ANALYTICITY OF STEKLOV EIGENVALUES OF NEARLY-CIRCULAR AND NEARLY-SPHERICAL DOMAINS

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Abstract. We consider the Dirichlet-to-Neumann operator (DNO) on nearly-circular and nearly-spherical domains in two and three dimensions, respectively. Treating such domains as perturbations of the ball, we prove the analyticity of the DNO with respect to the domain perturbation parameter. Consequently, the Steklov eigenvalues are also shown to be analytic in the domain perturbation parameter. To obtain these results, we use the strategy of Nicholls and Nigam (2004); we transform the equation on the perturbed domain to a ball and geometrically bound the Neumann expansion of the transformed Dirichlet-to-Neumann operator.

1. Introduction

Let $\Omega_\varepsilon \subset \mathbb{R}^d$ for $d = 2, 3$ be a nearly-circular or nearly-spherical domain of the form

$$\Omega_\varepsilon = \{(r, \hat{\theta})\colon 0 \le r \le 1 + \varepsilon\rho(\hat{\theta}),\ \hat{\theta} \in S^{d-1}\}, \tag{1}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$, $\rho \in C^{s+1}(S^{d-1})$ is the *domain perturbation function* for some $s \in \mathbb{N}$, [1] and $\varepsilon \ge 0$ is the *perturbation parameter*, which is assumed to be small in magnitude. We consider the Steklov eigenproblem on the perturbed domain Ω_ε,

$$\Delta u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon \tag{2a}$$

$$\partial_{n_\varepsilon} u_\varepsilon = \sigma_\varepsilon u_\varepsilon \qquad \text{on } \partial\Omega_\varepsilon. \tag{2b}$$

Here Δ is the Laplacian on $H^2(\Omega_\varepsilon)$ and $\partial_{n_\varepsilon} = \hat{n}_\varepsilon \cdot \nabla$ denotes the outward normal derivative on the boundary of Ω_ε. It is well-known that the Steklov spectrum is discrete, real, and non-negative; we enumerate the eigenvalues in increasing order, $0 = \sigma_0(\Omega_\varepsilon) < \sigma_1(\Omega_\varepsilon) \le \sigma_2(\Omega_\varepsilon) \cdots \to \infty$. The Steklov spectrum coincides with the spectrum of the Dirichlet-to-Neumann operator (DNO), $G_\varepsilon\colon H^{\frac{3}{2}}(\partial\Omega_\varepsilon) \to H^{\frac{1}{2}}(\partial\Omega_\varepsilon)$, which maps

$$\xi \mapsto G_\varepsilon \xi = \partial_{n_\varepsilon} u_\varepsilon,$$

where u_ε is the *harmonic extension* of ξ to Ω_ε, satisfying

$$\Delta u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon \tag{3a}$$

$$u_\varepsilon(\hat{\theta}) = \xi(\hat{\theta}) \qquad \text{on } \partial\Omega_\varepsilon. \tag{3b}$$

We refer the reader to [GP17] for a general description of the Steklov spectrum.

The *goal of this paper* is to prove the analyticity of the Steklov eigenvalues, σ_ε, in the perturbation parameter ε. Our main result is the following theorem.

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[1] Throughout this paper, we use the notation $\mathbb{N}$ to denote the positive integers.

Theorem 1.1. *Let $d = 2$ or 3 and $s \in \mathbb{N}$. If $\rho \in C^{s+1}(S^{d-1})$, then the Dirichlet-to-Neumann operator (DNO), $G_\varepsilon\colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s-\frac{1}{2}}(\partial\Omega_\varepsilon)$, is analytic in the domain parameter ε. More precisely, if $\rho \in C^{s+2}(S^{d-1})$, then there exists an isomorphism $L\colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s+\frac{1}{2}}(S^{d-1})$ and a Neumann series, $LG_\varepsilon L^{-1} = \sum_{n=0}^{\infty} \varepsilon^n G_n$, that converges strongly as an operator from $H^{s+\frac{1}{2}}(S^{d-1})$ to $H^{s-\frac{1}{2}}(S^{d-1})$. That is, there exists constants K_1 and C such that*

$$\|G_n\xi\|_{H^{s-\frac{1}{2}}(S^{d-1})} \leq K_1\|\xi\|_{H^{s+\frac{1}{2}}(S^{d-1})}B^n$$

for any $B > C|\rho|_{C^{s+1}}$.

We prove Theorem 1.1 in two and three dimensions separately; these proofs can be found in Sections 2.2 and 3.2, respectively. In both dimensions, our proof of Theorem 1.1 follows the strategy in [NN04, Thm.1]. We first show the analyticity of the harmonic extension, that is, for fixed $\xi(\hat{\theta})$ the solution u_ε in (3) is analytic in ε. Using this, we then prove that the DNO, G_ε, is also analytically dependent on ε, establishing Theorem 1.1.

Using an analyticity result in [Kat76], we obtain the analytic dependence of the Steklov eigenvalues $\{\sigma_j(\varepsilon)\}_{j\in\mathbb{N}}$ on ε within the same disc of convergence as in Theorem 1.1, as stated in the following corollary. We recall that an analytic function $f\colon \mathbb{C}\setminus\{z_0\} \to \mathbb{C}$ has an *algebraic singularity* at $z = z_0$ if there exists integers $n_0 \geq 1$ and $p \geq 1$ such that the function admits an expansion of the form $f(z) = \displaystyle\sum_{n=-n_0}^{\infty} c_n (z - z_0)^{\frac{n}{p}}$ near $z = z_0$ with $c_{n_0} \neq 0$ (see, *e.g.*, [Kno47, p.131]).

Corollary 1.2. *The Steklov eigenvalues, σ_ε, consist of branches of one or several analytic functions which have at most algebraic singularities near $\varepsilon = 0$. The same is true of the corresponding eigenprojections.*

A proof of Corollary 1.2 is given in Section 4.

Corollary 1.2 justifies Assumption 1.1 in [VO18]. Here, the first two terms of the asymptotic series for σ_ε are computed for reflection-symmetric nearly-circular domains. Corollary 1.2 also justifies the computation of the shape derivative that appears in [AKO17]. Here, numerical methods are developed for the eigenvalue optimization problem of maximizing the k-th Steklov eigenvalue as a function of the domain with an area constraint.

Finally, we remark that a similar boundary perturbation expansion as (1) has been used to develop an accurate and stable numerical method to solve the Helmholtz equation on two- and three-dimensional domains exterior to a bounded obstacle [NS06; FNS07].

2. Two-dimensional nearly-circular domains

Here we consider the Steklov eigenproblem (2) in $\mathbb{R}^2$. We will identify $\hat{\theta}$ with its corresponding angle θ made with the positive x-axis, as usual. We write the Fourier series for $f\colon S^1 \to \mathbb{C}$ as

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ik\theta}, \qquad \text{where } \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta)e^{-ik\theta}\, d\theta.$$

Denoting $\langle k\rangle = \sqrt{1+k^2}$, we introduce the spaces $L^2(S^1)$ and $H^1(S^1)$ with norms

$$\|f\|^2_{L^2(S^1)} = \int_0^{2\pi} |f|^2\, d\theta = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$$

$$\|f\|^2_{H^1(S^1)} = \int_0^{2\pi} |f(\theta)|^2 + |f'(\theta)|^2\, d\theta = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 + k^2|\hat{f}(k)|^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^2 |\hat{f}(k)|^2.$$

Similarly, for $s \in \mathbb{R}$, we define the space $H^s(S^1)$ with norm $\|f\|^2_{H^s(S^1)} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\hat{f}(k)|^2$.

2.1. **Analyticity of the harmonic extension for nearly-circular domains.** We first consider the problem of harmonically extending a function $\xi(\theta)$ from $\partial\Omega_\varepsilon$ to Ω_ε,

$$\left[r^{-1}\partial_r r\partial_r + r^{-2}\partial_\theta^2\right] v = 0 \tag{4a}$$

$$v(1+\varepsilon\rho(\theta),\theta) = \xi(\theta). \tag{4b}$$

Mapping Ω_ε to the unit disk, $D=\Omega_0$, we make the change of variables

$$(r',\theta') = \left((1+\varepsilon\rho(\theta))^{-1}r,\theta\right). \tag{5}$$

The partial derivatives in the new coordinates are given by

$$\frac{\partial}{\partial r} = \frac{1}{1+\varepsilon\rho(\theta')}\frac{\partial}{\partial r'} \qquad \text{and} \qquad \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta'} - \frac{\varepsilon r'\rho'(\theta')}{1+\varepsilon\rho(\theta')}\frac{\partial}{\partial r'}. \tag{6}$$

Applying this change of coordinates to the Laplace equation (4) and setting

$$u_\varepsilon(r',\theta') = v((1+\varepsilon\rho(\theta'))r',\theta'),$$

we obtain the problem

$$\frac{1}{r'\left(1+\varepsilon\rho(\theta')\right)^2}\frac{\partial}{\partial r'}r'\frac{\partial u_\varepsilon}{\partial r'} + \frac{1}{(r')^2\left(1+\varepsilon\rho(\theta')\right)^2}\left(\frac{\partial}{\partial\theta'} - \frac{\varepsilon r'\rho'(\theta')}{1+\varepsilon\rho(\theta')}\frac{\partial}{\partial r'}\right)^2 u_\varepsilon = 0.$$

Multiplying both sides by $\left(1+\varepsilon\rho(\theta')\right)^2$ and dropping the primes on the transformed variables yields

$$r^{-1}\partial_r r\partial_r u_\varepsilon + r^{-2}\left(\partial_\theta - \varepsilon r(1+\varepsilon\rho(\theta))^{-1}\rho'(\theta)\partial_r\right)^2 u_\varepsilon = 0.$$

Expanding the operator in the second term on the left hand side, we obtain

$$\begin{aligned}\left(r^{-1}\partial_r r\partial_r + r^{-2}\partial_\theta^2\right)u_\varepsilon - \varepsilon r^{-1}\left(1+\varepsilon\rho(\theta)\right)^{-1}\left(2\rho'(\theta)\partial_\theta + \rho''(\theta)\right)\partial_r u_\varepsilon& \\ + \varepsilon^2 r^{-1}\left(1+\varepsilon\rho(\theta)\right)^{-2}\left(\rho'(\theta)\right)^2\partial_r\left(2+r\partial_r\right)u_\varepsilon &= 0.\end{aligned}$$

Again multiplying both sides by $\left(1+\varepsilon\rho(\theta)\right)^2$, we obtain the transformed Laplace equation,

$$\Delta u_\varepsilon = \varepsilon L_1 u_\varepsilon + \varepsilon^2 L_2 u_\varepsilon \tag{7a}$$

$$u_\varepsilon(1,\theta) = \xi(\theta), \tag{7b}$$

where

$$\begin{aligned}\Delta &= \left(r^{-1}\partial_r r\partial_r + r^{-2}\partial_\theta^2\right)\\ L_1 &= 2\rho'(\theta)r^{-1}\partial_\theta\partial_r + \rho''(\theta)r^{-1}\partial_r - 2\rho(\theta)\left[\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2\right]\\ L_2 &= 2\rho(\theta)\rho'(\theta)r^{-1}\partial_\theta\partial_r + \rho(\theta)\rho''(\theta)r^{-1}\partial_r - (\rho'(\theta))^2\partial_r^2\\ &\quad - 2(\rho'(\theta))^2 r^{-1}\partial_r - \rho^2(\theta)\left[\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2\right].\end{aligned}$$

We formally expand the solution, u_ε, in powers of ε,

$$u_\varepsilon(r,\theta) = \sum_{n=0}^{\infty}\varepsilon^n u_n(r,\theta). \tag{8}$$

Next, we collect terms in powers of ε. At $O(\varepsilon^0)$, we obtain

$$\begin{aligned}\Delta u_0(r,\theta) &= 0\\ u_0(1,\theta) &= \xi(\theta).\end{aligned}$$

At $O(\varepsilon^n)$ for $n>0$, we obtain

$$\begin{aligned}\Delta u_n(r,\theta) &= L_1 u_{n-1} + L_2 u_{n-2}\\ u_n(1,\theta) &= 0.\end{aligned}$$

We next show that there exists a unique solution of (7) of the form in (8). The following Lemma is analogous to [NN04, Lemma 4].

Lemma 2.1. *[Elliptic Estimate.] For $s \in \mathbb{N}$, there exists a constant $K_0 > 0$ such that for any $F \in H^{s-2}(D)$ and $\xi \in H^{s-\frac{1}{2}}(S^1)$, the solution of*

$$\begin{aligned} \Delta w(r,\theta) &= F(r,\theta) && (r,\theta) \in D \\ w(1,\theta) &= \xi(\theta) && \theta \in S^1 \end{aligned}$$

satisfies

$$\|w\|_{H^s(D)} \le K_0 \left(\|F\|_{H^{s-2}(D)} + \|\xi\|_{H^{s-\frac{1}{2}}(S^1)} \right).$$

Proof. We will prove the result for $s = 1$. Since $\xi \in H^{\frac{1}{2}}(S^1)$, we have the Fourier series

$$\xi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{\xi}(k) e^{ik\theta}, \qquad \text{where } \hat{\xi}(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \xi(\theta) e^{-ik\theta}\, d\theta.$$

Setting $v = w - \Phi$, where $\Phi(r,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{\xi}(k) r^{|k|} e^{ik\theta}$, we have that

$$\begin{aligned} \Delta \Phi(r,\theta) &= 0 && (r,\theta) \in D \\ \Phi(1,\theta) &= \xi(\theta) && \theta \in \mathbb{S}^1 \end{aligned}$$

and

$$\Delta v(r,\theta) = F(r,\theta) \qquad (r,\theta) \in D \tag{9a}$$

$$v(1,\theta) = 0 \qquad \theta \in \mathbb{S}^1. \tag{9b}$$

Using $\hat{\Phi}(r,k) = \hat{\xi}(k) r^{|k|}$, a straightforward integration yields

$$\|\Phi\|^2_{H^1(D)} = \sum_{k\in\mathbb{Z}} \int_0^1 \left[\left(1 + r^{-2}|k|^2\right) |\hat{\Phi}(r,k)|^2 + |\partial_r \hat{\Phi}(r,k)|^2 \right] \, rdr \tag{10a}$$

$$= \sum_{k\in\mathbb{Z}} |\hat{\xi}(k)|^2 \int_0^1 \left[\left(1 + 2r^{-2}|k|^2\right) r^{2|k|} \right] \, rdr \tag{10b}$$

$$\le C^2_{\frac{1}{2}} \sum_{k\in\mathbb{Z}} \langle k \rangle |\hat{\xi}(k)|^2 \tag{10c}$$

$$= C^2_{\frac{1}{2}} \|\xi\|^2_{H^{\frac{1}{2}}(S^1)}, \tag{10d}$$

for some constant $C_{\frac{1}{2}} > 0$.

Multiplying (9a) by $\overline{v}$, integrating by parts, and using (9b) yields

$$\|v\|^2_{H^1_0(D)} = \int_D \nabla v \cdot \nabla \overline{v} = -\int_D F \overline{v}.$$

By the duality of $H^1_0(D)$ and $H^{-1}(D)$, we have $\int_D |F\overline{v}| \le \|F\|_{H^{-1}(D)} \|v\|_{H^1_0(D)}$. Since $v \in H^1_0(D)$, by the Poincaré inequality, there exists a constant C_D such that $\|v\|_{H^1(D)} \le C_D \|v\|_{H^1_0(D)}$ and we conclude that

$$\|v\|_{H^1(D)} \le C_D \|v\|_{H^1_0(D)} \le C_D \|F\|_{H^{-1}(D)}. \tag{11}$$

Using the decomposition $w = v + \Phi$ and using (10) and (11), we obtain

$$\|w\|_{H^1(D)} \le \|v\|_{H^1(D)} + \|\Phi\|_{H^1(D)} \le C_D \|F\|_{H^{-1}(D)} + C_{\frac{1}{2}} \|\xi\|_{H^{\frac{1}{2}}(S^1)}.$$

Taking $K_0 = \max\{C_{\frac{1}{2}}, C_D\}$ yields the desired result for $s = 1$. The proof for $s \ge 2$ is similar. □

The next Lemma will be used to prove the inductive step in the proof of Theorem 2.3 and is analogous to [NN04, Lemma 5]. In the proof, we use the following result [NR01; NN04]. Let B^d and S^{d-1} denote the unit-ball and unit-sphere in $\mathbb{R}^d$. For $\delta \geq 0$, $s \in \mathbb{N}$, $f \in C^s(S^{d-1})$, $u \in H^s(B^d)$, $g \in C^{s+\frac{1}{2}+\delta}(S^{d-1})$, and $\mu \in H^{s+\frac{1}{2}}(S^{d-1})$, there exists a constant $M = M(s,d)$ so that

$$\|fu\|_{H^s(B^d)} \leq M(s,d)\ |f|_{C^s(S^{d-1})}\ \|u\|_{H^s(B^d)} \tag{12a}$$

$$\|g\mu\|_{H^{s+\frac{1}{2}}(S^{d-1})} \leq M(s,d)\ |g|_{C^{s+\frac{1}{2}+\delta}(S^{d-1})}\ \|\mu\|_{H^{s+\frac{1}{2}}(S^{d-1})}. \tag{12b}$$

Lemma 2.2. *[Recursive estimates.] Let $s \in \mathbb{N}$ and let $\rho \in C^{s+1}(S^1)$. Assume that K_1 and B are constants so that*

$$\|u_n\|_{H^{s+1}(D)} \leq K_1 B^n \qquad \text{for all} \quad n < N.$$

If $B > |\rho|_{C^{s+1}}$, then there exists a constant C_0 such that

$$\|L_1 u_{N-1}\|_{H^{s-1}(D)} \leq K_1 |\rho|_{C^{s+1}} C_0 B^{N-1}$$
$$\|L_2 u_{N-2}\|_{H^{s-1}(D)} \leq K_1 |\rho|_{C^{s+1}} C_0 B^{N-1}.$$

Proof. First, we measure $L_1 u_{N-1}$ in $H^{s-1}(D)$ and use the triangle inequality and (12) to obtain:

$$\begin{aligned}
\|L_1 u_{N-1}\|_{H^{s-1}} &\leq 2\|\rho'(\theta) r^{-1} \partial_\theta \partial_r u_{N-1}\|_{H^{s-1}} + \|\rho''(\theta) r^{-1} \partial_r u_{N-1}\|_{H^{s-1}} + 2\|\rho(\theta)\Delta u_{N-1}\|_{H^{s-1}} \\
&\leq 2M(s)|\rho|_{C^s}\|r^{-1}\partial_\theta\partial_r u_{N-1}\|_{H^{s-1}} + M(s)|\rho|_{C^{s+1}}\|r^{-1}\partial_r u_{N-1}\|_{H^{s-1}} \\
&\quad + 2M(s)|\rho|_{C^{s-1}}\|\Delta u_{N-1}\|_{H^{s-1}} \\
&\leq 2M(s)|\rho|_{C^s}\|u_{N-1}\|_{H^{s+1}} + M(s)|\rho|_{C^{s+1}}\|u_{N-1}\|_{H^{s+1}} + 2M(s)|\rho|_{C^{s-1}}\|u_{N-1}\|_{H^{s+1}} \\
&\leq K_1|\rho|_{C^{s+1}} C_0 B^{N-1}.
\end{aligned}$$

Here, in the third inequality, we have used that all operators acting on u_{N-1} are second order.

Similarly, we estimate $L_2 u_{N-2}$ in $H^{s-1}(D)$:

$$\begin{aligned}
\|L_2 u_{N-2}\|_{H^{s-1}} &\leq 2\|\rho(\theta)\rho'(\theta) r^{-1}\partial_\theta\partial_r u_{N-2}\|_{H^{s-1}} + \|\rho(\theta)\rho''(\theta) r^{-1}\partial_r u_{N-2}\|_{H^{s-1}} + \|(\rho'(\theta))^2\partial_r^2 u_{N-2}\|_{H^{s-1}} \\
&\quad + 2\|(\rho'(\theta))^2 r^{-1}\partial_r u_{N-2}\|_{H^{s-1}} + \|\rho^2(\theta)\Delta u_{N-2}\|_{H^{s-1}} \\
&\leq 2M(s)|\rho|_{C^s}|\rho|_{C^{s-1}}\|r^{-1}\partial_\theta\partial_r u_{N-2}\|_{H^{s-1}} + M(s)|\rho|_{C^{s-1}}|\rho|_{C^{s+1}}\|r^{-1}\partial_r u_{N-2}\|_{H^{s-1}} \\
&\quad + M(s)|\rho|^2_{C^s}\|\partial_r^2 u_{N-2}\|_{H^{s-1}} + 2M(s)|\rho|^2_{C^s}\|r^{-1}\partial_r u_{N-2}\|_{H^{s-1}} + M(s)|\rho|^2_{C^{s-1}}\|\Delta u_{N-2}\|_{H^{s-1}} \\
&\leq 2M(s)|\rho|_{C^s}|\rho|_{C^{s-1}}\|u_{N-2}\|_{H^{s+1}} + M(s)|\rho|_{C^{s-1}}|\rho|_{C^{s+1}}\|u_{N-2}\|_{H^{s+1}} \\
&\quad + M(s)|\rho|^2_{C^s}\|u_{N-2}\|_{H^{s+1}} + 2M(s)|\rho|^2_{C^s}\|u_{N-2}\|_{H^{s+1}} + M(s)|\rho|^2_{C^{s-1}}\|u_{N-2}\|_{H^{s+1}} \\
&\leq K_1|\rho|_{C^{s+1}} C_0 B^{N-1}.
\end{aligned}$$

□

The following Theorem justifies the convergence of (8) for sufficiently small $\varepsilon > 0$ and is analogous to [NN04, Theorem 3].

Theorem 2.3. *Given $s \in \mathbb{N}$, if $\rho \in C^{s+1}(S^1)$ and $\xi \in H^{s+\frac{1}{2}}(S^1)$, there exists constants C_0 and K_0 and a unique solution of (7) such that*

$$\|u_n\|_{H^{s+1}(D)} \leq K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^n \tag{13}$$

for any $B > 2K_0 C_0 |\rho|_{C^{s+1}}$.

Proof. We proceed by induction. For $n = 0$, we use Lemma 2.1 to we see

$$\|u_0\|_{H^{s+1}(D)} \leq K_0 \|\xi\|_{H^{s+\frac{1}{2}}(\mathbb{S}^1)} B^0,$$

as desired to show (13). We now define $K_1 = K_0\|\xi\|_{H^{s+\frac{1}{2}}(S^1)}$ for the remainder of the proof to be used in Lemma 2.2.

Suppose inequality (13) holds for $n < N$. Then by Lemma 2.1,

$$\|u_N\|_{H^{s+1}} \leq K_0 \left(\|L_1 u_{N-1}\|_{H^{s-1}} + \|L_2 u_{N-2}\|_{H^{s-1}}\right).$$

By Lemma 2.2, we may bound $\|L_1 u_{N-1}\|_{H^{s-1}}$ and $\|L_2 u_{N-2}\|_{H^{s-1}}$ so that

$$\begin{aligned}\|u_N\|_{H^{s+1}} &\leq 2K_0 K_1 C_0 |\rho|_{C^{s+1}} B^{N-1} \\ &= 2K_0^2 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} C_0 |\rho|_{C^{s+1}} B^{N-1} \\ &\leq K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^N\end{aligned}$$

provided $B > 2K_0 C_0 |\rho|_{C^{s+1}}$. □

2.2. **Proof of Theorem 1.1 in two dimensions: Analyticity of the Dirichlet to Neumann operator.** The Dirichlet to Neumann operator (DNO), $G_\varepsilon \colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s-\frac{1}{2}}(\partial\Omega_\varepsilon)$, is given by

$$G_\varepsilon \xi = \left[1 + 2\varepsilon\rho(\theta) + \varepsilon^2\left(\rho^2(\theta) + (\rho'(\theta))^2\right]^{-\frac{1}{2}} \left[(1+\varepsilon\rho(\theta))\frac{\partial v}{\partial r} - \frac{\varepsilon\rho'(\theta)}{1+\varepsilon\rho(\theta)}\frac{\partial v}{\partial\theta}\right],$$

where v is the harmonic extension of ξ from $\partial\Omega_\varepsilon$ to Ω_ε, satisfying (4). Making the change of coordinates given in (5), we obtain the transformed DNO, $\tilde{G}_\varepsilon \colon H^{s+\frac{1}{2}}(S^1) \to H^{s-\frac{1}{2}}(S^1)$, given by

$$\begin{aligned}\tilde{G}_\varepsilon \xi &= \left[1 + 2\varepsilon\rho(\theta') + \varepsilon^2\left(\rho^2(\theta') + (\rho'(\theta'))^2\right)\right]^{-\frac{1}{2}} \left[\left(1 + \frac{\varepsilon^2(\rho'(\theta'))^2}{(1+\varepsilon\rho(\theta'))^2} r'\right)\frac{\partial u_\varepsilon}{\partial r'} - \frac{\varepsilon\rho'(\theta')}{1+\varepsilon\rho(\theta')}\frac{\partial u_\varepsilon}{\partial\theta'}\right] \\ &= M_\rho(\varepsilon)\hat{G}_{\rho,\varepsilon}\xi,\end{aligned}$$

where u_ε satisfies (7) and

$$\begin{aligned}M_\rho(\varepsilon) &= \left[1 + 2\varepsilon\rho(\theta') + \varepsilon^2\left(\rho^2(\theta') + (\rho'(\theta'))^2\right]^{-\frac{1}{2}}, \\ \hat{G}_{\rho,\varepsilon}\xi &= \left[\left(1 + \frac{\varepsilon^2(\rho'(\theta'))^2}{(1+\varepsilon\rho(\theta'))^2} r'\right)\frac{\partial u_\varepsilon}{\partial r'} - \frac{\varepsilon\rho'(\theta')}{1+\varepsilon\rho(\theta')}\frac{\partial u_\varepsilon}{\partial\theta'}\right].\end{aligned}$$

We note that the change of coordinates between G_ε and $\tilde{G}_\varepsilon$ is conjugation by an isomorphism, as follows. Let $T_{\varepsilon,\rho} H^s(\Omega_\varepsilon) \to H^s(B^2)$ denote the change of variables in (5), *i.e.*,

$$u_\varepsilon(r',\theta') = [T_{\varepsilon,\rho}u](r',\theta') = u\left((1+\varepsilon\rho(\theta')r',\theta')\right).$$

By the change of variables formula, $T_{\varepsilon,\rho} \colon H^s(\Omega_\varepsilon) \to H^s(B^2)$ is an isometry. For $s \in \mathbb{N}$, we recall the trace operators,

$$\begin{aligned}\gamma_0 &\colon H^s(B^2) \to H^{s-\frac{1}{2}}(S^1) \\ \gamma_{\varepsilon,\rho} &\colon H^s(\Omega_\varepsilon) \to H^{s-\frac{1}{2}}(\partial\Omega_\varepsilon),\end{aligned}$$

These are bounded, surjective operators with bounded right inverses. Define the map $L \colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s+\frac{1}{2}}(S^{d-1})$ by

$$L = \gamma_0 T_{\varepsilon,\rho}\gamma_{\varepsilon,\rho}^{-1}.$$

Lemma 2.4. *$L \colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s+\frac{1}{2}}(S^1)$ is a Hilbert space isomorphism and $LG_\varepsilon = \tilde{G}_\varepsilon L$.*

Proof. We first show that L is injective. Let $\phi \in H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon)$ with $L\phi = 0$. Let $u = \gamma_{\varepsilon,\rho}^{-1}\phi$. Then $\tilde{u} = T_{\varepsilon,\rho}u \in H_0^{s+1}(B)$. But

$$0 = \tilde{u}(1,\theta) = u\left((1+\varepsilon\rho(\theta)),\theta\right) = \phi$$

and so L is injective.

We next show that L is surjective. Let $\phi \in H^{s+\frac{1}{2}}(S^1)$ and set

$$\psi = \gamma_{\varepsilon,\rho}T_{\varepsilon,\rho}^{-1}\gamma_0^{-1}\phi \in H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon).$$

Let $\tilde{u} = \gamma_0^{-1}\phi$, so that $\tilde{u}(1,\theta) = \phi$. Then $u(r,\theta) := T_{\varepsilon,\rho}^{-1}\tilde{u} = \tilde{u}(\frac{r}{1+\varepsilon\rho(\theta)},\theta)$, and

$$\psi(\theta) = u(1+\varepsilon\rho(\theta),\theta) = \tilde{u}(1,\theta).$$

Set $w = \gamma_{\varepsilon,\rho}^{-1}\psi$ so that $w(1+\varepsilon\rho(\theta),\theta) = \tilde{u}(1,\theta)$. If we define $\tilde{w}(r',\theta) := T_{\varepsilon,\rho}w = w((1+\varepsilon\rho(\theta))r',\theta)$, then $L\psi = \gamma_0\tilde{w} = w(1+\varepsilon\rho(\theta),\theta) = \tilde{u}(1,\theta) = \phi$. Thus L is surjective.

$L = \gamma_0 T_{\varepsilon,\rho}\gamma_{\varepsilon,\rho}^{-1}$ is bounded since $T_{\varepsilon,\rho}$ is an isometry and $\gamma_0, \gamma_{\varepsilon,\rho}$ are both bounded with bounded right inverse. By the Bounded Inverse Theorem, since L is a bijective bounded linear operator between Hilbert spaces, its inverse is bounded and hence L is an isomorphism.

It remains to verify that

$$LG_\varepsilon\phi = \tilde{G}_\varepsilon L\phi \qquad \forall\phi \in H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon).$$

Let $\phi \in H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon)$. Then $G_\varepsilon\phi = \partial_n u$ where u is the harmonic extension of ϕ into Ω_ε. Let $v = \gamma_{\varepsilon,\rho}^{-1}\partial_n u$, *i.e.*, the harmonic function in Ω_ε with *Dirichlet* data $v|_{\partial\Omega_\varepsilon} = \partial_n u$. Then the above calculations show that

$$\begin{aligned} LG_\varepsilon\phi &= \gamma_0 T_{\varepsilon,\rho}v = \partial_n u|_{\partial\Omega_\varepsilon} \\ &= M_\rho(\varepsilon)\left[(1+\varepsilon\rho(\theta))\frac{\partial u}{\partial r} - \frac{\varepsilon\rho'(\theta)}{1+\varepsilon\rho(\theta)}\frac{\partial u}{\partial\theta}\right]. \end{aligned}$$

A similar calculation shows that

$$\tilde{G}_\varepsilon L\phi = M_\rho(\varepsilon)\left[\left(1+\frac{\varepsilon^2(\rho'(\theta'))^2}{(1+\varepsilon\rho(\theta'))^2}\right)\frac{\partial\tilde{u}_\varepsilon}{\partial r'} - \frac{\varepsilon\rho'(\theta')}{1+\varepsilon\rho(\theta')}\frac{\partial\tilde{u}_\varepsilon}{\partial\theta'}\right]$$

where $\tilde{u}_\varepsilon(r',\theta') = u\left((1+\varepsilon\rho(\theta'))r',\theta'\right)$. The identities in (6) can be used to verify that these two expressions are equal and the proof is complete. ☐

We have now established that

$$LG_\varepsilon L^{-1} = \tilde{G}_\varepsilon = M_\rho(\varepsilon)\hat{G}_{\rho,\varepsilon}.$$

Since $M_\rho(\varepsilon)$ is clearly analytic in ε, we need only show the analyticity of $\hat{G}_{\rho,\varepsilon}$. Dropping the prime notation on the new variables, we obtain

$$(1+\varepsilon\rho(\theta))^2\,\hat{G}_{\rho,\varepsilon}\xi = \left[\left((1+\varepsilon\rho(\theta))^2 + \varepsilon^2(\rho'(\theta))^2\right)\partial_r u - \varepsilon\left(1+\varepsilon\rho(\theta)\right)\rho'(\theta)\partial_\theta u\right].$$

We expand the non-normalized DNO, $\hat{G}_{\rho,\varepsilon}$, as a power series in ε

$$\hat{G}_{\rho,\varepsilon}\xi = \sum_{n=0}^{\infty}\varepsilon^n\hat{G}_{\rho,n}\xi, \tag{14}$$

which yields the following recursive formula:

$$\hat{G}_{\rho,n}\xi = \partial_r u_n + 2\rho\partial_r u_{n-1} + \left((\rho')^2+\rho^2\right)\partial_r u_{n-2} - \rho'\partial_\theta u_{n-1} - \rho\rho'\partial_\theta u_{n-2} - 2\rho\hat{G}_{\rho,n-1}\xi - \rho^2\hat{G}_{\rho,n-2}\xi.$$

We now prove the following theorem, which proves Theorem 1.1 and guarantees the uniform convergence of the series (14) for suitably small ε.

Theorem 2.5. *Let $\xi \in H^{s+\frac{1}{2}}(S^1)$ for $s \in \mathbb{N}$. Then*

$$\|\hat{G}_{\rho,n}\xi\|_{H^{s-\frac{1}{2}}(S^1)} \le K_1 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^n \tag{15}$$

for $B > C|\rho|_{C^{s+1}}$.

Proof. We will proceed via induction. First, we show (15) fo $n = 0$:

$$\begin{aligned}\|\hat{G}_{\rho,0}\xi\|_{H^{s-\frac{1}{2}}(S^1)} &\le \|\partial_r u_0\|_{H^{s-\frac{1}{2}}(S^1)} \le C_1 \|\partial_r u_0\|_{H^s(D)} \\ &\le C_1 \|u_0\|_{H^{s+1}(D)} \le C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)}.\end{aligned}$$

In the second inequality of the first line, we have used the trace theorem, while Theorem 2.3 is used in the second line. Now suppose that (15) holds for $n < N$. Then we have the following estimate:

$$\begin{aligned}\|\hat{G}_{\rho,N}\xi\|_{H^{s-\frac{1}{2}}(S^1)} &\le \|\partial_r u_N\|_{H^{s-\frac{1}{2}}(S^1)} + 2\|\rho\partial_r u_{N-1}\|_{H^{s-\frac{1}{2}}(S^1)} + \|(\rho')^2\partial_r u_{N-2}\|_{H^{s-\frac{1}{2}}(S^1)} \\ &\quad + \|\rho^2\partial_r u_{N-2}\|_{H^{s-\frac{1}{2}}(S^1)} + \|\rho'\partial_\theta u_{N-1}\|_{H^{s-\frac{1}{2}}(S^1)} + \|\rho\rho'\partial_\theta u_{N-2}\|_{H^{s-\frac{1}{2}}(S^1)} \\ &\quad + 2\|\rho\hat{G}_{\rho,N-1}\xi\|_{H^{s-\frac{1}{2}}(S^1)} + \|\rho^2\hat{G}_{\rho,N-2}\xi\|_{H^{s-\frac{1}{2}}(S^1)} \\ &\le C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^N + 2|\rho|_{C^{s-\frac{1}{2}+\delta}} C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^{N-1} + \\ &\quad + |\rho|^2_{C^{s+\frac{1}{2}+\delta}} C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^{N-2} \\ &\quad + |\rho|^2_{C^{s-\frac{1}{2}+\delta}} C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^{N-2} + |\rho|_{C^{s+\frac{1}{2}+\delta}} C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^{N-1} \\ &\quad + |\rho|_{C^{s-\frac{1}{2}+\delta}} |\rho|_{C^{s+\frac{1}{2}+\delta}} C_1 K_0 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^{N-2} \\ &\quad + 2|\rho|_{C^{s-\frac{1}{2}+\delta}} \|\hat{G}_{\rho,N-1}\xi\|_{H^{s-\frac{1}{2}}(S^1)} + |\rho|^2_{C^{s-\frac{1}{2}+\delta}} \|\hat{G}_{\rho,N-2}\xi\|_{H^{s-\frac{1}{2}}(S^1)} \\ &\le K_1 \|\xi\|_{H^{s+\frac{1}{2}}(S^1)} B^N,\end{aligned}$$

for $B > C|\rho|_{C^{s+1}}$, where $C = C(s)$ is independent of u, N, ξ, and ρ. Here we have used the second inequality in (12), as well as the trace theorem, Theorem 2.3, and the inductive hypothesis on $\hat{G}_{\rho,N-1}$ and $\hat{G}_{\rho,N-2}$. □

3. Three-dimensional nearly-spherical domains

Here we consider (2) in dimension $d = 3$. We identify $\hat{\theta} \in S^2$ with the inclination, $\theta \in [0, \pi]$, and azimuth, $\phi \in [0, 2\pi]$. Let Ω_ε be a nearly-spherical domain where the perturbation function, $\rho(\theta, \phi)$, is expanded in the basis of real spherical harmonics,

$$\Omega_\varepsilon = \{(r, \theta, \phi) \colon 0 \le r \le 1 + \varepsilon\rho(\theta, \phi)\}, \qquad \text{where} \;\; \rho(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell,m} Y_{\ell,m}(\theta, \phi). \tag{16}$$

Here, $Y_{\ell,m}$ denote the real spherical harmonics, which are obtained from the complex spherical harmonics as follows [NIS]. Define the *complex spherical harmonic* by

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos(\theta)) e^{im\phi}, \qquad \ell \ge 0, \;\; |m| \le \ell, \tag{17}$$

where P_ℓ^m is the *associated Legendre polynomial*, which can be defined through the Rodrigues formula, $P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1 - x^2)^{\frac{m}{2}} \frac{d^{m+\ell}}{dx^{m+\ell}}(x^2 - 1)^\ell$. For $\ell \ge 0$ and $|m| \le \ell$, the *real spherical* *harmonics* are then defined by

$$Y_{\ell,m}(\theta,\phi) = \begin{cases} \frac{i}{\sqrt{2}}\left[Y_\ell^m(\theta,\phi) - (-1)^m Y_\ell^{-m}(\theta,\phi)\right] & \text{if } m<0 \\ Y_\ell^0(\theta,\phi) & \text{if } m=0 \\ \frac{1}{\sqrt{2}}\left[Y_\ell^{-m}(\theta,\phi) + (-1)^m Y_\ell^{m}(\theta,\phi)\right] & \text{if } m>0 \end{cases}. \tag{18}$$

We define the Sobolev space on the sphere, $H^s(S^2)$ for $s \in \mathbb{R}$, with squared norm

$$\|f\|^2_{H^s(S^2)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \langle \ell \rangle^{2s} |\hat{f}(\ell,m)|^2, \qquad \text{where} \quad \hat{f}(\ell,m) = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi) Y_\ell^m(\theta,\phi)\ \sin\theta d\theta d\phi;$$

properties of this norm can be found in [BD13].

3.1. **Analyticity of the harmonic extension for nearly-spherical domains.** As in Section 2.1, we first consider the problem of harmonically extending a function $\xi(\theta,\phi)$ from $\partial\Omega_\varepsilon$ to Ω_ε,

$$\Delta v = r^{-2}\partial_r\left(r^2\partial_r v\right) + r^{-2}\sin^{-1}(\theta)\partial_\theta\left(\sin(\theta)\partial_\theta v\right) + r^{-2}\sin^{-2}(\theta)\partial_\phi^2 v = 0 \tag{19a}$$

$$v(1+\varepsilon\rho(\theta,\phi),\theta,\phi) = \xi(\theta,\phi). \tag{19b}$$

Mapping Ω_ε to the unit ball, $B = \Omega_0$, we make the change of variables

$$(r',\theta',\phi') = \left((1+\varepsilon\rho(\theta,\phi))^{-1}r, \theta, \phi\right). \tag{20}$$

The partial derivatives in the new coordinates are given by

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{1}{1+\varepsilon\rho(\theta',\phi')}\frac{\partial}{\partial r'} \\ \frac{\partial}{\partial \theta} &= \frac{\partial}{\partial \theta'} - \frac{\varepsilon r'\rho_\theta(\theta',\phi')}{1+\varepsilon\rho(\theta',\phi')}\frac{\partial}{\partial r'} \\ \frac{\partial}{\partial \phi} &= \frac{\partial}{\partial \phi'} - \frac{\varepsilon r'\rho_\phi(\theta',\phi')}{1+\varepsilon\rho(\theta',\phi')}\frac{\partial}{\partial r'}. \end{aligned}$$

Applying this change of coordinates to the Laplace equation (19a), setting

$$u_\varepsilon(r',\theta',\phi') = v((1+\varepsilon\rho(\theta'))r',\theta',\phi'),$$

multiplying by $(1+\varepsilon\rho)^4$, and dropping the primes on the transformed variables yields

$$\begin{aligned} (1+\varepsilon\rho)^4\Delta v &= (1+\varepsilon\rho)^2\Delta u_\varepsilon - \varepsilon(1+\varepsilon\rho)\Big[\sin^{-1}(\theta)\partial_\theta\Big(\sin(\theta)\partial_\theta\rho\Big) + \sin^{-2}(\theta)\partial_\phi^2\rho\Big](r^{-1}\partial_r u_\varepsilon) \\ &\quad - 2\varepsilon(1+\varepsilon\rho)\Big[\rho_\theta r^{-1}\partial_\theta\partial_r u_\varepsilon + \Big(\sin^{-1}(\theta)\partial_\phi\rho\Big)\Big(r^{-1}\sin^{-1}(\theta)\partial_\phi\Big)\partial_r u_\varepsilon\Big] \\ &\quad + 2\varepsilon^2\Big[(\rho_\theta^2 + \sin^{-2}(\theta)\rho_\phi^2\Big](r^{-1}\partial_r u_\varepsilon) + \varepsilon^2\Big(\sin^{-2}(\theta)\rho_\phi^2\Big)\partial_r^2 u_\varepsilon \\ &= 0. \end{aligned}$$

Define the operators:

$$\Delta_S u = \sin^{-1}(\theta)\partial_\theta\left(\sin(\theta)\partial_\theta u\right) + \sin^{-2}(\theta)\partial_\phi^2 u$$

$$L_1 u = 2\rho\Delta u + (\Delta_S\rho)r^{-1}\partial_r u - 2\left(r^{-1}\rho_\theta\partial_\theta\partial_r u + \frac{\rho_\phi}{\sin\theta}\frac{\partial_\phi\partial_r u}{r\sin\theta}\right)$$

$$L_2 u = \rho^2\Delta u + (\rho\Delta_S\rho)r^{-1}\partial_r u - 2\rho\left(r^{-1}\rho_\theta\partial_\theta\partial_r u + \frac{\rho_\phi}{\sin\theta}\frac{\partial_\phi\partial_r u}{r\sin\theta}\right)$$
$$+ 2\left((\rho_\theta^2 + \frac{\rho_\phi^2}{\sin^2\theta})(r^{-1}\partial_r u) + \frac{\rho_\phi^2}{\sin^2\theta}\partial_r^2 u\right)$$

The function u_ε satisfies

(21a) $$\Delta u_\varepsilon = \varepsilon L_1 u_\varepsilon + \varepsilon^2 L_2 u_\varepsilon \qquad \text{in } B$$

(21b) $$u_\varepsilon(1,\theta,\phi) = \xi(\theta,\phi) \qquad \text{on } S^2.$$

Lemma 3.1. *[Elliptic Estimate.] For $s \in \mathbb{N}$, there exists a constant $K_0 > 0$ such that for any $F \in H^{s-2}(B)$ and $\xi \in H^{s-\frac{1}{2}}(S^2)$, the solution of*

$$\Delta w(r,\theta,\phi) = F(r,\theta,\phi) \qquad (r,\theta,\phi) \in B$$
$$w(1,\theta,\phi) = \xi(\theta,\phi) \qquad (\theta,\phi) \in S^2$$

satisfies

$$\|w\|_{H^s(B)} \le K_0\left(\|F\|_{H^{s-2}(B)} + \|\xi\|_{H^{s-\frac{1}{2}}(S^2)}\right).$$

Proof. We will prove the result for $s = 1$. Since $\xi \in H^{\frac{1}{2}}(S^2)$, we have the spherical harmonic transform,

$$\xi(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\hat{\xi}(\ell,m)Y_\ell^m(\theta,\phi), \qquad \text{where } \hat{\xi}(\ell,m) = \int_0^{2\pi}\int_0^{\pi}\xi(\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\sin\theta d\theta d\phi.$$

and Y_ℓ^m are the (complex) spherical harmonics. Set $v = w - \Phi$, where Φ solves

$$\Delta\Phi = 0 \qquad \text{in } B$$
$$\Phi = \xi \qquad \text{on } S^2.$$

Then v satisfies

(22a) $$\Delta v = F \qquad \text{in } B$$

(22b) $$v = 0 \qquad \text{on } S^2.$$

We have

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} r^\ell\hat{\xi}(\ell,m)Y_\ell^m(\theta,\phi),$$

and defining

$$\hat{\Phi}(r,\ell,m) = r^\ell\int_0^{2\pi}\int_0^{\pi}\Phi(1,\theta,\phi)\overline{Y_\ell^m(\theta,\phi)}\ \sin\theta d\theta d\phi$$

we see that

$$\hat{\Phi}(r,\ell,m) = r^\ell\hat{\xi}(\ell,m).$$

We thus calculate:

$$\|\Phi\|^2_{H^1(B)} = \sum_{\ell,m} \Big[\iiint_B \left(r^{2\ell} |\hat{\xi}(\ell,m)|^2 |Y_\ell^m(\theta,\phi)|^2 + |\nabla r^\ell \hat{\xi}(\ell,m) Y_\ell^m(\theta,\phi)|^2 \right) \; r^2 \sin\theta d\theta d\phi dr \tag{23a}$$

$$= \sum_{\ell,m} \iiint_B |\hat{\xi}(\ell,m)|^2 |Y_\ell^m(\theta,\phi)|^2 \left(r^{2\ell}(1+\ell(\ell+1)) + \ell^2 r^{2\ell-2} \right) \; r^2 \sin\theta d\theta d\phi dr \tag{23b}$$

$$= \sum_{\ell,m} |\hat{\xi}(\ell,m)|^2 \left(\frac{\ell^2+\ell+1}{2\ell+3} + \frac{\ell^2}{2\ell+1} \right) \tag{23c}$$

$$\leq C_{\frac{1}{2}} \|\xi\|^2_{H^{\frac{1}{2}}(S^2)} \tag{23d}$$

for some constant $C_{\frac{1}{2}} > 0$.

Multiplying (22) by $\overline{v}$ and integrating by parts yields

$$\|v\|^2_{H^1_0(B)} = \int_B \nabla v \cdot \nabla \overline{v} = -\int_B F\overline{v}.$$

By the duality of $H^1_0(B)$ and $H^{-1}(B)$, we have $\int_B |F\overline{v}| \leq \|F\|_{H^{-1}(B)} \|v\|_{H^1_0(B)}$. Since $v \in H^1_0(B)$, by the Poincaré inequality, there exists a constant C_B such that $\|v\|_{H^1(B)} \leq C_B \|v\|_{H^1_0(B)}$ and we conclude that

$$\|v\|_{H^1(B)} \leq C_B \|v\|_{H^1_0(B)} \leq C_B \|F\|_{H^{-1}(B)}. \tag{24}$$

Using the decomposition $w = v + \Phi$ and using (23) and (24), we obtain

$$\|w\|_{H^1(B)} \leq \|v\|_{H^1(B)} + \|\Phi\|_{H^1(B)} \leq C_B \|F\|_{H^{-1}(B)} + C_{\frac{1}{2}} \|\xi\|_{H^{\frac{1}{2}}(S^2)}.$$

Taking $K_0 = \max\{C_{\frac{1}{2}}, C_B\}$ yields the desired result for $s = 1$. The proof for $s \geq 2$ is similar. ☐

Let us make the ansatz

$$u_\varepsilon(r,\theta,\phi) = \sum_{n=0}^{\infty} \varepsilon^n u_n(r,\theta,\phi). \tag{25}$$

Then, by (21), we have the recursive formula

$$\Delta u_n = L_1 u_{n-1} + L_2 u_{n-2} \qquad \text{in } B \tag{26a}$$

$$u_n = \begin{cases} \xi & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \qquad \text{on } S^2. \tag{26b}$$

Lemma 3.2. *[Recursive Estimates.] Let $s \in \mathbb{N}$ and let $\rho \in C^{s+1}(S^2)$. Assume that K_1 and A are constants so that*

$$\|u_n\|_{H^{s+1}(B)} \leq K_1 A^n \qquad \textit{for all} \quad n < N.$$

If $A > |\rho|_{C^{s+1}}$, then there exists a constant C_0 such that

$$\|L_1 u_{N-1}\|_{H^{s-1}(B)} \leq K_1 |\rho|_{C^{s+1}} C_0 A^{N-1}$$
$$\|L_2 u_{N-2}\|_{H^{s-1}(B)} \leq K_1 |\rho|_{C^{s+1}} C_0 A^{N-1}.$$

Proof. Using the triangle inequality and (12), we calculate:

$$\begin{aligned}\|L_1u_{N-1}\|_{H^{s-1}(B)} &\le \|2\rho\Delta u_{N-1}\|_{H^{s-1}(B)} + \|(\Delta_{S^2}\rho)r^{-1}\partial_r u_{N-1}\|_{H^{s-1}(B)} + 2\|r^{-1}\rho_\theta\partial_\theta\partial_r u_{N-1}\|_{H^{s-1}(B)}\\ &\quad + 2\left\|\frac{\rho_\phi}{\sin\theta}\cdot\frac{\partial_\phi\partial_r u_{N-1}}{r\sin\theta}\right\|_{H^{s-1}(B)}\\ &\le M(s)\Big(2|\rho|_{C^{s-1}}\|u_{N-1}\|_{H^{s+1}(B)} + |\rho|_{C^{s+1}}\|u_{N-1}\|_{H^{s+1}(B)} + 4|\rho|_{C^s}\|u_{N-1}\|_{H^{s+1}(B)}\Big)\\ &\le 7M(s)|\rho|_{C^{s+1}}K_1A^{N-1}.\end{aligned}$$

In the second inequality, we have also used that all operators acting on u_{N-1} are second order. We similarly estimate $\|L_2u_{N-2}\|_{H^{s-1}(B)}$:

$$\begin{aligned}\|L_2u_{N-2}\|_{H^{s-1}(B)} &\le \|\rho^2\Delta u_{N-2}\|_{H^{s-1}(B)} + \|(\rho\Delta_{S^2}\rho)r^{-1}\partial_r u_{N-2}\|_{H^{s-1}(B)} + 2\|\rho\rho_\theta r^{-1}\partial_\theta\partial_r u_{N-2}\|_{H^{s-1}(B)}\\ &\quad + \left\|\frac{\rho\rho_\phi}{\sin\theta}\cdot\frac{\partial_\phi\partial_r u_{N-2}}{r\sin\theta}\right\|_{H^{s-1}(B)} + 2\left\|\rho_\theta^2 r^{-1}\partial_r u_{N-2}\right\|_{H^{s-1}(B)}\\ &\quad + 2\left\|\frac{\rho_\phi^2}{\sin^2\theta}r^{-1}\partial_r u_{N-2}\right\|_{H^{s-1}(B)} + 2\left\|\frac{\rho_\phi^2}{\sin^2\theta}\partial_r^2 u_{N-2}\right\|_{H^{s-1}(B)}\\ &\le M(s)\Big(|\rho|_{C^2}^2\|u_{N-2}\|_{H^{s+1}(B)} + |\rho|_{C^{s-1}}|\rho|_{C^{s+1}}\|u_{N-2}\|_{H^{s+1}(B)}\\ &\quad + 4|\rho|_{C^{s-1}}|\rho|_{C^s}\|u_{N-2}\|_{H^{s+1}(B)} + 6|\rho|_{C^s}\|u_{N-2}\|_{H^{s+1}(B)}\Big)\\ &\le 12M(s)|\rho|_{C^{s+1}}^2K_1A^{N-2}\\ &\le 12M(s)|\rho|_{C^{s+1}}K_1A^{N-1}.\end{aligned}$$

Taking $C_0 = 12M(s)$ completes the proof. ∎

The following theorem justifies the convergence of (25) for suitably small $\varepsilon > 0$.

Theorem 3.3. *Given $s \in \mathbb{N}$, if $\rho \in C^{s+1}(S^2)$ and $\xi \in H^{s+\frac{1}{2}}(S^2)$, there exists constants C_0 and K_0 and a unique solution u_ε of (21) satisfying (25) such that*

$$\|u_n\|_{H^{s+1}(B)} \le K_0\|\xi\|_{H^{s+\frac{1}{2}}(S^2)}A^n \tag{27}$$

for any $A > 2K_0C_0|\rho|_{C^{s+1}}$.

Proof. We proceed by induction. For $n = 0$, we use Lemma 3.1 to we see

$$\|u_0\|_{H^{s+1}} \le K_0\|\xi\|_{H^{s+\frac{1}{2}}}A^0,$$

as desired to show (27). We now define $K_1 = K_0\|\xi\|_{H^{s+\frac{1}{2}}(S^2)}$ for the remainder of the proof to be used in Lemma 3.2.

Suppose inequality (27) holds for $n < N$. Then by Lemma 3.1,

$$\|u_N\|_{H^{s+1}} \le K_0\left(\|L_1u_{N-1}\|_{H^{s-1}} + \|L_2u_{N-2}\|_{H^{s-1}}\right).$$

By Lemma 3.2, we may bound $\|L_1u_{N-1}\|_{H^{s-1}}$ and $\|L_2u_{N-2}\|_{H^{s-1}}$ so that

$$\begin{aligned}\|u_N\|_{H^{s+1}} &\le 2K_0K_1C_0|\rho|_{C^{s+1}}A^{N-1}\\ &= 2K_0^2\|\xi\|_{H^{s+\frac{1}{2}}(S^2)}C_0|\rho|_{C^{s+1}}A^{N-1}\\ &\le K_0\|\xi\|_{H^{s+\frac{1}{2}}(S^2)}A^N\end{aligned}$$

provided $A > 2K_0C_0|\rho|_{C^{s+1}}$. ∎

3.2. **Proof of Theorem 1.1 in three dimensions: Analyticity of the Dirichlet to Neumann operator.** Denote the Dirichlet-to-Neumann operator $G_\varepsilon\colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s-\frac{1}{2}}(\partial\Omega_\varepsilon)$ which is defined

$$G_\varepsilon\xi = \vec{n}_\varepsilon \cdot \nabla v$$

where v satisfies (19) and

$$\begin{aligned}\vec{n}_\varepsilon &= \left((1+\varepsilon\rho)^2 + \varepsilon^2\rho_\theta^2 + \varepsilon^2\frac{\rho_\phi^2}{\sin^2\theta}\right)^{-\frac{1}{2}} \left((1+\varepsilon\rho)\hat{r} - \varepsilon\rho_\theta\hat{\theta} - \varepsilon\frac{\rho_\phi}{\sin\theta}\hat{\phi}\right)\\ &= M_\rho(\varepsilon)\left((1+\varepsilon\rho)\hat{r} - \varepsilon\rho_\theta\hat{\theta} - \varepsilon\frac{\rho_\phi}{\sin\theta}\hat{\phi}\right)\end{aligned}$$

is the unit-length normal vector on $\partial\Omega_\varepsilon$. Here the spherical coordinate vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ are given by

$$\hat{r} = \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix}, \qquad \hat{\theta} = \begin{pmatrix}\cos(\theta)\cos(\phi)\\ \cos(\theta)\sin(\phi)\\ -\sin(\theta)\end{pmatrix}, \quad \text{and} \quad \hat{\phi} = \begin{pmatrix}-\sin(\phi)\\ \cos(\phi)\\ 0\end{pmatrix}.$$

Making the change of variables in (20), we obtain the transformed DNO, $\tilde{G}_\varepsilon\colon H^{s+\frac{1}{2}}(S^2) \to H^{s-\frac{1}{2}}(S^2)$, given by

$$\tilde{G}_\varepsilon\xi = M_\rho(\varepsilon)\left[(1 + \frac{\varepsilon^2}{(1+\varepsilon\rho)^2}\left(\rho_\theta^2 + \frac{\rho_\phi^2}{\sin^2\theta}\right)\partial_r u_\varepsilon - \frac{\varepsilon\rho_\theta}{1+\varepsilon\rho}\partial_\theta u_\varepsilon - \frac{\varepsilon\rho_\phi}{(1+\varepsilon\rho)\sin^2\theta}\partial_\phi u_\varepsilon\right] \tag{28a}$$

$$= M_\rho(\varepsilon)\hat{G}_{\rho,\varepsilon}\xi \tag{28b}$$

where u_ε satisfies (21).

The following Lemma shows that the change of coordinates between G_ε and $\tilde{G}_\varepsilon$ is conjugation by an isomorphism.

Lemma 3.4. *Let*

$$\gamma_0\colon H^s(B^3) \to H^{s-\frac{1}{2}}(S^2)$$
$$\gamma_{\varepsilon,\rho}\colon H^s(\Omega_\varepsilon) \to H^{s-\frac{1}{2}}(\partial\Omega_\varepsilon),$$

be the trace operators and $T_{\varepsilon,\rho}\colon H^s(\Omega_\varepsilon) \to H^s(B^3)$ denotes the change of variables in (20). Then $L\colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s+\frac{1}{2}}(S^2)$ defined by

$$L = \gamma_0 T_{\varepsilon,\rho}\gamma_{\varepsilon,\rho}^{-1}$$

is a Hilbert space isomorphism and $LG_\varepsilon = \tilde{G}_\varepsilon L$.

Proof. The proof of Lemma 3.4 is similar to the proof of Lemma 2.4. ∎

We have now established that

$$LG_\varepsilon L^{-1} = \tilde{G}_\varepsilon = M_\rho(\varepsilon)\hat{G}_{\rho,\varepsilon}.$$

Since $M_\rho(\varepsilon)$ is clearly analytic near $\varepsilon = 0$, we need only show the analyticity of $\hat{G}_{\rho,\varepsilon}$ near $\varepsilon = 0$ to verify that $\tilde{G}_\varepsilon$ is analytic as well. Note that $\hat{G}_{\rho,\varepsilon}$ satisfies

$$\begin{aligned}\hat{G}_{\rho,\varepsilon}\xi = \partial_r u &+ \varepsilon\left(2\rho\partial_r u - \rho_\theta\partial_\theta u - \frac{\rho_\phi}{\sin^2\theta}\partial_\phi u - 2\rho\hat{G}_{\rho,\varepsilon}\xi\right)\\ &+ \varepsilon^2\left[\left(\rho^2 + \rho_\theta^2 + \frac{\rho_\phi^2}{\sin^2\theta}\right)\partial_r u - \rho\rho_\theta\partial_\theta u - \frac{\rho\rho_\phi}{\sin^2\theta}\partial_\phi u - \rho^2\hat{G}_{\rho,\varepsilon}\xi\right].\end{aligned} \tag{29}$$

We now make a power series ansatz for the non-normalized DNO $\hat{G}_{\rho,\varepsilon}$,

$$\hat{G}_{\rho,\varepsilon}\xi = \sum_{n=0}^{\infty} \varepsilon^n \hat{G}_{\rho,n}\xi, \tag{30}$$

for $\xi \in H^{s+\frac{1}{2}}(S^2)$ and $s \in \mathbb{N}$. By (25) and (29), we obtain the recursive relationship,

$$\begin{aligned}\hat{G}_{\rho,n}\xi = \partial_r u_n + 2\rho\partial_r u_{n-1} - \rho_\theta\partial_\theta u_{n-1} - \frac{\rho_\phi}{\sin^2\theta}\partial_\phi u_{n-1} + \left(\rho^2 + \rho_\theta^2 + \frac{\rho_\phi^2}{\sin^2\theta}\right)\partial_r u_{n-2} \\ - \rho\rho_\theta\partial_\theta u_{n-2} - \frac{\rho\rho_\phi}{\sin^2\theta}\partial_\phi u_{n-2} - 2\rho\hat{G}_{\rho,n-1}\xi - \rho^2\hat{G}_{\rho,n-2}\xi.\end{aligned} \tag{31}$$

The following theorem proves Theorem 1.1 in three dimensions and justifies the convergence of (30) for suitably small $\varepsilon > 0$.

Theorem 3.5. *Let $\xi \in H^{s+\frac{1}{2}}(S^2)$. Then*

$$\|\hat{G}_{\rho,n}\xi\|_{H^{s-\frac{1}{2}}(S^2)} \leq K_1\|\xi\|_{H^{s+\frac{1}{2}}(S^2)} A^n \tag{32}$$

for $A > C|\rho|_{C^{s+1}}$.

Proof. We will proceed via induction. First, we show (32) fo $n = 0$:

$$\begin{aligned}\|\hat{G}_{\rho,0}\xi\|_{H^{s-\frac{1}{2}}(S^2)} &\leq \|\partial_r u_0\|_{H^{s-\frac{1}{2}}(S^2)} \leq C_1\|\partial_r u_0\|_{H^s(B)} \\ &\leq C_1\|u_0\|_{H^{s+1}(B)} \leq C_1K_0\|\xi\|_{H^{s+\frac{1}{2}}(S^2)}.\end{aligned}$$

In the second inequality of the first line, we have used the standard trace theorem, while Theorem 3.3 is used in the second line. Now suppose that (32) holds for $n < N$. Then we have the following estimate:

$$\begin{aligned}\|\hat{G}_{\rho,N}\xi\|_{s-\frac{1}{2}} \leq{}& \|\partial_r u_N\|_{s-\frac{1}{2}} + 2\left\|\rho\partial_r u_{N-1}\right\|_{s-\frac{1}{2}} + \|\rho_\theta\partial_\theta u_{N-1}\|_{s-\frac{1}{2}} + \left\|\frac{\rho_\phi}{\sin^2\theta}\partial_\phi u_{N-1}\right\|_{s-\frac{1}{2}} \\ &+ \left\|\rho^2\partial_r u_{N-2}\right\|_{s-\frac{1}{2}} + \left\|\rho_\theta^2\partial_r u_{N-2}\right\|_{s-\frac{1}{2}} + \left\|\frac{\rho_\phi^2}{\sin^2\theta}\partial_r u_{N-2}\right\|_{s-\frac{1}{2}} + \|\rho\rho_\theta\partial_\theta u_{N-2}\|_{s-\frac{1}{2}} \\ &+ \left\|\frac{\rho\rho_\phi}{\sin^2\theta}\partial_\phi u_{N-2}\right\|_{s-\frac{1}{2}} + 2\left\|\rho\hat{G}_{\rho,N-1}\xi\right\|_{s-\frac{1}{2}} + \left\|\rho^2\hat{G}_{\rho,N-2}\xi\right\|_{s-\frac{1}{2}} \\ \leq{}& C_1K_0\|\xi\|_{s+\frac{1}{2}} A^N + M(s)C_1K_0\Big(2|\rho|_{C^{s-\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}} A^{N-1} \\ &+ |\rho|_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-1} + |\rho|_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-1} \\ &+ |\rho|^2_{C^{s-\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2} + |\rho|^2_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2} + |\rho|^2_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2} \\ &+ |\rho|_{C^{s-\frac{1}{2}+\delta}}|\rho|_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2} + |\rho|_{C^{s-\frac{1}{2}+\delta}}|\rho|_{C^{s+\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2} \\ &+ K_1|\rho|_{C^{s-\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-1} + K_1|\rho|^2_{C^{s-\frac{1}{2}+\delta}}\|\xi\|_{s+\frac{1}{2}}A^{N-2}\Big) \\ \leq{}& K_1\|\xi\|_{s+\frac{1}{2}}A^N,\end{aligned}$$

for $K_1 = \max\{2C_1K_0, 2C_1K_0M(s)\}$ and $A > C|\rho|_{C^{s+1}}$. □

4. Proof of Corollary 1.2: Analyticity of the Steklov eigenvalues

We now have all of the ingredients to prove Corollary 1.2.

Proof of Corollary 1.2. For dimensions $d = 2, 3$ respectively, Theorems 2.5 and 3.5 show that $\tilde{G}_\varepsilon \colon H^{s+\frac{1}{2}}(S^{d-1}) \to H^{s-\frac{1}{2}}(S^{d-1})$ is analytic for small ε. By Lemmas 2.4 and 3.4, we have an isomorphism $L \colon H^{s+\frac{1}{2}}(\partial\Omega_\varepsilon) \to H^{s+\frac{1}{2}}(S^{d-1})$ satisfying

$$G_\varepsilon = L^{-1}\tilde{G}_\varepsilon L.$$

The DNO operator $G_\varepsilon \colon L^2(\partial\Omega_\varepsilon) \to L^2(\partial\Omega_\varepsilon)$ is self-adjoint [Are+14], hence closed. Thus, $L^{-1}\tilde{G}_\varepsilon L$ is closed and analytic for small ε. Since the spectrum of the left and right hand sides are equal, the result follows from [Kat76, Ch. 7, Thm 1.8, p. 370]. □

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