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Manakov system with parity symmetry on nonzero background and associated boundary value problems

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Abstract

We characterize initial value problems for the defocusing Manakov system (coupled two-component nonlinear Schrödinger equation) with nonzero background and well-defined spatial parity symmetry (i.e., when each of the components of the solution is either even or odd), corresponding to boundary value problems on the half line with Dirichlet or Neumann boundary conditions at the origin. We identify the symmetries of the eigenfunctions arising from the spatial parity of the solution, and we determine the corresponding symmetries of the scattering data (reflection coefficients, discrete spectrum and norming constants). All parity induced symmetries are found to be more complicated than in the scalar (i.e., one-component) case. In particular, we show that the discrete eigenvalues giving rise to dark solitons arise in symmetric quartets, and those giving rise to dark–bright solitons in symmetric octets. We also characterize the differences between the purely even or purely odd case (in which both components are either even or odd functions of x) and the 'mixed parity' cases (in which one component is even while the other is odd). Finally, we show how, in each case, the spatial symmetry yields a constraint on the possible existence of self-symmetric eigenvalues, corresponding to stationary solitons, and we study the resulting behavior of solutions.

Keywords: parity, Manakov system, nonzero background, inverse scattering, boundary value problems, solitons

(Some figures may appear in colour only in the online journal)

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1. Introduction

In many physical contexts, the dynamics of multi-component nonlinear physical systems is governed by coupled systems of evolution equations of nonlinear Schrödinger (NLS) type. These include, for example, Bose–Einstein condensates (BECs) [31, 37] and nonlinear optics [32, 36], among others. The special case of coupled two-component NLS equations in which the strength of self-phase modulation terms equals that of cross-phase modulation terms is referred to as the Manakov system. The Manakov system is also completely integrable and, as such, its initial value problem (IVP) can in principle be solved by the inverse scattering transform (IST), a nonlinear analogue of the Fourier transform. For these reasons, the Manakov system continues to receive considerable attention from the research community.

The IVP for the scalar NLS equation and for the Manakov system with localized fields (i.e., zero boundary conditions at infinity) was first formulated in 1972 and 1974, respectively [34, 45], and was later generalized to multi-component and matrix NLS systems [6]. In many physical situations, the typical spatial scales of the dynamical structures are much narrower than those of the nonlinear medium in which they appear (for example, see [8, 29, 30, 35, 43, 44] for BECs). These situations are appropriately modeled by fields with finite (i.e., nonzero) background densities, which in turn are described mathematically by *nonzero boundary conditions* (NZBC) at space infinities.

It is well known that the IST with NZBC is more challenging than with localized fields, even for the scalar NLS equation (e.g., see [14, 16, 23, 24, 46]). The IST for the defocusing Manakov system with NZBC was first formulated in [38], and later revisited in [15, 21], and the IST for the focusing Manakov system with NZBC was formulated in [33]. The main motivation for the present work is provided by recent analytical and numerical studies [39, 40] on repulsive scalar and two-component BECs in which the initial system configurations possess a definite parity symmetry. These kinds of configurations are the easiest ones to generate experimentally, and they also give rise to a variety of interesting phenomena.

The two-component repulsive BECs mentioned above can be modeled, under appropriate assumptions (e.g., the absence of a trapping potential), by the defocusing Manakov system. The goal of the present paper is therefore to characterize the defocusing Manakov system with spatial parity. We write the defocusing Manakov system in the form

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} - 2(\|\mathbf{q}\|^2 - q_o^2)\,\mathbf{q} = 0, \tag{1.1}$$

where $\mathbf{q}(x,t) = (q_1, q_2)^{\mathrm{T}}$ is a complex-valued column vector function, and subscripts x and t denote partial differentiation. Here, the quantity $q_o > 0$ is the background amplitude of the medium (but its explicit presence in (1.1) can be removed by a simple gauge transformation). Specifically, we study solutions of (1.1) satisfying the following NZBC at infinity:

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm = \mathbf{q}_o\,\mathrm{e}^{\pm\mathrm{i}\theta}, \tag{1.2}$$

with $q_o = \|\mathbf{q}_o\| > 0$. Note that we have restricted the boundary conditions $\mathbf{q}_\pm$ to be parallel to each other. Moreover, the asymptotic phases were chosen as $\theta_+ = -\theta_- = \theta$, which can be done without loss of generality thanks to the phase invariance of (1.1); the reflection symmetry then allows us to restrict ourselves to the range $0 \leqslant \theta \leqslant \pi/2$.

The condition that $\mathbf{q}_\pm$ be parallel does not preclude the possible presence of a relative phase between the two components. Such a phase can always be removed using the invariance of the Manakov system (1.1) under unitary transformations. In fact, hereafter we will often take advantage of this invariance to set $\mathbf{q}_\pm = (0, q_\pm)^{\mathrm{T}}$, i.e., we will restrict ourselves to the case in which the first component vanishes as $x \to \pm\infty$ while the second component satisfies NZBC, which can always be done without loss of generality whenever (1.2) holds. In that case will then refer to q_1 and q_2 respectively as the 'bright' and 'dark' components, since the soliton solutions then comprise bright solitons in the first component and dark solitons in the second component. (For consistency with the literature, however, we will still call these solutions 'dark–bright' solitons.)

When both components of the initial condition (IC) $\mathbf{q}(x, 0)$ have a well-defined spatial parity, i.e., if $q_j(-x, 0) = \nu_j q_j(x, 0)$ for all $x \in \mathbb{R}$, with $\nu_j = \pm 1$ for $j = 1, 2$, this parity is preserved by the time evolution, i.e., $q_j(-x, t) = \nu_j q_j(x, t)$ for all $x \in \mathbb{R},\ t > 0$. (The constants ν_1 and ν_2 disappear from the term $\|\mathbf{q}(x, t)\|^2$ in (1.1). As a result, $q_{j,t}(x, t)$ inherits the same symmetries as $q_j(x, 0)$.) Note that ν_1 and ν_2 need not be the same. Accordingly, we can distinguish four scenarios:

(a) Even ICs: $\nu_1 = \nu_2 = 1$.
(b) Odd ICs: $\nu_1 = \nu_2 = -1$.
(c) Even–odd ICs: $\nu_1 = 1$ and $\nu_2 = -1$.
(d) Odd–even ICs: $\nu_1 = -1$ and $\nu_2 = 1$.

Obviously the last two scenarios do not arise in the scalar NLS equation, and are therefore novel features of the two-component setting. We will see that these 'mixed parity' cases are not equivalent to cases (a) or (b), and yield different results.

We point out that in principle the mixed cases (c) and (d) could be viewed as equivalent upon a simple switch of the components of the solution, which is of course allowed in general for the Manakov system. However, the symmetry between the components is broken when one requires the first component to vanish at infinity, i.e., when identifying q_1 and q_2 respectively as the bright and dark components (as we will do through parts of section 3 and throughout sections 4 and 5), since the choice of gauge breaks the unitary invariance of the system. Thus, with this constraint, the two mixed cases are not equivalent, and must be therefore treated separately.

Importantly, each of the above four types of ICs can be associated to a particular boundary value problem (BVP) for the Manakov system on the half line $0 < x < \infty$ with a specific boundary condition (BC) at $x = 0$, with the equivalence obtained by extending the solution of the BVP to the full line via the corresponding odd or even extension for each component. Specifically:

(a) Even ICs are equivalent to BVPs with homogeneous Neumann BCs, i.e., $\mathbf{q}_x(0, t) = \mathbf{0}$.
(b) Odd ICs are equivalent to BVPs with homogeneous Dirichlet BCs, i.e., $\mathbf{q}(0, t) = \mathbf{0}$.
(c) Even–odd ICs are equivalent to BVPs with homogeneous Neumann and Dirichlet BCs for the first and second component, respectively, i.e., $q_{1,x}(0, t) = q_2(0, t) = 0$.
(d) Odd–even ICs are equivalent to BVPs with homogeneous Dirichlet and Neumann BCs for the first and second component, respectively, i.e., $q_1(0, t) = q_{2,x}(0, t) = 0$.

Therefore, the results of the present work also provide for the first time a characterization of certain BVPs for the Manakov system with NZBC at space infinity.

The connection between BVPs and IVPs with spatial parity via the IST was first explored in [7], where the odd and even extensions to the full line were used to solve BVPs for the scalar NLS equation with homogeneous Dirichlet and Neumann BC. Several researchers then expanded on that first work to study BVPs for the scalar NLS equation with homogeneous Robin BCs using a more general extension to the full line [9, 10, 13, 22, 28, 41]. Similar techniques were also generalized to study BVPs for the Manakov system with zero BC at infinity in [19, 20, 27, 42, 47]. In all the above cases, the spatial parity induces additional symmetries in the scattering data, and in the present work we will show that the same is true for the Manakov system with NZBC. We should mention that much more general kinds of BCs can be studied using the so-called Fokas method, or unified transform method [12, 25, 26]. However, to the best of our knowledge, BVPs for the Manakov system with NZBC at space infinity have not been considered with either approach before. The present work aims at filling this gap.

The structure of this paper is the following. In section 2 we briefly review some essential elements of the solution of the Manakov system with NZBC via the IST, which will be used in the subsequent sections. In section 3 we identify the symmetries of the eigenfunctions and scattering coefficients induced by the spatial parity of the initial conditions. In section 4 we characterize the corresponding symmetries of the discrete spectrum and norming constants. Finally, in section 5 we describe the resulting behavior of the solutions. Several calculations and technical considerations are relegated to various appendices.

2. Essential elements of the IST for the Manakov system with NZBC

In this section we briefly review the essential elements of the IST for the Manakov system with NZBC, in order to define the quantities that will be used in the later sections. We omit all calculations, referring the reader to [15, 38] for all details (keeping in mind that here we use the normalizations of [38], not those of [15]). Recall that the Lax pair for the Manakov system is given by

$$v_x = \mathbf{X}v, \qquad v_t = \mathbf{T}v, \tag{2.1}$$

where

$$\mathbf{X}(x,t,k) = -\mathrm{i}kJ + Q, \qquad \mathbf{T}(x,t,k) = -2\mathrm{i}k^2 J + \mathrm{i}J\left(Q_x - Q^2 + q_o^2\right) - 2kQ, \tag{2.2a}$$

$$J = \mathrm{diag}(1,-1,-1), \qquad Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^{\mathrm{T}} \\ \mathbf{q}^* & 0_{2\times 2} \end{pmatrix}, \tag{2.2b}$$

and the superscript * denotes complex conjugate throughout. Namely, $\mathbf{q}$ satisfies (1.1) if and only if the overdetermined system (2.1) is compatible (i.e., $v_{tx} = v_{xt}$). The Jost eigenfunctions are defined as

$$\phi_\pm(x,t,k) = Y_\pm(k)\mathrm{e}^{\mathrm{i}\Theta(x,t,k)} + o(1), \quad x \to \pm\infty, \tag{2.3}$$

where

$$\Theta = \mathbf{\Lambda}x - \mathbf{\Omega}t = \mathrm{diag}(\theta_1, \theta_2, -\theta_1), \tag{2.4a}$$

$$\mathbf{\Lambda} = \mathrm{diag}(-\lambda, k, \lambda), \qquad \mathbf{\Omega} = \mathrm{diag}(2k\lambda, -(k^2+\lambda^2), -2k\lambda), \tag{2.4b}$$

and

$$\lambda^2 = k^2 - q_o^2. \tag{2.4c}$$

Here, $Y_\pm(k)$ are the matrices of asymptotic eigenvectors of the Lax pair, which can be chosen as follows:

$$Y_\pm(z) = \begin{pmatrix} z & 0 & -q_o^2/z \\ \mathrm{i}\mathbf{q}_\pm^* & \mathrm{i}\mathbf{q}_\pm^\perp & -\mathrm{i}\mathbf{q}_\pm^*, \end{pmatrix}, \tag{2.5}$$

where for any two-component vector $\mathbf{p} = (p_1, p_2)^{\mathrm{T}}$ we write $\mathbf{p}^{\perp} = (p_2, -p_1)^{\mathrm{T}}$. Like in the scalar case, in order to remove the branching of the eigenfunctions in the spectral parameter k due to their dependence on λ, it is convenient to introduce a uniformization variable z defined by the conformal mapping

$$z = k + \lambda(k)$$

whose inverse is given by

$$k = \frac{1}{2}(z + \hat{z}^*), \qquad \lambda = \frac{1}{2}(z - \hat{z}^*),$$

with $\hat{z} = q_o^2/z^*$, which maps the two sheets of the Riemann surface for k, λ into a single copy of the complex z-plane.

If $\mathbf{q}(x,t)$ approaches $\mathbf{q}_{\pm}$ sufficiently rapidly as $x \to \pm\infty$ for all t, the Jost eigenfunctions are well-defined and continuous for all $x, t, z \in \mathbb{R}$. Furthermore, since $\det Y_{\pm}(z) = -2q_o^2\lambda(z)$, both $\phi_+(x,t,z)$ and $\phi_-(x,t,z)$ are fundamental matrix solutions of the Lax pair (2.1) for all $z \in \mathbb{R}\backslash\{\pm q_o\}$, and hence there exists a 3×3 invertible matrix $A(z)$ such that

$$\phi_-(x,t,z) = \phi_+(x,t,z)A(z), \quad z \in \mathbb{R}\backslash\{\pm q_o\}. \tag{2.6}$$

As usual, $A(z) = (a_{ij}(z))$ is referred to as the *scattering matrix*. For future reference, we also introduce the inverse matrix $B(z) := A^{-1}(z) = (b_{ij}(z))$.

Two of the columns of the Jost eigenfunctions cannot be analytically continued off the real z-axis, so one needs to introduce auxiliary eigenfunctions to obtain a complete set of eigenfunctions which are analytic in either the upper or the lower z-plane. This is done by using the formal 'adjoint' of the Lax pair (2.1):

$$\tilde{v}_x = \mathbf{X}^*\tilde{v}, \qquad \tilde{v}_t = \mathbf{T}^*\tilde{v}. \tag{2.7}$$

One can define the adjoint Jost solutions as the simultaneous solutions $\tilde{\phi}_{\pm}(x,t,z)$ of (2.7) such that

$$\tilde{\phi}_{\pm}(x,t,z) = \tilde{Y}_{\pm}(z)\mathrm{e}^{-\mathrm{i}\Theta(x,t,z)} + o(1), \quad x \to \pm\infty, \tag{2.8}$$

with $\tilde{Y}_{\pm}(z) = Y_{\pm}^*(z)$. Two additional solutions of the original Lax pair (2.1) can then be obtained from the adjoint Jost solutions as:

$$\chi(x,t,z) = -\mathrm{e}^{\mathrm{i}\theta_2(x,t,z)}J[\tilde{\phi}_{-,3}(x,t,z) \times \tilde{\phi}_{+,1}(x,t,z)], \tag{2.9}$$

$$\bar{\chi}(x,t,z) = -\mathrm{e}^{\mathrm{i}\theta_2(x,t,z)}J[\tilde{\phi}_{-,1}(x,t,z) \times \tilde{\phi}_{+,3}(x,t,z)]. \tag{2.10}$$

Moreover, one can show that, for $z \in \mathbb{R}\backslash\{\pm q_o\}$:

$$\chi(z) = 2\lambda(z)[b_{33}(z)\phi_{+,2}(z) - b_{23}(z)\phi_{+,3}(z)] = 2\lambda(z)\left[a_{11}(z)\phi_{-,2}(z) - a_{21}(z)\phi_{-,1}(z)\right], \tag{2.11a}$$

$$\bar{\chi}(z) = 2\lambda(z)[b_{21}(z)\phi_{+,1}(z) - b_{11}(z)\phi_{+,2}(z)] = 2\lambda(z)\left[a_{23}(z)\phi_{-,3}(z) - a_{33}(z)\phi_{-,2}(z)\right], \tag{2.11b}$$

where the (x,t)-dependence in the eigenfunctions was omitted for simplicity. One can now define two complete sets of eigenfunctions $\Psi^{\pm}(x,t,z)$ for the system (2.1) as:

$$\Psi^+(x,t,z) = (\phi_{-,1}(x,t,z),\ \chi(x,t,z),\ \phi_{+,3}(x,t,z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R}, \tag{2.12a}$$

$$\Psi^-(x,t,z) = (\phi_{+,1}(x,t,z),\ \bar{\chi}(x,t,z),\ \phi_{-,3}(x,t,z)), \quad z \in \mathbb{C}^- \cup \mathbb{R}, \tag{2.12b}$$

with $\Psi^{\pm}(x,t,z)$ analytic for $z \in \mathbb{C}^{\pm}$, respectively. Furthermore, one can show that

$$\det\ \Psi^{+}(x,t,z) = -4q_o^2\lambda^2(z)a_{11}(z)b_{33}(z)\mathrm{e}^{\mathrm{i}\theta_2(x,t,z)}, \quad \mathbb{C}^{+}\cup\mathbb{R}, \tag{2.13a}$$

$$\det\ \Psi^{-}(x,t,z) = 4q_o^2\,\lambda^2(z)a_{33}(z)b_{11}(z)\mathrm{e}^{\mathrm{i}\theta_2(x,t,z)}, \quad \mathbb{C}^{-}\cup\mathbb{R}, \tag{2.13b}$$

where $a_{11}(z)$ and $b_{33}(z)$ are analytic in $\mathbb{C}^{+}$, while $b_{11}(z)$ and $a_{33}(z)$ are analytic in $\mathbb{C}^{-}$.

For the NLS equation and the Manakov system with NZBC, the Lax pair admits two symmetries, which in terms of the uniformization variable z can be expressed as follows. The first symmetry corresponds to the mapping $z \mapsto z^*$ ($\mathbb{C}^{+}/\mathbb{C}^{-}$), under which one has

$$A^{\dagger}(z) = \Gamma(z)B(z)\Gamma^{-1}(z), \quad z\in\mathbb{R}, \tag{2.14a}$$

$$\Gamma(z) = \mathrm{diag}(-q_o^2/z, 2\lambda(z), z), \tag{2.14b}$$

where the superscript '†' denotes the conjugate transpose. The symmetries for the analytic entries of the scattering matrix are then extended off the real z-axis using Schwartz reflection principle. The second symmetry corresponds to the mapping $z \mapsto q_o^2/z$, and the corresponding symmetries for the eigenfunctions are:

$$\phi_{\pm}(x,t,z) = \phi_{\pm}(x,t,\hat{z})\Pi, \qquad \chi(x,t,z) = \bar{\chi}(x,t,\hat{z}), \quad z\in\mathbb{R}, \tag{2.15a}$$

where

$$\Pi = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{2.15b}$$

In turn, (2.15a) induce the following relations for the scattering coefficients:

$$A(\hat{z}^*) = \Pi A(z)\Pi^{-1}, \quad z\in\mathbb{R}. \tag{2.15c}$$

The columns of $\Psi^{\pm}(x,t,z)$ become linearly dependent at the zeros of $a_{11}(z)$ and $b_{33}(z)$ and those of $a_{33}(z)$ and $b_{11}(z)$, respectively (cf (2.13)). These zeros are the discrete eigenvalues of the scattering problem. In the following we assume that none of these zeros lie on the real z-axis (i.e., we exclude spectral singularities/embedded eigenvalues). It is convenient to introduce the circle $C_o := \{z\in\mathbb{C} : |z| = q_o\}$ and the open disk $D_o = \{z\in\mathbb{C} : |z| < q_o\}$ of radius q_o, respectively. Then we label ζ_n for $n = 1,\ldots,N_1$ the zeros of $a_{11}(z)$ on $C_o\cap\mathbb{C}^{+}$, and z_n for $n = 1,\ldots,N_2$ the zeros of $a_{11}(z)$ in $D_o\cap\mathbb{C}^{+}$. For each pair ζ_n, ζ_n^* of discrete eigenvalues on the circle, one then has $a_{11}(\zeta_n) = a_{33}(\zeta_n^*) = 0$, and

$$\phi_{-,1}(x,t,\zeta_n) = c_n\phi_{+,3}(x,t,\zeta_n), \qquad \phi_{-,3}(x,t,\zeta_n^*) = \bar{c}_n\phi_{+,1}(x,t,\zeta_n^*), \quad n = 1,\ldots,N_1, \tag{2.16a}$$

with

$$\bar{c}_n = c_n, \qquad c_n^* = \frac{\zeta_n^*}{\zeta_n}\frac{b'_{11}(\zeta_n^*)}{a'_{33}(\zeta_n^*)}\bar{c}_n. \tag{2.16b}$$

Similarly, for each quartet of eigenvalues $z_n, z_n^*, \hat{z}_n, \hat{z}_n^*$ off the circle, one has $a_{11}(z_n) = b_{11}(z_n^*) = a_{33}(\hat{z}_n^*) = b_{33}(\hat{z}_n) = 0$, and

$$\phi_{-,1}(x,t,z_n) = d_n\chi(x,t,z_n) \equiv d_n\bar{\chi}(x,t,\hat{z}_n^*), \quad \bar{\chi}(x,t,z_n^*) = \bar{d}_n\phi_{+,1}(x,t,z_n^*) \equiv -\bar{d}_n\phi_{+,3}(x,t,\hat{z}_n), \tag{2.17a}$$

for all $n = 1, \ldots, N_2$, with

$$\bar{d}_n^* = 2\,\hat{z}_n^*\,\lambda(\hat{z}_n^*) b_{11}(\hat{z}_n^*)\,d_n. \tag{2.17b}$$

Discrete eigenvalues on the circle C_o correspond to dark–dark (or simply dark) solitons (i.e., solitons that appear as localized dips of intensity on the background field q_o in both components), which we also refer to simply as dark solitons, while discrete eigenvalues off C_o give rise to dark–bright solitons (i.e., solitons that appear as dark solitons in one component and bright solitons in the other component). The soliton velocity V_j is determined by the real part of the discrete eigenvalue z_j. (Explicitly, $V_j = -2\,\mathrm{Re}\, z_j$, cf reference [38] but note that (1.1) is obtained by letting $t \mapsto -t$ in [38].) Purely imaginary eigenvalues correspond to stationary solitons, and the special case of a stationary dark–dark soliton, produced by a discrete eigenvalue at $z = \mathrm{i}q_o$, is called a 'black' soliton.

The starting point for reconstructing the solution $\mathbf{q}(x, t)$ from the scattering data is the scattering relation (2.6), which yields a jump condition for a matrix Riemann–Hilbert problem. One can then reduce the solution of the inverse problem to a set of integral equations for the following three modified vector eigenfunctions:

$$M_1(z) = \phi_{+,3}(z)\,\mathrm{e}^{\mathrm{i}\theta_1(z)}, \quad z \in \mathbb{C}^+, \tag{2.18a}$$

$$M_2(z) = \phi_{+,1}(z)\,\mathrm{e}^{-\mathrm{i}\theta_1(z)}, \quad M_3(z) = \frac{\bar{\chi}(z)}{2\lambda(z)\,b_{11}(z)}\,\mathrm{e}^{-\mathrm{i}\theta_2(z)}, \quad z \in \mathbb{C}^-. \tag{2.18b}$$

Note that $\bar{\chi}(z)/(2\lambda(z)\,b_{11}(z))$ remains analytic also at the zeros of $b_{11}(z)$. Moreover, these eigenfunctions satisfy the integral equations

$$\begin{aligned} M_1(z) = &-\begin{pmatrix}\hat{z}^* \\ \mathrm{i}\mathbf{q}_+^*\end{pmatrix} + \sum_{n=1}^{N_1} \frac{\bar{\gamma}_n\,\mathrm{e}^{-2\mathrm{i}\theta(\zeta_n)}}{z-\zeta_n^*} M_2(\zeta_n^*) - \sum_{n=1}^{N_2} \frac{\bar{\delta}_n^*\,\mathrm{e}^{\mathrm{i}(\theta_2(z_n)-\theta_1(z_n))}}{z-\hat{z}_n^*} M_3(\hat{z}_n^*) \\ &+ \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta - z}\left[\rho_1(\zeta) M_2(\zeta)\,\mathrm{e}^{2\mathrm{i}\theta_1(\zeta)} - \rho_2(\hat{\zeta}^*) M_3(\zeta)\,\mathrm{e}^{\mathrm{i}(\theta_2(\zeta)+\theta_1(\zeta))}\right], \quad z \in \mathbb{C}^+, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} M_2(z) = &\begin{pmatrix} z \\ \mathrm{i}\mathbf{q}_+^*\end{pmatrix} + \sum_{n=1}^{N_1} \frac{\gamma_n z\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta_n)}}{z-\zeta_n} M_1(\zeta_n) - \sum_{n=1}^{N_2} \frac{\bar{\delta}_n^* z\,\mathrm{e}^{\mathrm{i}(\theta_2(z_n)-\theta_1(z_n))}}{\hat{z}_n^*(z-z_n)} M_3(\hat{z}_n^*) \\ &- \frac{z}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta(\zeta - z)}\left[\rho_1(\hat{\zeta}^*)\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta)} M_1(\zeta) + \rho_2(\zeta) M_3(\hat{\zeta}^*)\mathrm{e}^{\mathrm{i}(\theta_2(\zeta)-\theta_1(\zeta))}\right], \quad z \in \mathbb{C}^-, \end{aligned} \tag{2.19b}$$

$$\begin{aligned} M_3(z) = &\begin{pmatrix} 0 \\ \mathrm{i}\mathbf{q}_+^\perp\end{pmatrix} - \sum_{n=1}^{N_2} \frac{\bar{\delta}_n z}{(z-\hat{z}_n)(z-z_n^*)} M_2(z_n^*)\mathrm{e}^{\mathrm{i}(\theta_1(z_n^*)-\theta_2(z_n^*))} \\ &- \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta - z}\left[\bar{\rho}_2(\zeta)\,\mathrm{e}^{\mathrm{i}(\theta_1(\zeta)-\theta_2(\zeta))} M_2(\zeta) + \bar{\rho}_2(\hat{\zeta}^*) M_1(\zeta)\mathrm{e}^{-\mathrm{i}(\theta_1(\zeta)+\theta_2(\zeta))}\right], \quad z \in \mathbb{C}^-, \end{aligned} \tag{2.19c}$$

where, as before, the (x, t)-dependence was omitted for brevity. Note that: (i) γ_n and $\bar{\gamma}_n$ are the norming constants associated to the discrete eigenvalues ζ_n, ζ_n^* as given in (2.21) below (the first symmetry can then be used to express γ_n in terms of $\bar{\gamma}_n$, see (2.22)); (ii) δ_n and $\bar{\delta}_n$ are the norming constants associated to the discrete eigenvalues z_n, z_n^* given in (2.21) (the second symmetry has been used to eliminate from the system the norming constants associated to the other 2 eigenvalues in the quartet; moreover, the first symmetry allows one to express $\bar{\delta}_n$ in terms of δ_n, cf (2.22) below); (iii) the reflection coefficients are given by

$$\rho_1(z) = \frac{b_{31}(z)}{b_{11}(z)}, \qquad \rho_2(z) = \frac{a_{12}(z)}{a_{11}(z)}, \qquad \bar{\rho}_2(z) = \frac{b_{21}(z)}{b_{11}(z)}. \tag{2.20a}$$

Note that only two of the above three coefficients are independent, since using the first symmetry we have

$$\bar{\rho}_2^*(z^*) = \frac{q_o^2}{q_o^2 - z^2}\rho_2(z). \tag{2.20b}$$

The norming constants are defined in terms of the coefficients in (2.17) and (2.16) as follows:

$$\gamma_n = \frac{c_n}{\zeta_n a'_{11}(\zeta_n)}, \qquad \bar{\gamma}_n = \frac{\bar{c}_n}{a'_{33}(\zeta_n^*)}, \tag{2.21a}$$

$$\delta_n = \frac{d_n}{z_n a'_{11}(z_n)}, \qquad \bar{\delta}_n = -\frac{\bar{d}_n}{z_n^* b'_{11}(z_n^*)}, \tag{2.21b}$$

and they can be shown to satisfy the symmetries

$$\gamma_n = -\bar{\gamma}_n/\zeta_n^*, \qquad \bar{\gamma}_n^* = -\bar{\gamma}_n, \qquad \bar{\delta}_n^* = 2\hat{z}_n^*\lambda(z_n)b_{11}(\hat{z}_n^*)\delta_n. \tag{2.22}$$

Note in particular that the first two conditions require $\bar{\gamma}_n$ to be purely imaginary.

One can reconstruct the scattering coefficient $a_{11}(z)$ in terms of scattering data (discrete eigenvalues, norming constants and reflections coefficients) via the so-called 'trace formula', and also obtain as a byproduct the so-called 'theta condition' relating the scattering data to the asymptotic phase difference of the potential, respectively given by:

$$a_{11}(z) = \prod_{n=1}^{N_1}\frac{z-\zeta_n}{z-\zeta_n^*}\prod_{n=1}^{N_2}\frac{z-z_n}{z-z_n^*}\,\exp\left\{-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log[1-R(\zeta)]}{\zeta - z}\,\mathrm{d}\zeta\right\}, \tag{2.23a}$$

$$\mathrm{e}^{\mathrm{i}\Delta\,\theta} = \prod_{n=1}^{N_1}\frac{\zeta_n}{\zeta_n^*}\prod_{n=1}^{N_2}\frac{z_n}{z_n^*}\,\exp\left\{-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log[1-R(\zeta)]}{\zeta}\,\mathrm{d}\zeta\right\}, \tag{2.23b}$$

with $\rho_1(z), \rho_2(z)$ as in (2.20a) and

$$R(z) = \frac{z^2}{q_o^2}|\rho_1(z)|^2 + \frac{q_o^2}{(z^2-q_o^2)}|\rho_2(z)|^2, \quad z\in\mathbb{R}. \tag{2.24}$$

The trace formula (2.23a) and the symmetry $b_{11}(z) = a_{11}^*(z^*)$ (cf (2.14)) then allow to express δ_n in terms of $\bar{\delta}_n$ (cf (2.22)) in the linear system (2.19). (Note that $\lim_{z\to\pm q_o}\rho_2(z) = 0$, so $R(z)$ remains finite at the branch points. This was shown in detail in [15], and the same result holds with our choice of normalizations.) For future reference, we also observe that (2.23a) yields

$$a'_{11}(\alpha_j) = \frac{1}{(\alpha_j - \alpha_j^*)} \prod_{n=1}^{N_1+N_2}{}' \frac{\alpha_j - \alpha_n}{\alpha_j - \alpha_n^*} \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\log[1-R(\zeta)]}{\zeta - \alpha_j} \mathrm{d}\zeta \right\}, \qquad j = 1, \ldots, N_1 + N_2, \tag{2.25}$$

where $\alpha_k = \zeta_k$ when $k = 1, \ldots, N_1$ and $\alpha_k = z_{k-N_1}$ for $k = N_1 + 1, \ldots, N_1 + N_2$, and prime indicates that the term with $n = j$ is omitted from the product.

One can reconstruct the potential $\mathbf{q}(x, t)$ of the Manakov system from M_1, M_2, M_3 by comparing their asymptotics to that of the eigenfunctions obtained from the direct scattering problem, obtaining

$$\mathbf{q}(x,t) = \mathbf{q}_+ + \mathrm{i} \sum_{n=1}^{N_1} \frac{\bar{\gamma}_n}{\zeta_n} M_1^{(\mathrm{dn},*)}(x,t,\zeta_n)\, \mathrm{e}^{-2\mathrm{i}\theta_1(x,t,\zeta_n)} - \mathrm{i} \sum_{n=1}^{N_2} \frac{\bar{\delta}_n}{\hat{z}_n} M_3^{(\mathrm{dn},*)}(x,t,\hat{z}_n^*) \mathrm{e}^{\mathrm{i}(\theta_1(x,t,z_n^*) - \theta_2(x,t,z_n^*))}$$
$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \rho_1^*(\hat{\zeta}) M_1^{(\mathrm{dn},*)}(x,t,\zeta) \mathrm{e}^{2\mathrm{i}\theta_1(x,t,\zeta)} - \rho_2^*(\zeta) M_3^{(\mathrm{dn},*)}(x,t,\hat{\zeta}) \mathrm{e}^{\mathrm{i}(\theta_1(x,t,\zeta) - \theta_2(x,t,\zeta))} \right\} \frac{\mathrm{d}\zeta}{\zeta}, \tag{2.26}$$

where the superscript '(dn)' denotes the lower two components of the vector eigenfunctions, and '(dn,*)' their complex conjugate.

In the reflectionless case, the integral terms in (2.26) are absent, and one can obtain a closed-form determinantal expression for the multi-soliton solutions of the Manakov system with NZBC:

$$\mathbf{q}(x,t) = \frac{1}{\det \mathbf{R}^*} \begin{pmatrix} \det N_1^{\mathrm{aug}} \\ \det N_2^{\mathrm{aug}} \end{pmatrix}, \tag{2.27}$$

where the explicit expressions for the $(N_1 + N_2) \times (N_1 + N_2)$ matrix $\mathbf{R}$ and for the augmented $(N_1 + N_2 + 1) \times (N_1 + N_2 + 1)$ matrices N_1^{aug} and N_2^{aug} are given in appendix A.4. A determinantal expression for the multisoliton solutions of the Manakov system was first given in [15]. Since [15] used a different normalization for the eigenfunctions, however, for completeness we present the derivation of (2.27) in appendix A.4.

3. Solutions with parity: eigenfunctions and scattering coefficients

We now begin our study of solutions of the Manakov system with spatial parity. In this section we characterize the additional symmetries of eigenfunctions and scattering coefficients. In sections 4 and 5 we will then use the results of this section to obtain several results about the behavior of solutions of the Manakov system.

It is convenient for what follows to introduce the matrices $D_{\nu_1\nu_2\nu_3} = \mathrm{diag}(\nu_1, \nu_2, \nu_3)$. For example, with this notation one has $J = D_{+--} = \mathrm{diag}(1, -1, -1)$. Explicitly, we will use $D_{++-} = \mathrm{diag}(1, 1, -1)$ and $D_{+-+} = \mathrm{diag}(1, -1, 1)$.

3.1. Even initial conditions

Here we assume the IC $\mathbf{q}(x, 0)$ to be an even function of x, which, as mentioned above, implies $Q(x, t) = Q(-x, t)$ for all $x, t \in \mathbb{R}$, as well as $\mathbf{q}_+ = \mathbf{q}_-$. In this case, one can show that if $v(x, t, k)$ is any solution of the Lax pair, then so is $w(x, t, k) = Jv(-x, t, -k)$.

Moreover, if $k \mapsto -k$ and $\lambda \mapsto -\lambda$, then $z \mapsto -z$, and if $(x,t,z) \mapsto (-x,t,-z)$, then $e^{i(\Lambda x - \Omega t)}$ remains invariant. As to the asymptotic eigenvectors, if $\mathbf{q}_+ = \mathbf{q}_-$, then $Y_+(-z) = -JY_-(z)$. Therefore, comparing the boundary conditions of the Jost eigenfunctions one can verify that

$$\phi_{\pm,j}(-x,t,-z) = -J\phi_{\mp,j}(x,t,z), \quad j = 1,2,3. \tag{3.1}$$

Or, in matrix form:

$$\phi_{\mp}(-x,t,-z) = -J\phi_{\pm}(x,t,z). \tag{3.2}$$

The adjoint Jost eigenfunctions satisfy the same symmetry (3.1) as $\phi_\pm$, and therefore from (2.9) it follows that

$$\bar{\chi}(-x,t,-z) = -J\chi(x,t,z). \tag{3.3}$$

Now, from $\phi_-(x,t,z) = \phi_+(x,t,z)\mathbf{A}(z)$ and $\phi_-(-x,t,-z) = \phi_+(-x,t,-z)\mathbf{A}(-z)$ we then find

$$\mathbf{A}(-z) = \mathbf{A}^{-1}(z) \tag{3.4}$$

or, entry-wise, $a_{ij}(-z) = b_{ij}(z)$.

Next, we compute the symmetries of the reflection coefficients. Recall that the parity symmetry of the scattering coefficients for even ICs is $a_{ij}(-z) = b_{ij}(z)$ for $i, j = 1,2,3$ (cf (3.4)). Using the first and the second symmetry of the scattering matrix, one can obtain the following induced parity symmetry of the reflection coefficients:

$$\rho_1(-z) = -\frac{q_o^2}{z^2}\frac{a_{11}(\hat{z}^*)}{a_{11}(z)}\rho_1^*(\hat{z}), \qquad \rho_2(-z) = -\frac{a_{11}(z)a_{11}(\hat{z}^*)}{a_{11}^*(z)}\left[\rho_2(z) - \frac{z^2}{q_o^2}\rho_1^*(z^*)\rho_2(\hat{z}^*)\right], \quad z \in \mathbb{R}, \tag{3.5}$$

with $a_{11}(z)$ as in (2.23a). Note that the parity-induced symmetry of the reflection coefficients involves the discrete eigenvalues through $a_{11}(z)$. Note that, even though it was not explicitly noted in earlier works, this is not a novel feature of the problem for the Manakov system, since it also arises the scalar NLS equation, both with zero background [9, 13] and nonzero background (see [18, 41] and appendix A.1).

3.2. *Odd initial conditions*

Now we assume the IC $\mathbf{q}(x,0)$ is an odd function of x for all $x \in \mathbb{R}$, and as a consequence $\mathbf{q}_+ = -\mathbf{q}_-$. Direct computation shows that if $v(x,t,k)$ is any solution of the Lax pair, then so is $w(x,t,k) = v(-x,t,-k)$.

As to the asymptotic eigenvectors, if $\mathbf{q}_+ = \mathbf{q}_-$, then $Y_+(-z) = -Y_-(z)$. Therefore, comparing the boundary conditions of the Jost eigenfunctions it follows that

$$\phi_{\pm,j}(-x,t,-z) = -\phi_{\mp,j}(x,t,z), \quad j = 1,2,3, \tag{3.6}$$

or, in matrix form

$$\phi_{\mp}(-x,t,-z) = -\phi_{\pm}(x,t,z). \tag{3.7}$$

The adjoint eigenfunctions satisfy the same symmetry (3.6) as $\phi_\pm$, and therefore from (2.9) it follows

$$\bar{\chi}(-x,t,-z) = -\chi(x,t,z). \tag{3.8}$$

Now, from $\phi_-(x,t,z) = \phi_+(x,t,z)\mathbf{A}(z)$ and $\phi_-(-x,t,-z) = \phi_+(-x,t,-z)\mathbf{A}(-z)$ we then find

$$\mathbf{A}(-z) = \mathbf{A}^{-1}(z) \tag{3.9}$$

or, entry-wise, $a_{ij}(-z) = b_{ij}(z)$ where again $b_{ij}(z)$ denote the entries of $\mathbf{A}^{-1}(z)$. As before, one can easily verify the above symmetries are consistent with the representations of the supplemental analytic eigenfunctions $\chi, \bar{\chi}$ in terms of the Jost eigenfunctions on the real axis.

Importantly, although the parity-induced symmetries for the eigenfunctions are different for even or odd initial conditions, the scattering matrix $A(z)$ satisfies the same symmetry in both cases. Therefore, the symmetries of the reflection coefficients in the odd case are also the same as in the even case, namely (3.5). This may appear surprising, since the reflection coefficients are the nonlinear analogue of the Fourier transform, and the symmetries of the Fourier transforms of even and odd functions are different. In fact, even the reflection coefficient in the case of BVPs with Neumann or Dirichlet BCs (corresponding to even or odd ICs, respectively) with zero BC at infinity are different [7, 9, 13]. The fact that the situation is different in the case of NZBC stems from the choice of normalization for the eigenfunctions.

3.3. Even–odd initial conditions

Next, we assume the IC $\mathbf{q}(x,0)$ consists of components with mixed parities. We begin with the case of an even first component and an odd second component w.r.t. x, i.e., $q_1(-x,0) = q_1(x,0)$ and $q_2(-x,0) = -q_2(x,0)$ for all $x \in \mathbb{R}$, which we call an even–odd IC. In vector form, one has:

$$\mathbf{q}(x,0) = \sigma_3\, \mathbf{q}(-x,0). \tag{3.10}$$

Equation (3.10) implies a more complicated symmetry for the matrix potential compared to the case of even or odd ICs, namely:

$$Q(x) = D_{++-}Q(-x)D_{++-}. \tag{3.11}$$

Then one can show that if $v(x,t,k)$ is any solution of the Lax pair (2.1), then so is $w(x,t,k) = D_{+-+}v(-x,t,-k)$.

As to the asymptotic eigenvectors, one can easily verify that if $\mathbf{q}_\pm = \sigma_3\mathbf{q}_\mp$, then

$$Y_{\pm,j}(-z) = -D_{+-+}Y_{\mp,j}(z), \quad j = 1, 3, \quad Y_{\pm,2}(-z) = D_{+-+}Y_{\mp,2}(z). \tag{3.12}$$

Therefore, comparing the boundary conditions we find that the eigenfunctions satisfy the following symmetry:

$$\phi_\mp(-x,t,-z) = -D_{+-+}\phi_\pm(x,t,z)\,D_{+-+}. \tag{3.13}$$

Now, from the scattering relation (2.6) evaluated at (x,t,z) and $(-x,t-z)$, we find

$$\mathbf{A}(-z) = D_{+-+}\mathbf{A}^{-1}(z)D_{+-+} \tag{3.14}$$

or entry-wise

$$a_{11}(-z) = b_{11}(z), \qquad a_{22}(-z) = b_{22}(z), \qquad a_{33}(-z) = b_{33}(z), \tag{3.15a}$$

$$a_{12}(-z) = -b_{12}(z), \qquad a_{23}(-z) = -b_{23}(z), \qquad a_{31}(-z) = b_{31}(z), \tag{3.15b}$$

$$a_{21}(-z) = -b_{21}(z), \qquad a_{32}(-z) = -b_{32}(z), \qquad a_{13}(-z) = b_{13}(z), \tag{3.15c}$$

where again $b_{ij}(z)$ denote the entries of $\mathbf{A}^{-1}(z)$. The supplemental analytic eigenfunctions satisfy

$$\bar{\chi}(-x,t,-z) = D_{+-+}\chi(x,t,z), \tag{3.16}$$

which again on the real k-axis are consistent with equation (2.11).

Recall that the parity symmetry of the scattering coefficients for even–odd ICs is given by equation (3.15). Using the first and the second symmetry of the scattering matrix, one can obtain the following induced parity symmetry of the reflection coefficients,

$$\rho_1(-z) = -\frac{q_o^2}{z^2}\frac{a_{11}(\hat{z}^*)}{a_{11}(z)}\rho_1^*(\hat{z}), \qquad \rho_2(-z) = \frac{a_{11}(z)a_{11}(\hat{z}^*)}{a_{11}^*(z)}\left[\rho_2(z) - \frac{z^2}{q_o^2}\rho_1^*(z^*)\rho_2(\hat{z}^*)\right], \quad z \in \mathbb{R}, \tag{3.17}$$

with $a_{11}(z)$ as in (2.23a).

In order for (3.10) to be compatible with the requirement (1.2) that $\mathbf{q}_\pm$ be parallel, one of the components of $\mathbf{q}_\pm$ needs to be zero. (Even though the direct problem carries through when $\mathbf{q}_\pm$ are not parallel, the formulation of the inverse problem require this assumption.) Without loss of generality, we can take $q_{\pm,1} = 0$, implying $\mathbf{q}_+ = -\mathbf{q}_-$. Importantly, note $q_{\pm,1} = 0$ does not imply $q_1(x,t) \equiv 0$. Nonetheless, requiring $q_{\pm,1} = 0$ means that $q_1(x,t)$ and $q_2(x,t)$ are respectively the bright and dark components of the solution. With the additional constraint $\mathbf{q}_+ = -\mathbf{q}_-$, the symmetry (3.12) for the asymptotic eigenvectors simplifies to $Y_+(-z) = -Y_-(z)$, namely, the same symmetry as in the case of odd ICs. We emphasize, however, that the symmetries of the eigenfunctions $\phi_\pm(x,t,z)$ are still given by (3.13), because they are determined by the relation between $v(x,t,z)$ and $w(x,t,z)$ above. In turn, the latter is determined by the relation between $Q(x,t)$ and $Q(-x,t)$ *for all* x (i.e., (3.11)), not just its limiting value as $x \to \pm\infty$. As a result, *the symmetries with even–odd ICs do not coincide with those for purely odd ICs, even when* $q_{\pm,1} = 0$.

3.4. Odd–even initial conditions

Finally, we consider initial conditions corresponding to an odd first component and an even second component w.r.t. x, i.e.,

$$\mathbf{q}(x,0) = -\sigma_3\,\mathbf{q}(-x,0), \tag{3.18}$$

which we call an odd–even IC, and for which $Q(x,t) = D_{+-+}Q(-x,t)D_{+-+}$. In this case, one can show that if $v(x,t,k)$ is any solution of the Lax pair (2.1), then so is $w(x,t,k) = D_{++-}v(-x,t,-k)$.

Also, the asymptotic eigenvectors are such that

$$D_{++-}Y_{\pm,j}(-z) = -Y_{\mp,j}(z), \quad j = 1,3, \qquad D_{++-}Y_{\pm,2}(-z) = Y_{\mp,2}(z), \tag{3.19}$$

i.e., $Y_+(z) = -D_{++-}Y_-(z)D_{+-+}$, and therefore the eigenfunctions satisfy the symmetry

$$\phi_\mp(x,t,z) = -D_{++-}\phi_\pm(-x,t,-z)D_{+-+}. \tag{3.20}$$

The adjoint eigenfunctions satisfy the same symmetry (3.20) as $\phi_\pm$, and therefore from (2.9) it follows that

$$\bar{\chi}(-x,t,-z) = D_{++-}\chi(x,t,z). \tag{3.21}$$

Now, from $\phi_-(x,t,z) = \phi_+(x,t,z)\mathbf{A}(z)$ and $\phi_-(-x,t,-z) = \phi_+(-x,t,-z)\mathbf{A}(-z)$ we then find

$$\mathbf{A}(-z) = D_{+-+}\mathbf{A}^{-1}(z)D_{+-+}. \tag{3.22}$$

As before, one can check that the above symmetries are consistent with the representations of the supplemental analytic eigenfunctions $\chi, \bar{\chi}$ in terms of the Jost eigenfunctions on the real axis. Note that the scattering matrix satisfies the same symmetry in both even–odd and odd–even ICs. Therefore, the symmetries of the reflection coefficients in the odd–even case are also the same as in the even–odd case, namely (3.17).

Restricting ourselves to parallel BCs, in combination with (3.18), requires one of the components of $\mathbf{q}_\pm$ to be zero. As before, we assume $q_{\pm,1} = 0$ and hence $\mathbf{q}_+ = \mathbf{q}_-$. With this constraint on the boundary conditions, the symmetry in the asymptotic eigenvectors (3.19) simplifies to $Y_+(-z) = JY_-(z)$, i.e., the same asymptotic symmetry satisfied in the case of even ICs. Again, we stress that the symmetries of the eigenfunctions $\phi_\pm(x,t,z)$ are different in the two cases, and therefore the symmetries with odd–even ICs and $q_{\pm,1} = 0$ do not coincide with those with purely even ICs.

4. Solutions with parity: discrete spectrum and norming constants

In this section we use the symmetries of the eigenfunctions and scattering coefficients that we derived in section 3 to characterize the symmetries of the discrete spectrum and norming constants. Then in section 5 we will use all these results to characterize the solutions of the Manakov system.

4.1. General considerations

Theorem 4.1. *If the ICs for the Manakov system have a well-defined spatial parity, the discrete eigenvalues of the scattering problem come in symmetric quartets* $\{\zeta_j, -\zeta_j^*, \zeta_j^*, -\zeta_j\}$ *on the circle* C_o*, with two in each half plane, and in symmetric octets off the circle* C_o*, with a quartet* $\{z_j, -z_j^*, q_o^2/z_j^*, -q_o^2/z_j\}$ *in the upper half-plane and a symmetric one* $\{z_j^*, -z_j, q_o^2/z_j, -q_o^2/z_j^*\}$ *in the lower half-plane.*

Proof. Recall from section 2 that the discrete eigenvalues are zeros of the scattering coefficients $a_{11}(z)$, $a_{33}(z)$, $b_{11}(z)$ and $b_{33}(z)$. Also recall that in all four parity cases considered in section 3 we have $a_{11}(-z) = b_{11}(z)$. Finally, recall that the first symmetry yields $b_{11}(z) = a_{11}^*(z^*)$, and the second symmetry yields $a_{33}(z) = a_{11}(\hat{z}^*)$, and similarly for the other analytic coefficients. Combining the above symmetries, we then have

$$a_{11}(-z) = a_{11}^*(z^*). \tag{4.1}$$

Hence any zero z_o of $a_{11}(z)$ in the right half plane is accompanied by a corresponding zero $-z_o^*$ in the left half plane. The same holds for the other analytic scattering coefficients, namely $a_{33}(z), b_{11}(z), b_{33}(z)$.

Remark 4.2. If an eigenvalue $z_j = \mathrm{i}Z_j$ is purely imaginary (i.e., $Z_j \in \mathbb{R}$), then $-z_j^* = \mathrm{i}Z_j = z_j$. Accordingly, we refer to discrete eigenvalues on the imaginary axis as self-symmetric. Importantly, the relation between the location of the discrete eigenvalue and the speed and the amplitudes (for both the bright and the dark component) of the associated soliton (cf section 2 and [38]) implies that the soliton generated by a symmetric eigenvalue has opposite speed but equal amplitudes (in each component) as the original soliton. The relation between eigenvalues and soliton speed then also implies that self-symmetric eigenvalues correspond to stationary solitons.

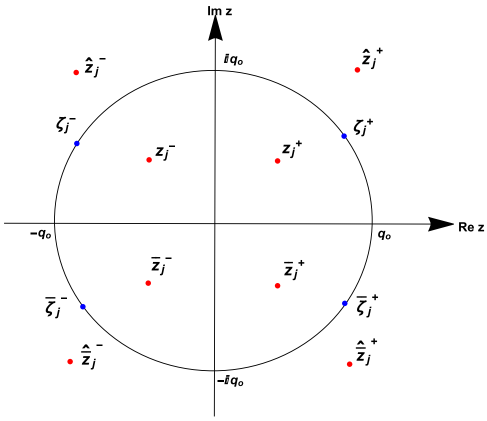

Figure 1. A symmetric quartet and a symmetric octet of discrete eigenvalues resulting from the symmetries of the eigenfunctions and scattering data. The norming constants associated with $\zeta_j^\pm, \bar{\zeta}_j^\pm$ are $\gamma_j^\pm, \bar{\gamma}_j^\pm$, while those associated with $z_j^\pm, \hat{z}_j^\pm, \bar{z}_j^\pm, \hat{\bar{z}}_j^\pm$ are respectively $\delta_j^\pm, \hat{\delta}_j^\pm, \bar{\delta}_j^\pm, \hat{\bar{\delta}}_j^\pm$.

For the discrete eigenvalues on the circle C_o, the only possible self-symmetric eigenvalue is at $z = \mathrm{i}q_o$, together with its counterpart at $z = -\mathrm{i}q_o$. More generally, for any quartet $\{\pm\zeta_j, \pm\zeta_j^*\}$ of eigenvalues on the circle C_o, we will denote $\zeta_j^+ = \zeta_j$ and $\zeta_j^- = -\zeta_j^*$ the eigenvalues in $\mathbb{C}^+$, and $\bar{\zeta}_j^+ = \zeta_j^*$ and $\bar{\zeta}_j^- = -\zeta_j$ those in $\mathbb{C}^-$, with a similar notation for the associated proportionality constants. Similarly, for any octet $\{\pm z_j, \pm z_j^*, \pm q_o^2/z_j, \pm q_o^2/z_j^*\}$ of eigenvalues off the circle C_o, we will use the notation

$$z_j^+ = z_j, \qquad z_j^- = -z_j^*, \qquad \hat{z}_j^+ = q_o^2/z_j^*, \qquad \hat{z}_j^- = -q_o^2/z_j, \tag{4.2a}$$

$$\bar{z}_j^+ = z_j^*, \qquad \bar{z}_j^- = -z_j, \qquad \hat{\bar{z}}_j^+ = q_o^2/z_j, \qquad \hat{\bar{z}}_j^- = -q_o^2/z_j^*, \tag{4.2b}$$

where the overbar identifies the eigenvalues in $\mathbb{C}^-$ (see figure 1). A similar notation will be used for the associated proportionality constants, denoted, correspondingly, as $d_j^\pm, \hat{d}_j^\pm, \bar{d}_j^\pm, \hat{\bar{d}}_j^\pm$. For the self-symmetric eigenvalues (i.e. eigenvalues on the imaginary axis), one has $z_j^+ = z_j^-$, $\hat{z}_j^+ = \hat{z}_j^-$, $\bar{z}_j^+ = \bar{z}_j^-$ and $\hat{\bar{z}}_j^+ = \hat{\bar{z}}_j^-$, which then implies that $d_j^+ = d_j^-$ and similarly for all the other proportionality constants. Moreover, to fix the notation we will order the discrete eigenvalues on the circle as $\{\zeta_1^\pm, \ldots, \zeta_{N_1}^\pm\}$ on $C_o \cap \mathbb{C}^+$, with the first m_1 being self-symmetric, with $m_1 = 0$ or 1 (implying $\zeta_1^+ = \zeta_1^- = \mathrm{i}q_o$ when $m_1 = 1$). Similarly, we will order the discrete eigenvalues off the circle as $\{z_1^\pm, \ldots, z_{m_2}^\pm, z_{m_2+1}^\pm, \ldots, z_{N_2}^\pm\}$ in $D_o \cap \mathbb{C}^+$, with the first m_2 of them being self-symmetric eigenvalues (implying $\arg z_n^\pm = \pi/2$, or, equivalently, $z_n^+ = z_n^-$).

As we will show below, the symmetries of the eigenfunctions obtained in section 3 imply that the norming constants associated with symmetric eigenvalues ζ_j and $-\zeta_j^*$ are related, and similarly for those associated to eigenvalues z_j and $-z_j^*$. Moreover, we will also show that the norming constant associated to self-symmetric eigenvalues must satisfy an extra constraint, resulting from the fact that in this case the norming constant associated with z_j and the one associated with $-z_j^*$ are one and the same, and the same is true for the pair of self-symmetric eigenvalues in $\mathbb{C}^-$.

For the scalar defocusing NLS equation, discrete eigenvalues are confined to the circle C_o, and they can be shown to always be simple [24]. In the Manakov system, however, one can have eigenvalues off the circle C_o, associated to zeros of $a_{11}(z)$ inside the disk D_o and their symmetric counterparts in the quartet (see [38] for details). Furthermore, discrete eigenvalues off the circle can have higher multiplicity (examples of solutions corresponding to multiple zeros were given in [15]), and eigenvalues on the circle C_o could also in principle have higher multiplicity if the solution is not a trivial 'vectorization' of a solution of the scalar NLS equation (e.g., see appendices A.2 and A.3). For simplicity, in this work we assume that all discrete eigenvalues are simple. We then have:

Lemma 4.3. *If the ICs of the Manakov system have a well-defined spatial parity, the function $R(z)$ defined in* (2.24) *is even:*

$$R(-z) = R(z), \quad z \in \mathbb{R}. \tag{4.3}$$

Moreover, the asymptotic phase difference $\Delta\theta$ of the solution is related to the numbers m_1 and m_2 of self-symmetric eigenvalues on and off the circle as follows:

$$e^{i\Delta\theta} = (-1)^{m_1+m_2}. \tag{4.4}$$

Proof. The symmetry (4.3) for the function $R(z)$ can be obtained using the extra symmetry of the scattering coefficients derived in section 3 and the cofactor expansion for both $A(z)$ and $B(z)$ along with the first symmetry (cf appendix A.5). Accounting for the parity symmetry of the eigenvalues, the 'theta condition' (2.23b) becomes

$$e^{i\Delta\theta} = (-1)^{m_1+m_2} \exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log[1-R(\zeta)]}{\zeta}\,d\zeta\right\}. \tag{4.5}$$

As a result of the symmetry of $R(z)$, the integrand in (4.5) is odd, which implies that the radiation does not contribute to the asymptotic phase difference of the potential. Consequently, (4.5) reduces simply to (4.4).

Remark 4.4. Lemma 4.3 implies that, for ICs with spatial parity, *the only spectral data that contribute to the asymptotic phase difference of the potential are the self-symmetric eigenvalues* (i.e., *stationary solitons) on or inside the circle C_o. This is in contrast to solutions without spatial parity, for which it is known that the radiation can produce a nonzero contribution to the asymptotic phase difference even for the scalar NLS equation [*16*].*

For later reference, it is also convenient to evaluate the trace formula (2.23a) at the self-symmetric eigenvalues. After straightforward manipulations, one finds (see appendix A.5)

$$\begin{aligned} a_{11}(\hat{z}_j^+) = {} & \left(\frac{q_o - Z_j}{q_o + Z_j}\right)^{m_1} \prod_{n=1}^{m_2} \frac{q_o^2 - Z_n Z_j}{q_o^2 + Z_n Z_j} \prod_{n=m_1+1}^{(N_1+m_1)/2} \times \frac{\left(|\hat{z}_j^+|^2 - q_o^2\right)^2 + \left(2|\hat{z}_j^+|\mathrm{Re}(\zeta_n^+)\right)^2}{|(\hat{z}_j^+ - \bar{\zeta}_n^+)(\hat{z}_j^+ - \bar{\zeta}_n^-)|^2} \\ & \times \prod_{n=m_2+1}^{(N_2+m_2)/2} \frac{\left(|\hat{z}_j^+|^2 - |z_n^+|^2\right)^2 + \left(2|\hat{z}_j^+|\mathrm{Re}(z_n^+)\right)^2}{|(\hat{z}_j^+ - \bar{z}_n^+)(\hat{z}_j^+ - \bar{z}_n^-)|^2} \\ & \times \exp\left\{-\frac{q_o^2}{Z_j\pi}\int_0^{\infty}\frac{Z_j\log[1-R(\zeta)]}{Z_j\zeta^2 + q_o^2}\,d\zeta\right\}, \\ j = {} & 1,\dots,m_2, \end{aligned} \tag{4.6a}$$

where we denoted the self-symmetric eigenvalues off the circle by $z_n = \mathrm{i}Z_n$ with $0 < Z_n < q_o$ for $n = 1, \ldots, m_2$. Similarly, from (2.25) with $m_1 = 1$ evaluated at $\zeta_1^+ \equiv \mathrm{i}q_o$ we have:

$$a'_{11}(\mathrm{i}q_o) = \frac{1}{2\mathrm{i}q_o}\prod_{n=1}^{m_2}\frac{q_o - Z_n}{q_o + Z_n}\prod_{n=2}^{(N_1+1)/2} \times \frac{(2q_o\mathrm{Re}(\zeta_n^+))^2}{|(\mathrm{i}q_o - \bar{\zeta}_n^+)(\mathrm{i}q_o - \bar{\zeta}_n^-)|^2}$$
$$\times \prod_{n=m_2+1}^{(N_2+m_2)/2} \frac{\left(q_o^2 - |z_n^+|^2\right)^2 + \left(2q_o\,\mathrm{Re}(z_n^+)\right)^2}{|(\mathrm{i}q_o - \bar{z}_n^+)(\mathrm{i}q_o - \bar{z}_n^-)|^2} \times \exp\left\{-\frac{q_o}{\pi}\int_0^\infty \frac{\log[1 - R(\zeta)]}{\zeta^2 + q_o^2}\,\mathrm{d}\zeta\right\}. \tag{4.6b}$$

Next we discuss the parity symmetries of the norming constants for eigenvalues on and off the circle.

4.2. Symmetries of the norming constants for the eigenvalues on the circle

Recall that eigenvalues on the circle arise in symmetric quartets. For the associated eigenfunctions, one has:

$$\phi_{-,1}(x,t,\zeta_n^\pm) = c_n^\pm \phi_{+,3}(x,t,\zeta_n^\pm), \qquad \phi_{-,3}(x,t,\bar{\zeta}_n^\pm) = \bar{c}_n^\pm \phi_{+,1}(x,t,\bar{\zeta}_n^\pm), \tag{4.7}$$

$$\gamma_n^\pm = \frac{c_n^\pm}{\zeta_n^\pm a'_{11}(\zeta_n^\pm)}, \qquad \bar{\gamma}_n^\pm = \frac{\bar{c}_n^\pm}{a'_{33}(\bar{\zeta}_n^\pm)}. \tag{4.8}$$

Using the parity symmetries of the eigenfunctions one finds:

$$\phi_{-,1}(x,t,\zeta_n^+) = c_n^+\phi_{+,3}(x,t,\zeta_n^+) \quad \Leftrightarrow \quad \phi_{+,1}(-x,t,-\zeta_n^+) = c_n^+\phi_{-,3}(-x,t,-\zeta_n^+)$$
$$\Leftrightarrow \quad \phi_{+,1}(x,t,\bar{\zeta}_n^-) = c_n^+\phi_{-,3}(x,t,\bar{\zeta}_n^-) \tag{4.9}$$

and comparing with (4.7) we obtain

$$c_n^\pm \bar{c}_n^\mp = 1. \tag{4.10}$$

(The symmetry corresponding to the upper sign is obtained directly from the above equations; the one for the lower sign is obtained in a similar way.) Using the extra symmetries we obtained in (4.10), as well as $a'_{33}(z) = -q_o^2 a'_{11}(q_o^2/z)/z^2$, we find from (2.21) the following symmetries for the norming constants:

$$\gamma_n^\pm \bar{\gamma}_n^\mp = \frac{1}{\zeta_n^\pm a'_{11}(\zeta_n^\pm)a'_{33}(\bar{\zeta}_n^\mp)} = \frac{1}{\zeta_n^\mp a'_{11}(\zeta_n^\pm)a'_{11}(\zeta_n^\mp)}. \tag{4.11}$$

As mentioned above, an extra symmetry applies for self-symmetric eigenvalues. Recall that there can only be at most one self-symmetric eigenvalue on the upper-half of the circle C_o, which we have chosen to correspond to $m_1 = 1$, i.e., $\zeta_1^+ = \mathrm{i}q_o$. Using (4.7) and the fact that $\zeta_1^+ = \zeta_1^-$ and $\bar{\zeta}_1^+ = \bar{\zeta}_1{}^-$, we obtain the constraints $c_1^+ = c_1^-$ and $\bar{c}_1^+ = \bar{c}_1^-$. Consequently, it follows from (2.21) that $\gamma_1^+ = \gamma_1^-$ and $\bar{\gamma}_1^+ = \bar{\gamma}_1^-$, which yield

$$\gamma_1^+\ \bar{\gamma}_1^+ = \frac{1}{\zeta_1^+(a'_{11}(\zeta_1^+))^2}. \tag{4.12a}$$

On the other hand, the first two symmetries (2.22) imply

$$\bar{\gamma}_1^+ = \zeta_1^+ \gamma_1^+, \tag{4.13}$$

and combining two equations above finally yields

$$\bar{\gamma}_1^+ = \pm \frac{1}{a'_{11}(\zeta_1^+)}. \tag{4.14}$$

To obtain a regular solution, one must pick the appropriate sign in (4.14) to make $\mathrm{Im}\,\bar{\gamma}_1^+ > 0$. Indeed, we can see using (4.6b) that $a'_{11}(\zeta_1^+)$ is purely imaginary, with a negative imaginary part.

Therefore, to have regular solutions we take

$$\bar{\gamma}_1^+ = \frac{1}{a'_{11}(\zeta_1^+)}. \tag{4.15}$$

4.3. Symmetries of the norming constants for the eigenvalues off the circle

Recall that eigenvalues off the circle arise in symmetric octets, labeled as in (4.2). For the associated eigenfunctions, one has

$$\bar{\chi}(x,t,\bar{z}_n^{\pm}) = \bar{d}_n^{\pm} \phi_{+,1}(x,t,\bar{z}_n^{\pm}), \qquad \phi_{-,1}(x,t,z_n^{\pm}) = d_n^{\pm} \chi(x,t,z_n^{\pm}), \tag{4.16}$$

$$\delta_n^{\pm} = \frac{d_n^{\pm}}{z_n^{\pm} a'_{11}(z_n^{\pm})}, \qquad \bar{\delta}_n^{\pm} = -\frac{\bar{d}_n^{\pm}}{\bar{z}_n^{\pm} b'_{11}(\bar{z}_n^{\pm})}. \tag{4.17}$$

Next, we derive the parity induced symmetries of the norming constants. Unlike the case of eigenvalues on the circle, however, here we need to consider separately the case of purely even or purely odd ICs, and the case of ICs with mixed parity.

Case 1. Purely odd or purely even ICs. Using (3.2) and (3.7) we have

$$\begin{aligned} \phi_{1,-}(x,t,z_n) = d_n^+ \chi(x,t,z_n) \quad &\Leftrightarrow \quad \phi_{+,1}(-x,t,-z_n) = d_n^+ \bar{\chi}(-x,t,-z_n) \\ &\Leftrightarrow \quad \phi_{+,1}(x,t,\bar{z}_n^-) = d_n^+ \bar{\chi}(x,t,\bar{z}_n^-) \end{aligned} \tag{4.18}$$

and comparing with (4.16) we obtain

$$d_n^{\pm} \bar{d}_n^{\mp} = 1. \tag{4.19}$$

(The symmetry corresponding to the upper sign is obtained directly from (4.18); the one for the lower sign is obtained in a similar way.) From (4.19) and (4.17), using $a'_{11}(z_n^{\pm}) = (b'_{11}(\bar{z}_n^{\pm}))^*$, one obtains

$$\delta_n^{\pm} \bar{\delta}_n^{\mp} = -\frac{1}{z_n^{\pm} \bar{z}_n^{\mp} a'_{11}(z_n^{\pm}) b'_{11}(\bar{z}_n^{\mp})} = -\frac{1}{z_n^{\pm} \bar{z}_n^{\mp} a'_{11}(z_n^{\pm})(a'_{11}(z_n^{\mp}))^*}. \tag{4.20}$$

We now discuss self-symmetric eigenvalues off the circle, i.e., eigenvalues on the imaginary axis. Using (4.16) and the fact that $z_j^+ = z_j^-$ and $\bar{z}_j^+ = \bar{z}_j^-$, for $j = 1, \ldots, m_2$, we obtain

$$d_j^+ = d_j^-, \qquad \bar{d}_j^+ = \bar{d}_j^-. \tag{4.21}$$

Thus the norming constants of self-symmetric eigenvalues satisfy the additional symmetry

$$\delta_j^+ = \delta_j^-, \qquad \bar{\delta}_j^+ = \delta_j^-, \quad j = 1, \dots, m_2, \tag{4.22a}$$

which, combined with (4.20), yields

$$\delta_j^+ \bar{\delta}_j^+ = -\frac{1}{|z_j^+ a_{11}'(z_j^+)|^2}. \tag{4.22b}$$

Now (2.22) implies

$$\bar{\delta}_j^+ = 2\hat{z}_j^+ \lambda(\bar{z}_j^+) a_{11}(\hat{z}_j^+)(\delta_j^+)^*, \tag{4.22c}$$

and therefore the norming constants associated with the self-symmetric eigenvalues must be such that

$$|\delta_j^+|^2 = -\frac{1}{2\hat{z}_j^+ \lambda(\bar{z}_j^+) a_{11}(\hat{z}_j^+)\, |z_j^+ a_{11}'(z_j^+)|^2}, \quad j = 1, \dots, m_2. \tag{4.23}$$

One can now use (4.6a) to examine $a_{11}(\hat{z}_j^+)$. It is straightforward to show that, for any ICs with defined spatial parity, $a_{11}(z) > 0$ for all $z \in \mathrm{i}\mathbb{R}^+$. But this implies that the rhs of (4.23) is always negative! We therefore conclude that *purely even and purely odd ICs cannot admit any self-symmetric eigenvalues off the circle*. That is, $m_2 = 0$ in those cases.

Since self-symmetric eigenvalues off the circle correspond to dark–bright solitons with zero velocity (recall that the velocity of the a dark–bright soliton is given by the real part of the discrete eigenvalue [38]), we can also state the above result by saying that for even or odd ICs no pure stationary dark–bright soliton solution can exist. Importantly, this results holds for all even or odd solutions of the Manakov system, not just for pure soliton solutions. That is, the result applies independently of whether or not the reflection coefficients are zero.

Case 2. Mixed parities: even–odd or odd–even ICs. Using (3.13) and (3.20) we have

$$\begin{aligned} \phi_{1,-}(x,t,z_n) = d_n^+ \chi(x,t,z_n) \quad &\Leftrightarrow \quad \phi_{+,1}(-x,t,-z_n) = -d_n^+ \bar{\chi}(-x,t,-z_n) \\ &\Leftrightarrow \quad \phi_{+,1}(x,t,\bar{z}_n^-) = -d_n^+ \bar{\chi}(x,t,\bar{z}_n^-) \end{aligned} \tag{4.24}$$

and comparing with (4.16) we obtain

$$d_n^\pm \bar{d}_n^\mp = -1. \tag{4.25}$$

(As before, the symmetry corresponding to the upper sign is obtained directly from (4.24); the one for the lower sign is obtained in a similar way.) From (4.19) and (4.17), using $a_{11}'(z_n^\pm) = (b_{11}'(\bar{z}_n^\pm))^*$, we obtain

$$\delta_n^\pm \bar{\delta}_n^\mp = \frac{1}{z_n^\pm \bar{z}_n^\mp a_{11}'(z_n^\pm)(a_{11}'(z_n^\mp))^*}. \tag{4.26}$$

The norming constants of self-symmetric eigenvalues must satisfy

$$\delta_j^+ = \delta_j^-, \qquad \bar{\delta}_j^+ = \bar{\delta}_j^-, \quad j = 1, \dots, m_2, \tag{4.27a}$$

and the latter, combined with (4.27), in this case yield

$$\delta_j^+ \bar{\delta}_j^+ = \frac{1}{|z_j^+ a_{11}'(z_j^+)|^2}. \tag{4.27b}$$

Equation (2.22) still requires (4.22c) to hold, and therefore the norming constants associated with the self-symmetric eigenvalues in the case of mixed parity ICs must be such that

$$|\delta_j^+|^2 = \frac{1}{2\hat{z}_j^+ \lambda(\bar{z}_j^+) a_{11}(\hat{z}_j^+)\, |z_j^+ a_{11}'(z_j^+)|^2}, \quad j = 1, \dots, m_2. \tag{4.28}$$

As before, one can now use (4.6a) to show that $a_{11}(\hat{z}_j^+) > 0$ and therefore the rhs of (4.28) is positive. Therefore, contrary to the case of purely even or purely odd ICs, we conclude that in the case of mixed parities (even–odd or odd–even) ICs, self-symmetric eigenvalues (i.e., stationary dark–bright solitons) are allowed. The associated norming constants satisfy the constraint (4.28).

5. Solutions with parity: solution behavior

Having derived the symmetries of the eigenfunctions and scattering data in sections 3 and 4, we now use those results to characterize the behavior of the corresponding solutions.

We distinguish three cases: (i) solutions with only eigenvalues on the circle (i.e., with only dark solitons and without dark–bright solitons), (ii) solutions with only eigenvalues off the circle (i.e., with only dark–bright solitons and without dark solitons), and (iii) solutions a combination of eigenvalues on and off the circle (i.e., with both dark and dark–bright solitons). For clarity, we find it convenient to discuss these three cases separately.

5.1. Solutions with only dark–dark solitons

We first consider the case in which only eigenvalues on the circle C_o appear (i.e. $N_2 = 0$). In other words, we look at the case in which only dark–dark solitons are present in the solution.

Theorem 5.1 (Even ICs without dark–bright solitons). *If the ICs are even and no discrete eigenvalues off the circle are present, no black soliton (i.e., a self-symmetric eigenvalue at $z = \mathrm{i}q_o$) can be present in the solution. Moreover, the reflection coefficients obey the parity symmetry (3.5) and the norming constants of the non-self-symmetric eigenvalues satisfy the parity symmetry (4.11).*

Proof. For even ICs we have $\mathbf{q}_+ = \mathbf{q}_-$, which implies $e^{\mathrm{i}\Delta\theta} = 1$. Since $N_2 = 0$, we have $m_2 = 0$. Then (4.4) becomes $(-1)^{m_1} = 1$, which yields $m_1 = 0$ (since m_1 can only be either 0 or 1). The symmetries for the reflection coefficients and norming constants were derived in sections 3.1 and 4.2, respectively.

One could also exclude the existence of a black soliton with even ICs by looking at the constraint (4.14) for the norming constants associated to the self-symmetric eigenvalue $\mathrm{i}q_o$ (as was done for the NLS equation with zero background in [13]). Here, however, the result is also a direct consequence of the relation between discrete eigenvalues and the asymptotic phase difference of the solution.

Theorem 5.2 (Odd ICs without dark–bright solitons). *If the ICs are odd and no discrete eigenvalues off the circle are present, the solution always contains a black soliton (i.e., a*

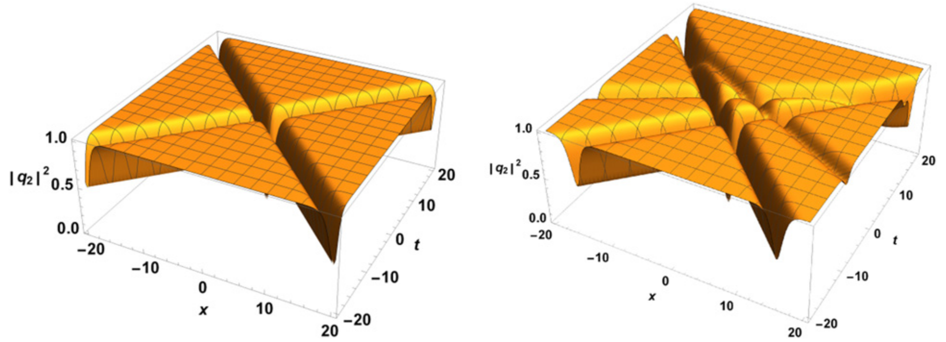

Figure 2. Soliton solutions of the Manakov system with even ICs and only dark solitons. Left: $\zeta_1^+ = \mathrm{e}^{\mathrm{i}\pi/3}, \bar{\gamma}_1^+ = 1 - \mathrm{i}$, right: $\zeta_1^+ = \mathrm{e}^{\mathrm{i}\pi/3}, \zeta_2^+ = \mathrm{e}^{\mathrm{i}\pi/6}, \bar{\gamma}_1^+ = \mathrm{e}^{-5}\mathrm{i}, \bar{\gamma}_2^+ = \mathrm{e}^{5}\mathrm{i}$. Here and in figure 3 we only plot $|q_2(x,t)|$, since $q_1(x,t)$ is identically zero in these cases.

self-symmetric eigenvalue at $z = \mathrm{i}q_o$), *whose norming constant satisfies the constraint* (4.14), *while the reflection coefficients obey the parity symmetry* (3.5) *and the norming constants of all other eigenvalues satisfy the parity symmetry* (4.11).

Proof. We have $m_2 = 0$, as above. Since the IC is odd, we have $\mathbf{q}_+ = -\mathbf{q}_-$ and hence $\mathrm{e}^{\mathrm{i}\Delta\theta} = -1$. Then (4.4) yields $-1 = (-1)^{m_1}$ and therefore $m_1 = 1$, which implies that the solution always exhibits a black soliton. The constraint (4.14) and the symmetry (4.11) for the norming constants were derived in section 4, while the symmetries for the reflection coefficients and norming constants were derived in sections 3.2 and 4.2, respectively.

Theorem 5.3 (Mixed-parity ICs without dark–bright solitons). *If the ICs are odd–even with* $q_{\pm,1} = 0$, *and no discrete eigenvalues off the circle are present, the same conclusions as in theorem* 5.1 *apply, except that the reflection coefficients now satisfy the parity symmetries* (3.17). *On the other hand, if the ICs are even–odd, the same conclusions as in theorem* 5.2 *apply, except that the reflection coefficients now satisfy the parity symmetries* (3.17).

Proof. Recall that even–odd and odd–even ICs imply that the BCs satisfy $\mathbf{q}_+ = \pm\sigma_3\mathbf{q}_-$, and in section 3 we took the first component of both $\mathbf{q}_\pm$ to be zero to maintain compatibility with the case of parallel BCs. Therefore, the constraints on the self-symmetric eigenvalues with odd–even ICs coincide with those for purely even ICs and follow theorem 5.1, whereas the constraints for even–odd ICs coincide with those for purely odd ICs and follow theorem 5.2. The constraints on the norming constants of the non-self-symmetric eigenvalues are the same in all four cases. On the other hand, the symmetries of the reflection coefficients, derived in sections 3.3 and 3.4, differ from those for even and odd ICs (derived in sections 3.1 and 3.2).

Remark 5.4. We emphasize that all the results of theorems 5.1–5.3 apply independently of whether or not the solution is purely solitonic. In other words, all of these results also hold for solutions containing nontrivial dispersive components i.e., solutions with contributions arising from nonzero reflection coefficients. Importantly, the fact that the symmetries of the reflection coefficients in the cases of mixed-parity ICs differ from those in the case of even or odd ICs demonstrates that these mixed parity cases are not simply reducible to odd or even ICs, even when one of the components of the potential vanishes asymptotically. The same considerations also imply that the results of the mixed-parity cases are a novel feature of the Manakov system compared to the scalar NLS equation.

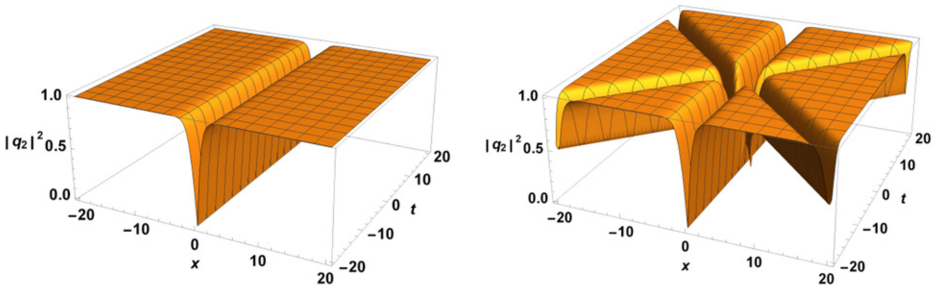

Figure 3. Soliton solutions of the Manakov system with odd ICs and only dark solitons. Left: $\zeta_1^+ = i$, right: $\zeta_1^+ = i, \zeta_2^+ = \mathrm{e}^{\mathrm{i}\pi/3}, \bar{\gamma}_2^+ = i$. Note that since $q_1(x,t)$ is identically zero here, we only plot $|q_2(x,t)|$.

Examples. We illustrate the above results by presenting explicit solutions. For simplicity we restrict ourselves to showing pure soliton solutions, since in this case one can obtain solution of the Manakov system in closed form via the determinantal expression (2.27). (On the other hand, when the reflection coefficients are not zero one must solve a mixed system of algebraic-integral equations. See section 2 for details.)

Figure 2 shows two pure soliton solutions corresponding to even ICs. The first solution (left) is generated by a single discrete eigenvalue plus its symmetric counterpart, while the second one (right) is generated by two dark solitons plus their symmetric counterparts. Figure 3 shows two pure soliton solutions corresponding to odd ICs. The first solution (left) is generated by a single (self-symmetric) black soliton, while the second solution (right) is generated by the black soliton, a dark soliton and its symmetric counterpart.

Since the solutions presented are reflectionless, $q_1(x,t)$ is identically zero in both cases (and is thus not shown for brevity). Therefore, $q_2(x,t)$ is actually a solution of the scalar defocusing NLS equation. (To the best of our knowledge, explicit soliton solutions of the defocusing NLS equation parity symmetry and NZBC were first presented in [18].) For the same reason, for reflectionless solutions with $q_{\pm,1} = 0$ and only dark–dark solitons, there is no difference between the case of even ICs and that of odd–even ICs, or between the case of odd ICs and that of even–odd ICs.

It is important to realize that, with any parity symmetry, the moduli $|q_1(x,t)|$ and $|q_2(x,t)|$ of each component are always even, and it is the parity symmetries of the phases $\arg q_1(x,t)$ and $\arg\ q_2(x,t)$ that determine whether the full, complex-valued solution is even or odd. Recall that, when traversing a soliton generated by an eigenvalue ζ_j, the phase of the solution experiences a jump of $2\arg\zeta_j$, and, in particular, a black soliton contributes a phase jump of π to the solution, i.e., it changes the solution's overall sign. Interestingly, even though each non-self-symmetric eigenvalue gives the same contribution to the local phase both in the case of even and that of odd ICs, the difference between even and odd ICs is that the additional sign change due to the black soliton with odd ICs effectively changes the direction of the local phase change generated by the non-self-symmetric eigenvalues. This phenomenon is markedly different from what happens in the focusing case with zero background, where the parity of the solution is determined by the sign of the norming constants [13]. Similarly, the fact that $q_2(0,t) = 0$ with odd and even–odd ICs is a consequence of the fact that the symmetries of the norming constants ensure that the black soliton is always centered at $x = 0$.

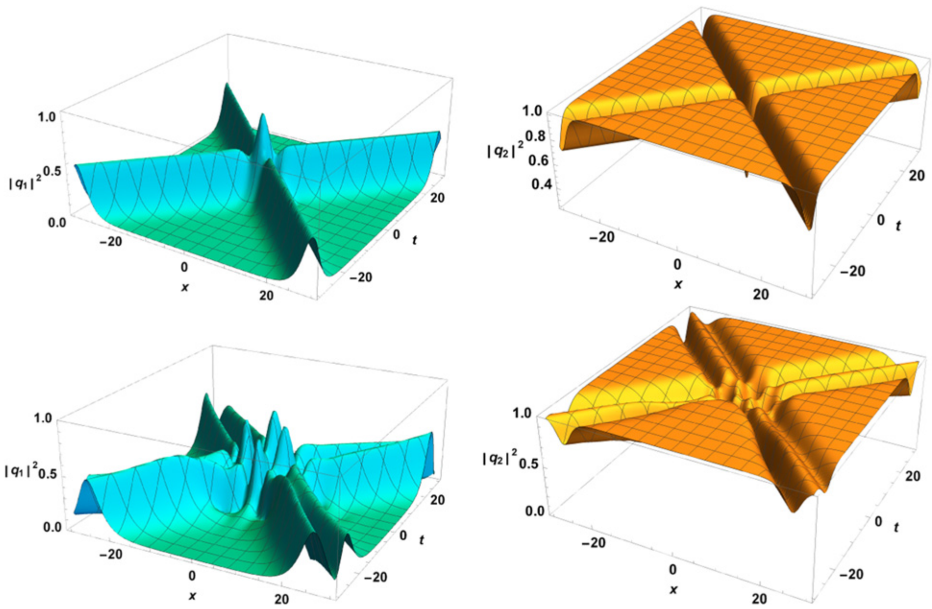

Figure 4. Soliton solutions of the Manakov system with even ICs and only dark–bright solitons. Top row: $z_1^+ = (1 + \mathrm{i})/2$, $\delta_1^+ = 1 - \mathrm{i}$, bottom row: $z_1^+ = (1 + \mathrm{i})/2$, $z_2^+ = (3 + 2\mathrm{i})/6$, $\delta_1^+ = 10 + 10\mathrm{i}$, $\delta_2^+ = 10 - 10\mathrm{i}$. Hereafter, the left column shows $|q_1(x, t)|$, the right column $|q_2(x, t)|$.

5.2. Solutions with only dark–bright solitons

Next we consider the case in which only eigenvalues off the circle C_o are present (i.e., $N_1 = 0$), implying that only dark–bright solitons (i.e., no dark–dark solitons) are present in the solution. Recall that these eigenvalues come in symmetric octets, labeled as in (4.2).

Theorem 5.5 (Even ICs without dark–dark solitons). *If the ICs are even and no discrete eigenvalues on the circle are present, the solution cannot contain any stationary dark–bright solitons. That is, the number m_2 of self-symmetric eigenvalues off the circle is zero. The reflection coefficients obey the symmetry* (3.5)*, and the norming constants of the non-self-symmetric eigenvalues obey the symmetry* (4.20)*.*

Proof. In section 4.3 we showed that no self-symmetric eigenvalues off the circle can exist (implying $m_2 = 0$) when the ICs are odd or even, because their norming constants cannot satisfy the self-consistency constraint (4.23), and we also derived the symmetry (4.20) of the norming constants of the non-self-symmetric eigenvalues. The symmetry (3.5) of the reflection coefficients was derived in section 3.1.

Theorem 5.6 (Odd ICs without dark–dark solitons). *No odd solutions of Manakov system* (1.1) *without discrete eigenvalues on the circle can exist.*

Proof. If the solution does not contain dark–dark solitons, the number m_1 of self-symmetric eigenvalues on the circle is necessarily zero. However, the results of section 4.3 imply that when the ICs are odd or even, no self-symmetric eigenvalues off the circle can exist, and therefore m_2 is also zero. Thus the relation (4.4) in lemma 4.3 (which ties the overall number of self-symmetric eigenvalues to the asymptotic phase difference of the potential) implies

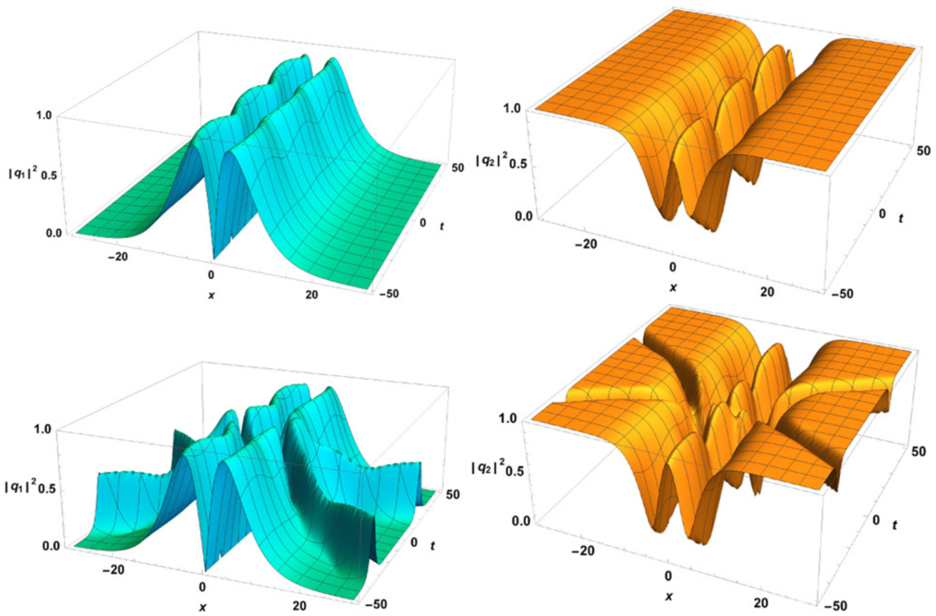

Figure 5. Soliton solutions of the Manakov system with odd–even ICs and only dark–bright solitons. Top row: $z_1^+ = \mathrm{i}/2, z_2^+ = \mathrm{i}/4$, $(\delta_1^+) = \pi/6$, $\arg(\delta_2^+) = \pi/3$, bottom row: $z_1^+ = \mathrm{i}/2, z_2^+ = \mathrm{i}/4$, $z_3^+ = (1+\mathrm{i})/2$, $\arg(\delta_1^+) = \pi/3$, $\arg(\delta_2^+) = \pi/2$, $\delta_3^+ = 1 + \mathrm{i}$.

$\exp[\mathrm{i}\Delta\theta] = 1$. But odd ICs require $\Delta\theta = \pi$. Thus the constraint the can never be satisfied, which means that no odd solutions without dark–dark solitons can exist.

Theorem 5.7 (Mixed parity ICs without dark–dark solitons). *If the ICs are odd–even, $q_{\pm,1} = 0$, and no discrete eigenvalues on the circle are present, the number m_2 of self-symmetric eigenvalues off the circle must be even. On the other hand, if the ICs are even–odd, m_2 must be odd. In both cases, the norming constants of the self-symmetric eigenvalues satisfy the constraint* (4.23)*, those of the non-self-symmetric eigenvalues the symmetry* (4.20) *and the reflection coefficients the symmetry* (3.17).

Proof. Recall that, for ICs with mixed parity, $\mathbf{q}_+ = \pm\sigma_3\mathbf{q}_-$, and to satisfy the requirement that $\mathbf{q}_\pm$ are parallel we took $q_{\pm,1} = 0$. As before, the constraint on the number of self-symmetric eigenvalues off the circle then arises from the constraint (4.4). The norming constants of the non-self-symmetric eigenvalues off the circle must satisfy the symmetry (4.26), derived in section 4.3. However, self-symmetric eigenvalues off the circle (corresponding to stationary dark–bright solitons) are now possible, as long as the associated norming constants satisfy the parity symmetry (4.27). As before, the symmetry (3.17) of the reflection coefficients was derived in sections 3.3 and 3.4.

Remark 5.8. The fact that self-symmetric eigenvalues off the circle (i.e., stationary dark–bright solitons) are allowed for mixed-parity ICs (i.e., even–odd or odd–even) is an important difference between these cases and those of purely even or odd ICs. Another important difference is the fact that even–odd and odd–even solutions without dark–dark solitons can exist even though purely odd solutions without dark–dark solitons cannot. Finally, like

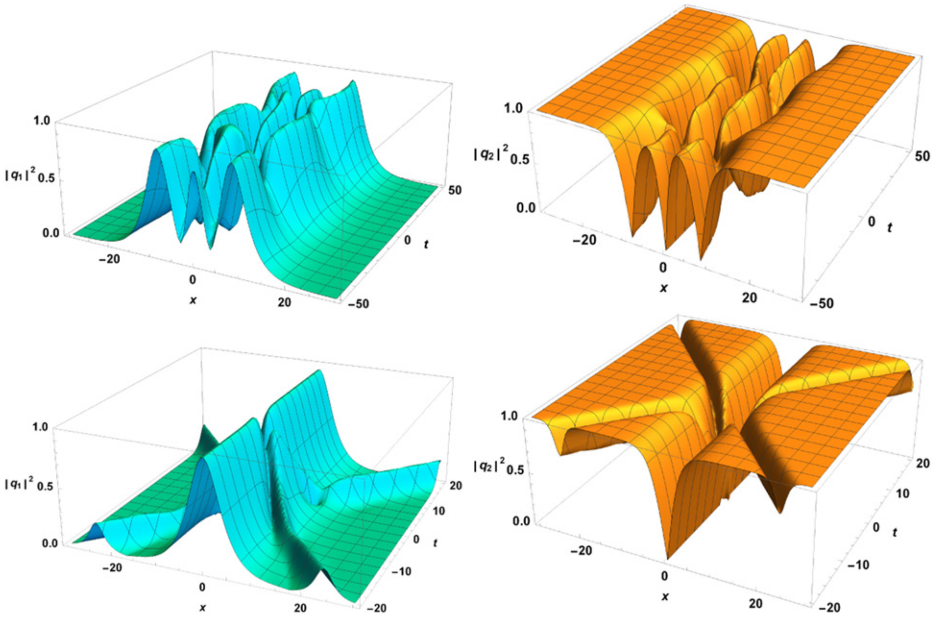

Figure 6. Soliton solutions of the Manakov system with even–odd ICs and only dark–bright solitons. Top row: $z_1^+ = \mathrm{i}/3, z_2^+ = 2\mathrm{i}/3$, $z_3^+ = \mathrm{i}/2$, $\arg(\delta_1^+) = \pi/3$, $\arg(\delta_2^+) = \pi/2$, $\arg(\delta_3^+) = \pi/6$, bottom row: $z_1^+ = \mathrm{i}/3$, $z_2^+ = 2(1+\mathrm{i})/3$, $\arg(\delta_1^+) = \pi/3$, $\delta_2^+ = 1 + \mathrm{i}$.

in section 5.1, all of the results of this sections hold both for pure soliton solutions and for solutions with non-trivial dispersive components.

Examples. As before, we illustrate the above results by presenting some explicit soliton solutions of the Manakov system. Figure 4 shows two pure soliton solutions corresponding to even ICs. The first one (top row) is generated by a single non-self-symmetric eigenvalue plus its symmetric counterpart; the second one (bottom row) by two non-self-symmetric eigenvalues plus their symmetric counterparts. Figure 5 shows two pure soliton solutions corresponding to odd–even ICs. The first one (top row) is generated by two self-symmetric eigenvalues plus their symmetric counterparts; the second one (bottom row) by two self-symmetric eigenvalues and two non-self-symmetric ones, plus their symmetric counterparts. Finally, figure 6 shows two pure soliton solutions corresponding to even–odd ICs. The first one (top row) is generated by three self-symmetric eigenvalues plus their symmetric counterparts; the second one (bottom row) by one self-symmetric eigenvalue and two non-self-symmetric eigenvalues, plus their symmetric counterparts. No odd solutions are presented here because no such solutions without a black soliton (and thus an eigenvalue on the circle) can exist.

5.3. Solutions with combinations of dark–dark and dark–bright solitons

Now we consider the most general case, in which both dark–dark and dark–bright solitons are simultaneously present in the solution, i.e., the case in which both N_1 and N_2 are nonzero. All

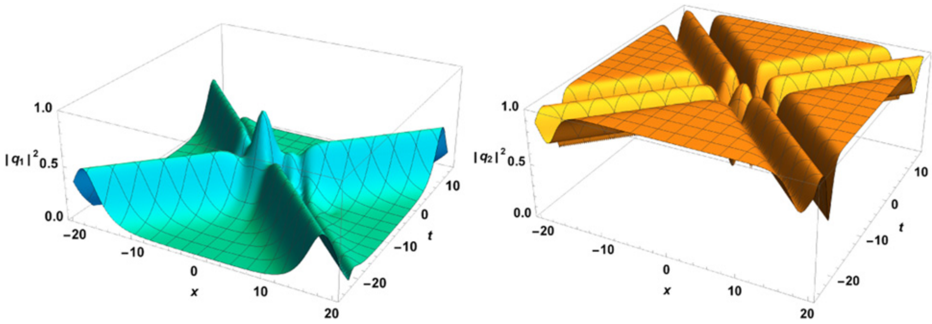

Figure 7. Soliton solutions of the Manakov system with even ICs a combination of dark–dark and dark–bright solitons: $\zeta_1^+ = \mathrm{e}^{\mathrm{i}\pi/3}$, $\bar{\gamma}_1^+ = i$, $z_1^+ = (1+\mathrm{i})/2$, $\delta_1^+ = 1+\mathrm{i}$.

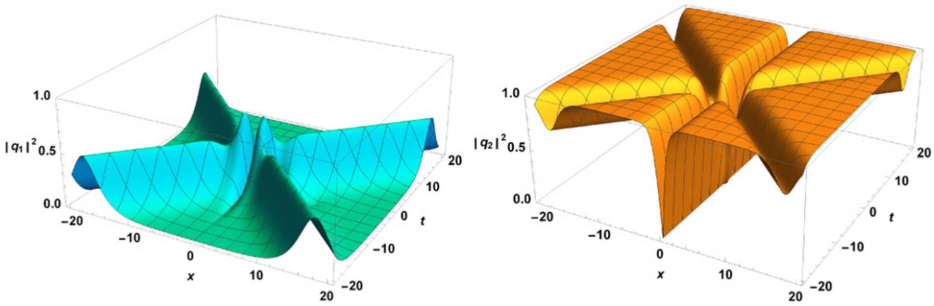

Figure 8. Soliton solutions of the Manakov system with odd ICs a combination of dark–dark and dark–bright solitons: $\zeta_1^+ = i$, $z_1^+ = (1+\mathrm{i})/2$, $\delta_1^+ = 1+\mathrm{i}$.

the results in this section can be obtained in a similar way as those in sections 5.1 and 5.2. Therefore, for brevity we omit the proofs.

Theorem 5.9. (Even ICs with both dark–dark and dark–bright solitons.) *If the ICs are even and both eigenvalues on and off the circle are present, no self-symmetric eigenvalues are present (i.e.,* $m_1 = m_2 = 0$*, all eigenvalues have non-zero real part, and the solution has no stationary solitons). The norming constants of the eigenvalues on and off the circle satisfy the symmetries* (4.11) *and* (4.20)*, respectively, and the reflection coefficients obey the symmetry* (3.5).

Theorem 5.10 (Odd ICs with both dark–dark and dark–bright solitons). *If the ICs are odd, the solution always contains a black soliton (self-symmetric eigenvalue at* $z = \mathrm{i}q_o$*) and no self-symmetric eigenvalues off the circle. The norming constants of the eigenvalues on and off the circle satisfy the symmetries* (4.11) *and* (4.20)*, respectively, with the extra constraint* (4.14) *for the norming constant of the black soliton, and the reflection coefficients obey the symmetry* (3.5).

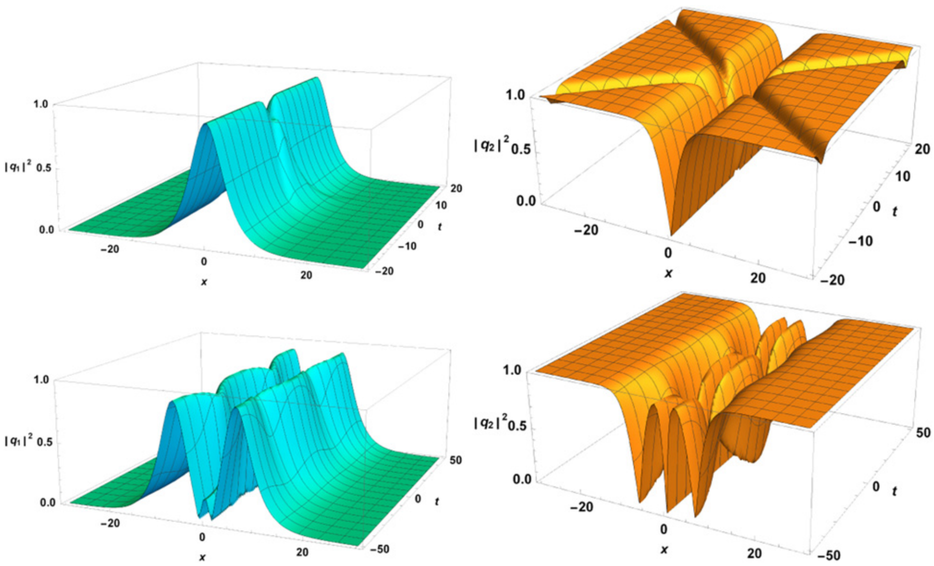

Figure 9. Soliton solutions of the Manakov system with even–odd ICs and combination of dark and dark–bright solitons. Top row: $\zeta_1^+ = \mathrm{e}^{\mathrm{i}\pi/6}$, $\bar{\gamma}_1^+ = i$, $z_1^+ = \mathrm{i}/3$, $\arg(\delta_1^+) = \pi/3$. Bottom row: $\zeta_1^+ = i$, $z_1^+ = \mathrm{i}/2$, $z_2^+ = \mathrm{i}/3$, $\arg(\delta_1^+) = \pi/6$, $\arg(\delta_2^+) = \pi/3$.

Theorem 5.11 (Even–odd ICs with both dark–dark and dark–bright solitons). *If the ICs are even–odd, $q_{\pm,1} = 0$, and both eigenvalues on and off the circle are present, the following situations can arise:*

(*a*) *$m_1 = 1$ and m_2 is zero or even; that is, there is a black soliton (self-symmetric eigenvalue at $z = \mathrm{i}q_o$) and an even (possibly zero) number of self-symmetric eigenvalues off the circle.*

(*b*) *$m_1 = 0$ and $m_2 > 0$ is odd; that is, all eigenvalues on the circle have non-zero real parts, and there is an odd number of self-symmetric eigenvalues off the circle.*

In all cases, the norming constants satisfy the symmetries (4.11) and (4.26), with the extra constraints (4.14) and (4.28) for the norming constants of the self-symmetric eigenvalues (if present), and the reflection coefficients satisfy the symmetry (3.17).

Theorem 5.12 (Odd–even ICs with both dark–dark and dark–bright solitons). *If the ICs are odd–even, $q_{\pm,1} = 0$, and both eigenvalues on and off the circle are present, the following situations can arise:*

(*a*) *$m_1 = m_2 = 0$; that is, no self-symmetric eigenvalues are present (i.e., all eigenvalues have non-zero real parts, and the solution has no stationary solitons).*

(*b*) *$m_1 = 0$ and $m_2 > 0$ is even; that is, all eigenvalues on the circle have non-zero real parts, and there is an odd number of self-symmetric eigenvalues off the circle.*

(*c*) *$m_1 = 1$ and $m_2 > 0$ is odd; that is, there is a black soliton (self-symmetric eigenvalue at $z = \mathrm{i}q_o$) and an odd number of self-symmetric eigenvalues off the circle.*

In all cases, the norming constants satisfy the symmetries (4.11) and (4.26), with the extra constraints (4.14) and (4.28) for the norming constants of the self-symmetric eigenvalues (if present), and the reflection coefficients satisfy the symmetry (3.17).

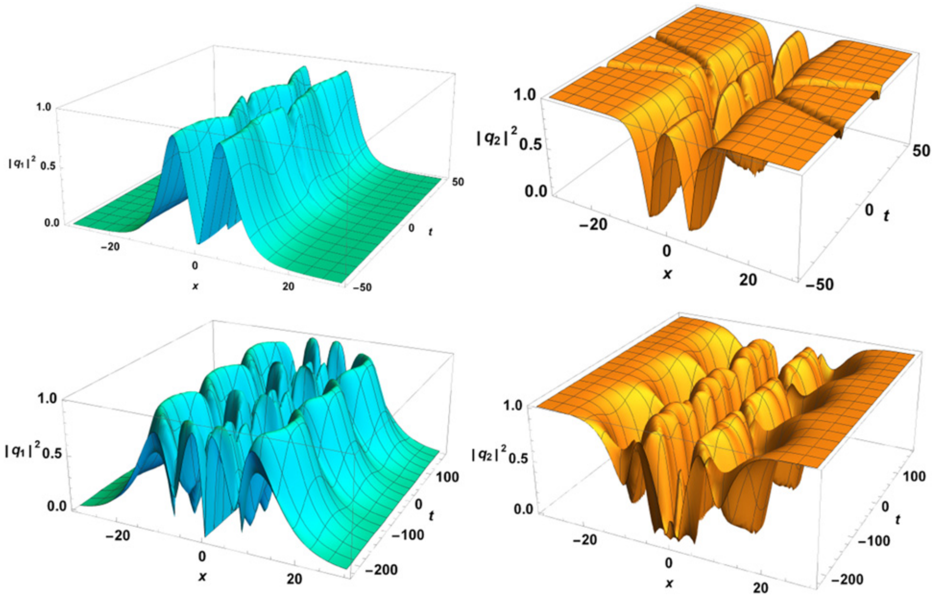

Figure 10. Soliton solutions of the Manakov system with odd–even ICs and combination of dark and dark–bright solitons. Top row: $\zeta_1^+ = \mathrm{e}^{\mathrm{i}\pi/6}$, $\bar{\gamma}_1^+ = i$, $z_1^+ = \mathrm{i}/2$, $z_2^+ = \mathrm{i}/3$, $\arg(\delta_1^+) = \pi/3$, $\arg(\delta_2^+) = \pi/6$. Bottom row: $\zeta_1^+ = i$, $z_1^+ = \mathrm{i}/2$, $z_2^+ = \mathrm{i}/3$, $z_3^+ = \mathrm{i}/4$, $\arg(\delta_1^+) = \pi/3$, $\arg(\delta_2^+) = \pi/6$, $\arg(\delta_3^+) = \pi/2$.

Remark 5.13. Like in sections 5.1 and 5.2, all of the results of this section hold both for reflectionless (i.e., pure soliton) solutions and for solutions with nonzero reflection coefficients (i.e., with non-trivial dispersive components).

Examples. As in sections 5.1 and 5.2, we illustrate the above results by presenting a few explicit soliton solutions combining both dark–dark and dark–bright solitons. Figure 7 shows an even soliton solution that combines a non-stationary dark soliton and a non-stationary dark–bright soliton, plus their symmetric counterparts. Note how, even though only one bright soliton (plus its symmetric counterpart) is present, there is a nontrivial interaction between the bright soliton and the dark soliton in the other component. Figure 8 shows an odd soliton solution that combines a black soliton and a non-stationary dark–bright soliton, plus its symmetric counterpart. Note how $q_1(0,t) = 0$, consistently with the odd parity (while $q_2(0,t) = 0$ because of the black soliton centered at $x = 0$). Figure 9 shows two even–odd soliton solutions. The first one (top row) is generated by a single self-symmetric eigenvalue off the circle (stationary dark–bright soliton) and a non-self-symmetric dark soliton plus its symmetric counterpart. The second one (bottom row) is generated by a black soliton plus two self-symmetric eigenvalues off the circle (corresponding to two stationary dark–bright solitons). Finally, figure 10 shows two odd–even soliton solutions. The first one (top row) is generated by two self-symmetric eigenvalues off the circle (stationary dark–bright soliton), and a non-self-symmetric eigenvalue on the circle (moving dark soliton), plus its symmetric counterpart. The second one (bottom row) is generated by a black soliton (self-symmetric eigenvalue on the circle) plus three self-symmetric eigenvalues off the circle (stationary dark–bright solitons).

6. Final remarks

In this work we investigated IVPs for the defocusing Manakov system with NZBC in which the initial conditions possess a well-defined spatial parity symmetry. These problems are in one-to-one correspondence with BVPs on the half line with Dirichlet or Neumann boundary conditions at the origin.

In particular, we used the IST to identify the symmetries of the eigenfunctions arising from the spatial parity of the solution, and we determined the corresponding symmetries of the scattering data, which are more complicated than in the scalar NLS equation. In particular, we showed that the discrete eigenvalues giving rise to dark solitons arise in symmetric quartets, and those giving rise to dark–bright solitons in symmetric octets. We also characterized the differences between the purely even or purely odd case and the 'mixed parity' (even–odd or odd–even) cases. Finally, we showed how, in each case, the spatial symmetry yields a constraint on the possible existence of self-symmetric eigenvalue, corresponding to stationary solitons, and we studied the resulting behavior of solutions.

We note that solutions exhibiting parity-time (PT) symmetry, i.e., $\mathbf{q}(-x,t) = \mathbf{q}^*(x,t)$ have also received considerable attention in recent years (e.g., see [2, 3, 5] and references therein). However, PT symmetry induces very different symmetries for the eigenfunctions and scattering data compared to those arising from pure spatial symmetry (e.g., see [4] for the scalar NLS equation). Therefore, the study of PT-symmetric solutions requires a different kind of analysis, and leads to very different conclusions for the behavior of solutions. For those reasons, PT symmetry was not considered in this work.

It is straightforward to show using (3.10) and (3.18) that the mixed parity cases satisfy $\sigma_3\mathbf{q}_o = \pm\exp[\mathrm{i}\Delta\theta]\mathbf{q}_o$, which means that $\mathbf{q}_o$ is an eigenvector of σ_3 with eigenvalue $\pm\exp[\mathrm{i}\Delta\theta]$ (with the plus and minus signs for the even–odd and the odd–even case, respectively). Choosing $q_{\pm,1} = 0$ is then equivalent to taking $\mathbf{q}_o$ to always be the eigenvector associated to the eigenvalue -1. In principle, one could restore the symmetry between the two components and formulate the results in a unified way by discussing just one mixed case which covers both of the current odd–even and even–odd cases. The current two cases would then correspond to whether $\mathbf{q}_o$ is an eigenvector associated with the eigenvalue $+1$ or -1. However, the resulting nominal simplification would likely be negated by the subtleties associated to which particular eigenvalue is selected. At the same time, the reason for our choice is that taking $q_{\pm,1} = 0$ allows one to identify the first component of the solution with a bright field and the second one with a dark field. This identification reflects a physical difference, and therefore makes our results more directly applicable to describe actual experiments. For this reason, we believe that our presentation of the results is more useful for practical purposes.

We anticipate that an interesting direction for further study could be the effort to extend the results of the present work to the case of non-parallel BCs, i.e., the case when $\mathbf{q}_+^\dagger\mathbf{q}_- < \|\mathbf{q}_\pm\|^2$. The IST for the IVP with non-parallel BCs was recently presented in [1]. There are several complications arising in that case, however. Some of these complications are of a mathematical nature, and require one to deal with various new technical issues. In turn, some of these difficulties reflect a fundamental difference in the properties of the problem, namely the fact that no reflectionless (i.e., purely solitonic) solutions exist in that problem. (The same phenomenon arises for the defocusing NLS equation with asymmetric BCs [11, 17].) On the other hand, the case of parallel BCs is also the easiest one to generate experimentally.

We also expect that the results of this work will be useful to further characterize recent analytical and numerical studies [39, 40] on repulsive scalar and two-component BECs in which the initial system configurations possess a definite parity symmetry. We also anticipate that the behavior of solutions characterized in this work can be observed experimentally in suitable realizations of repulsive two-component BECs.

Acknowledgments

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Data availability statement

This is a theoretical work, and no data were produced as part of it.

Appendix A

A.1. Boundary value problems for the scalar defocusing NLS equation with NZBC

For completeness, in this appendix we discuss the reduction of the Manakov system to the scalar defocusing NLS equation:

$$\mathrm{i}q_t + q_{xx} - 2(|q|^2 - q_o^2)q = 0 \tag{A.1}$$

with the following NZBC at infinity:

$$\lim_{x\to\pm\infty} q(x,t) = q_\pm = q_o\,\mathrm{e}^{\pm\mathrm{i}\theta}, \tag{A.2}$$

and we derive all the symmetries in the scattering data induced by spatial parity of the ICs. The Lax pair for the scalar NLS equation is a reduction of (2.1), given by

$$v_x = \mathbf{X}\,v, \qquad v_t = \mathbf{T}\,v, \tag{A.3a}$$

where

$$\mathbf{X}(x,t,k) = -\mathrm{i}k\sigma_3 + Q, \quad \mathbf{T}(x,t,k) = -2\mathrm{i}k^2\sigma_3 + \mathrm{i}\sigma_3(Q_x - Q^2 + q_o^2) + 2kQ, \tag{A.3b}$$

$$Q(x,t) = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.3c}$$

The Jost eigenfunctions are defined as

$$\phi_\pm(x,t,z) = W_\pm(k)\mathrm{e}^{\mathrm{i}\theta(x,t,k)} + o(1), \quad x \to \pm\infty,$$

where $\theta = -\lambda x - 2k\lambda t$, $\lambda^2 = k^2 - q_o^2$, and $W_\pm(k)$ are the matrices of asymptotic eigenvectors, which can be chosen as reductions of (2.5):

$$W_\pm(z) = \begin{pmatrix} z & -q_o^2/z \\ \mathrm{i}q_\pm^* & -\mathrm{i}q_\pm^* \end{pmatrix}. \tag{A.4}$$

Since $\phi_\pm(x,t,z)$ are two fundamental solution of the Lax pair for any $z \neq q_o$, there exists a 2×2 scattering matrix $A(z)$ such that

$$\phi_-(x,t,z) = \phi_+(x,t,z)A(z), \quad z \in \mathbb{R}\backslash \pm q_o. \tag{A.5}$$

Using the symmetries derived in [23], we have

$$A(z) = \begin{pmatrix} a(z) & \mathrm{e}^{\mathrm{i}\Delta\theta} b^*(z) q_o^2/z^2 \\ b(z) & \mathrm{e}^{\mathrm{i}\Delta\theta} a^*(z) \end{pmatrix}, \tag{A.6a}$$

$$B(z) := A^{-1}(z) = \begin{pmatrix} a^*(z) & -b^*(z)q_o^2/z^2 \\ -\mathrm{e}^{-\mathrm{i}\Delta\theta} b(z) & \mathrm{e}^{-\mathrm{i}\Delta\theta} a(z) \end{pmatrix}, \tag{A.6b}$$

where $\Delta\theta = \theta_+ - \theta_-$. One can define two complete sets of analytic eigenfunctions for the system (A.3a):

$$\Phi^+(x,t,z) = (\phi_{-,1}(x,t,z), \phi_{+,2}(x,t,z)), \quad z \in \mathbb{C}^+, \tag{A.7a}$$

$$\Phi^-(x,t,z) = (\phi_{+,1}(x,t,z), \phi_{-,2}(x,t,z)), \quad z \in \mathbb{C}^-, \tag{A.7b}$$

and show that

$$\det\, \Phi^+(x,t,z) = -2\mathrm{i}q_+^* \lambda(z) a(z), \qquad z \in \mathbb{C}^+ \cap \mathbb{R}, \tag{A.8a}$$

$$\det\, \Phi^-(x,t,z) = -2\mathrm{i}q_+^*\, \lambda(z) a^*(z^*) \mathrm{e}^{\mathrm{i}\Delta\theta}, \quad z \in \mathbb{C}^- \cap \mathbb{R}, \tag{A.8b}$$

where $a(z)$ is analytic in $\mathbb{C}^+$. For the scalar defocusing NLS equation, discrete eigenvalues are confined to the circle $C_o\{z \in \mathbb{C} : |z| = q_o\}$, and they can be shown to always be simple [24]. If one considers a pair of eigenvalues ζ_n, ζ_n^* on the circle C_o, such that $a(\zeta_n) = 0$, then the columns of both $\Phi^+(x,t,\zeta_n)$ and $\Phi^-(x,t,\zeta_n^*)$ become linearly independent, and one has

$$\phi_{-,1}(x,t,\zeta_n) = c_n \phi_{+,2}(x,t,\zeta_n), \qquad \phi_{-,2}(x,t,\zeta_n^*) = \bar{c}_n \phi_{+,1}(x,t,\zeta_n^*). \tag{A.9}$$

The norming constants are defined in terms of the coefficients in (A.9) as follows:

$$\gamma_n = \frac{c_n}{\zeta_n a'(\zeta_n)}, \qquad \bar{\gamma}_n = \frac{\bar{c}_n}{\mathrm{e}^{\Delta\theta}(a'(\zeta_n))^*}. \tag{A.10}$$

Moreover, like in the Manakov case, the asymptotic phases and the scattering data are related by the theta condition:

$$\mathrm{e}^{\mathrm{i}\Delta\,\theta} = \prod_{n=1}^{N} \frac{\zeta_n}{\zeta_n^*} \exp\left\{-\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log[1 - q_o^2|\rho(\zeta)|^2/\zeta^2]}{\zeta}\, \mathrm{d}\zeta\right\}, \tag{A.11}$$

where $\rho(z) = b(z)/a(z)$. Furthermore, the scattering coefficient $a(z)$ can be recovered using the scattering data via the trace formula:

$$a(z) = \prod_{n=1}^{N} \frac{z - \zeta_n}{z - \zeta_n^*} \exp\left\{-\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log[1 - q_o^2|\rho(\zeta)|^2/\zeta^2]}{\zeta - z}\, \mathrm{d}\zeta\right\}. \tag{A.12}$$

Next we discuss the symmetries induced in the scattering data by the spatial parity of the IC. As we did before for the Manakov system, one can show that the Jost solutions and scattering matrix satisfy an additional symmetry for both even and odd ICs. For even ICs, namely, when $q(x,0) = q(-x,0)$ for all $x \in \mathbb{R}$, we obtain

$$\phi_\mp(x,t,z) = -\sigma_3 \phi_\pm(-x,t,-z), \qquad A(z) = A^{-1}(-z), \quad z \in \mathbb{R}, \tag{A.13}$$

and odd ICs (i.e., $q(x,0) = -q(-x,0)$) give

$$\phi_{\mp}(x,t,z) = -\,\phi_{\pm}(-x,t,-z), \qquad A(z) = A^{-1}(-z), \quad z \in \mathbb{R}. \tag{A.14}$$

Equations (A.13) and (A.14) imply the following induced parity symmetry of the reflection coefficient:

$$\rho(-z) = -\mathrm{e}^{-\mathrm{i}\Delta\theta}\,\frac{a(z)}{a^*(z)}\rho(z), \quad z \in \mathbb{R}. \tag{A.15}$$

Moreover, using the Schwartz reflection principle, we also have $a(z) = a^*(-z^*)$ for any $z \in \mathbb{C}^+$. Thus the eigenvalues appear in symmetric quartets, $\{\pm\zeta_j, \pm\zeta_j^*\}$ on the circle C_o. As before, it will be convenient to denote these eigenvalues as $\zeta_j^+ = \zeta_j$ and $\zeta_j^- = -\zeta_j^*$ in $\mathbb{C}^+$, and $\bar{\zeta}_j^+ = \zeta_j^*$ and $\bar{\zeta}_j^- = -\zeta_j$ in $\mathbb{C}^-$, with corresponding notation for the associated proportionality constants and norming constants.

As before, we refer to self-symmetric eigenvalues as those eigenvalues for which $\zeta_j^+ = \zeta_j^-$ and $\bar{\zeta}_j^+ = \bar{\zeta}_j^-$. In the scalar, the only self-symmetric eigenvalues are $\pm\mathrm{i}q_o$. Let m_1 be the number of self-symmetric eigenvalues ($m_1 = 0$ or $m_1 = 1$), and N_1 be the number of non-self symmetric eigenvalues on the circle C_o. Then (A.11) becomes

$$\mathrm{e}^{\mathrm{i}\Delta\,\theta} = (-1)^{m_1}\,\exp\left\{-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\log[1 - q_o^2|\rho(\zeta)|^2/\zeta^2]}{\zeta}\,\mathrm{d}\zeta\right\}. \tag{A.16}$$

Moreover, using the parity symmetry (A.15) of the reflection coefficient we obtain

$$|\rho(-z)| = |\rho(z)|, \quad z \in \mathbb{R}. \tag{A.17}$$

As a result, the contribution of the radiation to the theta condition vanishes because the reflection coefficient is even, and therefore one simply has

$$\mathrm{e}^{\mathrm{i}\Delta\theta} = (-1)^{m_1}. \tag{A.18}$$

Next we show that the norming constants of the symmetric eigenvalues are related. Indeed, one has

$$\phi_{-,1}(x,t,\zeta_n^+) = c_n^+\phi_{+,2}(x,t,\zeta_n^+) \quad \Leftrightarrow \quad \phi_{+,1}(-x,t,-\zeta_n^+) = c_n^+\phi_{-,2}(-x,t,-\zeta_n^+)$$
$$\Leftrightarrow \quad \phi_{+,1}(x,t,\bar{\zeta}_n^-) = c_n^+\phi_{-,3}(x,t,\bar{\zeta}_n^-), \tag{A.19}$$

and comparing with (A.9) we obtain

$$c_n^{\pm}\bar{c}_n^{\mp} = 1. \tag{A.20}$$

(The symmetry corresponding to the upper sign is obtained directly from the above equations; the one for the lower sign is obtained in a similar way.) Using the extra symmetries we obtained in (4.10) it follows that

$$\gamma_n^{\pm}\bar{\gamma}_n^{\mp} = \frac{1}{\mathrm{e}^{\Delta\theta}\zeta_n^{\pm}a'(\zeta_n^{\pm})(a'(\bar{\zeta}_n^{\mp}))^*}. \tag{A.21}$$

As in the Manakov case, an extra symmetry holds for the self-symmetric eigenvalues $\pm\mathrm{i}\mathbf{q}_o$. Using (A.9) and the fact that $\zeta_1^+ = \zeta_1^-$ and $\bar{\zeta}_1^+ = \bar{\zeta}_1{}^-$ we obtain

$$c_1^+ = c_1^-, \qquad \bar{c}_1^+ = \bar{c}_1^-. \tag{A.22}$$

Thus the associated norming constants satisfy an additional symmetry

$$\gamma_1^+ = \gamma_1^-, \qquad \bar{\gamma}_1^+ = \bar{\gamma}_1^-, \tag{A.23a}$$

which in turn yields

$$\gamma_n^+ \bar{\gamma}_n^- = \frac{1}{\mathrm{e}^{\Delta\theta} \zeta_n^+ |a'(\zeta_n^+)|^2}. \tag{A.23b}$$

We next use the resulting parity symmetries to investigate the behavior of the corresponding soliton solutions.

Lemma A.1. (Even ICs). *In the case of even ICs, no stationary dark soliton (black solitons) can be present in the solution.*

Proof. For even ICs, we have $q_+ = q_-$, which implies $\mathrm{e}^{\mathrm{i}\Delta\theta} = 1$. Then (A.18) becomes

$$1 = (-1)^{m_1},$$

and then necessarily $m_1 = 0$, which yields the desired result. Moreover, in this case the norming constants satisfy the parity symmetry (A.21).

Lemma A.2 (Odd ICs). *In the case of odd ICs, the solution necessarily contains a black soliton, i.e., a self-symmetric eigenvalue pair at* $z = \pm \mathrm{i} q_o$.

Proof. For odd ICs, we have $q_+ = -q_-$ and $\mathrm{e}^{\mathrm{i}\Delta\theta} = -1$. Again (A.18) yields

$$-1 = (-1)^{m_1},$$

and therefore $m_1 = 1$, which implies the solution necessarily contains a black soliton. Similar to even ICs, the norming constants satisfy the parity symmetry (A.21) and the norming constants associated with the self-symmetric eigenvalues $\pm \mathrm{i} q_o$ follows an additional symmetry (A.23).

In summary, the above results imply that for even potentials no black solitons can exist, whereas for odd potentials a black soliton must always exist. To the best of our knowledge, these results were first presented in [18].

A.2. Invariances of the Manakov system

It is a well known fact that, if $\mathbf{q}(x,t)$ is any solution of the Manakov system and α, v, x_o, and t_o are any real constants, $\bar{\mathbf{q}}(x,t) = \mathrm{e}^{\mathrm{i}\alpha}\mathbf{q}(x,t)$ and $\hat{\mathbf{q}}(x,t) = \mathbf{q}(x - x_o, t - t_o)$ are solutions of the Manakov system as well. Finally, for any constant unitary 2×2 matrix $\mathbf{U}$ (i.e., $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}_2$), $\check{\mathbf{q}}(x,t) = \mathbf{U}\mathbf{q}(x,t)$ is also a solution. We next show how each transformation affects the IST.

Lemma A.3. *Let* $\bar{\phi}_\pm(x,t,z)$ *and* $\bar{A}(z)$ *be the Jost solutions and scattering matrix corresponding to* $\bar{\mathbf{q}}(x,t)$. *We have*

$$\bar{\phi}_\pm(x,t,z) = \mathrm{e}^{\mathrm{i}(\alpha/2)\mathbf{J}} \phi_\pm(x,t,z) \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\alpha/2}, \mathrm{e}^{3\mathrm{i}\alpha/2}, \mathrm{e}^{-\mathrm{i}\alpha/2}), \tag{A.24a}$$

$$\bar{A}(z) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha/2}, \mathrm{e}^{-3\mathrm{i}\alpha/2}, \mathrm{e}^{\mathrm{i}\alpha/2}) A(z) \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\alpha/2}, \mathrm{e}^{3\mathrm{i}\alpha/2}, \mathrm{e}^{-\mathrm{i}\alpha/2}). \tag{A.24b}$$

Proof. One can show by the direct computation that $\mathrm{e}^{-\mathrm{i}(\alpha/2)\mathbf{J}} \bar{\phi}_\pm(x,t,z)$ solves the asymptotic scattering problem. Then since $\mathrm{e}^{-\mathrm{i}(\alpha/2)\mathbf{J}} \bar{\phi}_\pm$ and $\phi_\pm$ are both fundamental matrix solutions of the asymptotic scattering problem, there exists an invertible 3×3 matrix $\bar{C}(z)$ such that $\phi_\pm(x,t,z) = \mathrm{e}^{-\mathrm{i}(\alpha/2)\mathbf{J}}\bar{\phi}_\pm(x,t,z)\bar{C}(z)$. Comparing the asymptotics as $x \to \pm\infty$ of $\phi_\pm$ with those of $\mathrm{e}^{-\mathrm{i}(\alpha/2)\mathbf{J}}\bar{\phi}_\pm\bar{C}$ yields $\bar{C}(z) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha/2}, \mathrm{e}^{-3\mathrm{i}\alpha/2}, \mathrm{e}^{\mathrm{i}\alpha/2})$. Combining (A.24a) with the fact that $\bar{\phi}_- = \bar{\phi}_+\bar{A}$ yields (A.24b).

The proofs of the remaining lemmas in this section are omitted since they are similar to the proof of lemma A.3.

Lemma A.4. *Let $\hat{\phi}_\pm(x,t,z)$ and $\hat{A}(z)$ be the Jost solutions and scattering matrix corresponding to $\hat{\mathbf{q}}(x,t)$. We have*

$$\hat{\phi}_\pm(x,t,z) = \phi_\pm(x,t,z)\mathrm{e}^{-\mathrm{i}\Theta(x_o,t_o,z)}, \tag{A.25a}$$

$$\hat{A}(z) = \mathrm{e}^{\mathrm{i}\Theta(x_o,t_o,z)}A(z)\mathrm{e}^{-\mathrm{i}\Theta(x_o,t_o,z)}. \tag{A.25b}$$

Lemma A.5. *Let $\check{\phi}_\pm(x,t,z)$ and $\check{A}(z)$ be the Jost solutions and scattering matrix corresponding to $\check{\mathbf{q}}(x,t)$. We have*

$$\check{\phi}_\pm(x,t,z) = \mathrm{diag}(1,\mathbf{U}^*)\phi_\pm(x,t,z)\mathrm{diag}(1,\mathrm{e}^{\mathrm{i}u},1), \tag{A.26a}$$

$$\check{A}(z) = \mathrm{diag}(1,\mathrm{e}^{-\mathrm{i}u},1)A(z)\mathrm{diag}(1,\mathrm{e}^{\mathrm{i}u},1), \tag{A.26b}$$

where $\det\mathbf{U} = \mathrm{e}^{\mathrm{i}u}$, *with* $u \in \mathbb{R}$.

A.3. Scalar reductions of the Manakov system

There are several ways in which solutions of the Manakov system can reduce to 'scalar' solutions, i.e., solutions of the scalar NLS equation. Indeed, if $q(x,t)$ is any solution of the scalar defocusing NLS equation (A.1), $\mathbf{q}(x,t) = (0, q(x,t))^{\mathrm{T}}$ as well as $\tilde{\mathbf{q}}(x,t) = (q(x,t),0)^{\mathrm{T}}$ and $\check{\mathbf{q}}(x,t) = \mathbf{p}q(x,t)$, where $\mathbf{p}$ is any complex unit-length vector (i.e., $\mathbf{p}^\dagger\mathbf{p} = 1$), are all solutions of the defocusing defocusing Manakov system (1.1). Note that $\tilde{\mathbf{q}}(x,t) = \sigma_1\,\mathbf{q}(x,t)$. Therefore if one can find the symmetries of the IST induced by the first reduction, then the effect of the second reduction will immediately follow from lemma A.5.

Next we find the reductions of the Jost solutions and the scattering matrix with respect to the first reduction, in which case the first component of $\mathbf{q}(x,t)$ is identically zero. As a direct consequence, we have $\mathbf{q}_\pm = (0, q_o\,\mathrm{e}^{\pm\mathrm{i}\theta})^{\mathrm{T}}$.

Let $\phi_{\pm,\mathrm{nls}}(x,t,z)$ and $A_{\mathrm{nls}}(z) = (a_{ij,\mathrm{nls}}(z))$ denote the 2×2 Jost solutions and the scattering matrix for the scalar case, respectively. Using (2.5) and the scattering relation we have

$$\phi_\pm(x,t,z) = \begin{pmatrix} \phi_{\pm,11,\mathrm{nls}} & 0 & \phi_{\pm,12,\mathrm{nls}} \\ 0 & \mathrm{i}q_o\,\mathrm{e}^{\mathrm{i}\theta_\pm}\,\mathrm{e}^{\mathrm{i}\theta_2(x,t,z)} & 0 \\ \phi_{\pm,21,\mathrm{nls}} & 0 & \phi_{\pm,22,\mathrm{nls}} \end{pmatrix}, \tag{A.27a}$$

$$A(z) = \begin{pmatrix} a_{11,\mathrm{nls}}(z) & 0 & a_{12,\mathrm{nls}}(z) \\ 0 & \mathrm{e}^{\mathrm{i}(\theta_- - \theta_+)} & 0 \\ a_{21,\mathrm{nls}}(z) & 0 & a_{22,\mathrm{nls}}(z) \end{pmatrix}, \tag{A.27b}$$

where $\theta_2(x,t,z) = kx + (k^2 + \lambda^2)t$ and for the scalar case we used the normalization in appendix A.1. Moreover, the cofactor representation of $B(z) = A^{-1}(z)$ yields

$$A^{\mathrm{T}}(z) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}\,b_{33}(z) & 0 & -\mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}b_{31}(z) \\ 0 & b_{11}(z)b_{33}(z) & 0 \\ -\mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}b_{13}(z) & 0 & \mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}b_{11}(z) \end{pmatrix}. \tag{A.28}$$

Recall from (2.20a) we have $\rho_2(z) = 0$.

We next show how to account for the second reduction in the IST by using lemma A.5. Recall that, $\tilde{\mathbf{q}}(x, t) = \sigma_1 \mathbf{q}(x, t)$ and $\det \sigma_1 = -1$. Then one can show by the direct computation that

$$\tilde{\phi}_\pm(x, t, z) = \mathrm{diag}(1, \sigma_1)\phi_\pm(x, t, z)\mathrm{diag}(1, -1, 1), \tag{A.29a}$$

$$\tilde{\mathbf{A}}(z) = A(z). \tag{A.29b}$$

Finally, note that $\check{\mathbf{q}}$ is related to $\mathbf{q}$ simply by the unitary transformation $\check{\mathbf{q}} = \mathbf{U}\mathbf{q}$, with $\mathbf{U} = ((\mathbf{p}^\perp)^*, \mathbf{p})$. Therefore changes to the IST caused by the third reduction will follow from lemma A.5 as a direct consequence.

Odd and even potentials for the scalar reductions. For odd and even ICs we have an additional symmetry for the scattering matrix, namely, component-wise $b_{i,j}(-z) = a_{i,j}(z)$ where $i, j = 1, 2, 3$. Using (A.28) and the above mentioned extra symmetry along with the first symmetry yields

$$|\rho_1(-z)| = \left|\frac{b_{31}(-z)}{b_{11}(-z)}\right| = \left|\frac{a_{31}(z)}{a_{11}(z)}\right| = \left|-\mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}\frac{b_{31}(z)}{b_{11}^*(z^*)}\right| = |\rho_1(z)|, \quad z \in \mathbb{R}. \tag{A.30}$$

As before, this symmetry in the reflection coefficient implies that the radiation does not contribute to the theta-condition. Using the symmetries of the discrete spectrum, it is easy to see that one recovers the results given in appendix A.1.

A.4. Determinantal formula for the multi-soliton solutions of the Manakov system with NZBC

Here we derive the expression (2.27) for the multi-soliton solutions of the defocusing Manakov system with NZBC. We start from the reconstruction formula (2.26) in the reflectionless case, in which case the integral term is absent. First of all we define the following quantities:

$$d_{n,j}^{(1)} = \frac{\bar{\gamma}_j}{\zeta_n - \zeta_j^*}\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta_j)}, \qquad d_{n,j}^{(2)} = \frac{\gamma_j \zeta_n^*}{\zeta_n^* - \zeta_j}\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta_j)}, \tag{A.31a}$$

$$d_{n,\ell}^{(3)} = -\frac{\bar{\delta}_\ell^*}{\zeta_n - \hat{z}_\ell^*}\,\mathrm{e}^{\mathrm{i}(\theta_2(z_\ell) - \theta_1(z_\ell))}, \qquad d_{n,\ell}^{(4)} = -\frac{\bar{\delta}_\ell^* \zeta_n^*}{\hat{z}_\ell^*(\zeta_n^* - z_\ell)}\,\mathrm{e}^{\mathrm{i}(\theta_2(z_\ell) - \theta_1(z_\ell))}, \tag{A.31b}$$

for $n, j = 1, 2, \ldots, N_1$ and $\ell = 1, 2, \ldots, N_2$, as well as

$$d_{n,\ell}^{(5)} = \frac{\gamma_\ell z_n^*}{z_n^* - \zeta_\ell}\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta_\ell)}, \quad d_{n,j}^{(6)} = -\frac{\bar{\delta}_j^* z_n^*}{\hat{z}_j^*(z_n^* - z_j)}\,\mathrm{e}^{\mathrm{i}(\theta_2(z_j) - \theta_1(z_j))}, \tag{A.31c}$$

$$d_{n,j}^{(7)} = \frac{\bar{\delta}_j \hat{z}_n^*}{(\hat{z}_n^* - \hat{z}_j)(\hat{z}_n^* - z_j^*)}\,\mathrm{e}^{\mathrm{i}(\theta_1(z_j^*) - \theta_2(z_j^*))}, \tag{A.31d}$$

for $n, j = 1, 2, \ldots, N_2$ and $\ell = 1, 2, \ldots, N_1$. As before, the x, t-dependence in the exponentials has been omitted for brevity. Now using the above defined quantities we can rewrite the equation (2.19) as follows

$$M_1^{(\mathrm{dn})}(\zeta_n) = -\mathrm{i}\mathbf{q}_+^* + \sum_{j=1}^{N_1} d_{n,j}^{(1)} M_2^{(\mathrm{dn})}(\zeta_j^*) + \sum_{j=1}^{N_2} d_{n,j}^{(3)} M_3^{(\mathrm{dn})}(\hat{z}_j^*), \tag{A.32a}$$

$$M_2^{(\mathrm{dn})}(\zeta_n^*) = \mathrm{i}\mathbf{q}_+^* + \sum_{j=1}^{N_1} d_{n,j}^{(2)} M_1^{(\mathrm{dn})}(\zeta_j) + \sum_{j=1}^{N_2} d_{n,j}^{(4)} M_3^{(\mathrm{dn})}(\hat{z}_j^*), \tag{A.32b}$$

$$M_2^{(\mathrm{dn})}(z_n^*) = \mathrm{i}\mathbf{q}_+^* + \sum_{j=1}^{N_1} d_{n,j}^{(5)} M_1^{(\mathrm{dn})}(\zeta_j) + \sum_{j=1}^{N_2} d_{n,j}^{(6)} M_3^{(\mathrm{dn})}(\hat{z}_j^*), \tag{A.32c}$$

$$M_3^{(\mathrm{dn})}(\hat{z}_n^*) = \mathrm{i}\mathbf{q}_+^\perp - \sum_{j=1}^{N_2} d_{n,j}^{(7)} M_2^{(\mathrm{dn})}(z_j^*). \tag{A.32d}$$

Substituting (A.32c) into (A.32d) yields

$$M_3^{(\mathrm{dn})}(\hat{z}_n^*) + \sum_{\ell=1}^{N_2}\left(\sum_{j=1}^{N_2} d_{n,j}^{(7)} d_{j,\ell}^{(6)}\right) M_3^{(\mathrm{dn})}(\hat{z}_\ell^*) + \sum_{\ell=1}^{N_1}\left(\sum_{j=1}^{N_2} d_{n,j}^{(7)} d_{j,\ell}^{(5)}\right) M_1^{(\mathrm{dn})}(\zeta_\ell) = \mathrm{i}\mathbf{q}_+^\perp - \mathrm{i}\mathbf{q}^* \sum_{j=1}^{N_2} d_{n,j}^{(7)}, \tag{A.33a}$$

and similarly using (A.32a) and (A.32b) we have

$$M_1^{(\mathrm{dn})}(\zeta_n) - \sum_{\ell=1}^{N_1}\left(\sum_{j=1}^{N_1} d_{n,j}^{(1)} d_{j,\ell}^{(2)}\right) M_1^{(\mathrm{dn})}(\zeta_\ell) - \sum_{j=1}^{N_2} d_{n,j}^{(3)} M_3^{(\mathrm{dn})}(\hat{z}_j^*) - \sum_{\ell=1}^{N_2}\sum_{j=1}^{N_1} d_{n,j}^{(1)} d_{j,\ell}^{(4)} M_3^{(\mathrm{dn})}(\hat{z}_\ell^*) \tag{A.33b}$$

$$= -\mathrm{i}\mathbf{q}_+^* + \mathrm{i}\mathbf{q}_+^* \sum_{j=1}^{N_1} d_{n,j}^{(1)}. \tag{A.33c}$$

First, we introduce $\mathbf{Y} = (Y_1, Y_2, \dots, Y_{N_1+N_2})^{\mathrm{T}}$ and $\mathbf{V} = (V_1, V_2, \dots, V_{N_1+N_2})^{\mathrm{T}}$ where

$$Y_n = \begin{cases} \left(M_3^{\mathrm{dn}}(\hat{z}_n^*)\right)^{\mathrm{T}} & n = 1, 2, ..., N_2, \\ \left(M_1^{\mathrm{dn}}(\zeta_{N_2-n})\right)^{\mathrm{T}} & n = N_2+1, ..., N_2+N_1, \end{cases} \tag{A.34a}$$

$$V_n = \begin{cases} \mathrm{i}(\mathbf{q}_+^\perp)^{\mathrm{T}} - \mathrm{i}(\mathbf{q}_+^*)^{\mathrm{T}} \displaystyle\sum_{j=1}^{N_2} d_{n,j}^{(7)} & n = 1, 2, \dots, N_2, \\ -\mathrm{i}(\mathbf{q}_+^*)^{T} + \mathrm{i}(\mathbf{q}_+^*)^{\mathrm{T}} \displaystyle\sum_{j=1}^{N_1} d_{N_2-n,j}^{(1)} & n = N_2+1, \dots, N_2+N_1. \end{cases} \tag{A.34b}$$

Note that both $\mathbf{Y}$ and $\mathbf{V}$ are $(N_1+N_2)\times 2$ matrices. Now we write the closed system (A.33) for $M_1^{(\mathrm{dn})}(\zeta_j)$ and $M_3^{(\mathrm{dn})}(\hat{z}_j^*)$ in matrix form as

$$\mathbf{RY} = \mathbf{V}, \tag{A.35}$$

where $\mathbf{R} = (R_{n,k})$ with $n, k = 1, 2, \ldots, N_1 + N_2$, with

$$R_{n,k} = \begin{cases} \delta_{n,k} + \sum_{j=1}^{N_2} d_{n,j}^{(7)} d_{j,k}^{(6)}, & n, k = 1, 2, \ldots, N_2, \\ \sum_{j=1}^{N_2} d_{n,j}^{(7)} d_{j,N_2-k}^{(5)}, & n = 1, 2, \ldots, N_2, \quad k = N_2 + 1, \ldots, N_2 + N_1, \\ -d_{n-N_2,k}^{(3)} - \sum_{j=1}^{N_1} d_{n-N_2,j}^{(1)} d_{j,k}^{(4)}, & k = 1, 2, \ldots, N_2, \quad n = N_2 + 1, \ldots, N_2 + N_1, \\ \delta_{n-N_2,k-N_2} - \sum_{j=1}^{N_1} d_{n-N_2,j}^{(1)} d_{j,k-N_2}^{(2)}, & n, k = N_2 + 1, \ldots, N_2 + N_1, \end{cases} \tag{A.36}$$

and where $\delta_{n,k}$ is the Kronecker delta. We can also rewrite the system (A.35) as

$$\mathbf{R}\mathbf{Y}_1 = \mathbf{V}_1, \qquad \mathbf{R}\mathbf{Y}_2 = \mathbf{V}_2, \tag{A.37}$$

where we now express $\mathbf{Y}$ and $\mathbf{V}$ as $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$ and $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2)$, and where $\mathbf{Y}_j$ and $\mathbf{V}_j$ for $j = 1, 2$ are $(N_1 + N_2)$-component column vectors. The solution to (A.37) gives, for each component of the vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$,

$$Y_{n,1} = \frac{\det \hat{R}_n^{\text{ext}}}{\det R}, \qquad Y_{n,2} = \frac{\det \check{R}_n^{\text{ext}}}{\det R} \quad n = 1, 2, \ldots, N_1 + N_2,$$

where

$$\hat{R}_n^{\text{ext}} = (R_1, R_2, \ldots, R_{n-1}, B_1, R_{n+1}, \ldots, R_N), \tag{A.38a}$$

$$\check{R}_n^{\text{ext}} = (R_1, R_2, \ldots, R_{n-1}, B_2, R_{n+1}, \ldots, R_N). \tag{A.38b}$$

It then follows that

$$M_3^{(\text{dn})}(\hat{z}_n^*) = \frac{1}{\det \mathbf{R}} \begin{pmatrix} \det \hat{R}_n^{\text{ext}} \\ \det \check{R}_n^{\text{ext}} \end{pmatrix}, \quad n = 1, 2, \ldots, N_2, \tag{A.39}$$

and

$$M_1^{(\text{dn})}(\zeta_n) = \frac{1}{\det \mathbf{R}} \begin{pmatrix} \det \hat{R}_{N_2+n}^{\text{ext}} \\ \det \check{R}_{N_2+N}^{\text{ext}} \end{pmatrix}, \quad n = 1, 2, \ldots, N_1. \tag{A.40}$$

Finally, upon substituting $Y_1, \ldots, Y_N$ into (2.26) in the reflectionless case, the resulting expression for the potential can be written compactly as (2.27), where the augmented $(N_1 + N_2 + 1) \times (N_1 + N_2 + 1)$ matrices are given by

$$N_j^{\text{aug}} = \begin{pmatrix} q_{+,j} & -\mathrm{i}\mathbf{v}^{\mathrm{T}} \\ \mathbf{V}_j^* & \mathbf{R}^* \end{pmatrix}, \quad j \in \{1, 2\},$$

with

$$\mathbf{v} = (v_1, v_2, \ldots, v_{N_1+N_2})^{\mathrm{T}}, \tag{A.41a}$$

and

$$v_n = \begin{cases} -\dfrac{\bar{\delta}_n}{\hat{z}_n}\,\mathrm{e}^{\mathrm{i}(\theta_1(z_n^*))-\theta_2(z_n^*)} & n = 1, 2, \ldots, N_2, \\ \dfrac{\bar{\gamma}_{n-N_2}}{\zeta_{n-N_2}^*}\,\mathrm{e}^{-2\mathrm{i}\theta_1(\zeta_{n-N_2})} & n = N_2+1, \ldots, N_2+N_1. \end{cases} \tag{A.41b}$$

A.5. Symmetries of the discrete spectrum, norming constants and trace formula

First, we show that the function $R(z)$ defined as in (2.24) is an even function for all parity ICs. Using parity symmetries for all the cases we considered and $b_{11}(z) = a_{11}^*(z^*)$, we have

$$R(-z) = \frac{z^2}{q_o^2}|\rho_1(-z)|^2 + \frac{q_o^2}{z^2-q_o^2}|\rho_2(-z)|^2 = \frac{z^2}{q_o^2}\left|\frac{a_{31}(z)}{b_{11}(z)}\right|^2 + \frac{q_o^2}{z^2-q_o^2}\left|\frac{b_{12}(z)}{a_{11}(z)}\right|^2, \quad z \in \mathbb{R}. \tag{A.42}$$

Applying the co-factor transformation along with the first symmetry yields,

$$\frac{a_{31}(z)}{b_{11}(z)} = \frac{b_{21}(z)b_{32}(z) - b_{31}(z)b_{22}(z)}{b_{11}(z)} = -\frac{q_o^2}{z^2}a_{23}^*(z)\rho_2^*(z) - \rho_1(z)a_{22}^*(z), \quad z \in \mathbb{R}, \tag{A.43a}$$

$$\frac{b_{12}(z)}{a_{11}(z)} = \frac{a_{13}(z)a_{32}(z) - a_{12}(z)a_{33}(z)}{a_{11}(z)} = -\frac{z^2}{q_o^2}\left(\frac{q_o^2}{z^2}a_{33}(z)\rho_2(z) + \rho_1^*(z)a_{32}(z)\right), \quad z \in \mathbb{R}. \tag{A.43b}$$

Now employing (A.43a) and (A.43b) on the equation for $R(-z)$, we obtain for $z \in \mathbb{R}$,

$$\begin{aligned} R(-z) &= \frac{z^2}{q_o^2}\left|\frac{q_o^2}{z^2}a_{23}^*(z)\rho_2^*(z) + \rho_1(z)a_{22}^*(z)\right|^2 + \frac{z^4}{q_o^2(z^2-q_o^2)}\left|\frac{q_o^2}{z^2}a_{33}(z)\rho_2(z) + \rho_1^*(z)a_{32}(z)\right|^2, \\ &= \frac{q_o^2}{z^2-q_o^2}|\rho_2(z)|^2\left(\frac{z^2-q_o^2}{q_o^2}|a_{23}|^2 + |a_{33}(z)|^2\right) + \frac{z^2}{q_o^2}|\rho_1(z)|^2\left(|a_{22}|^2 + \frac{z^2}{z^2-q_o^2}|a_{32}(z)|^2\right) \\ &\quad + 2\,\mathrm{Re}\left\{\rho_1(z)\rho_2(z)\left(a_{23}(z)a_{22}^*(z) + \frac{z^2}{(z^2-q_o^2)}a_{33}(z)a_{32}^*(z)\right)\right\}, \\ &= \frac{q_o^2}{z^2-q_o^2}|\rho_2(z)|^2\left(a_{23}(z)b_{32}(z) + a_{33}(z)b_{33}(z)\right) + \frac{z^2}{q_o^2}|\rho_1(z)|^2\left(a_{22}(z)b_{22}(z) + a_{32}(z)b_{23}(z)\right) \\ &\quad + 2\,\mathrm{Re}\{\rho_1(z)\rho_2(z)\left(a_{23}(z)b_{22}(z) + a_{33}(z)b_{23}(z)\right)\}. \end{aligned} \tag{A.44a}$$

Using the fact that $B(z)A(z) = A(z)B(z) = I_{3\times3}$, we have, when $z \in \mathbb{R}$,

$$a_{23}(z)b_{32}(z) + a_{33}(z)b_{33}(z) = 1 - b_{31}(z)a_{13}(z), \tag{A.44b}$$

$$a_{22}(z)b_{22}(z) + a_{32}(z)b_{23}(z) = 1 - b_{21}(z)a_{12}(z), \tag{A.44c}$$

$$a_{23}(z)b_{22}(z) + a_{33}(z)b_{23}(z) = -b_{21}(z)a_{13}(z). \tag{A.44d}$$

Collecting all together and simplify further yields,

$$R(-z) = \frac{q_o^2}{z^2-q_o^2}|\rho_2(z)|^2 + \frac{z^2}{q_o^2}|\rho_1(z)|^2 = R(z), \quad z \in \mathbb{R}, \tag{A.45}$$

i.e., (4.3).

Next we show that the scattering coefficient $a_{11}(z)$ simplifies considerably when z becomes purely imaginary. The trace formula (2.23a) yields

$$a_{11}(z) = \left(\frac{z - \mathrm{i}q_o}{z + \mathrm{i}q_o}\right)^{m_1} \prod_{n=1}^{m_2} \frac{z - \mathrm{i}Z_n}{z + \mathrm{i}Z_n} \prod_{n=m_1+1}^{(N_1+m_1)/2} \times \frac{(z - \zeta_n^+)(z - \zeta_n^-)}{(z - \bar{\zeta}_n^+)(z - \bar{\zeta}_n^-)} \prod_{n=m_2+1}^{(N_2+m_2)/2}$$

$$\times \frac{(z - z_n^+)(z - z_n^-)}{(z - \bar{z}_n^+)(z - \bar{z}_n^-)} \times \exp\left\{-\frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\log[1 - R(\zeta)]}{\zeta - z}\,\mathrm{d}\zeta\right\}, \quad z \in \mathbb{C}^+. \qquad (A.46)$$

Using (A.45), the following identities can be obtained by direct computation:

$$\frac{(z - \zeta_n^+)(z - \zeta_n^-)}{(z - \bar{\zeta}_n^+)(z - \bar{\zeta}_n^-)} = \frac{\left(|z|^2 - q_o^2\right)^2 + \left(2|z|\mathrm{Re}(\zeta_n^+)\right)^2}{|(z - \bar{\zeta}_n^+)(z - \bar{\zeta}_n^-)|^2}, \quad n = m_1 + 1, \ldots, (N_1 + m_1)/2, \quad z \in \mathrm{i}\mathbb{R}^+, \qquad (A.47a)$$

$$\frac{(z - z_n^+)(z - z_n^-)}{(z - \bar{z}_n^+)(z - \bar{z}_n^-)} = \frac{\left(|z|^2 - |z_n^+|^2\right)^2 + \left(2|z|\,\mathrm{Re}(z_n^+)\right)^2}{|(z - \bar{z}_n^+)(z - \bar{z}_n^-)|^2}, \quad n = m_2 + 1, \ldots, (N_2 + m_2)/2, \quad z \in \mathrm{i}\mathbb{R}^+, \qquad (A.47b)$$

$$\int_{-\infty}^{\infty} \frac{\log[1 - R(\zeta)]}{\zeta - z}\,\mathrm{d}\zeta = 2z \int_0^{\infty} \frac{\log[1 - R(\zeta)]}{\zeta^2 - z^2}\,\mathrm{d}\zeta, \quad z \in \mathrm{i}\mathbb{R}^+. \qquad (A.47c)$$

Hence we have

$$a_{11}(z) = \left(\frac{z - \mathrm{i}q_o}{z + \mathrm{i}q_o}\right)^{m_1} \prod_{n=1}^{m_2} \frac{z - \mathrm{i}Z_n}{z + \mathrm{i}Z_n} \prod_{n=m_1+1}^{(N_1+m_1)/2} \frac{\left(|z|^2 - q_o^2\right)^2 + \left(2|z|\,\mathrm{Re}(\zeta_n^+)\right)^2}{|(z - \bar{\zeta}_n^+)(z - \bar{\zeta}_n^-)|^2}$$

$$\times \prod_{n=m_2+1}^{(N_2+m_2)/2} \frac{\left(|z|^2 - |z_n^+|^2\right)^2 + \left(2|z|\,\mathrm{Re}(z_n^+)\right)^2}{|(z - \bar{z}_n^+)(z - \bar{z}_n^-)|^2}$$

$$\times \exp\left\{-\frac{z}{\pi\mathrm{i}} \int_0^{\infty} \frac{\log[1 - R(\zeta)]}{\zeta^2 - z^2}\,\mathrm{d}\zeta\right\}, \quad z \in \mathrm{i}\mathbb{R}^+, \qquad (A.48)$$

from which (4.6a) and (4.6b) follow.

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