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A Bounded-Confidence Model of Opinion Dynamics on Hypergraphs*

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Abstract. People's opinions evolve with time as they interact with their friends, family, colleagues, and others. In the study of opinion dynamics on networks, one often encodes interactions between people in the form of dyadic relationships, but many social interactions in real life are polyadic (i.e., they involve three or more people). In this paper, we extend an asynchronous bounded-confidence model (BCM) on graphs, in which nodes are connected pairwise by edges, to an asynchronous BCM on hypergraphs, in which arbitrarily many nodes can be connected by a single hyperedge. We show that our hypergraph BCM converges to consensus for a wide range of initial conditions for the opinions of the nodes, including for nonuniform and asymmetric initial opinion distributions. We also show that, under suitable conditions, echo chambers can form on hypergraphs with community structure. We demonstrate that the opinions of nodes can sometimes jump from one opinion cluster to another in a single time step; this phenomenon (which we call "opinion jumping") is not possible in standard dyadic BCMs. Additionally, we observe a phase transition in the convergence time of our BCM on a complete hypergraph when the variance σ^2 of the initial opinion distribution equals the confidence bound c. We prove that the convergence time grows at least exponentially fast with the number of nodes when $\sigma^2 > c$ and the initial opinions are normally distributed. Therefore, to determine the convergence properties of our hypergraph BCM when the variance and the number of hyperedges are both large, it is necessary to use analytical methods instead of relying only on Monte Carlo simulations.

Key words. opinion models, hypergraphs, bounded-confidence models, polyadic interactions

AMS subject classifications. 91D30, 05C65, 05C82

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1. Introduction.

Social interactions with friends and acquaintances can persuade people to change their opinions about public figures [30], social issues [47], economic policy [25], and more. In opinion dynamics, researchers study how opinions about one or more topics evolve with time as people interact and influence each other [19]. Traditionally, one models entities as nodes in a graph and one models their social relationships as edges that encode pairwise relationships between them [6, 39]. The opinions of these entities can change as a result of such relationships. In the present paper, we build on these ideas by studying the effects of group

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interactions on opinion formation by modeling relationships as hyperedges in a hypergraph. We find that polyadic interactions play a key role in whether or not a group reaches consensus and in how long it takes to reach it.

We focus on *continuous-opinion dynamics*, in which nodes have continuous-valued opinions. In our model, nodes hold opinions in $\mathbb{R}$. We denote the *opinion state* of the system by $\boldsymbol{x} \in \mathbb{R}^N$, where N is the number of nodes. This is an appropriate model for opinions, such as the strength of support for a political candidate, that lie on a spectrum. By contrast, opinions such as whether one supports the Los Angeles Dodgers or the San Francisco Giants may leave little or no room for any middle ground.

Bounded-confidence models (BCMs) are models with continuous-valued opinions in which individuals are influenced only by neighbors who hold opinions that are within some *confidence bound* c of their own opinion [33, 41]. Individuals who disagree with each other too much do not influence each other [49]. This models the concept of *selective exposure* from social psychology; according to this principle, individuals tend to ignore information that is contrary to their current viewpoint [31, 48]. In a traditional BCM, each individual is a node in a graph and its neighbors are its adjacent nodes. A BCM is *asynchronous* if only one pair of neighbors can interact at a time and is *synchronous* if all pairs of neighboring nodes interact in each time step. The two most commonly studied BCMs are the (asynchronous) Deffuant–Weisbuch (DW) model [9, 51] and the (synchronous) Hegselmann–Krause (HK) model [22]. See [33] for a review and a comparison of these two models, and see the introduction of [36] for a recent summary of research on BCMs.

An important limitation of graphs is that they force one to consider only pairwise (i.e., "dyadic") interactions between nodes (as well as self-interactions, if one allows self-edges), whereas many social interactions involve three or more individuals at once [1, 4, 43]. One example of such a polyadic (i.e., "higher-order") social interaction is group messaging, such as group texting or e-mails with more than one recipient. We seek to examine the effects of polyadic interactions on opinion dynamics, so we develop and analyze an extension of BCMs to hypergraphs. In a hypergraph, a hyperedge can connect an arbitrary number of nodes, rather than just two of them. In the context of opinion dynamics, one way to interpret such interactions is as a form of "peer pressure" [26, 35], but other interpretations are also possible. Importantly, it is not possible to reduce the higher-order interactions in our hypergraph BCM to an aggregation of pairwise interactions.

Hypergraph extensions of opinion models have attracted much attention in the last few years [37]. Sahasrabuddhe, Neuhaüser, and Lambiotte [45] proposed a synchronous[1] opinion model on hypergraphs, and they conducted numerical simulations of their model on complete hypergraphs, random hypergraphs, and real-world hypergraphs. Their model has continuous-valued opinions, uses continuous time, and does not involve a confidence bound. By contrast, our model is a hypergraph extension of an asynchronous[2] BCM. We both conduct numerical simulations and derive analytical results about our model. Our hypergraph BCM produces

[1]We define a *synchronous hypergraph model* to be a model in which each node interacts with all of its incident hyperedges at once.

[2]We define an *asynchronous hypergraph model* to be a model in which only nodes in a single hyperedge can interact in one time step.

qualitatively different dynamics from the model of Sahasrabuddhe, Neuhaüser, and Lambiotte. For example, consensus occurs in different circumstances in the two models. Additionally, in our model, the mean opinion is constant in time. To the best of our knowledge,[3] other hypergraph extensions of continuous-opinion models have only considered interactions with three or fewer nodes (i.e., hypergraphs in which the hyperedge "sizes" are no larger than 3)[4] [38, 40]. In our hypergraph BCM, we find that "large" hyperedges (i.e., hyperedges that are incident to many nodes) are crucial for reaching consensus and that hypergraphs that have large hyperedges yield rather different dynamics than hypergraphs with only small hyperedges.

A key issue in opinion dynamics is how model parameters, hypergraph structure (or the structure of other types of networks), and initial opinion states influence the opinion state to which a model converges. By applying the results of [32], we show that the opinion state always converges to some "limit state." In the standard dyadic DW and HK models, the number of opinion clusters in the limit state depends on the confidence bound c. We say that the opinion state is at *consensus* if there is a single opinion cluster; that is, every node has the same opinion. Most work on dyadic BCMs has drawn initial opinions uniformly at random from $[0, 1]$. In this case, the opinion state converges to consensus only when c is above a certain threshold value [9, 16, 18, 34, 50, 51]. By contrast, as we show in section 3, our hypergraph BCM has no such confidence-bound threshold. In subsection 3.1, we prove this result for complete hypergraphs. In fact, we prove the following stronger statement: if the initial opinion distribution is bounded, then the opinion state on a complete hypergraph converges to consensus almost surely if the hypergraph has sufficiently many nodes. The case in which the initial opinion distribution is bounded includes nonuniform probability distributions, asymmetric probability distributions, and probability distributions in which one draws initial opinions uniformly at random from an interval. When one draws the initial opinions of the nodes in a complete hypergraph from a distribution with variance $\sigma^2 < c$, we prove that the probability of consensus approaches 1 as the number of nodes approaches infinity. For the special case in which the initial opinions are normally distributed, we also present numerical evidence that the opinion state converges to consensus even when $c \leq \sigma^2$. We give a heuristic argument to explain this observation.

We also explore the behavior of our hypergraph BCM when the initial opinions depend on community structure [17, 44], in which dense sets of nodes are connected sparsely to other dense sets of nodes. A recent study of a dyadic BCM showed heuristically on ordinary graphs with community structure that separate opinion clusters tend to emerge for each community if the communities are not well-connected to each other [14] (also see [23]). We use the term *polarization* for this phenomenon,[5] and we use the term *echo chambers* [8, 12, 15] for these different opinion clusters. In subsection 3.2, we study our BCM on hypergraphs with community structure. We prove that polarization can occur if there is an upper bound on the sizes of the hyperedges that connect different communities. This provides a possible mechanism for the formation of echo chambers in hypergraphs. However, if there is no upper bound on the size of intercommunity hyperedges and each community forms a complete hypergraph (i.e., a

[3] A new preprint about hypergraph BCMs [46] appeared during the proofing stage of our paper.

[4] A few opinion models that consider interactions with more than three nodes have been developed for frameworks (such as voter models and contagion models) with discrete-valued opinions [7, 21, 24].

[5] Some researchers refer to this phenomenon as "opinion fragmentation" and use the term "polarization" only when there are exactly two opinion clusters.

hyperclique) and has sufficiently many nodes, we prove that our hypergraph BCM converges to consensus.

Using numerical simulations of our BCM, we demonstrate that our theoretical results about reaching consensus on hypergraphs are robust. The theorems in subsection 3.1 require that the hypergraph is complete, and some of the results in subsection 3.2 require that the communities form hypercliques. However, in practice, we can relax these conditions and the nodes' opinions still eventually reach consensus on the hypergraph in the former case and on individual communities in the latter case. In subsection 3.3, we study the behavior of our BCM on sparse Erdős–Rényi-like hypergraphs using Monte Carlo simulations. In subsection 3.4, we study the behavior of our model on the Enron e-mail hypergraph [3], in which the nodes are Enron employees and hyperedges encode e-mails between these employees. Hypergraphs that one constructs from empirical data are interesting examples both because typically they are sparse and because their hyperedges are usually small in comparison to the number of nodes.

The convergence time of our BCM is a significant factor to consider when we numerically simulate it. In section 4, we partially characterize the conditions for which our hypergraph BCM converges in finite time. In particular, we prove that it almost surely converges in finite time on a complete hypergraph. By comparison, the dyadic DW model for ordinary graphs usually does not converge in finite time, although the HK model always does [13]. We also observe that there is a phase transition in the convergence time of our BCM on a complete hypergraph when the variance σ^2 of the initial opinion distribution equals the confidence bound c. We prove that the expected convergence time of our BCM grows at least exponentially fast with the number N of nodes when $\sigma^2 > c$ and the initial opinions are normally distributed. When $\sigma^2 < c$, our numerical experiments with complete hypergraphs converge much faster than when $\sigma^2 > c$. Meng, Van Gorder, and Porter [36] demonstrated numerically that the standard dyadic DW model also has a phase transition in the convergence time. It is important to understand this phase transition because when one is running a Monte Carlo simulation of a BCM, one chooses a finite cutoff time to stop the simulation. Without analysis of the convergence time, one may accidentally cut off the numerical simulations too early and mistakenly conclude that there is a phase transition in the limit state when what has actually occurred is a phase transition in the convergence time.

When studying opinion dynamics, it is also desirable to understand the evolution of the opinion state before reaching a limit state. In section 5, we investigate a phenomenon, which we call *opinion jumping*, in which the opinion of a node changes by more than c in a single time step. Opinion jumping allows nodes with extreme opinions to jump close to the mean of the opinion distribution in a single time step. This behavior cannot occur in the classical dyadic DW or HK models because nodes in those BCMs interact only with neighbors whose opinions are sufficiently similar to their own.

Our paper proceeds as follows. In section 2, we give a formal definition of our hypergraph BCM. In section 3, we present our results about its limit state. These are the main results of our paper. In section 4, we discuss convergence time. In section 5, we examine opinion jumping and quantify how often it occurs. We conclude and discuss future work in section 6. Our code is available at https://bitbucket.org/ahickok/hypergraph-bcm.

2. A BCM on hypergraphs.

In this section, we develop an extension of the DW model to hypergraphs. We start by presenting the standard DW model on graphs.

In the standard dyadic DW model, opinion dynamics occur on an unweighted and undirected graph whose edges encode social ties. At each discrete time t, one chooses an edge $e = \{i, j\}$ uniformly at random. If the difference $|x_i(t) - x_j(t)|$ of opinions between nodes i and j is below some confidence bound $c_{i,j}$, then nodes i and j adjust their opinions as follows:

$$x_i(t+1) = x_i(t) + m_{i,j}(x_j(t) - x_i(t)),$$
$$x_j(t+1) = x_j(t) + m_{j,i}(x_i(t) - x_j(t)),$$

where $m_{i,j}$ is an entry of the matrix of convergence parameters. Otherwise, the opinions of nodes i and j are too far apart at time t, so $x_i(t+1) = x_i(t)$ and $x_j(t+1) = x_j(t)$. The opinions of all other nodes do not change when we do this update. With this type of update rule, the mean opinion of the nodes in a network is a conserved quantity. The confidence bounds $c_{i,j} \in [0, \infty)$ model the level of open-mindedness of individuals to the opinions of others [42]. The convergence parameters $m_{i,j} \in [0, 0.5]$ (which resemble the trust parameters in DeGroot models [41]) control the rate at which individuals adjust their opinions [9, 51]. Using a single value of c and a single value of m for all pairs leads to what is sometimes called the "homogeneous" DW model.

We now define our BCM on hypergraphs as an extension of the homogeneous DW model. A hypergraph is a generalization of a graph that allows interactions between arbitrarily many nodes. That is, interactions between multiple nodes can be either dyadic or polyadic. The space of possible hyperedges is the power set $\mathcal{P}(V)$ of the set V of nodes. Let $N := |V|$ denote the number of nodes. In the hypergraphs (V, E) that we consider, we restrict the hyperedge set $E \subseteq \{e \in \mathcal{P}(V) \mid |e| \geq 2\}$ so that each hyperedge is incident to at least two nodes. Prohibiting hyperedges that are attached to only a single node (these are called "self-hyperedges") affects only the convergence time; it does not affect the limit state. All of our hypergraphs are unweighted and undirected. In our BCM, there is a time-dependent opinion state $\boldsymbol{x}(t) \in O^N$, where we take the opinion space O to be the real line $\mathbb{R}$. We use $x_i(t)$ to denote the opinion of node i at time t.

To generalize the notion of a confidence bound to hyperedges, we define a *discordance function* $d : E \times O^N \to \mathbb{R}_{\geq 0}$ that maps a hyperedge and opinion state to a real number. We use this function, which quantifies the level of disagreement among the nodes that are incident to a hyperedge, to determine whether or not these nodes update their opinions. We consider the following family of discordance functions:

$$d_\alpha(e, \boldsymbol{x}) = \left(\frac{1}{|e| - 1}\right)^\alpha \sum_{i \in e} (x_i - \bar{x}_e)^2 , \tag{2.1}$$

which is parametrized by the scalar α, with $\bar{x}_e = (\sum_{i \in e} x_i)/|e|$. If the discordance $d_\alpha(e, \boldsymbol{x}(t))$ is less than the confidence bound c, we say that the hyperedge e is *concordant* at time t. Otherwise, we say that e is *discordant*.

The choice $\alpha = 1$ is a noteworthy special case. The function $d_1(e, \boldsymbol{x})$ is equal to the unbiased sample variance of the opinions of the nodes that are incident to e. This models a situation in which nodes with moderate opinions can mediate between nodes in a group with extreme opinions as long as the overall disagreement within the group is not too large. For example, let $\boldsymbol{x} = (0, 1, 0.5)$ and consider the hyperedges $e = \{1, 2\}$ and $e' = \{1, 2, 3\}$.

We see that $d_1(e, \boldsymbol{x}) = 0.5 > 0.25 = d_1(e', \boldsymbol{x})$, even though $e \subset e'$. One can interpret node 3's role in the interaction as that of a mediator who reduces the amount of discordance, thereby potentially yielding an opinion update that otherwise would not occur. The scaling by $1/(|e|-1)$ in (2.1) prevents advantaging hyperedges with few nodes over hyperedges with many nodes when we update opinions. Specifically, if the opinions are independent and identically distributed, then the expected d_1-discordance of any subset of nodes is

$$\mathbb{E}[d_1(e, \boldsymbol{x})] = \mathbb{E}[d_1(e', \boldsymbol{x})] \text{ for all } e, e' \in E\,.$$

We set the discordance function to $d = d_1$ for the remainder of this paper.

Another noteworthy special case, which we do not consider further in the present paper, is $\alpha = 0$. The function $d_0(e, \boldsymbol{x})$ penalizes large hyperedges, in the sense that

$$d(e, \boldsymbol{x}) \leq d(e', \boldsymbol{x}) \text{ if } e \subseteq e'\,, \tag{2.2}$$

with equality holding if and only if $x_i = \bar{x}_e$ for all $i \in e' \setminus e$. We use the term *hyperedge monotonic* for a discordance function that satisfies (2.2). This models situations in which large groups tend to be less effective than small groups at changing opinions.

We employ asynchronous updates (as in the DW model). At each discrete time, we randomly select a hyperedge $e \in E$ according to some probability distribution. For mathematical convenience, we use the uniform distribution over E. If the discordance $d(e, \boldsymbol{x})$ is less than the confidence bound c, the nodes $i \in e$ update their opinions x_i to the mean opinion $\bar{x}_e$; otherwise, their opinions do not change. One way to think of this update is that nodes $i \in e$ are "peer-pressured" into conforming to the mean opinion of the group when the overall discordance of the group is sufficiently small. More precisely, if we select hyperedge e at time t, the update rule for each node i is

$$x_i(t+1) = \begin{cases} \bar{x}_e(t)\,, & \text{if } i \in e \text{ and } d(e, \boldsymbol{x}) < c \\ x_i(t)\,, & \text{otherwise}\,. \end{cases} \tag{2.3}$$

The sequence $\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \ldots$ of opinion states is a discrete-time Markov chain with a continuous state space.

For the special case of a hypergraph that is a graph (i.e., if $|e| = 2$ for all $e \in E$), our generalized BCM reduces to a standard DW model with a rescaled confidence bound c. This rescaling arises from the difference in discordance functions: the standard DW model uses the absolute value $|x_i - x_j|$ of the difference of opinions, whereas our model uses $\frac{1}{2}(x_i - x_j)^2$ (see (2.1)). On hypergraphs that are graphs, our generalized BCM with confidence bound $\frac{1}{2}c^2$ is equivalent to the standard DW model with confidence bound c. Therefore, our generalized BCM does not exactly reduce to the standard DW model. However, the two models are still easy to compare. The advantage of our choice is that the discordance d_1 is equal to the unbiased sample variance; this is helpful for deriving our analytical results.

3. The limit state of our hypergraph BCM.

We say that the opinion state *converges* to $\boldsymbol{x}^*$ if $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*$. We refer to $\boldsymbol{x}^*$ as the *limit state*. An *opinion cluster* in the limit state is a collection of nodes that all have the same opinion in the limit state. The *opinion value* of an opinion cluster is the opinion $\gamma \in \mathbb{R}$ such that $x_i^* = \gamma$ for all nodes i in that cluster.

The opinion state converges to *consensus* if there is a $\gamma \in \mathbb{R}$ such that $x_i^* = \gamma$ for all i. Equivalently, the opinion state converges to consensus if there is exactly one opinion cluster in the limit state. If the opinion state converges to consensus, it is necessarily true that $\gamma = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$ because the mean opinion of the nodes is constant with respect to time.

An opinion state $\boldsymbol{x}$ is an *absorbing* state if for all $e \in E$, either $d(e, \boldsymbol{x}) \geq c$ or $d(e, \boldsymbol{x}) = 0$ (i.e., $x_i = x_j$ for all $i, j \in e$). If $\boldsymbol{x}(T)$ is an absorbing state, then $\boldsymbol{x}(t) = \boldsymbol{x}(T)$ for all $t \geq T$. We will prove in Lemma 3.2 that the limit state is almost surely an absorbing state.

We now show that for any initial opinion state $\boldsymbol{x}(0)$, the opinion state of our hypergraph BCM converges in the limit $t \to \infty$.

Theorem 3.1. *Let $\boldsymbol{x}(0)$ be the initial opinion state and update the opinion state $\boldsymbol{x}(t)$ according to (2.3). It follows that the limit state $\boldsymbol{x}^* := \lim_{t\to\infty} \boldsymbol{x}(t)$ exists.*

Proof. Let $A(\boldsymbol{x}(t), t)$ be the $N \times N$ matrix such that $\boldsymbol{x}(t+1) = A(\boldsymbol{x}(t), t)\boldsymbol{x}(t)$, and let e_t denote the hyperedge that we choose at discrete time t. If e_t is discordant, then $A(\boldsymbol{x}(t), t) = I_N$, where I_N denotes the $N \times N$ identity matrix. If e_t is concordant, then $A(\boldsymbol{x}(t), t)$ is the matrix with entries

$$A(\boldsymbol{x}(t), t)_{ij} = \begin{cases} 1/|e_t|\,, & i, j \in e_t \\ \delta_{ij}\,, & \text{otherwise}\,. \end{cases}$$

The matrix $A(\boldsymbol{x}(t), t)$ satisfies the following conditions [32]:

(1) *Every agent has self-confidence*: The diagonal entries of $A(\boldsymbol{x}(t), t)$ are positive.
(2) *Confidence is mutual*: That is, for all pairs $i, j \in \{1, \ldots, N\}$, we have that $A(\boldsymbol{x}(t), t)_{ij} > 0$ if and only if $A(\boldsymbol{x}(t), t)_{ji} > 0$.
(3) *Positive weights do not converge to* 0: There is a $\delta > 0$ such that every positive entry of $A(\boldsymbol{x}(t), t)$ is at least δ. In our model, every positive entry is at least $1/N$.

For any two times t_0 and t_1 with $t_0 < t_1$, Lorenz [32] defined the *accumulation matrix*

$$A(t_0, t_1) := A(\boldsymbol{x}(t_1 - 1), t_1 - 1)A(\boldsymbol{x}(t_1 - 2), t_1 - 2) \times \cdots \times A(\boldsymbol{x}(t_0 + 1), t_0 + 1)A(\boldsymbol{x}(t_0), t_0)\,.$$

Using this notation, $\boldsymbol{x}(t) = A(0, t)\boldsymbol{x}(0)$. He showed that if conditions (1)–(3) are satisfied, then there is a time t_0 and an ordering of the nodes such that

$$\lim_{t\to\infty} A(0, t) = \begin{bmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_p \end{bmatrix} A(0, t_0)\,, \tag{3.1}$$

where each K_i is a row-stochastic matrix whose rows are all the same. The DeGroot model [10], the dyadic DW model [9], and the HK model [22] all satisfy conditions (1)–(3).

Let $\mathcal{I}_i$ be the set of nodes in the block K_i. Equation (3.1) implies that $\boldsymbol{x}(t)$ converges to some opinion state $\boldsymbol{x}^*$ such that $x_j^* = x_k^*$ for all $j, k \in \mathcal{I}_i$. ∎

We will use the following lemma repeatedly in the subsections that follow.

Lemma 3.2. *Let $\boldsymbol{x}(0)$ be the initial opinion state, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). It follows that the limit state $\boldsymbol{x}^* := \lim_{t\to\infty} \boldsymbol{x}(t)$ is almost surely an absorbing state.*

Proof. By Theorem 3.1, we know that $\boldsymbol{x}^*$ exists. If $\boldsymbol{x}^*$ is not an absorbing state, then there is a hyperedge $e \in E$ such that $d(e, \boldsymbol{x}^*) < c$ and $x_i^* \neq x_j^*$ for some $i, j \in e$. Let $\bar{x}_e^* = \frac{1}{|e|} \sum_{k \in e} x_k^*$ and note that $x_i^* \neq \bar{x}_e^*$. For all $\epsilon > 0$, there is a time T such that

$$|x_i(t+1) - x_i(t)| < \epsilon\,, \tag{3.2}$$

$$|x_i(t) - x_i^*| < \epsilon\,, \tag{3.3}$$

$$|d(e, \boldsymbol{x}(t)) - d(e, \boldsymbol{x}^*)| < \epsilon\,, \tag{3.4}$$

$$|\bar{x}_e(t) - \bar{x}_e^*| < \epsilon \tag{3.5}$$

for all $t \geq T$. Choose $\epsilon < \min\{c - d(e, \boldsymbol{x}^*), |\bar{x}_e^* - x_i^*|/3\}$, and let T be a time that satisfies (3.2)–(3.5). With probability 1, we choose every hyperedge in E infinitely often (by the Borel–Cantelli lemma). Therefore, we choose e at some time $t \geq T$ almost surely. If this happens, then $d(e, \boldsymbol{x}(t)) < c$ by (3.4), and we update the nodes of e to obtain

$$|x_i(t+1) - x_i(t)| = |\bar{x}_e(t) - x_i(t)| > |\bar{x}_e^* - x_i^*| - 2\epsilon > \epsilon$$

by (3.3) and (3.5); this contradicts (3.2). ∎

Lemma 3.2 implies that almost surely there are no hyperedges that are possible to update in the limit state.

3.1. Our hypergraph BCM on complete hypergraphs. In this subsection, we study the limit state of our hypergraph BCM on complete hypergraphs. On a complete hypergraph, every possible subset of nodes can interact with one another. Some of our results apply more generally to any hypergraph that includes the hyperedge $e = V$. We begin by presenting several lemmas that we then use to prove Theorem 3.5.

Lemma 3.3. *If the opinion distribution at time t has finite variance σ^2, then*

$$\lim_{n \to \infty} \mathbb{P}[d(e, \boldsymbol{x}(t)) < c \mid |e| = n] = \begin{cases} 1\,, & c > \sigma^2 \\ \frac{1}{2}\,, & c = \sigma^2 \\ 0\,, & c < \sigma^2. \end{cases} \tag{3.6}$$

Proof. The discordance of a hyperedge e at time t is the sample variance of the opinions $\{x_j(t) \mid j \in e\}$. Let s_n^2 denote the sample variance of n opinions. By the definition of the discordance function d, it follows that $\mathbb{P}[d(e, \boldsymbol{x}(t)) < c \mid |e| = n] = \mathbb{P}[s_n^2 < c]$. Because $\mathbb{E}[s_n^2] = \sigma^2$ and $\lim_{n \to \infty} \mathrm{Var}[s_n^2] = 0$, Chebyshev's inequality implies that

$$\lim_{n \to \infty} \mathbb{P}[s_n^2 < c] \geq \lim_{n \to \infty} \left(1 - \frac{\mathrm{Var}[s_n^2]}{c - \sigma^2}\right) = 1\,, \qquad c > \sigma^2\,,$$

$$\lim_{n \to \infty} \mathbb{P}[s_n^2 < c] \leq \lim_{n \to \infty} \frac{\mathrm{Var}[s_n^2]}{\sigma^2 - c} = 0\,, \qquad c < \sigma^2\,.$$

Note that s_n^2 converges asymptotically to the normal distribution $\mathcal{N}(\sigma^2, \sigma^2(\kappa - 1)/n)$, where κ is the kurtosis of the initial opinion distribution. Because a normal distribution is symmetric, $\lim_{n \to \infty} \mathbb{P}[s_n^2 < c] = \lim_{n \to \infty} \mathbb{P}[s_n^2 < \mathbb{E}[s_n^2]] = \frac{1}{2}$ if $c = \sigma^2$. ∎

The following lemma says that if a hyperedge e has a nontrivial update at time t, then the discordance of each hyperedge $e' \supseteq e$ decreases (i.e., $d(e', \boldsymbol{x}(t+1)) < d(e', \boldsymbol{x}(t))$). As a direct consequence of Lemma 3.4, the discordance of the hyperedge $e = V$ is nonincreasing as the system evolves.

Lemma 3.4. *Let $A = \{x_1, x_2, \dots, x_n\}$ be a collection of n real numbers, and let $A' = \{x_{i_1}, x_{i_2}, \dots, x_{i_\ell}\}$ be some subcollection of A. Construct a new collection B by taking the union of $A \setminus A'$ and ℓ copies of the mean $\bar{x}_{A'} = \frac{1}{\ell}(\sum_{j=1}^{\ell} x_{i_j})$ of A'. The sample variances satisfy $s^2(B) \leq s^2(A)$, where equality holds if and only if $A = B$.*

Proof. The collections A and B have the same mean $\bar{x}_A$. The following equality holds:

$$(n-1)(s^2(A) - s^2(B)) = \sum_{j=1}^{\ell} (x_{i_j} - \bar{x})^2 - \ell(\bar{x}_{A'} - \bar{x})^2 \,. \tag{3.7}$$

We expand the second term of the right-hand side of (3.7) and write

$$\sum_{j=1}^{\ell} (x_{i_j} - \bar{x})^2 - \frac{1}{\ell} \left(\sum_{j=1}^{\ell} (x_{i_j} - \bar{x}) \right)^2 .$$

We then define $y_j := x_{i_j} - \bar{x}$ to simplify our notation and write

$$\sum_{j=1}^{\ell} y_j^2 - \frac{1}{\ell} \left(\sum_{j=1}^{\ell} y_j \right)^2 .$$

Expanding the second term of the right-hand side of (3.7) further and simplifying yields

$$\frac{1}{\ell} \left((\ell - 1) \sum_{j=1}^{\ell} y_j^2 - 2 \sum_{j=1}^{\ell} \sum_{k=j+1}^{\ell} y_j y_k \right) = \frac{1}{\ell} \left(\sum_{j=1}^{\ell} \sum_{k=j+1}^{\ell} (y_j - y_k)^2 \right) \geq 0 \,.$$

Equality occurs if and only if $y_1 = \cdots = y_\ell$, which proves the lemma. ∎

Theorem 3.5. *Suppose that H is an N-node hypergraph that includes the hyperedge $e_N = V$. Let $\boldsymbol{x}(0)$ be the initial opinion state, with opinions drawn independently from a distribution with a variance of $\sigma^2 < c$, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). It then follows that the probability of reaching consensus approaches 1 as $N \to \infty$.*

Proof. By Lemma 3.4, the discordance function $d(e_N, \boldsymbol{x}(t))$ is nonincreasing with time. Therefore, if e_N is concordant at time 0, it is concordant for all t. Additionally, if e_N is concordant, then H converges to consensus the first time that one selects the hyperedge e_N. With probability 1, this selection occurs at some finite time. This shows that

$$\begin{aligned} \mathbb{P}[\text{consensus}] &\geq \mathbb{P}[\text{consensus} \mid d(e_N, \boldsymbol{x}(0)) < c] \cdot \mathbb{P}[d(e_N, \boldsymbol{x}(0)) < c] \\ &= \mathbb{P}[d(e_N, \boldsymbol{x}(0)) < c] \,. \end{aligned}$$

By Lemma 3.3, $\lim_{N \to \infty} \mathbb{P}[\text{consensus}] = 1$. ∎

Remark 3.6. In particular, Theorem 3.5 applies to a complete hypergraph.

Remark 3.7. We emphasize that the probability distribution from which we draw the initial opinions need not be uniform or symmetric. The only condition on the distribution is that $\sigma^2 < c$.

3.1.1. Bounded initial opinions. In this subsubsection, we assume that the probability distribution from which we draw initial opinions is supported on a bounded interval $[a, b]$. We present results about the limit state for this important case, which includes drawing the initial opinions uniformly at random from $[0, 1]$ (a focus of much prior work on the standard dyadic DW model [9, 33, 51]) as a special case. In Theorem 3.11, which is the main result of this subsubsection, the probability distribution from which we draw the initial opinions need not be uniform or symmetric; it only needs to be bounded.

In the standard dyadic DW model on a complete graph (with dyadic relationships only), it has been observed using Monte Carlo simulations that when we draw the initial opinions from the uniform distribution $\mathcal{U}(0, 1)$, there is a threshold confidence bound $c^* \approx \frac{1}{2}$ such that (1) the system converges to consensus with high probability for $c > c^*$ and (2) the system converges to approximately $\lfloor \frac{1}{2c} \rfloor$ opinion clusters for $c < c^*$ [9, 50, 51]. A consensus threshold also exists for the standard dyadic HK model, but it occurs at a smaller confidence bound (of about $c^* = 0.19$) [18, 34]. In Theorem 3.11, we prove that no such threshold exists for our hypergraph BCM and that the opinion state converges to consensus almost surely for sufficiently large N whenever the initial opinion distribution is bounded.

Lemma 3.8. *Suppose that H is the complete hypergraph with N nodes and that $c \neq 0$. Let $\boldsymbol{x}(0)$ be the initial opinion state, where we draw $x_i(0)$ for each i from a bounded distribution that is supported on $[a, b]$, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). It is then the case that the number of opinion clusters in the limit state is almost surely less than or equal to $\frac{b-a}{\sqrt{2c}} + 1$.*

Proof. Let $\boldsymbol{x}^* = \lim_{t\to\infty} \boldsymbol{x}(t)$ be the limit state, and let $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ be the opinion values of the m opinion clusters. Let $\psi : V \to \{1, \dots, m\}$ map each node to its associated opinion cluster, so that $x_i^* = \gamma_{\psi(i)}$. By Lemma 3.2, it suffices to show that $m > \frac{b-a}{\sqrt{2c}}$ implies that $\boldsymbol{x}^*$ is not an absorbing state.

It must be the case that $\gamma_i \in [a, b]$ for all i because $x_j(t) \in [a, b]$ for all t and all j. If $m > \frac{b-a}{\sqrt{2c}}$, then there is a pair γ_i, γ_j such that $\gamma_i \neq \gamma_j$ and $|\gamma_i - \gamma_j| < \sqrt{2c}$. Let k_i be a node in the ith opinion cluster $\psi^{-1}(i)$, let k_j be a node in the jth opinion cluster $\psi^{-1}(j)$, and let $e = \{k_i, k_j\}$. The limit state $\boldsymbol{x}^*$ is not an absorbing state because

$$0 < d(e, \boldsymbol{x}^*) = \frac{(\gamma_i - \gamma_j)^2}{2} < c\,. \qquad \blacksquare$$

The following lemma says that as $N \to \infty$, it is almost surely the case that at least one of the following three outcomes occurs: (1) the opinion state converges to consensus, (2) the number of opinion clusters approaches infinity, or (3) the difference between the opinion values of different opinion clusters approaches infinity.

Lemma 3.9. *Suppose that H is the complete hypergraph with N nodes and that $c \neq 0$. Let $\boldsymbol{x}(0)$ be the initial opinion state, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3).*

Let $\boldsymbol{x}^*$ be the limit state (which exists by Theorem 3.1), and let $\{\gamma_1, \dots, \gamma_m\}$ be the opinion values of the opinion clusters. It then follows that the number m of opinion clusters almost surely is either $m = 1$ (i.e., consensus) or satisfies

$$m \times \left(\max_{k,j} \frac{(\gamma_k - \gamma_j)^2}{c} - 1 \right) \geq N \,.$$

Proof. By Lemma 3.2, it suffices to show that if $N > m \times (\max_{k,j} \frac{(\gamma_k-\gamma_j)^2}{c} - 1)$ and $m > 1$, then $\boldsymbol{x}^*$ is not an absorbing state. Let $\psi : V \to \{1, \dots, m\}$ map each node to its associated opinion cluster, so that $x_i^* = \gamma_{\psi(i)}$, and let $N_i = |\psi^{-1}(i)|$ be the size of the ith opinion cluster. If $N > m \times (\max_{k,j} \frac{(\gamma_k-\gamma_j)^2}{c} - 1)$, then there exists an opinion cluster i such that $N_i > \max_{k,j} \frac{(\gamma_k-\gamma_j)^2}{c} - 1$. If $m > 1$, there is an opinion γ_j such that $\gamma_j \neq \gamma_i$. Let e be a hyperedge of size $N_i + 1$ that is incident to the N_i nodes in $\psi^{-1}(i)$ and 1 node in $\psi^{-1}(j)$. The limit-state sample mean of the nodes that are incident to e is

$$\bar{x}_e^*(t) = \frac{N_i}{N_i + 1}\gamma_i + \frac{1}{N_i + 1}\gamma_j \,.$$

The limit state $\boldsymbol{x}^*$ is not an absorbing state because

$$0 < d(e, \boldsymbol{x}^*) = \frac{N_i(\gamma_i - \bar{x}_e^*(t))^2 + (\gamma_j - \bar{x}_e^*(t))^2}{N_i + 1} = \frac{(\gamma_i - \gamma_j)^2}{N_i + 1} < c \,.$$

■

Remark 3.10. We only apply Lemma 3.9 to the case in which the initial opinion distribution is bounded, but it holds for any initial opinion distribution.

The following theorem says that if the initial opinion distribution is bounded, then the system reaches consensus almost surely for sufficiently large N.

Theorem 3.11. *Suppose that H is the complete hypergraph with N nodes and that $c \neq 0$. Let $\boldsymbol{x}(0)$ be the initial opinion state, where we draw $x_i(0)$ for each i from a bounded distribution that is supported on $[a, b]$, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). If $N > (\frac{b-a}{\sqrt{c}} + 1)(\frac{(b-a)^2}{c} - 1)$, then the opinion state converges to consensus almost surely.*

Proof. Let $\boldsymbol{x}^*$ be the limit state (which exists by Theorem 3.1), and let $\{\gamma_1, \dots, \gamma_m\}$ be the opinion values of the opinion clusters. By Lemma 3.8, $m \leq \frac{b-a}{\sqrt{c}} + 1$ almost surely. It is necessarily true that $\gamma_i \in [a, b]$ for all i, so $\max_{k,j}(\gamma_k - \gamma_j)^2 < (b-a)^2$. By Lemma 3.9, $m = 1$ almost surely if $N > (\frac{b-a}{\sqrt{c}} + 1)(\frac{(b-a)^2}{c} - 1)$. ■

3.1.2. Normally distributed initial opinions. Assume that the initial opinions are normally distributed with mean μ and variance σ^2. When $\sigma^2 < c$, Theorem 3.5 implies that the probability that our BCM reaches consensus on the N-node complete hypergraph approaches 1 as $N \to \infty$. Based on numerical evidence, we conjecture that this probability approaches 1 as $N \to \infty$ even when $\sigma^2 \geq c$, unless $c = 0$. Because the hypergraph is complete, Lemma 3.9 also applies.

In Figure 1, we show a typical simulation with $\sigma^2 > c$. We have reduced the number of time steps by requiring that the hyperedge that we select at time 0 is concordant. This requirement

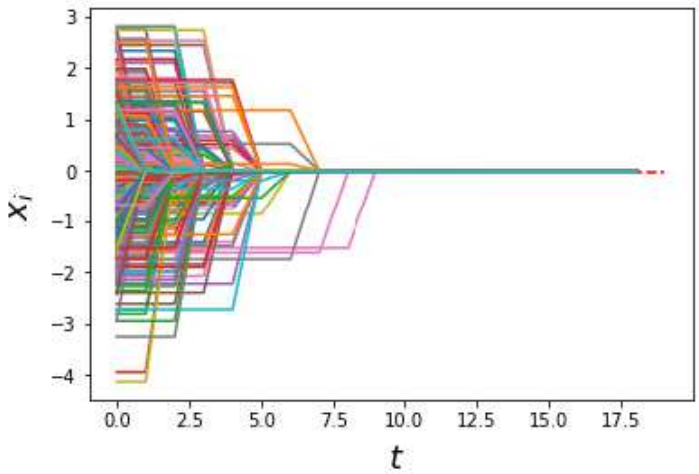


Figure 1. *A typical simulation of our hypergraph BCM on the complete hypergraph with $N = 500$ nodes. Each curve traces the evolution of one node's opinion. We draw the initial opinions from the normal distribution $\mathcal{N}(0, \sigma^2)$ with $\sigma = 1.2$. We set the confidence bound to $c = 1$. We have reduced the number of time steps by requiring that the hyperedge that we choose at time 0 is concordant. The opinion state converges to consensus.*

has no effect on the subsequent behavior or on the system's limit, but it significantly reduces the number of time steps; we will see why this is true in our proof of Theorem 4.7. Observe that the opinion state converges to consensus. In 1000 trials of our BCM on the complete 200-node hypergraph with a confidence bound of $c = 1$ and a normal initial opinion distribution with standard deviation $\sigma = 1.2$, we find that the opinion state converges to consensus in every trial.

Based on the results of our Monte Carlo simulations, we conjecture that when $c \neq 0$ and the initial opinion distribution is normal, the probability of consensus approaches 1 even when $\sigma^2 > c$. We now provide a heuristic explanation of this conjecture, although we do not have a mathematically rigorous proof of it.

We fix the variance $\sigma^2 > c$. At each time step, we select a hyperedge uniformly at random. Let e^* be the first concordant hyperedge that we choose. In section 4, we will show that if e is an arbitrary hyperedge, then $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] \leq r^{n-1}$, where $r = e^{\frac{1}{2}(1-c/\sigma^2)}\sqrt{(c/\sigma^2)} < 1$. If it is true (and we suspect that it is) that $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] = ar^n$ for some constants a and $r < 1$, then one can calculate that

$$\mathbb{E}[|e^*|] = \frac{N(1+r)^{N-1} - N - 1}{\frac{1}{r}(1+r)^N - N - \frac{1}{r}} \sim \frac{r}{r+1}N \quad \text{as } N \to \infty\,.$$

That is, if we assume that $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] = ar^n$, then the expected size of e^* grows linearly with N. In Figure 2, we show the results of Monte Carlo simulations to estimate $\mathbb{E}[|e^*|]$ as a function of N without assuming that $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] = ar^n$. As hypothesized, we observe a linear relationship.

Because e^* is the first concordant hyperedge that we select, we update the nodes of e^* to the opinion $\bar{x}_{e^*}(0)$, which is the mean of the initial opinions of the nodes that are incident to e_0. Let μ be the mean of the opinion distribution. If e is an arbitrary hyperedge, then $\bar{x}_e(0) \to \mu$ in

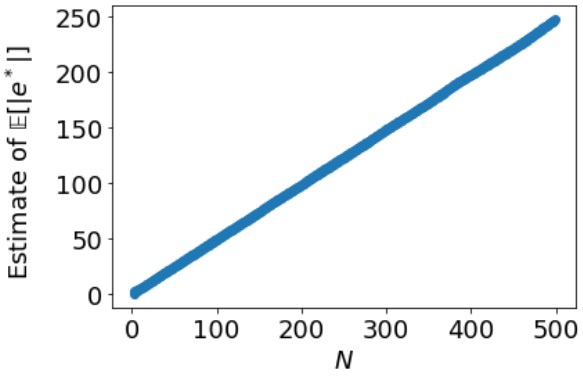


Figure 2. *An estimate of $\mathbb{E}[|e^*|]$ as a function of the number N of nodes, where e^* is the first concordant hyperedge that we select and the initial opinions are normally distributed with a standard deviation of $\sigma = 1.2$. The confidence bound is $c = 1$. For each possible hyperedge size $n \in \{2, \dots, N\}$, we run 10,000 trials. For each trial, we randomly draw n opinions from $\mathcal{N}(0, \sigma^2)$ and calculate the sample variance, which equals the discordance of those n opinions. We approximate $a_n := \mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n]$ by letting $\widehat{a_n}$ be the fraction of trials that result in a concordant set of n opinions. For each $N \in \{2, \dots, 500\}$, we have that $\mathbb{E}[|e^*|] \approx \frac{\sum_{n=2}^{N} n\widehat{a_n}\binom{N}{n}}{\sum_{n=2}^{N} \widehat{a_n}\binom{N}{n}}$.*

distribution as $|e| \to \infty$. It is necessarily also true that $\bar{x}_{e^*}(0) \to \mu$ in distribution as $|e^*| \to \infty$ because the sample mean and sample variance of a normal distribution are independent of each other (by Basu's theorem). Because $\mathbb{E}[|e^*|] \to \infty$ as $N \to \infty$, it follows that $\bar{x}_{e^*}(0) \to \mu$ in distribution as $N \to \infty$. That is, we are updating the opinions of the nodes in e^* to a value that is approximately equal to μ.

The observations above imply that (1) the first concordant hyperedge that we update includes a fraction of the nodes that is approximately constant as $N \to \infty$ (even for large σ^2) and (2) when we update all of those nodes, we are updating them to a value that is approximately equal to μ, which is the mean of the distribution. This decreases the total sample variance of the opinions of the N nodes and increases the clustering of opinions near μ, making it even more likely that the next hyperedge that we update will also be large and have a mean opinion that is centered near μ. Eventually, the opinions converge to a consensus value that is near μ.

3.2. Our BCM on hypergraphs with community structure. In this subsection, we examine our BCM on hypergraphs with planted community structure.

Suppose that we partition the set of nodes in a hypergraph into communities and that each community has its own independent distribution of initial opinions. Specifically, we study our BCM on hypergraphs that we generate using the hypergraph stochastic block model (HSBM) of [20]. An HSBM is a generative model for producing hypergraphs with community structure. Like a traditional stochastic block model for ordinary graphs [17], the probability that a hyperedge exists depends on the community memberships of its nodes. In this HSBM, the probability that a hyperedge exists also depends on the size of the hyperedge and on the number of nodes in the hypergraph. More precisely, let $\psi : V \to \{1, \dots, k\}$ be a partition of the set of nodes into k communities, where we assume without loss of generality that every community is nonempty. We denote the ith community by $C_i := \psi^{-1}(i)$. For $N = |V|$

and $n \in \{2, \dots, N\}$, let $\alpha_{n,N} \in [0,1]$. For each $n \in \{2, \dots, N\}$, let B^n be a symmetric k-dimensional tensor of order n whose entries take values in $[0,1]$. We generate a hypergraph as follows. For each subset $e = \{i_1, \dots, i_n\} \in \mathcal{P}(V)$ of nodes, we include e in the hypergraph with a probability of $\alpha_{n,N} B^n_{\psi(i_1),\dots,\psi(i_n)}$.

In the simplest version of this HSBM, each intracommunity hyperedge exists with an independent and uniform probability p, and each intercommunity hyperedge exists with an independent and uniform probability q.

Definition 3.12. *Consider the HSBM of* [20] *with parameters* $\alpha_{n,N} = 1$ *for all* n, N *and*

$$B^n_{\psi(i_1),\dots,\psi(i_n)} = \begin{cases} p\,, & \psi(i_1) = \cdots = \psi(i_n) \\ q\,, & \textit{otherwise} \end{cases}$$

for some $p, q \in [0,1]$. *We will refer to this HSBM as a* (p,q)-HSBM.

If $q = 0$, the communities in a (p,q)-HSBM are disjoint, so the opinions in a community cannot influence the opinions in other communities.

Definition 3.13. *Let* $\psi : V \to \{1, \dots, k\}$ *be a partition of the set of nodes into* k *nonempty communities. The opinion state* $\boldsymbol{x}^*$ *is* polarized *if there are opinions* $\gamma_1, \dots, \gamma_k \in \mathbb{R}$, *not all equal, such that* $x_i^* = \gamma_{\psi(i)}$ *for all* i.

Definition 3.13 says that $\boldsymbol{x}^*$ is polarized if each community is at consensus but the communities are not at consensus with each other. For example, if $q = 0$ and it is not the case that every community has the same initial mean opinion, then the opinion state converges either to a polarized limit state or to a limit state that includes at least one community whose nodes are not at consensus within the community.

Remark 3.14. As we mentioned previously, some researchers refer to this situation as "opinion fragmentation" and reserve the term "polarization" for situations in which there are exactly two opinion clusters in an opinion state.

The following theorem says that if $q \neq 0$, then the probability that the limit state of our BCM on a (p,q)-HSBM hypergraph is polarized approaches 0 as $N \to \infty$.

Theorem 3.15. *Suppose that we generate* H *from a* (p,q)*-HSBM with partition* $\psi : V \to \{1, \dots, k\}$ *and that* $q \neq 0$. *Additionally, suppose that* $c \neq 0$. *Let* $\boldsymbol{x}(0)$ *be the initial opinion state, where we draw* $x_i(0)$ *for each* i *from a bounded distribution that is supported on* $[a,b]$, *and let* $\boldsymbol{x}(t)$ *be the opinion state that is determined by* (2.3). *Let* $\boldsymbol{x}^*$ *be the limit state, which exists by Theorem* 3.1. *It then follows that the probability that* $\boldsymbol{x}^*$ *is polarized approaches* 0 *as* $N \to \infty$.

Proof. Because $\boldsymbol{x}^*$ is almost surely an absorbing state by Lemma 3.2, it suffices to show that $\mathbb{P}[\boldsymbol{x}^* \text{ polarized and absorbing}] \to 0$ as $N \to \infty$. Suppose that $\boldsymbol{x}^*$ is a polarized absorbing state. Because $\boldsymbol{x}^*$ is polarized, there are opinions $\gamma_1, \dots, \gamma_k$ that are not all equal and that satisfy $x_i^* = \gamma_{\psi(i)}$. Without loss of generality, let C_1 be the largest community. It is necessarily true that $|C_1| \geq \lceil N/k \rceil$. Because $\gamma_1, \dots, \gamma_k$ are not all equal, there is a community j such that $\gamma_1 \neq \gamma_j$.

To find a contradiction, suppose that there is a hyperedge $e \in E$ of size $n > |\gamma_1 - \gamma_j|^2/c$ such that e is incident to $n-1$ nodes in C_1 and to 1 node in C_j. It follows that

$$0 < d(e, \boldsymbol{x}^*) = \frac{(\gamma_1 - \gamma_j)^2}{n} < \frac{(b-a)^2}{n} < c\,,$$

which contradicts the assumption that $\boldsymbol{x}^*$ is an absorbing state. As $N \to \infty$, the probability that there is no such $e \in E$ is

$$\lim_{N\to\infty} \mathbb{P}[\nexists\, e \in E] = \lim_{N\to\infty} (1-q)^{\sum_{i=\max\{0,1-n_0\}}^{|C_1|-n_0+1} \binom{|C_1|}{n_0+i-1}\times|C_j|} \leq \lim_{N\to\infty} (1-q)^{\lceil N/k\rceil} = 0\,,$$

where $n_0 = \lfloor \frac{|\gamma_1-\gamma_j|}{c} \rfloor$. ∎

The following theorem says that if we also impose the condition $p = 1$ (so that each community forms a hyperclique), then the probability of reaching consensus approaches 1 as $N \to \infty$ if all communities are sufficiently large.

Theorem 3.16. *Suppose that we generate a hypergraph H from a (p, q)-HSBM with partition $\psi : V \to \{1, \dots, k\}$ and that $p = 1$ and $q \neq 0$. Let $\boldsymbol{x}(0)$ be the initial opinion state, where we draw $x_i(0)$ for each i from a bounded distribution that is supported on $[a, b]$, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). Additionally, suppose that $|C_i| > (\frac{b-a}{\sqrt{c}}+1)(\frac{(b-a)^2}{c}-1)$ for all $i \in \{1, \dots, k\}$ and that $c \neq 0$. It then follows that $\mathbb{P}[\text{consensus}] \to 1$ as $N \to \infty$.*

Proof. By Theorem 3.1, the limit state $\boldsymbol{x}^* := \lim_{t\to\infty} \boldsymbol{x}(t)$ exists. By the same argument as in the proof of Theorem 3.11, the nodes in each community converge to consensus almost surely because $|C_i| > (\frac{b-a}{\sqrt{c}} + 1)(\frac{(b-a)^2}{c} - 1)$ for all $i \in \{1, \dots, k\}$. That is, there exist opinions $\gamma_1, \dots, \gamma_k \in [a, b]$ such that $x_i^* = \gamma_j$ for all $i \in C_j$. If $\gamma_i = \gamma_j$ for all communities i and j, then the opinion state converges to consensus. Otherwise, we assume without loss of generality that C_1 is the largest community. Let C_j be a community such that $\gamma_j \neq \gamma_1$. Suppose that there is a hyperedge $e \in E$ of size $n > \frac{(b-a)^2}{c}$ such that e is incident to $n-1$ nodes in C_1 and to 1 node in C_j. By the same argument as in the proof of Theorem 3.15, the probability that such a hyperedge exists approaches 1 as $N \to \infty$. If e does exist, then $\boldsymbol{x}^*$ is not an absorbing state. By Lemma 3.2, $\boldsymbol{x}^*$ is almost surely an absorbing state. ∎

In another version of the HSBM in [20], one requires that every intercommunity hyperedge is small.

Definition 3.17. *Consider the HSBM of [20] with parameters $\alpha_{n,N} = 1$ for all n, N and*

$$B^n_{\psi(i_1),\dots,\psi(i_n)} = \begin{cases} p\,, & \psi(i_1) = \cdots = \psi(i_n) \\ q\,, & \text{there exist } j \text{ and } k \text{ such that } \psi(i_j) \neq \psi(i_k) \text{ and } n \leq M \\ 0\,, & \text{otherwise} \end{cases}$$

*for some $M \geq 2$ and $p, q \in [0, 1]$. We will refer to this HSBM as a (p, q, M)-*HSBM.

For fixed p, q, M and $N \to \infty$, the communities are "almost" disjoint; that is, the number of intercommunity hyperedges divided by the total number of hyperedges approaches 0. Theorem 3.18, which one should contrast with Theorem 3.15, gives conditions for which a polarized opinion state is an absorbing state. Theorem 3.18 implies that echo chambers can form when all of the intercommunity hyperedges are sufficiently small.

Theorem 3.18. *Suppose that we generate a hypergraph H from a (p, q, M)-HSBM with the partition $\psi : V \to \{1, \dots, k\}$. Let $\gamma_1, \dots, \gamma_k \in \mathbb{R}$, and let $\boldsymbol{x}^*$ be the polarized opinion state with $x_i^* = \gamma_{\psi(i)}$. If $\min_{i,j}\{(\gamma_i - \gamma_j)^2 \mid \gamma_i \neq \gamma_j\}/M > c$, then $\boldsymbol{x}^*$ is an absorbing state.*

Proof. Without loss of generality, assume that $\gamma_i \neq \gamma_j$ if $i \neq j$. (If not, we combine any communities with the same opinion.) Let $e \in E$ be a hyperedge. Let $n = |e|$, and let $n_i = |\{j \in e \mid \psi(j) = i\}|$ be the number of nodes that are incident to e and belong to community i. We have that

$$\begin{aligned} d(e, \boldsymbol{x}^*) &= \frac{1}{n-1}\left(\sum_i n_i \left(\gamma_i - \frac{1}{n}\sum_j n_j\gamma_j\right)^2\right) \\ &= \frac{1}{n-1}\left(\sum_i n_i\gamma_i^2 - \frac{1}{n}\left(\sum_i n_i\gamma_i\right)^2\right) \\ &= \frac{1}{n(n-1)}\left(\sum_i n_i\gamma_i^2(n-n_i) - 2\sum_i\sum_{j<i} n_i n_j \gamma_i\gamma_j\right) \\ &= \frac{1}{n(n-1)}\left(\sum_i\sum_{j\neq i} n_i n_j\gamma_i^2 - 2\sum_i\sum_{j<i} n_i n_j\gamma_i\gamma_j\right) \\ &= \frac{1}{n(n-1)}\sum_i\sum_{j<i} n_i n_j(\gamma_i - \gamma_j)^2 . \end{aligned} \tag{3.8}$$

Let $\{i_*, j_*\} = \operatorname{argmin}_{i,j}\{(\gamma_i - \gamma_j)^2 \mid \gamma_i \neq \gamma_j\}$. If there is an ℓ such that $n_\ell = n$, then it follows that $e \subseteq C_\ell$. Therefore, $x_i^* = \gamma_\ell$ for all $i \in e$, so $d(e, \boldsymbol{x}^*) = 0$. Otherwise, $n_\ell \neq n$ for all ℓ. Fixing n and enforcing the constraint that $n_\ell \neq n$ for all ℓ, it follows that (3.8) is minimized either when

$$n_\ell = \begin{cases} 0\,, & \ell \notin \{i_*, j_*\} \\ n-1\,, & \ell = i_* \\ 1\,, & \ell = j_* \end{cases}$$

or when

$$n_\ell = \begin{cases} 0\,, & \ell \notin \{i_*, j_*\} \\ 1\,, & \ell = i_* \\ n-1\,, & \ell = j_*\,. \end{cases}$$

Therefore, $d(e, \boldsymbol{x}^*) \geq \frac{\min\{(\gamma_i-\gamma_j)^2 | \gamma_i \neq \gamma_j\}}{M} > c$ if $n_\ell \neq n$ for all n. Therefore, for all $e \in E$, it must be the case that either $d(e, \boldsymbol{x}^*) = 0$ or $d(e, \boldsymbol{x}^*) > c$. ∎

In Figure 3, we show a typical simulation when the conditions of Theorem 3.18 are satisfied. In the limit state, the communities are polarized. All nodes in community C_i converge to the opinion $\gamma_i = \sum_{j \in C_i} x_j(0)$, which is the mean of the initial opinions of the nodes in C_i. By Theorem 3.18, we know that this polarized opinion state is an absorbing state.

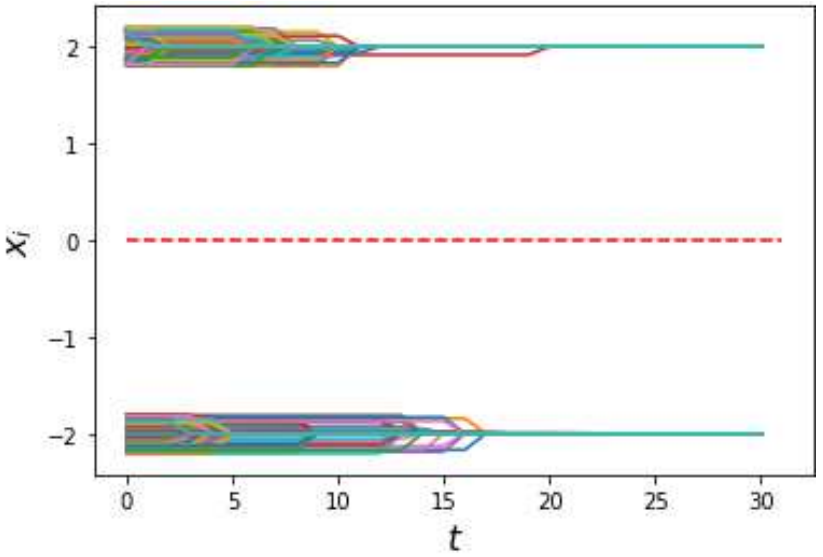


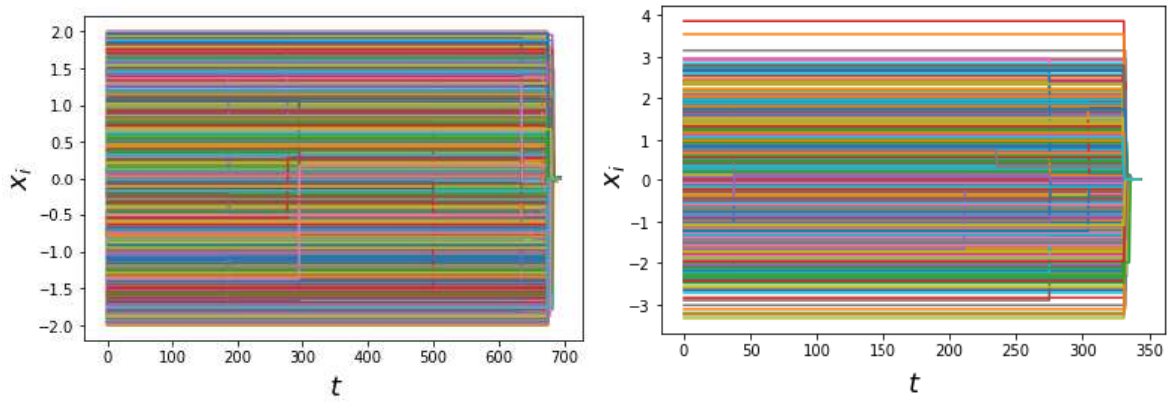
Figure 3. *A typical simulation when the conditions of Theorem 3.18 are satisfied. There are $N = 1000$ nodes, which we assign to two equal-sized communities. We generate the hypergraph on which we run our BCM from a (p, q, M)-HSBM with $p = 1$, $q = 1$, and $M = 2$. The confidence bound is $c = 1$. The initial opinions of the nodes in community 1 are $x_i(0) \sim \mathcal{U}(1.8, 2.2)$, and the initial opinions of the nodes in community 2 are $x_i(0) \sim \mathcal{U}(-2.2, -1.8)$. In the limit state, all nodes in community 1 have opinion $\gamma_1 \approx 2$ and all nodes in community 2 have opinion $\gamma_2 \approx -2$. This polarized opinion state is an absorbing state.*

Remark 3.19. By the same strategy as in the proofs of Theorem 3.15, Theorem 3.16, and Theorem 3.18, one can show that echo chambers do not usually form in a hypergraph that we generate using a (p, q, M)-HSBM if the initial opinions of the different communities are sufficiently close to each other. In particular, if we draw the initial opinions from a bounded interval $[a, b]$ with $(b-a)^2/M < c$ and if $|C_i| > (\frac{b-a}{\sqrt{c}})(\frac{(b-a)^2}{c} - 1)$ for all i, then the probability of reaching consensus approaches 1 as $N \to \infty$.

3.3. Our BCM on sparse hypergraphs. We now study our hypergraph BCM on sparse $G(N, \boldsymbol{m})$ hypergraphs, where $\boldsymbol{m}$ is a vector with entries m_i. The $G(N, \boldsymbol{m})$ model is a generative hypergraph model that is defined analogously to the Erdős–Rényi $G(N, m)$ generative graph model. Each hypergraph that one constructs from the $G(N, \boldsymbol{m})$ model has N nodes; additionally, for each possible hyperedge size $i \in \{2, \dots, N\}$, we choose m_i hyperedges of that size uniformly at random to include in the hypergraph.

In our Monte Carlo simulations, we set $N = 1000$ and $m_i = \max\{100, \binom{N}{i}\}$ for all i. We run 1000 simulations with a confidence bound of $c = 1$ and initial opinions that we draw uniformly at random from $[-2, 2]$. In all trials, we find that the opinion state converges to consensus. We also run 1000 trials with a confidence bound of $c = 1$ and initial opinions that we draw from the normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1.2$. All of these trials also converge to consensus. In Figure 4, we show a typical simulation for each of these two initial opinion distributions.

As we increase N or increase the variance of the initial opinions, we observe that the time that it takes to converge increases but that the opinion state still converges to consensus.



(a) $x_i(0) \sim \mathcal{U}(-2, 2)$ (b) $x_i(0) \sim \mathcal{N}(0, 1.2)$

Figure 4. *Typical simulations on sparse $G(N, \boldsymbol{m})$ hypergraphs with a confidence bound of $c = 1$ and $N = 1000$ nodes that have* (a) *uniformly distributed initial opinions and* (b) *normally distributed initial opinions. Both of these simulations converge to consensus.*

3.4. Our BCM on an Enron e-mail hypergraph. In subsection 3.3, we studied our BCM on sparse hypergraphs. However, it is typically also the case that hypergraphs that one constructs from empirical data are not merely sparse; they also often have the property that their hyperedges are small in size in comparison to the number of nodes. As one example, we use a hypergraph that Benson et al. [3] constructed from the well-known (and infamous) Enron e-mail data set [28]. In this Enron e-mail hypergraph, nodes represent Enron employees and hyperedges encode e-mails between them. Each hyperedge is incident to the sender and recipients of one e-mail message. There are $N = 143$ nodes, but the maximum hyperedge size is only 18.

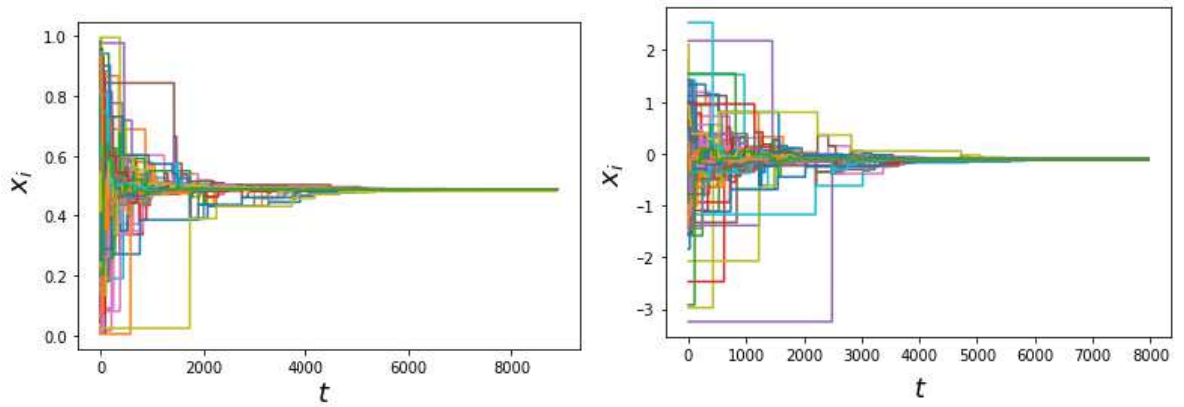
To examine our hypergraph BCM on the Enron e-mail hypergraph, we run 1000 simulations with initial opinions that are uniformly distributed in $[0, 1]$. In all 1000 trials, the opinion state converges to consensus. In Figure 5(a), we show the results of a typical simulation. We also run 1000 simulations on the Enron e-mail hypergraph with initial opinions that are normally distributed with a mean of $\mu = 0$ and a variance of $\sigma^2 = 1$. The confidence bound is $c = 1$. In all trials, the opinion state converges to consensus. In Figure 5(b), we show the results of a typical simulation.

4. Convergence time.

In this section, we analyze the convergence time of our hypergraph BCM.

4.1. Conditions for convergence in finite time. We say that the opinion state *converges in finite time* if there is a time T such that $\boldsymbol{x}(T)$ is an absorbing state. At time T, no further opinion updates can occur. We use the following lemma to prove Theorem 4.2.

Lemma 4.1. *Let $\tilde{e} \in \mathcal{P}(V)$ be a subset of the nodes in a hypergraph. There is a finite sequence $\{\tilde{e}_i\}_{i=1}^{m}$ in $\mathcal{P}(V)$ such that* (1) *$|\tilde{e}_i|$ is prime for all $i \geq 1$ and* (2) *consecutively updating the nodes that are incident to $\tilde{e}_1, \ldots, \tilde{e}_m$ to their respective mean opinions results in the same opinion state as updating the nodes that are incident to $\tilde{e}$ to the mean of their opinions.*



(a) $x_i(0) \sim \mathcal{U}(0,1)$ (b) $x_i(0) \sim \mathcal{N}(0,1)$

Figure 5. *A typical simulation on the Enron e-mail hypergraph with a confidence bound of $c = 1$ and nodes with* (a) *uniformly distributed initial opinions and* (b) *normally distributed initial opinions. Both of these simulations converge to consensus.*

Proof. Let $n = |\tilde{e}|$. If n is prime, we are done. This also proves the base case $n = 2$. If n is not prime, we can write $n = pm$, where p is prime and $m < n$. Without loss of generality, suppose that $\tilde{e} = \{1, \dots, n\}$. Updating the nodes of $\tilde{e}$ to the mean opinion in $\tilde{e}$ at time t results in the opinion state $\boldsymbol{x}$ with entries

$$x_i = \begin{cases} \frac{1}{n}\sum_{\ell\in\tilde{e}} x_\ell(t) = \frac{1}{p}\sum_{k=0}^{p-1}\left(\frac{1}{m}\sum_{j=1}^{m} x_{km+j}(t)\right), & i \le n \\ x_i(t)\,, & i > n\,. \end{cases} \tag{4.1}$$

Updating the nodes of $\{km+1, \dots, km+m\} \in \mathcal{P}(V)$ for each $k \in \{0, \dots, p-1\}$ to their respective mean opinions at time t results in the opinion state $\boldsymbol{x}$ with entries

$$x_i = \begin{cases} \frac{1}{m}\sum_{\ell=1}^{m} x_{km+\ell}(t)\,, & i = km+j\,,\ 1 \le j \le m\,,\ 0 \le k \le p-1 \\ x_i(t)\,, & i > n\,. \end{cases} \tag{4.2}$$

By induction (because $m < n$), there is a sequence of elements of prime size in $\mathcal{P}(V)$ such that updating this sequence results in the same opinion state as updating $\{km+1, \dots, km+m\}$. Concatenating these p sequences (there is one for each k) yields a sequence $\tilde{e}_1, \dots, \tilde{e}_\ell$ of elements of prime size in $\mathcal{P}(V)$ such that updating the nodes of $\tilde{e}_1, \dots, \tilde{e}_\ell$ to their respective mean opinions at time t results in (4.2). Updating the sequence $\{1, m+1, \dots, (p-1)m+1\}$, $\{2, m+2, \dots, (p-1)m+2\}$, $\dots$, $\{m, 2m, \dots, pm\}$ of elements in $\mathcal{P}(V)$ then results in (4.1). Therefore, $\tilde{e}_1, \dots, \tilde{e}_\ell, \{1, \dots, (p-1)m+1\}, \{2, \dots, (p-1)m+2\}, \dots, \{m, \dots, pm\}$ is the desired sequence of prime-sized elements in $\mathcal{P}(V)$. ∎

Theorem 4.2. *Let H be a hypergraph such that $\{e \in \mathcal{P}(V) \mid |e| \text{ is prime}\} \subseteq E$. Let $\boldsymbol{x}(0)$ be any initial opinion state, and let $\boldsymbol{x}(t)$ be the opinion state that is determined by (2.3). It follows that $\boldsymbol{x}(t)$ almost surely converges in finite time.*

Proof. Let the time t_0, the matrices $K_1, \dots, K_p$, and the node sets $\mathcal{I}_1, \dots, \mathcal{I}_p$ be defined as in (3.1) and the text that is immediately below it. This equation implies for all $j, k \in \mathcal{I}_i$ that

$$\lim_{t\to\infty} x_j(t) = \lim_{t\to\infty} x_k(t)\,. \tag{4.3}$$

Updating the nodes of $\mathcal{I}_1 \in \mathcal{P}(V)$ to the mean of their opinions results in consensus among the nodes of $\mathcal{I}_1$. By Lemma 4.1, there is a sequence $\{e_i\}_{i=1}^n$ of prime-sized elements in $\mathcal{P}(V)$ such that consecutively updating the hyperedges in the sequence $\{e_i\}_{i=1}^n$ results in the same opinion state as updating $\mathcal{I}_1$. By hypothesis, the hypergraph H includes e_i for all i because e_i is prime-sized. There is a time $t_1 \geq t_0$ such that $d(e_i, \boldsymbol{x}(t)) < c$ for all i for all times $t \geq t_1$. The probability of consecutively choosing the hyperedges in the sequence $\{e_i\}_{i=1}^n$ starting at a given time is $1/|E|^n > 0$, where E is the hyperedge set of H. Because this probability is positive, it is almost surely the case that the event of consecutively choosing these hyperedges occurs infinitely often. Therefore, there is almost surely some time after t_1 that we consecutively choose the hyperedges $\{e_i\}_{i=1}^n$ for updating. Let $t_2 > t_1$ be the time that we choose the last hyperedge e_n in the sequence. At this time, $x_j(t_2) = x_k(t_2)$ for all $j, k \in \mathcal{I}_1$. Similarly, we can find times $t_3 \leq \cdots \leq t_{p+1}$ for the nodes of $\mathcal{I}_2, \ldots, \mathcal{I}_p$. By (3.1), any hyperedge e_t that we choose at $t \geq t_1$ is discordant if e_t is not contained in some $\mathcal{I}_i$. Therefore, for $t \geq t_{p+1}$, it follows that $d(e, \boldsymbol{x}(t)) = 0$ if there is an i such that $e \subseteq \mathcal{I}_i$; otherwise, $d(e, \boldsymbol{x}(t)) > c$. Therefore, $\boldsymbol{x}(t_{p+1})$ is an absorbing state. ∎

Remark 4.3. Theorem 4.2 applies to a complete hypergraph.

The following theorem gives a partial converse of Theorem 4.2.

Theorem 4.4. *Let H be a hypergraph with N nodes and hyperedge set E. Suppose that there is a subset $e = \{i_1, \ldots, i_p\} \notin E$ of nodes such that e has prime size p, and suppose that the subhypergraph that is induced by $\{i_1, \ldots, i_p\}$ is connected. (The connectivity requirement implies that $p \geq 3$.) It then follows that there is an initial opinion state $\boldsymbol{x}(0)$ such that the opinion state $\boldsymbol{x}(t)$ that is determined by (2.3) does not converge in finite time.*

Proof. Without loss of generality, $e = \{1, \ldots, p\}$. By the continuity of the discordance function, there is an $\epsilon > 0$ such that if $x_1(0), \ldots, x_p(0) \in [0, \epsilon]$, then $d(e', \boldsymbol{x}(0)) < c$ for all $e' \subseteq e$. We choose r such that $1/2^r < \epsilon$, and we let $x_1(0) = 0$ and $x_i(0) = 1/2^r$ for all $i \in \{2, \ldots, p\}$. Additionally, there is an $M > \epsilon$ such that if $x_{p+1}(0), \ldots, x_N(0) > M$, then $d(e', \boldsymbol{x}(0)) > c$ for all $e' \in E$ that satisfy $e' \not\subseteq \{1, \ldots, p\}$ and $e' \not\subseteq \{p+1, \ldots, n\}$. Let $x_{p+1}(0) = \cdots = x_N(0) = M$. Because of these initial values, the nodes in $\{1, \ldots, p\}$ never interact with the nodes in $\{p+1, \ldots, n\}$. Therefore, $x_1(t), \ldots, x_p(t) \in [0, \epsilon]$ for all t and every $e' \subseteq e$ is concordant for all t. The subhypergraph that is induced by the nodes $\{1, \ldots, p\}$ is connected, so

$$\lim_{t\to\infty} x_i(t) = \frac{1}{p}\sum_{j=1}^{p} x_j(0) = \frac{p-1}{2^r p} \quad \text{for all} \quad i \in \{1, \ldots, p\}\,.$$

Let $e_t = \{i_1, \ldots, i_k\} \in E$ be the hyperedge that we choose at time t. If $e_t \not\subseteq e$, then $x_i(t+1) = x_i(t)$ for all $i \in e$. Otherwise, $e_t \subset e$ (recall that $e \notin E$ by hypothesis) and for all $i \in e_t$, we have

$$x_i(t+1) = \frac{1}{k}\sum_{j=1}^{k} x_j(t)\,,$$

with $k \in \{2, \dots, p-1\}$, because $|e| = p$ and e_t is a strict subset of e. By induction on t, there exist $a_{ij}(t) \in \mathbb{N} \cup \{0\}$ and $n_k(t) \in \mathbb{N} \cup \{0\}$ such that

$$x_i(t+1) = \frac{1}{\prod_{k=2}^{p-1} k^{n_k(t)}} \sum_{j=1}^{p} a_{ij}(t) x_j(0) = \frac{1}{2^r \prod_{k=2}^{p-1} k^{n_k(t)}} \sum_{j=2}^{p} a_{ij}(t) \quad \text{for all} \quad i \in e\,.$$

For all t, we have

$$(p-1)2^r \prod_{k=2}^{p-1} k^{n_k(t)} \neq 2^r p \sum_{j=2}^{p} a_{ij}(t) \tag{4.4}$$

because the left-hand side of (4.4) is a nonzero integer without p in its prime factorization and the right-hand side of (4.4) is either 0 (if $\sum a_{i_j}(t) = 0$) or a nonzero integer with p in its prime factorization. Consequently, $x_i(T) \neq \lim_{t\to\infty} x_i(t)$ for any $T \geq 0$ and for all $i \in e$. Therefore, $\boldsymbol{x}(t)$ does not converge in finite time. ∎

Together, Theorem 4.2 and Theorem 4.4 partially characterize the conditions for finite-time convergence of our BCM. If H is a hypergraph whose hyperedge set includes all prime-sized subsets of nodes, then the opinion state almost surely converges in finite time. However, if the set of hyperedges does not include some prime-sized subset of nodes and the subhypergraph that is induced by those nodes is connected, then it is not the case that the opinion state almost surely converges in finite time. We do not have a characterization of the convergence time when this connectivity condition is not satisfied for any of the prime-sized elements in $\mathcal{P}(V)$ that are not in the set E of hyperedges. However, we expect that "most" hypergraphs do not fall into this missing case. The connectivity condition is not hard to satisfy. For example, in the $G(N, \boldsymbol{m})$ model, we expect the subhypergraph that is induced by any set $\{i_1, \dots, i_p\}$ of nodes to be connected whenever p is sufficiently larger than the index of the first nonzero entry of $\boldsymbol{m}$. The vast majority of hypergraphs that are produced by the $G(N, \boldsymbol{m})$ model satisfy the conditions of either Theorem 4.2 or Theorem 4.4.

4.2. A phase transition at $\sigma^2 = c$. We now study the rate of convergence of the opinion state in our hypergraph BCM. We focus on complete hypergraphs, for which we observe that there is a phase transition in the convergence time when the confidence bound is $c = \sigma^2$. We prove that the convergence time grows at least exponentially fast with the number N of nodes when $\sigma^2 > c$ and the initial opinions are normally distributed.

In Figure 6, we simulate our BCM with $c = 1$ on the complete hypergraph with 50,000 nodes and initial opinions that we seed independently by setting $x_i(0) \sim \mathcal{N}(0, \sigma^2)$ for $\sigma \in [0.9, 1.1]$. We plot an empirical convergence time t^*, which we set to be the earliest time that the discordance function satisfies $d(e = V, \boldsymbol{x}) < 10^{-5}$. Our results are consistent with the existence of a phase transition in the convergence time.

In Theorem 4.7, we prove that the convergence time grows at least exponentially fast as a function of N if $c < \sigma^2$. The proof relies on Lemma 3.3, where we calculated the value of $\lim_{n\to\infty} \mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n]$ for any initial opinion distribution with finite variance σ^2 and showed that there is a transition at $\sigma^2 = c$. We also need a bound on the convergence rate in this limit. Using the inequalities that we derived in the proof of Lemma 3.3 and

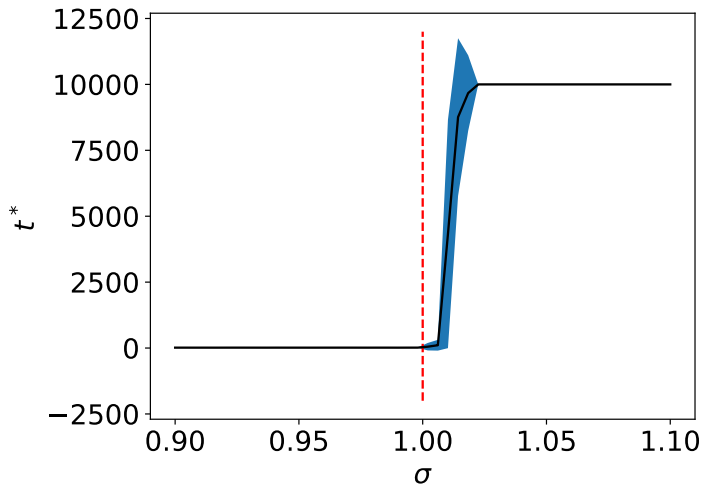


Figure 6. *Empirical convergence time of our BCM on the complete hypergraph with* 50,000 *nodes and opinions that we seed independently using* $x_i(0) \sim \mathcal{N}(0, \sigma^2)$. *We consider uniformly spaced values* $\sigma \in [0.9, 1.1]$ *with a step size of* $\Delta\sigma = 0.004$. *The empirical convergence time* t^* *is the first time that the discordance function of the opinion state satisfies* $d(V, \boldsymbol{x}) < 10^{-5}$. *If the system does not reach such an opinion state by* $t = 10^4$, *we record* t^* *as* 10^4. *We simulate* 20 *trials for each value of* σ. *The black curve gives the mean of* t^* *over the trials, and the blue area depicts one standard deviation from the mean. We include a dashed red line at* $\sigma^2 = 1 = c$ *for reference.*

the fact that $\mathrm{Var}[s_n^2] = O(\frac{1}{n})$, we see that the convergence rate is $O(\frac{1}{n})$ whenever $\sigma^2 \neq c$. When the initial opinions are normally distributed, we can derive a much tighter bound on the convergence rate.

Lemma 4.5. *Suppose that we draw the initial opinions from a normal distribution with variance* $\sigma^2 \neq c$, *and let* $\lambda = \frac{c}{\sigma^2}$. *It follows that*

$$\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] \geq 1 - \left(e^{\frac{1}{2}(1-\lambda)}\sqrt{\lambda}\right)^{n-1}, \qquad \lambda > 1\,,$$
$$\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] \leq \left(e^{\frac{1}{2}(1-\lambda)}\sqrt{\lambda}\right)^{n-1}, \qquad \lambda < 1\,.$$

Therefore, $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n]$ *converges exponentially fast as* $n \to \infty$.

Proof. The discordance of a hyperedge e at time 0 is the sample variance of the opinions $\{x_j(0) \mid j \in e\}$. Let s_n^2 denote the sample variance of n opinions. We have that $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n] = \mathbb{P}[s_n^2 < c]$.

Case 1. Suppose that $\lambda > 1$. By Chernoff's bound [11],

$$\mathbb{P}[s_n^2 \geq c] = \mathbb{P}\Big[\frac{n-1}{\sigma^2} s_n^2 \geq \lambda(n-1)\Big] \leq \frac{\mathbb{E}\left[e^{t\frac{n-1}{\sigma^2}s_n^2}\right]}{e^{t\lambda(n-1)}} \quad \text{for all} \quad t > 0\,. \tag{4.5}$$

By Cochran's theorem [29], $\frac{n-1}{\sigma^2}s_n^2 \sim \chi^2_{n-1}$, where χ^2_{n-1} denotes the chi-square distribution with $n-1$ degrees of freedom. When $t < \frac{1}{2}$, we have that

$$\begin{aligned}\mathbb{E}[e^{t\frac{n-1}{\sigma^2}s_n^2}] &= \frac{1}{2^{(n-1)/2}\Gamma(\frac{n-1}{2})}\int_0^\infty e^{tx}x^{\frac{n-1}{2}-1}e^{-x/2}dx \\ &= \frac{1}{\Gamma(\frac{n-1}{2})}\int_0^\infty x^{\frac{n-1}{2}-1}e^{-x(1-2t)}dx \\ &= \frac{1}{\Gamma(\frac{n-1}{2})}(1-2t)^{-\frac{n-1}{2}}\int_0^\infty x^{\frac{n-1}{2}-1}e^{-x}dx \\ &= (1-2t)^{-\frac{n-1}{2}}.\end{aligned}$$

Therefore, when $0 < t < \frac{1}{2}$, (4.5) becomes

$$\mathbb{P}[s_n^2 \geq c] \leq \left(\frac{1}{e^{t\lambda}\sqrt{1-2t}}\right)^{n-1}. \tag{4.6}$$

Setting $t = \frac{1}{2}(1-\frac{1}{\lambda})$ in (4.6) yields

$$\mathbb{P}[s_n^2 \geq c] \leq \left(e^{\frac{1}{2}(1-\lambda)}\sqrt{\lambda}\right)^{n-1}.$$

Case 2. Suppose that $\lambda < 1$. By Chernoff's bound,

$$\mathbb{P}[s_n^2 < c] = \mathbb{P}\Big[\frac{n-1}{\sigma^2}s_n^2 < \lambda(n-1)\Big] \leq \frac{\mathbb{E}\left[e^{-t\frac{n-1}{\sigma^2}s_n^2}\right]}{e^{-t\lambda(n-1)}} \quad \text{for all} \quad t > 0\,. \tag{4.7}$$

Similarly to Case 1, we compute that

$$\mathbb{E}\left[e^{-t\frac{n-1}{\sigma^2}s_n^2}\right] = \left(\frac{1}{1+2t}\right)^{\frac{n-1}{2}} \quad \text{for all} \quad t > 0\,.$$

Therefore, when $t > 0$, (4.7) becomes

$$\mathbb{P}[s_n^2 < c] \leq \left(\frac{e^{\lambda t}}{\sqrt{1+2t}}\right)^{n-1}. \tag{4.8}$$

Setting $t = \frac{1}{2}(\frac{1}{\lambda}-1)$ in (4.8) yields

$$\mathbb{P}[s_n^2 < c] \leq \left(e^{\frac{1}{2}(1-\lambda)}\sqrt{\lambda}\right)^{n-1}.$$

■

Remark 4.6. When $\sigma^2 = c$ and the initial opinions are normally distributed, we have numerical evidence that $\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n]$ converges to $\frac{1}{2}$ exponentially fast as $n \to \infty$, but we do not have a mathematical proof of the convergence rate.

We say that the opinion state of our hypergraph BCM *converges with threshold* ϵ *at time* T_ϵ if, for all $e \in E$, either $d(e, \boldsymbol{x}(T_\epsilon)) \leq \epsilon$ or $d(e, \boldsymbol{x}(T_\epsilon)) \geq c$. When $\epsilon = 0$, the time T_ϵ is exactly the convergence time.

Theorem 4.7. *Let H be the complete hypergraph with N nodes. Let $\boldsymbol{x}(0)$ be the initial opinion state and suppose that $x_i(0) \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 > c$ and $c \neq 0$. Finally, let $\epsilon < c$ and let $T_{\epsilon,N}$ be the convergence time with threshold ϵ. It then follows that*

$$\mathbb{E}[T_{\epsilon,N}] = \Omega\left(r\left[\frac{2}{r+1}\right]^N\right),$$

where $r = e^{\frac{1}{2}(1-c/\sigma^2)}\sqrt{\frac{c}{\sigma^2}} < 1$.

Proof. Let A_N be the Bernoulli random variable that equals 1 if there is a hyperedge $e \in E$ such that $\epsilon < d(e, \boldsymbol{x}(0)) < c$ and equals 0 if there is no such hyperedge. We have that

$$\mathbb{E}[T_{\epsilon,N}] = \mathbb{E}[T_{\epsilon,N} \mid A_N = 1]\mathbb{P}[A_N = 1] + \mathbb{E}[T_{\epsilon,N} \mid A_N = 0]\mathbb{P}[A_N = 0]\,.$$

If $A_N = 0$, then $T_{\epsilon,N} = 0$. As $N \to \infty$, we have

$$\lim_{N\to\infty} \mathbb{P}[A_N = 0] = \lim_{N\to\infty} \prod_{n=2}^{N} (1 - \mathbb{P}[\epsilon < d(e, \boldsymbol{x}(0)) < c \mid |e| = n])^{\binom{N}{n}} = 0\,,$$

where e denotes a hyperedge that we choose uniformly at random. Let $\tilde{t}_N$ be the first time that we select a concordant hyperedge. If $A_N = 1$, then $T_{\epsilon,N} \geq \tilde{t}_N$. Let X_N be the fraction of hyperedges in H that are concordant at time 0. We calculate

$$\begin{aligned}\mathbb{E}[T_{\epsilon,N}] &\geq \mathbb{E}[T_{\epsilon,N} \mid A_N = 1]\mathbb{P}[A_N = 1] \\ &\geq \mathbb{E}[\tilde{t}_N]\mathbb{P}[A_N = 1] \\ &= \frac{\mathbb{P}[A_N = 1]}{\mathbb{E}[X_N]}\,,\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[X_N] &= \frac{\sum_{n=2}^{N}\binom{N}{n}\mathbb{P}[d(e, \boldsymbol{x}(0)) < c \mid |e| = n]}{2^N - N - 1} \\ &\leq \frac{\sum_{n=2}^{N}\binom{N}{n} r^{n-1}}{2^N - N - 1} \qquad \text{(by Lemma 4.5)} \\ &= \frac{\frac{1}{r}((r+1)^N - Nr - 1)}{2^N - N - 1}\,.\end{aligned}$$

Therefore, as $N \to \infty$, we obtain

$$\mathbb{E}[T_{\epsilon,N}] \geq r\mathbb{P}[A_N = 1]\frac{2^N - N - 1}{(r+1)^N - Nr - 1} \sim r\left(\frac{2}{r+1}\right)^N. \qquad \blacksquare$$

Remark 4.8. Theorem 4.7 applies to the convergence time $T_{0,N}$, which is almost surely finite by Theorem 4.2.

5. Opinion jumping.

We now study "opinion jumping," a phenomenon that occurs in our hypergraph BCM that cannot occur in standard dyadic BCMs. An *opinion jump* occurs at time t if there is a node i such that $|x_i(t+1) - x_i(t)| > c$. The number of opinion jumps that occur at time t is the number of nodes i that satisfy $|x_i(t+1) - x_i(t)| > c$.

An opinion jump can occur at time t only if the size of the hyperedge that is selected at time t is at least 3. Therefore, this behavior requires polyadic interactions; it cannot occur in BCMs on ordinary graphs. Moreover, we believe that it is one of the driving behaviors that causes our hypergraph BCM to converge to consensus so much more easily than is the case for standard dyadic BCMs. For examples of opinion jumping, see Figure 1 and Figure 5. In this section, we quantify how common it is for an opinion jump to occur.

Lemma 5.1. *Let J_t be the number of opinion jumps that occur at time t. Suppose that the distribution of opinions at time t has a mean of μ and a variance of $\sigma^2 < \infty$. Let $p_n = \mathbb{P}[|x_i - \bar{x}_e| > c \mid i \in e\,,\, |e| = n\,,\, d(e, \boldsymbol{x}(t)) < c]$ be the probability that a node's opinion is farther than c from the mean opinion of the nodes in a concordant size-n hyperedge that is incident to the node, and let $p = \mathbb{P}[|x - \mu| > c]$. Let $a_n = \mathbb{P}[d(e, \boldsymbol{x}(t)) < c \mid |e| = n]$ be the probability that a size-n hyperedge is concordant, and let $a = \lim_{n\to\infty} a_n$ be the limiting probability of concordance. Finally, let e_t be the hyperedge that we select at time t. The expected number of opinion jumps is*

$$\mathbb{E}[J_t] = \left(pa\mathbb{E}[|e_t|] + p\sum_{n=2}^{N}(a_n - a)\mathbb{P}[|e_t| = n]n + \sum_{n=2}^{N}(p_n - p)a\mathbb{P}[|e_t| = n]n \right.$$
$$\left. + \sum_{n=2}^{N}(p_n - p)(a_n - a)\mathbb{P}[|e_t| = n]n \right).$$

Remark 5.2. The quantities p_n, p, a_n, and a depend on the distribution of opinions at time t. The value of a is given by (3.6).

Proof. For $j \geq 1$, we have

$$\begin{aligned}
\mathbb{P}[J_t = j] &= \sum_{n=2}^{N} \mathbb{P}[J_t = j \text{ and } |e_t| = n] \\
&= \sum_{n} \mathbb{P}[J_t = j \mid |e_t| = n]\mathbb{P}[|e_t| = n] \\
&= \sum_{n} \mathbb{P}[J_t = j \text{ and } d(e_t, \boldsymbol{x}(t)) < c \mid |e_t| = n]\mathbb{P}[|e_t| = n] \qquad \text{(because } j \geq 1\text{)} \\
&= \sum_{n} \mathbb{P}[J_t = j \mid d(e_t, \boldsymbol{x}(t)) < c, |e_t| = n]a_n\mathbb{P}[|e_t| = n] \\
&= \sum_{n} \binom{n}{j} p_n^j (1 - p_n)^j a_n \mathbb{P}[|e_t| = n]\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[J_t] &= \sum_{j=0}^{N} \mathbb{P}[J_t = j]j \\
&= \sum_{j}\sum_{n} j\binom{n}{j} p_n^j (1 - p_n)^j a_n \mathbb{P}[|e_t| = n] \\
&= \sum_{n} a_n \mathbb{P}[|e_t| = n] \sum_{j} j\binom{n}{j} p_n^j (1 - p_n)^j
\end{aligned}$$

$$= \sum_n a_n \mathbb{P}[|e_t| = n] p_n n$$
$$= \left(pa\mathbb{E}[|e_t|] + p\sum_{n=2}^{N}(a_n - a)\mathbb{P}[|e_t| = n]n + \sum_{n=2}^{N}(p_n - p)a\mathbb{P}[|e_t| = n]n \right.$$
$$\left. + \sum_{n=2}^{N}(p_n - p)(a_n - a)\mathbb{P}[|e_t| = n]n \right). \qquad \blacksquare$$

We use Lemma 5.1 to derive the asymptotic behavior of $\mathbb{E}[J_0]$. The following proposition says that, under certain conditions, $\mathbb{E}[J_0]$ grows linearly with the mean hyperedge size of the hypergraph on which we run our BCM.

Proposition 5.3. *Let $\{H_m\}$ be a sequence of hypergraphs with the associated sequence $\{V_m\}$ of nodes and sequence $\{E_m\}$ of hyperedge sets. Let $\{g_m\}$, where*

$$g_m(n) = \frac{|\{e \in E_m \mid |e| = n\}|}{|E_m|},$$

be the corresponding sequence of hyperedge-size distributions. Suppose that we draw the initial opinions from the same distribution for all H_m, and let p, p_n, a, and a_n be defined as in Lemma 5.1. Finally, let J_0^m be the number of opinion jumps that occur at time 0 for H_m. Suppose that $(a - a_n)n \to 0$ as $n \to \infty$, $g_m(n) \to 0$ for all n as $m \to \infty$, and $a(p_n - p)n \to 0$ as $n \to \infty$. It then follows that $\mathbb{E}[J_0^m] \sim pa\mathbb{E}[|e_0|]$ as $m \to \infty$.

Remark 5.4. When H_m is the complete hypergraph with m nodes, we have that $g_m(n) \to 0$ for all n as $m \to \infty$. The values of a, a_n, p, and p_n depend only on the opinion distribution at time t. The value of a is given by (3.6). If the initial opinions are normally distributed, then Lemma 4.5 implies that $(a_n - a)n \to 0$ when $\sigma^2 \neq c$. The exact value of p depends on the initial distribution, but it tends to increase with σ^2. Our numerical computations suggest that $a(p_n - p)n \to 0$ when the initial opinion distribution is normally distributed with variance $\sigma^2 \neq c$.

Proof. Lemma 5.1 implies that

$$\mathbb{E}[J_0^m] = pa\mathbb{E}[|e_0|] + a\sum_n (p_n - p)ng_m(n) + p\sum_n (a_n - a)ng_m(n)$$
$$+ \sum_n (p - p_n)(a_n - a)ng_m(n).$$

Let x_n be any sequence such that $x_n \to 0$. For any m, the quantity $\sum_n x_n g_m(n)$ is a weighted average of $\{x_n\}$. As $m \to \infty$, the weights concentrate at larger values of n. Therefore, because $x_n \to 0$, it follows that $\sum_n x_n g_m(n) \to 0$ as $m \to \infty$.

We apply the above argument to $x_n = a(p_n - p)n$, $x_n = (a_n - a)n$, and $x_n = (p - p_n)(a_n - a)n$ to prove the proposition. $\blacksquare$

In Figure 7, we present numerical results that support the claim that $\mathbb{E}[J_0] \approx pa\mathbb{E}[|e_0|]$ when the hyperedge-size distribution concentrates at large hyperedge sizes. We generate hypergraphs from the $G(N, \boldsymbol{m})$ hypergraph model for different values of $\boldsymbol{m}$. For each hypergraph,

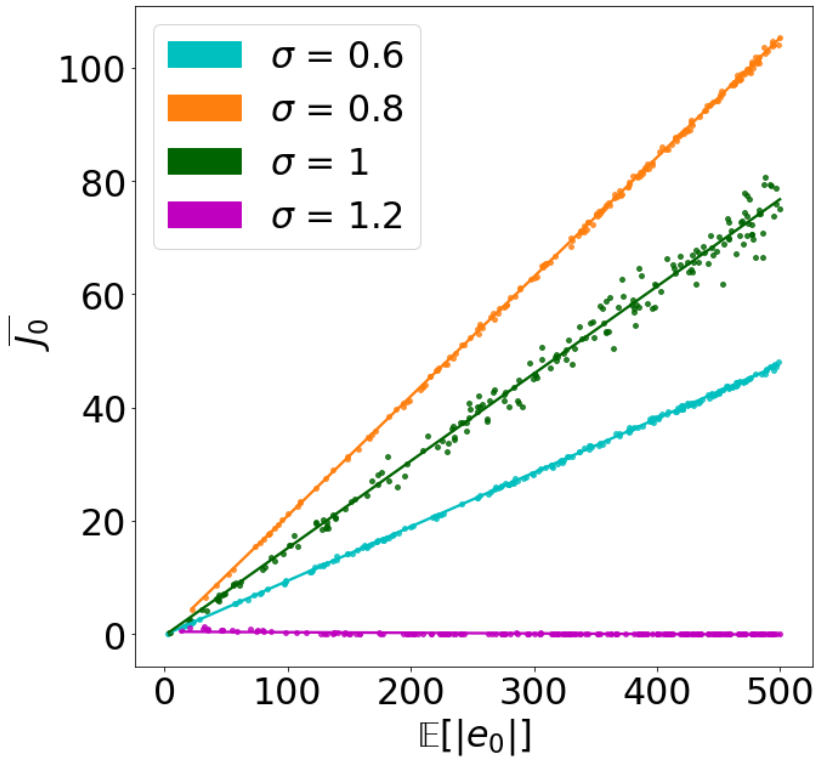


Figure 7. *Empirical evidence for the linear relationship between* $\mathbb{E}[J_0]$ *and* $\mathbb{E}[|e_0|]$*, where* J_0 *is the number of opinion jumps that occur at time* 0 *and the hyperedge* e_0 *is the hyperedge that we choose uniformly at random at time* 0*. In this numerical experiment, the initial opinions are normally distributed with a mean of* 0 *and standard deviation of* σ*, which takes values of* 0.6*,* 0.8*,* 1*, and* 1.2*. For each* σ*, we generate* 200 *hypergraphs with* $N = 1000$ *nodes. To construct the* ℓ*th hypergraph, we choose* $x_\ell \in [0, 1]$ *uniformly at random and set* $m_n^{(\ell)} = \binom{N}{i} x_\ell^n$*. We generate the* ℓ*th hypergraph from the* $G(N, \boldsymbol{m}^{(\ell)})$ *model, which has* N *nodes and* $m_n^{(\ell)}$ *hyperedges of size* n *that we choose uniformly at random from the set of all size-*n *elements in* $\mathcal{P}(V)$*. For each hypergraph, we run one step of our hypergraph BCM with confidence bound* $c = 1$ *and record the number* J_0 *of opinion jumps. We run* 500 *trials for each hypergraph; for each trial, we preserve the hypergraph structure, reset the initial opinions, and run one step of our hypergraph BCM. We record the mean value of* J_0 *over these* 500 *trials; we denote this mean value by* $\overline{J_0}$*. We plot* $\overline{J_0}$ *versus* $\mathbb{E}[|e_0|]$*, which is equal to the mean hyperedge size of the hypergraph. Each data point corresponds to the trial results for a single hypergraph for a given value of* σ*. For each* σ*, we plot the line of best fit.*

we run 500 trials of our hypergraph BCM and record the mean value of J_0 for the hypergraph. We plot the mean value of J_0 versus $\mathbb{E}[|e_0|]$, which is equal to the mean hyperedge size of the hypergraph. We show results for initial opinions that are normally distributed with standard deviations of $\sigma = 0.6$, $\sigma = 0.8$, $\sigma = 1$, and $\sigma = 1.2$. We use a confidence bound of $c = 1$ in all trials. The claim $\mathbb{E}[J_0] \approx pa\mathbb{E}[|e_0|]$ implies that for each σ, there should be a linear relationship with a slope of pa, where p and a depend on σ. Whenever $\sigma^2 < c$, the limiting probability of concordance is $a = 1$. The slope when $\sigma = 0.8$ is steeper than the slope when $\sigma = 0.6$ because p becomes larger for progressively larger values of σ. When $\sigma = 1$, the value of p is larger than it is for $\sigma = 0.6$ and $\sigma = 0.8$. However, the limiting probability of

concordance is only $a = \frac{1}{2}$ and the slope pa for the case $\sigma = 1$ is slightly less steep than the slope when $\sigma = 0.8$. We observe that the linear relationship between $\mathbb{E}[J_0]$ and $\mathbb{E}[|e_0|]$ is not as strong when $\sigma^2 = c = 1$ as when $\sigma^2 \neq c$. Based on numerical evidence, we suspect that this is because $(p_n - p)n \not\to 0$ when $\sigma^2 = c$. Finally, when $\sigma = 1.2$, we observe that $\mathbb{E}[J_0] \approx 0$ because the limiting probability of concordance is $a = 0$ whenever $\sigma^2 > c$.

6. Conclusions and discussion. We formulated a bounded-confidence model (BCM) of opinion dynamics on hypergraphs and explored its properties using both mathematical analysis and Monte Carlo simulations. We showed that polyadic (i.e., "higher-order") interactions play an important role in opinion dynamics and that one cannot reduce such interactions to pairwise interactions on a graph. In our hypergraph BCM, we also demonstrated a novel phenomenon, which we called "opinion jumping," that requires polyadic interactions to manifest. Therefore, opinion jumping cannot occur in standard dyadic BCMs.

We proved that our hypergraph BCM converges to consensus on complete hypergraphs for a wide range of initial conditions. This is very different from what occurs in standard dyadic BCMs, which usually converge to multiple opinion clusters. We also studied the effects of a range of initial opinion distributions on the dynamics of our BCM. In particular, we examined the convergence properties of our BCM when the initial opinion distribution is bounded (but not necessarily uniform), normally distributed, or has a variance σ^2 that is less than the confidence bound c. Based on our results, we expect that the limit states of dyadic BCMs also depend on the initial opinion distribution (although, to the best of our knowledge, this is not something that has been studied in detail in prior research) and that the number of opinion clusters depends not only on the confidence bound c but also on the relative sizes of c and on the variance σ^2 of the initial opinion distribution.

We also explored the dependence of the limit state of our hypergraph BCM on community structure. We proved that the opinion state can become polarized if the intracommunity hyperedges have sufficiently small sizes. This leads to the formation of echo chambers. We also showed that the opinion state converges to consensus if the intracommunity hyperedges are unbounded in size and if the communities are sufficiently large and form hypercliques.

We demonstrated that there is a phase transition in the convergence time of our BCM on complete hypergraphs when the confidence bound c equals the variance σ^2 of the initial opinion distribution. When $\sigma^2 > c$, the convergence time of our BCM on a complete hypergraph depends at least exponentially on the number N of nodes. This has implications for the feasibility of using Monte Carlo simulations for simulating our BCM on a complete hypergraph when $\sigma^2 > c$ and N is large (and, more generally, on any hypergraph in which large hyperedges constitute a significant proportion of the hyperedge set, because large hyperedges are likely to be discordant and it thus takes many time steps to choose a concordant hyperedge). It is fascinating that there is a phase transition in the convergence time but not in the limit state. By contrast, in the standard dyadic Deffuant–Weisbuch model, there is a phase transition in the convergence time at the same confidence-bound threshold c^* at which there is a phase transition in the limit state [36]. We also proved that our hypergraph BCM converges in finite time on complete hypergraphs. This is similar to what occurs in the standard dyadic Hegselmann–Krause model, which converges in finite time on complete graphs; however, it differs from the DW model, which tends not to converge in finite time on complete graphs.

Because of opinion jumping, which requires polyadic interactions, nodes with extreme opinions can move quickly towards the mean opinion in our hypergraph BCM. When the variance σ^2 satisfies $\sigma^2 < c$, we showed that the number of opinion jumps in the first time step grows roughly linearly with the mean hyperedge size of a hypergraph and that it becomes larger for progressively larger values of σ^2 up to the value c. It will be worthwhile to determine the precise necessary conditions for opinion jumping in hypergraph BCMs.

In our work, we made several modeling choices, and there are numerous alternatives that are also worth studying. For example, one can formulate a hypergraph BCM that has synchronous updates of opinions instead of asynchronous updates. For example, at each discrete time, suppose that each node updates its opinion to the mean of its incident hyperedges' mean opinions.[6] We believe that such a synchronous model has similar limit states as our asynchronous model, but we expect such a model to converge much more quickly to a limit state. One can also develop synchronous models in which each node updates its opinion to a weighted average of the incident hyperedges' mean opinions. Such heterogeneity models a situation in which some friendship groups exert more influence on a person than others. This extends the notion of trust from the dyadic DeGroot model [10]. Another of our modeling choices was our discordance function. Instead of choosing $d = d_1$ for the discordance function, it is worthwhile to study the entire family of discordance functions d_α (with $\alpha \in [0, 1]$) that we defined in (2.1). The case $\alpha = 0$ is particularly interesting because it models a scenario in which it is more difficult for large groups of people to agree than it is for small groups. Another variation of our model involves incorporating heterogeneous confidence bounds to model situations in which some individuals are persuaded more easily than others.

There are a variety of other avenues to explore. For example, it is worth conducting a deeper investigation of the role of hypergraph topology on the limit states of hypergraph BCMs, and one can also study BCMs on simplicial complexes (which entail various constraints on which polyadic interactions are permissible). We believe that the presence of large hyperedges that connect some subset of a hypergraph's nodes will facilitate the convergence of those nodes to consensus. One can also develop adaptive (i.e., coevolving) hypergraph BCMs, such as by modifying the hypergraph structure at each time step in response to the current opinion state. For example, as in recent studies of dyadic BCMs [5, 27], one can allow agents to strategically rewire in a way that maximizes their influence or to simply leave a hyperedge when the other nodes that are incident to it become too "annoying" (which can occur sometimes in discussion groups on social media). From a control-theoretic perspective, one can examine how much control the nodes in a hypergraph (or an outside controller) can have in steering an opinion state towards a particular limit state by choosing which hyperedges to update or rewire.

Appendix A. Continuum formalism. Instead of running Monte Carlo simulations of a BCM, which are costly, one can study a "continuum" formalism (that is similar to the one in [2]) of a BCM using numerical integration. Consider a hypergraph in which every hyperedge has size $\ell \in L \subseteq \{2, \dots, n\}$, and let $P(x, t)\, dx$ be the probability density function that indicates how many nodes have opinions in the interval $(x, x + dx)$ at time t. The distribution $P(x, t)$ evolves according to the rate equation

[6] The model of Sahasrabuddhe, Neuhaüser, and Lambiotte [45] incorporates a mechanism that is related to this idea. In their model, the amount of influence of a hyperedge depends on the opinion state of the system.

(A.1)
$$\frac{\partial}{\partial t}P(x,t) = \sum_{\ell \in L} \int_{\{\sum_{j=1}^{\ell}(y_j-\bar{y})^2 < c(\ell-1)\}} dy_1 \cdots dy_\ell P(y_1,t) \times \cdots \times P(y_\ell,t)\left[\delta(x-\bar{y}) - \delta(x-y_1)\right] .$$

The ℓ-fold integrals in the summand are over all ℓ-tuples of points whose sample variance is less than c. The delta functions reflect the gains (from nodes that update their opinion to $\bar{y}$) and losses (from nodes that update their opinions and thus change their current opinion) in the update process. We do not study (A.1) in the present paper, but it seems interesting to examine in future work.

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