n \in \mathbb{N}$ be arbitrarily large, and let $A, B, C \subseteq_n G$ be such that $|F_G \cap (A \times B \times C)| \geq cn^2$. We define $A' := \bigcup_{g \in A} \varphi(X, g), B' := \bigcup_{g \in B} \varphi(Y, g)$ and $C' := \bigcup_{g \in C} \varphi(Z, g)$. Then $|A'|, |B'|, |C'| \leq kn := n'$ by ($\dagger$). We define a function

$$f : F_G \cap (A \times B \times C) \rightarrow F \cap (A' \times B' \times C'),$$

where for $\bar{g} = (g_1, g_2, g_3) \in F_G \cap (A \times B \times C)$ we take $f(\bar{g})$ to be an arbitrary element $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in F \cap (\varphi_1(X, g_1) \times \varphi_2(Y, g_2) \times \varphi_3(Z, g_3))$, it exists by ($\dagger\dagger$). But as $|\varphi_i(\alpha_i, G)| \leq k$ for any $(\alpha_1, \alpha_2, \alpha_3) \in X \times Y \times Z$ by ($\dagger$), we have that $|f^{-1}(\{\bar{\alpha}\})| \leq k^3$ for every $\bar{\alpha} \in \text{Im}(f)$. Hence

$$|F \cap (A' \times B' \times C')| \geq \frac{1}{k^3} |F_G \cap (A \times B \times C)| \geq \frac{c}{k^3} n^2 \geq \frac{c}{k^5} (n')^2 \geq c'(n')^2$$

with $c' := \frac{c}{k^5} > 0$. As n' can be taken arbitrarily large, this shows that F doesn't satisfy (a). $\square$

The proof of Theorem 1.3 consists of three main ingredients: a bound on the number of edges for non-cartesian relations in our context (i.e. Theorem 2.17 established in Section 2 using local stability and the distal cutting lemma from [3]); Hrushovski's group configuration theorem in stable theories; and the construction of the group configuration in the cartesian case connecting the two aforementioned results. In Section 3 this last part is reduced to a certain dichotomy for binary relations between sets of rank 2, and this dichotomy is proved in Section 4).

Acknowledgements. We thank the referee for some very insightful suggestions on improving the paper. Chernikov was supported by the NSF Research Grant DMS-1600796, by the NSF CAREER grant DMS-1651321 and by an Alfred P. Sloan Fellowship.

Starchenko was supported by the NSF Research Grant DMS-1800806.

2. BOUNDS FOR NON-CARTESIAN RELATIONS

2.1. Zarankiewicz for distal relations. As is well-known, if $E \subseteq A \times B$ is a bipartite graph, with A, B finite, such that no two distinct points in A have more than s common neighbours in B (i.e. E is $K_{2,s}$ -free), then a simple argument via the Cauchy-Schwarz inequality implies that the number of edges of E is at most $O_s(|A||B|^{\frac{1}{2}} + |B|)$ — and this is optimal in general. The classical theorem of Szemerédi–Trotter [21] and its generalizations improve on this inequality, reducing the power by some $\varepsilon > 0$, in the situation when A, B are points in a Euclidean space and the graph E is given by some (semi-)algebraic relation (we refer to [19] for a general introduction to the area of incidence geometry). In particular, a higher-dimensional “point – algebraic variety” incidence bound due to Elekes and Szabó [8, Theorem 9] was crucial in their proof of the ε -gap in the exponent in Theorem 1.3(a) for $\mathcal{M} = \mathbb{C}$.

In this section we generalize (and strengthen) the incidence bound due to Elekes and Szabó in [8] to arbitrary graphs definable in *distal* structures. Distal structures constitute a subclass of purely unstable NIP structures [20] that contains all $\mathcal{o}$ -minimal structures, various expansions of the field $\mathbb{Q}_p$ and the valued differential field of transseries (we refer to the introduction of [6] or [1] for a general discussion of distality and references). It is demonstrated in [3, 6] that many of the results in semialgebraic incidence combinatorics generalize to relations definable in distal structures.

We recall some of the notions and results from [3], and refer to that article for further details. The following definition captures a combinatorial “shadow” of the existence of a nice topological cell decomposition (as e.g. in $\mathcal{o}$ -minimal theories or in the p -adics).

Definition 2.1. [3, Section 2] Let X, Y be infinite sets, and $E \subseteq X \times Y$ a binary relation.

- (1) Let $A \subseteq X$. For $b \in Y$, we say that E_b *crosses* A if $E_b \cap A \neq \emptyset$ and $(X \setminus E_b) \cap A \neq \emptyset$.
- (2) A set $A \subseteq X$ is *E -complete over* $B \subseteq Y$ if A is not crossed by any E_b with $b \in B$.

- (3) A family $\mathcal{F}$ of subsets of X is a *cell decomposition for E over $B \subseteq Y$* if $X \subseteq \bigcup \mathcal{F}$ and every $A \in \mathcal{F}$ is E -complete over B .
- (4) A *cell decomposition for E* is a map $\mathcal{T} : B \mapsto \mathcal{T}(B)$ such that for each finite $B \subseteq Y$, $\mathcal{T}(B)$ is a cell decomposition for E over B .
- (5) A cell decomposition $\mathcal{T}$ is *distal* if there exist $k \in \mathbb{N}$ and a relation $D \subseteq X \times Y^k$ such that for all finite $B \subseteq Y$, $\mathcal{T}(B) = \{D_{(b_1, \dots, b_k)} : b_1, \dots, b_k \in B \text{ and } D_{(b_1, \dots, b_k)} \text{ is } E\text{-complete over } B\}$.
- (6) The *exponent* of a cell decomposition $\mathcal{T}$ is the smallest $t \in \mathbb{N}$ for which there exists some $c \in \mathbb{R}_{>0}$ such that $|\mathcal{T}(B)| \leq c|B|^t$ for all finite sets $B \subseteq Y$.

Existence of “strong honest definitions” established in [5] shows that every relation definable in a distal structure admits a distal cell decomposition (of some exponent).

Fact 2.2. *Assume that E is definable in a distal structure. Then E admits a distal cell decomposition. Moreover, in this case the relation D in Definition 2.1(5) is definable in $\mathcal{M}$.*

The following definition abstracts from the notion of cuttings in incidence geometry (see the introduction of [3] for an extended discussion).

Definition 2.3. Let X, Y be infinite sets, $E \subseteq X \times Y$ and $t \in \mathbb{N}_{>0}$. We say that E *admits cuttings with exponent t* if there is some constant $c \in \mathbb{R}_{>0}$ satisfying the following. For any $B \subseteq_n Y$ and any $r \in \mathbb{R}$ with $1 < r < n$ there are some sets $X_1, \dots, X_s \subseteq X$ covering X with $s \leq cr^t$ and such that each X_i is crossed by at most $\frac{n}{r}$ of the fibers $\{E_b : b \in B\}$.

In the case $r > n$, an r -cutting is equivalent to a distal cell decomposition (sets in the covering are not crossed at all). And for r varying between 1 and n , r -cutting allows to control the trade-off between the number of cells in a covering and the number of times each cell is allowed to be crossed.

Fact 2.4. (*Distal cutting lemma*, [3, Theorem 3.2]) *Assume $E \subseteq X \times Y$ admits a distal cell decomposition $\mathcal{T}$ of exponent $t \in \mathbb{N}$. Then E admits cuttings with exponent t .*

The following classical fact generalizes the Cauchy-Schwarz bound discussed at the beginning of the section.

Fact 2.5. [12] *Assume $E \subseteq A \times B$ is $K_{d,s}$ -free, for some $d, s \in \mathbb{N}$ and A, B finite. Then $|E \cap A \times B| \leq s^{\frac{1}{d}}|A||B|^{1-\frac{1}{d}} + d|B|$.*

Cuttings allow to establish super-Cauchy-Schwarz estimates on the number of edges. Here we will need the following theorem, which is a slight generalization of [3, Theorem 5.7].

Theorem 2.6. *Assume that $E \subseteq X \times Y$ is $K_{k,k}$ -free for some $k \in \mathbb{N}$, admits cuttings $\mathcal{T}$ with exponent $t \in \mathbb{N}_{\geq 2}$, and satisfies the following with $d \in \mathbb{N}_{\geq 2}$: there exists some $\alpha_1 \in \mathbb{R}_{>0}$ such that*

$$(*) \quad |E \cap A \times B| \leq \alpha_1 \left(|A||B|^{1-\frac{1}{d}} + |B| \right) \text{ for all finite } A \subseteq X, B \subseteq Y.$$

Then there exists some $\alpha = \alpha(\alpha_1, k, \mathcal{T}, d) \in \mathbb{R}_{>0}$ such that

$$|E \cap A \times B| \leq \alpha \left(|A|^{\frac{(t-1)d}{td-1}} |B|^{\frac{td-t}{td-1}} + |A| + |B| \right)$$

for all finite set $A \subseteq U, B \subseteq V$. Moreover, for fixed $k, \mathcal{T}, d$, $\alpha(\alpha_1, k, \mathcal{T}, d)$ is a sub-linear function in α_1 .

Note that for both powers we clearly have $0 < \frac{(t-1)d}{td-1}, \frac{td-t}{td-1} < 1$.

Remark 2.7. The assumption $(*)$ is satisfied in the following cases:

- (1) The family of sets $\mathcal{F} = \{E_b : b \in Y\}$ has VC-density at most d and E is $K_{k,k}$ -free for some $k \in \mathbb{N}$.

Then $(*)$ holds for some $\alpha_1 > 0$ by [9, Theorem 2.1], and Theorem 2.6 specializes to [3, Theorem 5.7] (see [3, Section 5.1] for a discussion of VC-density).

- (2) E is $K_{d,s}$ -free for some $s \in \mathbb{N}$.

Indeed, in this case, by Fact 2.5, for all A, B we have $|E \cap A \times B| \leq \alpha_1(|A||B|^{1-\frac{1}{d}} + |B|)$, with $\alpha_1 = \alpha_1(d, s) := s^{\frac{1}{d}} + d$.

This case of Theorem 2.6 generalizes/strengthens [8, Theorem 9] (more precisely, its dual), removing the logarithmic factor.

It follows that for fixed $t, \mathcal{T}$ and d , α depends sub-linearly on s .

Proof of Theorem 2.6. The result follows from the proof of [3, Theorem 5.7]. The assumption on the VC-density in the statement of [3, Theorem 5.7] is only used to conclude that $(*)$ holds with some $\alpha_1 > 0$, by Remark 2.7(1). Similarly, the dual version of $(*)$ holds with some $\alpha_4 = \alpha_4(k, \mathcal{T}) > 0$:

$$|E \cap A \times B| \leq \alpha_4 \left(|B||A|^{1-\frac{1}{t}} + |A| \right) \text{ for all } A \subseteq U, B \subseteq V.$$

As explained in the proof there, this is true as E admits cuttings of exponent t , hence has dual VC-density $\leq t$ (by [3, Remark 5.5]), so Remark 2.7(1) applies to E with the roles of the variables exchanged.

The rest of the proof only relies on $(*)$, dual $(*)$ and $K_{k,k}$ -freeness. The fact that $\alpha(\alpha_1, k, \mathcal{T}, d)$ is a linear function in α_1 for fixed $k, \mathcal{T}, d$ follows by a careful inspection of the proof. $\square$

Corollary 2.8. *Assume that $E \subseteq X \times Y$ admits a distal cell decomposition $\mathcal{T}$ with exponent t for some $t \in \mathbb{N}_{\geq 2}$ and E is $K_{2,s}$ -free for some*

$s \in \mathbb{N}_{\geq 2}$. Then there is some $\delta = \delta(t) > 0$ and $c = c(\mathcal{T}) > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y$ we have $|E(A, B)| \leq cs n^{\frac{3}{2}-\delta}$.

We can take $\delta := \frac{1}{2(2t-1)}$.

Proof. By Remark 2.7(2), $(*)$ holds with $d = 2$, and by Theorem 2.6 with $d = 2$ we get that $E(A, B) \leq \alpha \left(n^{\frac{2t-2}{2t-1}} n^{\frac{t}{2t-1}} + 2n \right) \leq (\alpha + 2) \left(n^{\frac{3}{2}-\delta} \right)$ with $\delta = \frac{1}{2(2t-1)} > 0$. And, by Remark 2.7(2), α depends sublinearly on s , that is $\alpha \leq sc$ for some $c = c(\mathcal{T})$. $\square$

This corollary can be interpreted as follows in an incidence-like manner. Assume that, working in a distal structure $\mathcal{M}$, we are given definable subsets X, Y of some powers of $\mathcal{M}$, and a definable family $(E_a)_{a \in X}$ of subsets of Y parametrized by X , such that $E_a \cap E_{a'}$ has bounded finite size for every $a \neq a' \in X$ (such definable families are often called *normal* in model theory, e.g. the family of lines in $\mathbb{C}^2$ is a normal family of curves since any two distinct lines can have at most one point in common). Then Corollary 2.8 says that given n sets in the family and n elements in Y , the number of “incidences” between them is at most $O(n^{\frac{3}{2}-\delta})$ for some $\delta > 0$ — strictly better than the bound $O(n^{\frac{3}{2}})$ given by Cauchy-Schwarz, as explained in the beginning of the section.

In particular, this provides a version of the theorem of Tóth [23] with a weaker bound.

2.2. Local stability. For the rest of Section 2 we assume that $\mathcal{M} = (M, \dots)$ is a sufficiently saturated structure, $\tilde{Y}, \tilde{Z}$ are definable subsets, and that $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ is a stable relation.

As usual, by a Φ -definable set we mean a subset $B \subseteq \tilde{Y}$ that is a finite Boolean combination of sets defined by $\Phi(y, c)$, $c \in \tilde{Z}$. We write $\Phi^* \subseteq \tilde{Z} \times \tilde{Y}$ for the relation obtained from Φ by exchanging the roles of the variables. Similarly we have a notion of Φ^* -definable subsets of $\tilde{Z}$. We denote by $S_\Phi(M)$ the set of all complete Φ -types on $\tilde{Y}$ over M (equivalently, the set of all ultrafilters on the Boolean algebra of all Φ -definable subsets of $\tilde{Y}$), and similarly we denote by $S_{\Phi^*}(M)$ the set of all complete Φ^* -types on $\tilde{Z}$. If $\mathbb{U}$ is an elementary extension of $\mathcal{M}$, then for an M -definable set V we will denote by $V(\mathbb{U})$ the set of elements of $\mathbb{U}$ realizing a formula defining V . We say that a Φ -type $p(y)$ is *non-algebraic* if in some elementary extension of $\mathcal{M}$ it has a realization outside of M .

The following are some basic facts from local stability, all of which can be found in e.g. [15, Chapter 1, Sections 1–3].

Fact 2.9. (1) For any $p(y) \in S_\Phi(M)$, the set $\{c \in \tilde{Z} : \Phi(y, c) \in p\}$ is definable by some Φ^* -formula $d_p^\Phi(z)$.

Moreover, this definition is uniform, meaning that there is some $n \in \mathbb{N}$ and a formula $d^\Phi(\bar{y}, z)$, $\bar{y} = (y_1, \dots, y_n)$, given by a finite Boolean combination of formulas $\{\Phi(y_i, z) : i \in [n]\}$, such that for every $p \in S_\Phi(M)$ we have that $d_p^\Phi(z)$ is of the form $d^\Phi(\bar{c}, z)$ for some $\bar{c} \in (\tilde{Y})^n$.

(2) Similarly, for $q(z) \in S_{\Phi^*}(M)$, the set $\{b \in \tilde{Y} : \Phi(b, z) \in q\}$ is uniformly Φ -definable via $d_q^{\Phi^*}(y)$.

Fact 2.10. Let $\mathbb{U}$ be an elementary extension of $\mathcal{M}$, $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$. Then $\text{tp}_\Phi(\beta/M\gamma)$ is finitely satisfiable in M if and only if $\text{tp}_{\Phi^*}(\gamma/M\beta)$ is finitely satisfiable in M .

Definition 2.11. For an elementary extension $\mathbb{U}$ of $\mathcal{M}$, $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$ we say that β and γ are Φ -independent over $\mathcal{M}$ if $\text{tp}_\Phi(\beta/M\gamma)$ is finitely satisfiable in M .

The following is a consequence of the fundamental theorem of local stability (it was also used in e.g. [10]).

Fact 2.12. For types $p(y) \in S_\Phi(M)$, $q(z) \in S_{\Phi^*}(M)$ the following conditions are equivalent.

- (1) There are realizations $\beta \models p(y)$ and $\gamma \models q(z)$ that are Φ -independent over M and such that $\models \Phi(\beta, \gamma)$;
- (2) for any realizations $\beta \models p(y)$ and $\gamma \models q(z)$ that are Φ -independent over M we have $\models \Phi(\beta, \gamma)$;
- (3) $d_p^\Phi(z) \in q(z)$;
- (4) $d_q^{\Phi^*}(y) \in p(y)$.

For types $p(y) \in S_\Phi(M)$ and $q(z) \in S_{\Phi^*}(M)$ we write $\Phi(p, q)$ if one of the equivalent conditions of Fact 2.12 holds.

2.3. Cartesian relations and popular types.

Proposition 2.13. The relation Φ is cartesian if and only if $\models \Phi(p, q)$ for some non-algebraic types $p(y) \in S_\Phi(M)$ and $q(z) \in S_{\Phi^*}(M)$.

Proof. Assume that Φ is cartesian, and let $B \subseteq \tilde{Y}$, $C \subseteq \tilde{Z}$ be infinite sets with $B \times C \subseteq \Phi$. By compactness, there is a non-algebraic type $p(y) \in S_\Phi(M)$ with $\Phi(y, c) \in p$ for all $c \in C$. Hence the set $d_p^\Phi(z)$ is infinite, and we can take $q(z)$ to be any non-algebraic type containing this formula.

For the converse, assume that $\models \Phi(p, q)$ holds for some non-algebraic p, q . Then we can choose inductively a sequence $(\alpha_i, \beta_i)_{i \in \mathbb{N}}$ in $\mathcal{M}$ such that the following hold for all $i \in \mathbb{N}$:

- (1) $\alpha_i \models d_q^{\Phi^*}(y), \beta_i \models d_p^{\Phi}(z)$;
- (2) $\alpha_i \in \tilde{Y} \setminus \{\alpha_j : j < i\}, \beta_i \in \tilde{Z} \setminus \{\beta_j : j < i\}$;
- (3) $\models \Phi(\alpha_j, \beta_j)$ for all $j \leq i$.

Indeed, assume $(\alpha_j, \beta_j : j < i)$ satisfying (1)–(3) were already chosen. By Fact 2.12(4), $d_q^{\Phi^*}(y) \in p$, and $\{\Phi(y, \beta_j) : j < i\} \subseteq p$ as $\beta_j \models d_p^{\Phi}(z)$ for $j < i$ by the inductive assumption. As p is non-algebraic, the set

$$\{d_q^{\Phi^*}(y)\} \cup \{\Phi(y, \beta_j) : j < i\} \cup \{y \neq \alpha_j : j < i\}$$

of formulas with parameters in M is consistent, hence realized in $\mathcal{M}$ by some $\alpha_i \in \tilde{Y}$. Similarly, $d_p^{\Phi}(z) \in q$ by Fact 2.12(3), and $\{\Phi(\alpha_j, z) : j \leq i\} \subseteq q$ as $\alpha_j \models d_q^{\Phi^*}(y)$ for all $j \leq i$ by the inductive assumption and choice of α_i . As q is non-algebraic,

$$\{d_p^{\Phi}(z)\} \cup \{\Phi(\alpha_j, z) : j \leq i\} \cup \{z \neq \beta_j : j < i\}$$

is consistent, and realized by some $\beta_i \in \tilde{Z}$. Then $(\alpha_j, \beta_j : j \leq i)$ satisfy (1)–(3) by construction.

Finally, $\{\alpha_i : i \in \mathbb{N}\} \times \{\beta_j : j \in \mathbb{N}\} \subseteq \Phi$ witnesses that Φ is cartesian. $\square$

The following definition is inspired by [17].

Definition 2.14. A non-algebraic type $p(y) \in S_{\Phi}(M)$ is called *popular* if the set $\{c \in \tilde{Z} : \Phi(y, c) \in p(y)\}$ is infinite.

Similarly, a non-algebraic type $q(z) \in S_{\Phi^*}(M)$ is popular if the set $\{b \in \tilde{Y} : \Phi(b, z) \in q(z)\}$ is infinite.

Lemma 2.15. (1) A non-algebraic type $p(y) \in S_{\Phi}(M)$ is popular if and only if there is a non-algebraic type $q(z) \in S_{\Phi^*}(M)$ with $\models \Phi(p, q)$.

(2) A non-algebraic type $q(z) \in S_{\Phi^*}(M)$ is popular if and only if there is a non-algebraic type $p(y) \in S_{\Phi}(M)$ with $\models \Phi(p, q)$.

Proof. Assume that $p(y)$ is popular, then the definable set $d_p^{\Phi}(z)$ is infinite and we can take q to be any non-algebraic type containing this set.

Assume $\models \Phi(p, q)$ for some non-algebraic $q(z) \in S_{\Phi^*}(M)$. Since $q(z)$ contains $d_p^{\Phi}(z)$, the set defined by $d_p^{\Phi}(z)$ must be infinite. $\square$

Combining, we have the following equivalence.

Proposition 2.16. The following conditions are equivalent.

- (1) The relation Φ is cartesian.
- (2) There is a popular type $p(y) \in S_{\Phi}(M)$.
- (3) There is a popular type $q(z) \in S_{\Phi^*}(M)$.

2.4. Bounds on the number of edges in non-cartesian relations.

The following theorem is a generalization of Theorem 1.3 in [14].

Theorem 2.17. *Let $\mathcal{M}$ be a sufficiently saturated structure eliminating $\exists^\infty$, and let $\tilde{Y}$ and $\tilde{Z}$ be definable sets in $\mathcal{M}$. Let $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ be an $\mathcal{M}$ -definable set such that for every $\beta \in \tilde{Y}$ the fiber $\Phi_\beta = \{z \in \tilde{Z} : (\beta, z) \in \Phi\}$ has Morley rank at most 1. Then the following conditions are equivalent.*

- (1) *The relation Φ is not cartesian.*
- (2) *The relation Φ is $K_{k,k}$ -free for some $k \in \mathbb{N}$.*
- (3) *For all $B \subseteq_n \tilde{Y}, C \subseteq_n \tilde{Z}$ we have*

$$|\Phi \cap (B \times C)| = O(n^{3/2}).$$

If, in addition, Φ admits cuttings (with some exponent $t \in \mathbb{N}$), then we also have

- (4) *There is some $\delta > 0$ such that for all $B \subseteq_n \tilde{Y}, C \subseteq_n \tilde{Z}$ we have*

$$|\Phi \cap (B \times C)| = O(n^{\frac{3}{2}-\delta}).$$

Proof. (1) implies (3). Assume that Φ is not cartesian.

Assume first that there is some $b \in \tilde{Y}$ for which there are some pairwise distinct $(b_i : i \in \mathbb{N})$ in $\tilde{Y}$ such that $\text{RM}(\Phi_b \cap \Phi_{b_i}) \geq 1$ (hence $= 1$) for all $i \in \mathbb{N}$. Then each of these sets contains a complete Φ^* -type of Morley rank 1. By the definition of Morley rank, there are only finitely many complete Φ^* -types $q_1, \dots, q_s \in S_{\Phi^*}(M)$ with $\Phi_b \in q_i$ and $\text{RM}(q_i) = 1$. But then one of these types must contain Φ_{b_i} for infinitely many different i , hence it is a popular type — contradicting the assumption by Proposition 2.16.

Thus there is no $b \in \tilde{Y}$ as above. Using that T eliminates $\exists^\infty$, there is some $r \in \mathbb{N}$ such that for every $b \in \tilde{Y}$, there are at most r many $b' \in \tilde{Y}$ such that $\Phi_b \cap \Phi_{b'}$ is infinite.

Given a finite set $B \subseteq_n \tilde{Y}$, consider the graph with the vertex set B and the edge relation E defined by $bEb' \iff \Phi_b \cap \Phi_{b'}$ is infinite. Then the graph (B, E) has degree at most r by the previous paragraph, and so it is $r + 1$ colorable by a standard result in graph theory. Let $B_i \subseteq B$ be the set of vertices corresponding to the i th color. Then $B = \bigsqcup_{1 \leq i \leq r+1} B_i$ and, by elimination of $\exists^\infty$ again, there is some $s \in \mathbb{N}$ depending only of Φ such that $|\Phi_b \cap \Phi_{b'}| \leq s$ for any $b, b' \in B_i$ and $1 \leq i \leq r + 1$.

Now, given a finite set $C \subseteq_n \tilde{Z}$, we have that $\Phi \upharpoonright (B_i \times C)$ is $K_{2,s}$ -free for each $1 \leq i \leq r+1$. Then, using Fact 2.5 with $d = 2$, we have

$$|\Phi \cap (B \times C)| \leq \sum_{i=1}^{r+1} |\Phi \cap (B_i \times C)| \leq (r+1)cn^{\frac{3}{2}}.$$

Hence taking $c' := (r+1)c$ depending only on Φ does the job. When Φ admits cuttings, we use Corollary 2.8 instead of Fact 2.5.

Finally, (3) implies (2) and (2) implies (1) are straightforward. $\square$

Remark 2.18. Theorem 2.17(4) is the only place where the assumption of the existence of a distal expansion is used. It is necessary to get a bound strictly less than $n^{\frac{3}{2}}$, as the points-lines incidence relation on the plane in an algebraically closed field of characteristic p (a strongly minimal structure) demonstrates. However, this $\delta > 0$ improvement is crucial to obtain the $2 - \varepsilon$ bound on the power in Theorem 1.3(a).

3. REDUCING MAIN THEOREM TO A DICHOTOMY FOR BINARY RELATIONS

To prove the main theorem we introduce some notions and make some reductions. Since we are only interested in definable subsets of products of strongly minimal sets, we may and will assume that $\mathcal{M}$ has finite Morley rank and eliminates the quantifier $\exists^\infty$. In particular, Morley rank is *additive* — this will be used freely throughout the proof.

Assumption 1. *For the rest of this section (Section 3) we assume that $\mathcal{M}$ is a sufficiently saturated structure of finite Morley rank that eliminates the quantifier $\exists^\infty$.*

We fix strongly minimal sets X, Y, Z definable in $\mathcal{M}$.

We also fix an M -definable set $F \subseteq X \times Y \times Z$ of Morley rank 2.

We assume that X, Y, Z and F are definable over the empty set (naming some constants if necessary).

Notice that writing F as a union $F = \cup_{i=1}^k F_i$ and applying Theorem 1.3 to each F_i , it is sufficient to consider only the case when F has Morley degree 1.

Assumption 2. *In addition, for the rest of Section 3, we assume that F has Morley degree 1.*

3.1. Fiber-algebraic relations.

Definition 3.1. Let S_1, S_2, S_3 be sets and $E \subseteq S_1 \times S_2 \times S_3$ be a subset. We say that E is *fiber-algebraic* if there is some $d \in \mathbb{N}$ such

that for $\{i, j, k\} = \{1, 2, 3\}$ we have

$$\models \forall y_i \forall y_j \exists^{\leq d} y_k E(y_1, y_2, y_3).$$

Assume, in addition to Assumptions 1 and 2, that $F \subseteq X \times Y \times Z$ is not cylindrical. Then, since Z is strongly minimal, the set $\{(a, b) \in X \times Y : \exists^\infty z F(a, b, z)\}$ is finite (otherwise we can find infinite sequences $a_i \in X, b_i \in Y$ such that $F(a_i, b_i, Z)$ is cofinite, hence $\bigcap_{i \in \mathbb{N}} F(a_i, b_i, Z)$ is infinite using saturation of $\mathcal{M}$ — so F is cylindrical). Thus there are co-finite $X_0 \subseteq X, Y_0 \subseteq Y$ such that

$$\models \forall x \in X_0 \forall y \in Y_0 \exists^{< \infty} z F(x, y, z).$$

Applying the same argument for every partition of the coordinates of F we conclude that if F is not cylindrical then there are co-finite $X_0 \subseteq X, Y_0 \subseteq Y, Z_0 \subseteq Z$ such that the restriction of F to $X_0 \times Y_0 \times Z_0$ is fiber-algebraic.

It is not hard to see that if the relation $F \upharpoonright_{X_0 \times Y_0 \times Z_0} \subseteq X_0 \times Y_0 \times Z_0$ satisfies one of the clauses (a) or (b) in Theorem 1.3, then F satisfies the same clause. For example, assume F restricted to $X_0 \times Y_0 \times Z_0$ satisfies clause (a). Let $X_1 := X \setminus X_0, Y_1 := Y \setminus Y_0, Z_1 := Z \setminus Z_0$, we have

$$F = \bigsqcup_{i,j,k \in \{0,1\}} F \cap (X_i \times Y_j \times Z_k).$$

Then to show that $|F \cap A \times B \times C| = O(n^{2-\varepsilon})$ holds for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$, it is enough to establish the same bound for all $A \subseteq_n X_i, B \subseteq_n Y_j, C \subseteq_n Z_k$ for each choice of $i, j, k \in \{0, 1\}$ separately. The case $(i, j, k) = (0, 0, 0)$ holds by assumption. If at least two of i, j, k are 1, i.e. at least two of the sets X_i, Y_j, Z_k are finite, then the bound $O(n)$ obviously holds. So we only need to consider the case when only one of i, j, k is 1, for example $F \cap X_1 \times Y_0 \times Z_0$. For any fixed $x \in X_1$ and $y \in Y_0$ there are only finitely many $z \in Z_0$ with $(x, y, z) \in F$, since $\{(a, b) \in X \times Y : \exists^\infty z F(a, b, z)\} \subseteq X_1 \times Y_1$. Hence again the bound $O(n)$ holds for $F \cap (X_1 \times Y_0 \times Z_0)$ by elimination of $\exists^\infty$.

In view of this observation, Theorem 1.3 follows from the following theorem.

Theorem 3.2. *Assume, in addition to Assumptions 1 and 2, that F is fiber-algebraic and also that $\mathcal{M}$ is interpretable in a distal structure. Then one of the following holds.*

(a) *There is $\varepsilon > 0$ such that for all $A \subseteq_n X, B \subseteq_n Y, C \subseteq_n Z$ we have*

$$|F \cap A \times B \times C| = O(n^{2-\varepsilon}).$$

(b) *F is group-like.*

Assumption 3. For the rest of Section 3 we assume in addition that the relation F is fiber-algebraic and $d \in \mathbb{N}$ is as in Definition 3.1.

3.2. On acl-diagrams. We say that three elements p_1, p_2, p_3 of M form an *acl-triangle* if $\text{RM}(p_i/\emptyset) = 1$ for $i \in [3]$, $\text{RM}(p_1 p_2 p_3/\emptyset) = 2$, and for all $\{i, j, k\} = \{1, 2, 3\}$ we have $p_i \in \text{acl}(p_j p_k)$ (hence also $p_i \perp p_j$ for all $i \neq j \in \{1, 2, 3\}$).

Since F is fiber-algebraic of Morley rank 2, we have the following claim.

Claim 3.3. Let (a, b, c) be generic in F , i.e. $(a, b, c) \in F$ and $\text{RM}(abc/\emptyset) = 2$. Then a, b, c form an *acl-triangle*.

In particular, b and c are independent generics in Y and Z , respectively; and, by stationarity (as $Y \times Z$ has Morley degree 1), for any independent generics $b' \in Y$, $c' \in Z$ there is some $x \in X$ such that $(x, b', c') \in F$ and (x, b', c') is generic in F .

Similarly, for any independent generics $a' \in X$, $b' \in Y$ we have $\mathcal{M} \models \exists z F(a', b', z)$.

In this paper we will consider some simple diagrams, where by a *diagram* we mean a collection of elements of M and lines between them (subsets of the given elements).

Definition 3.4. We say that a given diagram is an *acl-diagram* if

- (1) $\text{RM}(p/\emptyset) = 1$ for every point p in the diagram;
- (2) Every three collinear points form an *acl-triangle*;
- (3) $\text{RM}(pqr/\emptyset) = 3$ for any three non-collinear points p, q, r .

3.3. The 4-ary relation Q . Our next goal is to restate Theorem 3.2 in term of a 4-ary relation Q . (We continue to use Assumptions 1–3.)

Let $Q \subseteq Y^2 \times Z^2$ be the definable relation

$$Q = \{(y, y', z, z') \in Y^2 \times Z^2 : \exists x \in X ((x, y, z) \in F \ \& \ (x, y', z') \in F)\}.$$

We first observe some basic properties of Q .

Claim 3.5. For $(b_1, b_2) \in Y^2$ and $c_1 \in Z$ the set

$$\{z \in Z : (b_1, b_2, c_1, z) \in Q\}$$

has size at most d^2 .

Similarly, for $(c_1, c_2) \in Z^2$ and $b_1 \in Y$ the set

$$\{y \in Y : (b_1, y, c_1, c_2) \in Q\}$$

has size at most d^2 .

Proof. Let b_1, b_2, c_1 be fixed. As F is fiber-algebraic, by the choice of d there are at most d elements $x \in X$ such that $(x, b_1, c_1) \in F$, and for each such x , there are at most d elements $z \in Z$ such that $(x, b_2, z) \in F$. Hence, by definition of Q , there are at most d^2 elements $z \in Z$ such that $(b_1, b_2, c_1, z) \in Q$. $\square$

Corollary 3.6. *For any $\bar{b} = (b_1, b_2) \in Y^2$ the Morley rank of the fiber $Q_{\bar{b}} = \{\bar{z} \in Z^2 : (\bar{b}, \bar{z}) \in Q\}$ is at most 1.*

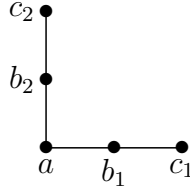
Similarly, for any $\bar{c} = (c_1, c_2) \in Z^2$ the Morley rank of the fiber $Q_{\bar{c}} = \{\bar{y} \in Y^2 : (\bar{y}, \bar{c}) \in Q\}$ is at most 1.

Corollary 3.7. *The Morley rank of Q is 3.*

Proof. It follows from the additivity of Morley rank and Corollary 3.6 that $\text{RM}(Q) \leq \text{RM}(Y^2) + 1 = 3$.

On the other hand, let $b_1, b_2 \in Y, c_1 \in Z$ be independent generics, hence $\text{RM}(b_1 b_2 c_1 / \emptyset) = 3$. By Claim 3.3, there is a generic $a \in X$ with $\mathcal{M} \models F(a, b_1, c_1)$. Applying Claim 3.3 to a and b_2 (as $a \in \text{acl}(b_1 c_1)$ and $b_1 c_1 \perp b_2$, we have $a \perp b_2$) we get $c_2 \in Z$ with $\mathcal{M} \models F(a, b_2, c_2)$. Since $\mathcal{M} \models Q(b_1, b_2, c_1, c_2)$ we have $\text{RM}(Q) \geq 3$. $\square$

Lemma 3.8. *Let (b_1, b_2, c_1, c_2) be generic in Q . Then there is $a \in X$ such that the diagram*



is an acl-diagram with $\mathcal{M} \models F(a, b_i, c_i)$.

Proof. By the definition of Q we can find $a \in X$ with

$$\mathcal{M} \models F(a, b_1, c_1) \ \& \ F(a, b_2, c_2).$$

We claim that for this a the diagram above is an acl-diagram.

First notice that $\text{RM}(p/\emptyset) \leq 1$ for any point p .

Secondly, by fiber-algebraicity of F , if p, q, r are three non-collinear points then every point of the diagram is in $\text{acl}(pqr)$; in particular, since $\text{RM}(b_1 b_2 c_1 c_2 / \emptyset) = 3$, we have $\text{RM}(pqr/\emptyset) = 3$.

Finally, given any two distinct non-collinear points (p, q) we can find a point r such that p, q, r are non-collinear (so $\text{RM}(pqr/\emptyset) = 3$), hence $\text{RM}(pq/\emptyset) = 2$ by additivity of Morley rank.

It follows then that any three collinear points, after reordering, form a generic realization of F , and hence, by Claim 3.3, an acl-triangle. $\square$

3.4. The clause (a) in Theorem 3.2. In this section we state a property of the relation Q that implies the clause (a) in Theorem 3.2.

First we need some basic counting properties.

Corollary 3.9. *For any $\bar{b} \in Y^2$ and a finite set $C \subseteq Z$ we have*

$$|Q \cap (\{\bar{b}\} \times C^2)| \leq d^2 |C|.$$

Similarly, for any $\bar{c} \in Z^2$ and a finite set $B \subseteq Y$ we have

$$|Q \cap (B^2 \times \{\bar{c}\})| \leq d^2 |B|.$$

Proof. Follows from Claim 3.5 □

The following bound is similar to [17, Lemma 2.2].

Proposition 3.10. *Let $A \subseteq X, B \subseteq Y, C \subseteq Z$ be finite. Then for $F' = F \cap (A \times B \times C)$ and $Q' = Q \cap (B^2 \times C^2)$ we have*

$$|F'| \leq d |A|^{1/2} |Q'|^{1/2}.$$

Proof. Let $W \subseteq X \times Y^2 \times Z^2$ be the definable set

$$W = \{(x, y, y', z, z') \in X \times Y^2 \times Z^2 : (x, y, z) \in F \text{ \& } (x, y' z') \in F\},$$

and let $W' = W \cap (A \times B^2 \times C^2)$.

As usual, for a set $S \subseteq A \times D$ and $a \in A$ we denote by S_a the fiber $S_a = \{u \in D : (a, u) \in S\}$.

Notice that $|F'| = \sum_{a \in A} |F'_a|$, and $|W'| = \sum_{a \in A} |F'_a|^2$. By the Cauchy-Schwarz inequality,

$$|F'| \leq |A|^{1/2} \left(\sum_{a \in A} |F'_a|^2 \right)^{1/2} = |A|^{1/2} |W'|^{1/2}.$$

For a point $g \in Q'$, the fiber W'_g has size at most d as F is fiber-algebraic, hence $|W'| \leq d |Q'|$ and $|F'| \leq d |A|^{1/2} |Q'|^{1/2}$. □

The next proposition shows that the bound $O(n^{3/2-\delta})$ for Q translates to the bound $O(n^{2-\delta})$ for F .

Proposition 3.11. *Let $\tilde{Y} := Y^2$, $\tilde{Z} := Z^2$, and we view Q as a subset of $\tilde{Y} \times \tilde{Z}$. Assume that there are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1 such that for some $0 < \delta < \frac{1}{2}$ for all $B' \subseteq_m \tilde{Y} \setminus Y_0, C' \subseteq_m \tilde{Z} \setminus Z_0$ we have $|Q \cap (A' \times B')| = O(m^{3/2-\delta})$. Then F satisfies the clause (a) of Theorem 3.2 (with $\varepsilon = \delta$).*

Proof. We fix $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1, $c_0 \in \mathbb{R}$ and $\delta > 0$ such that for all m large enough and for all $B' \subseteq_m \tilde{Y} \setminus Y_0, C' \subseteq_m \tilde{Z} \setminus Z_0$ we have $|Q \cap A' \times B'| \leq c_0 m^{3/2-\delta}$.

Since Y_0 has Morley rank at most 1, using elimination of $\exists^\infty$, it is not hard to see that there is $k_1 \in \mathbb{N}$ such that for any finite $B \subseteq Y$ we have $|B^2 \cap Y_0| \leq k_1|B|$.

Similarly, there is $k_2 \in \mathbb{N}$ such that for any finite $C \subseteq Z$ we have $|C^2 \cap Z_0| \leq k_2|C|$.

Given $A \subseteq_n X$, $B \subseteq_n Y$, $C \subseteq_n Z$, let $B' = B^2 \cap (\tilde{Y} \setminus Y_0)$ and $C' = C^2 \cap (\tilde{Z} \setminus Z_0)$. Obviously $|B'| \leq n^2$ and $|C'| \leq n^2$.

We have

$$\begin{aligned} & |Q \cap (B^2 \times C^2)| \leq \\ & |Q \cap (B' \times C')| + |Q \cap ((B^2 \cap Y_0) \times C^2)| + |Q \cap (B^2 \times (C^2 \cap Z_0))|. \end{aligned}$$

By our assumptions $|Q \cap (B' \times C')| \leq c_0(n^2)^{\frac{3}{2}-\delta} = c_0n^{3-2\delta}$. Since $|B^2 \cap Y_0| \leq k_1n$, from Corollary 3.9, we get $|Q \cap ((B^2 \cap Y_0) \times C^2)| \leq k_1d^2n^2$; and similarly $|Q \cap (B^2 \times (C^2 \cap Z_0))| \leq k_2d^2n^2$.

Thus $|Q \cap (B^2 \times C^2)| \leq c_1n^{3-2\delta}$, where $c_1 > 0$ does not depend on n, A, B, C .

Applying Proposition 3.10, we obtain

$$|F \cap (A \times B \times C)| \leq d(nc_1n^{3-2\delta})^{1/2} = O(n^{2-\delta}).$$

□

Combining this with Theorem 2.17, we obtain a property of Q that implies the clause (a) in Theorem 3.2.

Proposition 3.12. *Let $\tilde{Y} = Y^2$, $\tilde{Z} = Z^2$, and we view Q as a subset of $\tilde{Y} \times \tilde{Z}$. Assume in addition that Q admits cuttings (with some exponent). Assume also that there are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1 such that the restriction of Q to $(\tilde{Y} \setminus Y_0) \times (\tilde{Z} \setminus Z_0)$ is not cartesian. Then F satisfies the clause (a) of Theorem 3.2.*

3.5. The clause (b) of Theorem 3.2. We fix a saturated elementary extension $\mathbb{U}$ of $\mathcal{M}$.

Proposition 3.13. *Assume there are $\beta = (\beta_1, \beta_2) \in Y^2(\mathbb{U})$ and $\gamma = (\gamma_1, \gamma_2) \in Z^2(\mathbb{U})$ with $(\beta, \gamma) \in Q(\mathbb{U})$, such that $\text{RM}(\beta/M), \text{RM}(\gamma/M) > 0$, $\beta \perp_M \gamma$ and $\text{acl}(\beta) \cap \text{acl}(\gamma) \not\subseteq \text{acl}(\emptyset)$. Then F is group-like.*

Proof. Choose $t \in (\text{acl}(\beta) \cap \text{acl}(\gamma)) \setminus \text{acl}(\emptyset)$. We first list some properties of β, γ and t .

(i) Since $\beta \perp_M \gamma$, and $t \in (\text{acl}(\beta) \cap \text{acl}(\gamma))$ we have

$$t \in M.$$

(ii) Since $t \in \text{acl}(\beta) \setminus \text{acl}(\emptyset)$ we have

$$\beta \not\downarrow_{\emptyset} t;$$

and, similarly,

$$\gamma \not\downarrow_{\emptyset} t.$$

(iii) From (i) and (ii), since $\text{RM}(\beta/\emptyset) \leq 2$ and $\text{RM}(\beta/M) > 0$, we obtain

$$\text{RM}(\beta/\emptyset) = 2 \text{ and } \text{RM}(\beta/t) = \text{RM}(\beta/M) = 1;$$

and, similarly,

$$\text{RM}(\gamma/\emptyset) = 2 \text{ and } \text{RM}(\gamma/t) = \text{RM}(\gamma/M) = 1.$$

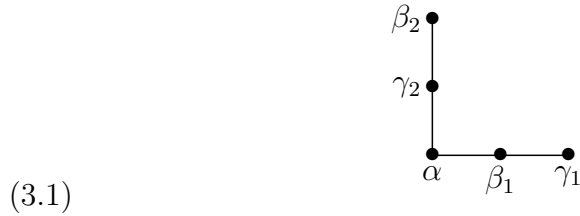
(iv) Since $t \in \text{acl}(\beta) \setminus \text{acl}(\emptyset)$ and $\beta \notin \text{acl}(t)$ we have $\text{RM}(t/\emptyset) = 1$.

(v) Since β and γ are independent over M we have $\text{RM}(\beta\gamma/M) = 2$, and since $\beta \not\downarrow_{\emptyset} M$, we have $\text{RM}(\beta\gamma/\emptyset) = 3$, i.e. (β, γ) is generic in $Q(\mathbb{U})$.

(vi) It follows from (v) and Lemma 3.8 that $\beta_i \notin \text{acl}(\gamma)$ and $\gamma_i \notin \text{acl}(\beta)$ for $i \in [2]$.

We also have that both $\{\beta_1, \beta_2, t\}$ and $\{\gamma_1, \gamma_2, t\}$ are acl-triangles. Indeed, for example, since $\text{RM}(\beta_1\beta_2t/\emptyset) = 2$, to show that $\{\beta_1, \beta_2, t\}$ is a triangle it is sufficient to check that $t \notin \text{acl}(\beta_i)$ for $i = 1, 2$. But if $t \in \text{acl}(\beta_i)$, then $\beta_i \in \text{acl}(t)$, hence $\beta_i \in \text{acl}(\gamma)$ — contradicting (vi).

By Lemma 3.8 there is $\alpha \in X(\mathbb{U})$ such that the diagram



is an acl-diagram with $\mathbb{U} \models F(\alpha, \beta_i, \gamma_i)$.

We claim that



is an acl-diagram.

We already have that every three colinear points form an acl-triangle, and it is sufficient to check that for three non-colinear points $\{p, q, r\}$ we have $\text{RM}(pqr/\emptyset) = 3$. If $t \notin \{p, q, r\}$ then it follows from the diagram (3.1). Assume $t \in \{p, q, r\}$, say $\{p, q, r\} = \{t, \alpha, \beta_1\}$. Then $\{t, \beta_1\}$ is inter-algebraic with $\{\beta_1, \beta_2\}$ and hence

$$\text{RM}(t\alpha\beta_1/\emptyset) = \text{RM}(\beta_2\alpha\beta_1/\emptyset) = 3.$$

The same argument works for any three non-collinear points containing t .

Thus the diagram (3.2) is an acl-diagram. It follows from the Group Configuration Theorem in stable theories (see [11, Theorem 6.1] and the discussion in [11, Section 6.2]) that F is group-like. $\square$

4. DICHOTOMY FOR BINARY RELATIONS

In this section we prove a dichotomy theorem for binary relations between sets of Morley rank 2. By Propositions 3.12 and 3.13, Theorem 3.2 follows from Theorem 4.1 applied with $\Phi := Q$, $\tilde{Y} := Y^2$, $\tilde{Z} := Z^2$ as in Section 3.

Theorem 4.1. *Let $\mathcal{M}$ be a sufficiently saturated structure of finite Morley rank that eliminates quantifier $\exists^\infty$.*

Let $\tilde{Y}$ and $\tilde{Z}$ be M -definable sets of Morley rank 2 and Morley degree 1. Let $\Phi \subseteq \tilde{Y} \times \tilde{Z}$ be a definable subset of Morley rank 3. Then one of the following holds.

- (a) *There are definable sets $Y_0 \subseteq \tilde{Y}$ and $Z_0 \subseteq \tilde{Z}$ of Morley rank at most 1 such that the restriction of Φ to $(\tilde{Y} \setminus Y_0) \times (\tilde{Z} \setminus Z_0)$ is not cartesian.*
- (b) *There are $\beta \in \tilde{Y}(\mathbb{U})$ and $\gamma \in \tilde{Z}(\mathbb{U})$ with $(\beta, \gamma) \in \Phi(\mathbb{U})$, such that $\text{RM}(\beta/M), \text{RM}(\gamma/M) > 0$, $\beta \perp_M \gamma$ and $\text{acl}(\beta) \cap \text{acl}(\gamma) \not\subseteq \text{acl}(\emptyset)$.*

Proof. As usual, for a Φ -type $p(y) \in S_\Phi(M)$, we denote by $\text{RM}(p(y))$ the Morley rank of p as an incomplete type.

We assume that (a) doesn't hold, and show that then (b) must hold. For a definable set $\tilde{Y}' \subseteq \tilde{Y}$ we say that $\tilde{Y}'$ is *large* in $\tilde{Y}$ if $\text{RM}(\tilde{Y} \setminus \tilde{Y}') \leq 1$; and the same for a subset $\tilde{Z}' \subseteq \tilde{Z}$. Notice that in the proof we can freely replace $\tilde{Y}$ and $\tilde{Z}$ by their large subsets.

Let $p^*(y) \in S(M)$ be the generic type on $\tilde{Y}$, it is the unique type on $\tilde{Y}$ of Morley rank 2 (as $\tilde{Y}$ has Morley degree 1 by assumption).

Since Φ has Morley rank 3, the set $\{c \in \tilde{Z} : \Phi_c \in p^*\}$ is definable (by Fact 2.9) and has Morley rank at most 1 by additivity of Morley rank. Thus we can throw away this set and assume that the Morley

rank of Φ_c is at most 1 for all $c \in \tilde{Z}$. Similarly, we may assume that the Morley rank of $\Phi_b \subseteq \tilde{Z}$ is at most 1 for all $b \in \tilde{Y}$.

Assume that p is a popular type for Φ (see Definition 2.14). Then $\text{RM}(p) = 1$ ($\text{RM}(p) \geq 1$ as p is non-algebraic, and $\text{RM}(p) \leq 1$ as $\Phi_c \in p$ for some c by definition of popular types, and $\text{RM}(\Phi_c) = 1$ by the previous paragraph). If there are only finitely many popular types for Φ , then we can throw away finitely many definable sets of Morley rank 1 (one in each of the popular types), and pass to a large subset on which there are no popular types (hence, obtaining (a) using Proposition 2.16). Thus we can assume that there are infinitely many popular types on $\tilde{Y}$.

Let $\mathcal{P}$ be the set of all popular types on $\tilde{Y}$ and $\mathcal{Q}$ be the set of all popular types on $\tilde{Z}$.

For $p \in S_\Phi(M)$ we denote by $[p] \in \mathcal{M}^{\text{eq}}$ the canonical parameter of the Φ^* -definable set $d_p^\Phi = \{c \in \tilde{Z} : \Phi_c \in p\}$. Recall that by Fact 2.9 the definition d_p^Φ is given by instances of the same formula for all p , hence $[p]$ is the canonical parameter of an instance of the same formula for all p , hence $\{[p] : p \in S_\Phi(M)\}$ is a subset of a fixed sort in $\mathcal{M}^{\text{eq}}$; similarly, for $q \in S_{\Phi^*}(M)$ we will denote by $[q] \in \mathcal{M}^{\text{eq}}$ the canonical parameter for $d_q^{\Phi^*}$.

Clearly both maps $p \mapsto [p]$ and $q \mapsto [q]$ are injective.

Claim 4.2. *The sets $\{[p] : p \in \mathcal{P}\}$ and $\{[q] : q \in \mathcal{Q}\}$ are $\emptyset$ -type-definable subsets of the corresponding sorts in $\mathcal{M}^{\text{eq}}$.*

Proof. Using that $\mathcal{M}$ eliminates $\exists^\infty$ (and uniform definability of types), the desired set $\{[p] : p \in \mathcal{P}\}$ is type-definable by

$$\{[p] : \exists^\infty z d_p^\Phi(z) \wedge \bigwedge_{n \in \mathbb{N}} (\forall z_1 \dots \forall z_n (\bigwedge_{i=1}^n d_p^\Phi(z_i) \rightarrow \exists^\infty y \bigwedge_{i=1}^n \Phi(y, z_i))\}.$$

And similarly for $\mathcal{Q}$. □

Claim 4.3. *If $p \in \mathcal{P}$ and $c \in d_p^\Phi$ then $[p] \in \text{acl}(c)$.*

Similarly, if $q \in \mathcal{Q}$ and $b \in d_q^{\Phi^}$ then $[q] \in \text{acl}(b)$.*

Proof. As for any $c \in \tilde{Z}$ there are only finitely many Φ -types of Morley rank 1 containing Φ_c . □

Claim 4.4. *For any $p \in \mathcal{P}$ there are only finitely many $q \in \mathcal{Q}$ with $\models \Phi(p, q)$, and vice versa.*

Proof. Let $\beta \in \mathbb{U}$ realize p . It is not hard to see that, for $q \in \mathcal{Q}$, if $\models \Phi(p, q)$ then the Morley rank of the partial type $\{\Phi(\beta, z)\} \cup q(z)$ is 1. Since the Morley rank of $\Phi(\beta, z)$ is 1, there only finitely many such q . □

Since the set $\mathcal{P}$ is infinite, we can find a popular type $p \in \mathcal{P}$ such that $[p] \notin \text{acl}(\emptyset)$. Choose $q \in \mathcal{Q}$ with $\models \Phi(p, q)$.

Choose some $\beta \models p(y)$ and $\gamma \models q(z)$ independent over M .

We have $[p] \in \text{acl}(\gamma)$, $[q] \in \text{acl}(\beta)$ with $[p]$ and $[q]$ inter-algebraic over the empty set. Hence (b) holds. $\square$

REFERENCES

- [1] Matthias Aschenbrenner, Artem Chernikov, Allen Gehret, and Martin Ziegler, *Distality in valued fields and related structures*, Preprint (2020).
- [2] Martin Bays and Emmanuel Breuillard, *Projective geometries arising from Elekes-Szabó problems*, Preprint, arXiv:1806.03422 (2018).
- [3] Artem Chernikov, David Galvin, and Sergei Starchenko, *Cutting lemma and Zarankiewicz’s problem in distal structures*, *Selecta Mathematica* **26** (2020), no. 2, 1–27.
- [4] Artem Chernikov, Ya’acov Peterzil, and Sergei Starchenko, *Model theoretic Elekes–Szabó for stable and o-minimal hypergraphs*, Preprint (2020).
- [5] Artem Chernikov and Pierre Simon, *Externally definable sets and dependent pairs II*, *Transactions of the American Mathematical Society* **367** (2015), no. 7, 5217–5235.
- [6] Artem Chernikov and Sergei Starchenko, *Regularity lemma for distal structures*, *Journal of the European Mathematical Society* **20** (2018), no. 10, 2437–2466.
- [7] Frank de Zeeuw, *A survey of Elekes–Rónyai-type problems*, *New trends in intuitive geometry*, 2018, pp. 95–124.
- [8] György Elekes and Endre Szabó, *How to find groups?(and how to use them in Erdős geometry?)*, *Combinatorica* (2012), 1–35.
- [9] Jacob Fox, János Pach, Adam Sheffer, Andrew Suk, and Joshua Zahl, *A semi-algebraic version of Zarankiewicz’s problem*, *Journal of the European Mathematical Society* **19** (2017), no. 6, 1785–1810.
- [10] Ehud Hrushovski, *Stable group theory and approximate subgroups*, *Journal of the American Mathematical Society* **25** (2012), no. 1, 189–243.
- [11] Ehud Hrushovski and Boris Zilber, *Zariski Geometries*, *Journal of the American Mathematical Society* **9** (January 1996), no. 1, 1–56.
- [12] Tamás Kovári, Vera Sós, and Pál Turán, *On a problem of K. Zarankiewicz*, *Colloquium mathematicae*, 1954, pp. 50–57.
- [13] David Marker, *Model theory: an introduction*, Vol. 217, Springer Science & Business Media, 2006.
- [14] Hossein Nassajian Mojarrad, Thang Pham, Claudiu Valculescu, and Frank de Zeeuw, *Schwartz–Zippel bounds for two-dimensional products*, *Discrete Anal.* (2017), Paper No. 20, 20.
- [15] Anand Pillay, *Geometric stability theory*, Oxford University Press, 1996.
- [16] Orit E Raz, Micha Sharir, and Frank De Zeeuw, *Polynomials vanishing on cartesian products: The Elekes–Szabó theorem revisited*, *Duke Mathematical Journal* **165** (2016), no. 18, 3517–3566.
- [17] Orit E. Raz, Micha Sharir, and József Solymosi, *Polynomials vanishing on grids: the Elekes–Rónyai problem revisited*, *Amer. J. Math.* **138** (2016), no. 4, 1029–1065.

- [18] Orit E Raz and Zvi Shem Tov, *Expanding polynomials: A generalization of the Elekes-Rónyai theorem to d variables*, Preprint, arXiv:1807.02238 (2018).
- [19] Adam Sheffer, *Polynomial methods and incidence theory*, Book draft, <http://faculty.baruch.cuny.edu/ASheffer/000book.pdf>.
- [20] Pierre Simon, *Distal and non-distal NIP theories*, Annals of Pure and Applied Logic **164** (2013), no. 3, 294–318.
- [21] Endre Szemerédi and William T. Trotter, *Extremal problems in discrete geometry*, Combinatorica **3** (1983), no. 3-4, 381–392.
- [22] Katrin Tent and Martin Ziegler, *A course in model theory*, Vol. 40, Cambridge University Press, 2012.
- [23] Csaba D Tóth, *The Szemerédi-Trotter theorem in the complex plane*, Combinatorica **35** (2015), no. 1, 95–126.
- [24] Hong Wang, *Exposition of Elekes Szabo paper*, arXiv preprint arXiv:1512.04998 (2015).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA LOS ANGELES,
 LOS ANGELES, CA 90095-1555
Email address: `chernikov@math.ucla.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME,
 IN, 46656, USA
Email address: `sstarche@nd.edu`