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Event-Driven H_∞-Constrained Control Using Adaptive Critic Learning

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***Abstract*—This article considers an event-driven H_∞ control problem of continuous-time nonlinear systems with asymmetric input constraints. Initially, the H_∞-constrained control problem is converted into a two-person zero-sum game with the discounted nonquadratic cost function. Then, we present the event-driven Hamilton–Jacobi–Isaacs equation (HJIE) associated with the two-person zero-sum game. Meanwhile, we develop a novel event-triggering condition making Zeno behavior excluded. The present event-triggering condition differs from the existing literature in that it can make the triggering threshold non-negative without the requirement of properly selecting the prescribed level of disturbance attenuation. After that, under the framework of adaptive critic learning, we use a single critic network to solve the event-driven HJIE and tune its weight parameters by using historical and instantaneous state data simultaneously. Based on the Lyapunov approach, we demonstrate that the uniform ultimate boundedness of all the signals in the closed-loop system is guaranteed. Finally, simulations of a nonlinear plant are presented to validate the developed event-driven H_∞ control strategy.**

***Index Terms*—Adaptive critic learning (ACL), adaptive dynamic programming (ADP), asymmetric constraints, event-driven H_∞ control, reinforcement learning (RL).**

I. Introduction

IN THE control community, many studies on H_∞ control problems of nonlinear systems have been done over the past decades [1]–[3]. This is because the H_∞ control method provides a promising way to design robust controllers for nonlinear systems, in particular, the robust optimal controllers. According to Basar and Bernhard's theory [4], the H_∞ optimal control problem can be transformed into the zero-sum game, which is in essence the minimax optimization problem. Thus, instead of directly solving the H_∞ optimal control problems, researchers often tend to solve the zero-sum games [5], [6], which are able to be solved via adaptive critic learning (ACL). ACL is an effective and powerful technique introduced to cope with optimization problems, which is built on the theories of dynamic programming and neural networks. As pointed out by [7], ACL, adaptive dynamic programming (ADP) [8], and reinforcement learning (RL) [9] are often considered synonyms because they have nearly the same characteristics in solving optimization problems. In this article, we view ADP and RL as the members of ACL's family. The past few years have witnessed various ACL (including ADP and RL) methods applied to handle H_∞ control problems or zero-sum games, such as policy iteration ADP [10], robust ADP [11], integral RL [12], and off-policy RL [13], [14].

Though the above-mentioned ACL (or rather, ADP and RL) could solve the H_∞ control problems or the zero-sum games, they were all implemented in a time-driven mechanism (*note:* the control policies developed in the time-driven mechanism were implemented *periodically*). As mentioned by [15], the periodic control policies often lead to inefficient use of limited resources, such as computation bandwidths and communication resources. To address this deficiency, the event-driven control methods were proposed (*note:* the event-driven control policies were often implemented *aperiodically* [16], [17]). According to [18], the core of designing event-driven controllers is to set an appropriate event-triggering condition. Specifically, the proposed event-triggering condition should not only have the non-negative triggering threshold but also prevent the Zeno behavior from happening. However, when designing the event-driven H_∞ controllers, one often sets the event-triggering condition corresponding to the prescribed level of disturbance attenuation [i.e., the parameter γ in later (4)]. Thus, one needs to properly choose γ to make the event-triggering threshold non-negative (*note:* the prescribed level of disturbance attenuation [i.e., γ] is often involved in the negative term constituting the triggering threshold (see [19, Theor. 1]). It is often challenging to provide such a parameter γ. How to avoid such a challenge is an issue to be addressed in this article. On the other hand, when solving the event-driven H_∞ control problems (or rather, the zero-sum games), one usually updates the control policy in the event-driven mechanism and tunes the disturbance policy in the time-driven mechanism. What if the disturbance policy is also updated in the event-driven mechanism? Is there any influence on designing the H_∞ controller when considering the event-driven disturbance? This article will also address this problem.

In industry applications, there are often restrictions imposed on the controllers/actuators because of the safety consideration (such as voltages and temperature) as well as the controllers'/actuators' physical characteristics. Though studies

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exist on designing event-driven H_∞ controllers for input-constrained nonlinear systems, they almost focused on nonlinear plants with *symmetric* input constraints (see the related work below). Few of them studied the event-driven H_∞ control problem of nonlinear systems with *asymmetric* input constraints. Actually, when considering asymmetric input constraints, we can see that the optimal control will not stay at zero when the steady state is obtained [see Remark 1 2) and Fig. 3 in Section IV]. This feature is totally different from the case with systems that have symmetric input constraints. Thus, we need novel approaches to handle this characteristic. This is another issue and will be addressed in this article.

A. Related Work

For continuous-time (CT) nonlinear systems without control constraints, Wang *et al.* [19] suggested an improving critic learning criterion to design the event-based H_∞ controller for affine-input nonlinear systems. After that, Wang *et al.* [20] extended the work of Wang *et al.* [19] to develop an event-driven H_∞ control strategy for unknown nonlinear systems. Apart from the time-driven identifier employed in [20], they used a unique critic network in both [19] and [20] to solve the event-driven Hamilton–Jacobi–Isaacs equation (HJIE). Later, by using neuro-dynamic programming, Zhao *et al.* [21] proposed a novel event-triggered H_∞ optimal control scheme for CT nonlinear systems. In [19]–[21], an exploration signal was added to make the persistency of excitation (PE) condition satisfied. To relax the PE condition, Zhang *et al.* [22] utilized the concurrent learning technique combined with the same structure as [19] to solve the nonlinear H_∞ optimal control problem. The disturbance policies in [19]–[22] were updated in the time-driven mechanism. Recently, Xue *et al.* [23] presented an event-triggered ADP to solve the zero-sum game of partially unknown nonlinear systems. To implement the event-triggered ADP, the control policy and the disturbance policy were tuned in the event-driven mechanism. However, just as mentioned before, the event-triggering thresholds given in [19]–[23] all depended on properly choosing the prescribed level of disturbance attenuation [i.e., the parameter γ in later (4)]. More recently, Yang *et al.* [24] proposed an H_∞ containment intermittent control scheme for a directed graph via solving the game algebraic Riccati equation. The event-triggering threshold also relied on selecting the appropriate parameter γ. Generally, it is very technical to select a suitable γ in the aforementioned literature to make the event-triggering threshold non-negative.

For CT nonlinear systems having control constraints, Wang *et al.* [25] suggested an ACL method to design the input-constrained nonlinear H_∞ feedback controller in the event-driven mechanism. After that, Yang *et al.* [26] obtained the robust event-driven control of input-constrained nonlinear systems by applying an RL to solve the H_∞-constrained control problem. Both Wang *et al.* [25] and Yang *et al.* [26] implemented their disturbance policies in the time-driven mechanism. In addition, similar to [19]–[24], they also needed to appropriately choose the prescribed level of disturbance attenuation to keep the event-triggering threshold non-negative. Recently, Wang *et al.* [27] developed an ADP approach to derive the robust optimal event-driven control of constrained nonlinear systems subject to the external disturbance. The crucial features distinguishing [27] and [25], [26] lie in that the work of [27] not only tunes the control policy and the disturbance policy in an event-driven mechanism but also has a non-negative event-triggering threshold. However, the condition that $g(x)g^{\mathsf{T}}(x) \geq \gamma^2 k(x)k^{\mathsf{T}}(x)$ must be satisfied (see [27, Theor. 1]). In general, it is challenging to make this condition hold. Furthermore, the control constraints given in [25]–[27] are all symmetric, that is, the symmetric input constraints.

B. Contribution

The contributions of this article have three aspects. First, we present a novel event-triggering condition for designing the nonlinear H_∞-constrained controller. The selection of the prescribed level of disturbance attenuation [i.e., the parameter γ in (4)] will not affect whether the event-triggering threshold is positive or not [see (15)]. Thus, we can overcome the difficulty in choosing a proper γ to make the event-triggering threshold non-negative, which is an advantage. Second, with the introduction of a discounted cost function, the present ACL approach can handle the event-driven H_∞ control problem of nonlinear systems with *asymmetric* input constraints. Therefore, the present ACL method has an advantage in applying for a wider scope of nonlinear systems, in particular, nonlinear plants suffering from asymmetric input constraints. Third, when implementing the event-driven ACL developed in this article, we tune both the control policy and the disturbance policy in the event-driven mechanism. Thus, the computational load can be remarkably reduced in comparison with those literature that only updates the control policy in the event-driven mechanism (see Table II). This is another advantage.

C. Notation

$\mathbb{R}$ is the set of real numbers. $\mathbb{R}^m$ and $\mathbb{R}^{n\times m}$ are the spaces of real m-vectors and $n \times m$ matrices, respectively. T is the transposition symbol. "$\triangleq$" and "C^1" mean "equal by definition" and "the function with continuous derivative," respectively. For the vector $x \in \mathbb{R}^m$, $\|x\|$ denotes its norm. For the constant matrix $Q \in \mathbb{R}^{n\times n}$, $\|Q\|$ and $\lambda_{\min}(Q)$ denote its Frobenius-norm and minimum eigenvalue, respectively.

II. Problem Formulation

We consider the CT nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + k(x(t))\omega(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable with its initial value $x_0 = x(t_0)$, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ is the control input, and $\mathcal{U}$ is the set consisting of control policies with asymmetric bounds, that is

$$\mathcal{U} = \big\{(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m : u_{\min} \leq u_i \leq u_{\max}$$
$$|u_{\min}| \neq |u_{\max}|, \quad i = 1, 2, \ldots, m\big\}$$

with $u_{\min} \in \mathbb{R}$ and $u_{\max} \in \mathbb{R}$ denoting the minimum and maximum bound of every $u_i \in \mathbb{R}$, respectively; $\omega(t) \in \mathbb{R}^q$ is an exogenous disturbance with $\omega(t) \in L_2[0,\infty)$; and $f(x) \in \mathbb{R}^n$, $g(x) \in \mathbb{R}^{n\times m}$, and $k(x) \in \mathbb{R}^{n\times q}$ are known continuously differentiable functions.

Assumption 1: $f(0) = 0$, that is, $x = 0$ is the equilibrium point of system (1) when $u = 0$ and $\omega = 0$ [or $g(0) = 0$ and $\omega = 0$]. Moreover, $f(x)+g(x)u+k(x)\omega$ has the Lipschitz property making that $x = 0$ is the unique equilibrium point over the compact set Ω (*note:* $0 \in \Omega \subset R^n$).

Assumption 2: For every $x \in \mathbb{R}^n$, $\|g(x)\| \le g_M$ and $\|k(x)\| \le k_M$ with $g_M > 0$ and $k_M > 0$ being the known constants. In addition, $g(0) = 0$.

Next, we introduce a necessary definition derived from [28].

Definition 1 [Uniform Ultimate Boundedness (UUB)]: The solution $x(t)$ of system (1) is said to be stable in the sense of UUB, if there exists a compact set $\Omega \subset \mathbb{R}^n$ such that, for each $x(t_0) = x_0 \in \Omega$, there exist a constant $\varepsilon > 0$ (independent of $t_0 \ge 0$) and a number $T = T(\varepsilon, x_0)$ such that $\|x(t)\| < \varepsilon$ for all $t \ge t_0 + T$.

Noticing that system (1) is subject to asymmetric input constraints, inspired by [2], we define the fictitious output as

$$\|z(t)\|^2 = Q(x(t)) + \mathcal{R}(u(t)) \tag{2}$$

where $Q(x) = x^{\mathsf{T}}Qx$, $Q \in \mathbb{R}^{n\times n}$ is a positive-definite matrix

$$\mathcal{R}(u) = 2\beta\sum_{i=1}^{m}\int_{b}^{u_i}\psi^{-1}\Big(\beta^{-1}(s_i-b)\Big)ds_i$$

with

$$\beta = (u_{\max}-u_{\min})/2, \quad b = (u_{\max}+u_{\min})/2 \tag{3}$$

and $\psi^{-1}(\cdot) \in C^1(\Omega)$ is an odd monotonic function (*note:* $\psi^{-1}(0) = 0$). In this article, we let $\psi^{-1}(\cdot)$ be the hyperbolic tangent function, that is, $\psi^{-1}(\cdot) = \tanh^{-1}(\cdot)$.

Similar to the classic H_∞ control problem stated in [1], the goal of this article is to find a suitable state-feedback control $u(x)$ such that system (1) not only is stable in the sense of UUB but also has L_2-gain no larger than γ, that is

$$\begin{aligned}\int_t^\infty e^{-\rho(\tau-t)}\|z(\tau)\|^2d\tau &= \int_t^\infty e^{-\rho(\tau-t)}(Q(x)+\mathcal{R}(u))d\tau\\ &\le \gamma^2\int_t^\infty e^{-\rho(\tau-t)}\|\omega(\tau)\|^2d\tau\end{aligned} \tag{4}$$

where $\rho > 0$ is called the discount factor and $\gamma > 0$ is the prescribed level of disturbance attenuation. Here, we introduce the decay term $e^{-\rho(\tau-t)}$ in order to guarantee the convergence of $\int_t^\infty e^{-\rho(\tau-t)}\|z(\tau)\|^2d\tau$.

To achieve the above-mentioned goal, according to [2], we convert the H_∞ control problem into a two-person zero-sum game as follows:

$$V^*(x(t)) = \min_u \max_\omega J(x(t), u, \omega) \tag{5}$$

where

$$J(x(t), u, \omega) = \int_t^\infty e^{-\rho(\tau-t)}\mathcal{S}(x(\tau), u(\tau), \omega(\tau))d\tau$$

with $\mathcal{S}(x,u,\omega) = Q(x) + \mathcal{R}(u) - \gamma^2\|\omega\|^2$. Here, $J(x(t),u,\omega)$ is the discounted cost function for system (1), and $V^*(x(t))$ denotes the optimal value of $J(x(t),u,\omega)$.

Applying Bellman's optimality principle to $V^*(x(t))$ in (5) and taking its time derivative, we obtain

$$\min_u \max_\omega H\big(x, V_x^*, u, \omega\big) = 0 \tag{6}$$

where

$$\begin{aligned}H\big(x, V_x^*, u, \omega\big) &= \big(V_x^*\big)^{\mathsf{T}}(f(x)+g(x)u+k(x)\omega)\\ &\quad - \rho V^*(x) + Q(x) + \mathcal{R}(u) - \gamma^2\|\omega\|^2\end{aligned} \tag{7}$$

with $V_x^* = \partial V^*(x)/\partial x$. According to [2], (6) is called the HJIE and $H(x, V_x^*, u, \omega)$ in (7) is called the Hamiltonian for u, ω, and V_x^*.

Using the stationarity condition [29, Theor. 5.8] [i.e., $\partial H(x, V_x^*, u, \omega)/\partial u = 0$ and $\partial H(x, V_x^*, u, \omega)/\partial\omega = 0$], we have the optimal control and the worst disturbance, respectively, formulated as

$$u^*(x) = -\beta\tanh\left(\frac{1}{2\beta}g^{\mathsf{T}}(x)V_x^*\right) + \ell_b \tag{8}$$

$$\omega^*(x) = \frac{1}{2\gamma^2}k^{\mathsf{T}}(x)V_x^* \tag{9}$$

where $\ell_b = [b, b, \ldots, b]^{\mathsf{T}} \in \mathbb{R}^m$ with $b \in \mathbb{R}$ being defined as in (3).

Remark 1: Two explanations related to $u^*(x)$ in (8) and $\omega^*(x)$ in (9) are given as follows.

1) $(u^*(x), \omega^*(x))$ is called the saddle point of two-person zero-sum game (5) if $\min_u \max_\omega H(x, V_x^*, u, \omega) = \max_\omega \min_u H(x, V_x^*, u, \omega)$ (i.e., Isaacs's condition holds). According to [2], such an Isaacs's condition holds when the optimal control $u^*(x)$ in (8) and the worst disturbance $\omega^*(x)$ in (9) are derived.
2) $u^*(x)$ in (8) implies that $u^*(0) \neq 0$ (*note*: $\ell_b \neq 0$). In order to make the equilibrium point of system (1) to zero, that is, $x = 0$, we have to impose a condition that $g(0) = 0$ (see Assumption 2).

Inserting (8) and (9) into (6), we find that the HJIE is able to be rewritten as

$$\begin{aligned}&\big(V_x^*\big)^{\mathsf{T}}f(x) - \rho V^*(x) + Q(x) + \big(V_x^*\big)^{\mathsf{T}}g(x)\ell_b\\ &- \beta\big(V_x^*\big)^{\mathsf{T}}g(x)\tanh\left(\frac{1}{2\beta}g^{\mathsf{T}}(x)V_x^*\right)\\ &+ \mathcal{R}\left(-\beta\tanh\left(\frac{1}{2\beta}g^{\mathsf{T}}(x)V_x^*\right) + \ell_b\right)\\ &+ \frac{1}{4\gamma^2}\big(V_x^*\big)^{\mathsf{T}}k(x)k^{\mathsf{T}}(x)V_x^* = 0.\end{aligned} \tag{10}$$

Due to the nature of nonlinearity in the HJIE (10), we often cannot obtain its analytical solution. Thus, many efforts were made to derive the numerical/approximate solutions of HJIEs like (10) (see [30]–[32]). Nonetheless, as illustrated in [30]–[32], the HJIE like (10) was generally solved in a time-driven mechanism. Due to the deficiencies of time-driven control methods stated in Section I, we will solve (10) approximately in an event-driven mechanism.

III. Event-Driven H_∞-Constrained Control

To make this section self-contained, we first present the event-driven mechanism introduced in [20]. Meanwhile, based on such a mechanism, we develop the event-driven HJIE. Then, we set an event-triggering condition and prove that it excludes the Zeno behavior. After that, we use an ACL to solve the event-driven HJIE. Finally, we conduct a stability analysis of the closed-loop system.

A. Event-Driven Mechanism and Related Event-Driven HJIE

We denote the jth triggering instant as t_j and write $t_j < t_{j+1}$, $j \in \{0, 1, 2, \ldots\}$. Letting all the triggering instants together, we obtain the sequence $\{t_j\}_{j=0}^{+\infty}$. At each triggering instant t_j, the system state is sampled and written as

$$\hat{x}_j = x(t_j), \quad j \in \{0, 1, 2, \ldots\}.$$

Generally, prior to releasing the next triggering instant t_{j+1}, there occurs a gap between the sampled state $\hat{x}_j$ and the current state $x(t)$. We describe the gap via an error function formulated as

$$e_j(t) = \hat{x}_j - x(t), \quad t \in [t_j, t_{j+1}). \tag{11}$$

Based on (11), we can briefly introduce the event-driven mechanism as follows. If the event is triggered, that is, $t = t_j$, then $e_j(t_j) = 0$ holds. In this situation, we update the control policies. If the event is not triggered (or rather, the event-triggering threshold is not overrode), that is, $t \neq t_j$, then $e_j(t) \neq 0$. In this circumstance, we keep the control policies unchanged over the interval $[t_j, t_{j+1})$, $j \in \{0, 1, 2, \ldots\}$. This technique is known as the zero-order hold [18] and described as

$$\mu(\hat{x}_j, t) = u(\hat{x}_j) = u(x(t_j)), \quad t \in [t_j, t_{j+1}).$$

Using the above-stated event-driven mechanism, we obtain the optimal event-driven control from (8) as (*note:* $t \in [t_j, t_{j+1})$)

$$\mu^*(\hat{x}_j, t) = u^*(\hat{x}_j) = -\beta \tanh\left(\frac{1}{2\beta} g^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j}\right) + \ell_b \tag{12}$$

where $V^*_{\hat{x}_j} = (\partial V^*(x)/\partial x)|_{x=\hat{x}_j}$.

Likewise, according to the aforementioned event-driven mechanism and using (9), we have the worst event-driven disturbance formulated as

$$\nu^*(\hat{x}_j, t) = \omega^*(\hat{x}_j) = \frac{1}{2\gamma^2} k^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j}, \quad t \in [t_j, t_{j+1}). \tag{13}$$

Remark 2: From (12) and (13), we can find that $\mu^*(\hat{x}_j, t)$ and $\nu^*(\hat{x}_j, t)$ [or rather, $u^*(\hat{x}_j)$ and $\omega^*(\hat{x}_j)$] are in the essence of the discretized values of $u^*(x)$ in (8) and $\omega^*(x)$ in (9) at the triggering instant t_j, respectively. For convenience, we write $\mu^*(\hat{x}_j, t)$ and $\nu^*(\hat{x}_j, t)$ as $\mu^*(\hat{x}_j)$ and $\nu^*(\hat{x}_j)$ without emphasizing $t \in [t_j, t_{j+1})$ in subsequent discussion.

Let u and ω in (6) be replaced with $u^*(\hat{x}_j)$ in (12) and $\omega^*(\hat{x}_j)$ in (13), respectively. Then, at the triggering instants $t = t_j$, $j \in \{0, 1, 2, \ldots\}$, we have the event-driven HJIE formulated as

$$\begin{aligned}
&\left(V_x^*\right)^{\mathsf{T}} f(x) - \rho V^*(x) + Q(x) + \left(V_x^*\right)^{\mathsf{T}} g(x)\ell_b \\
&\quad - \beta\left(V_x^*\right)^{\mathsf{T}} g(x) \tanh\left(\frac{1}{2\beta} g^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j}\right) \\
&\quad + \frac{1}{2\gamma^2}\left(V_x^*\right)^{\mathsf{T}} k(x) k^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j} \\
&\quad + \mathcal{R}\left(-\beta \tanh\left(\frac{1}{2\beta} g^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j}\right) + \ell_b\right) \\
&\quad - \frac{1}{4\gamma^2}\left(V^*_{\hat{x}_j}\right)^{\mathsf{T}} k(\hat{x}_j) k^{\mathsf{T}}(\hat{x}_j) V^*_{\hat{x}_j} = 0.
\end{aligned} \tag{14}$$

B. Event-Triggering Condition and Zeno Behavior Analysis

Before solving the event-driven HJIE (14), we provide the event-triggering condition. First, we impose an assumption having the same feature as it used in [33]–[35].

Assumption 3: There exist two Lipschitz constants $K_{u^*} > 0$ and $K_{\omega^*} > 0$ such that for every $x, \hat{x}_j \in \Omega$

$$\begin{aligned}
\left\|u^*(x) - u^*(\hat{x}_j)\right\| &\leq K_{u^*}\|x - \hat{x}_j\| = K_{u^*}\|e_j(t)\| \\
\left\|\omega^*(x) - \omega^*(\hat{x}_j)\right\| &\leq K_{\omega^*}\|x - \hat{x}_j\| = K_{\omega^*}\|e_j(t)\|.
\end{aligned}$$

Since $\mu^*(\hat{x}_j, t) = u^*(\hat{x}_j)$ and $\nu^*(\hat{x}_j, t) = \omega^*(\hat{x}_j)$ (see (12) and (13), respectively), we further have

$$\begin{aligned}
\left\|u^*(x) - \mu^*(\hat{x}_j)\right\| &\leq K_{u^*}\|e_j(t)\| \\
\left\|\omega^*(x) - \nu^*(\hat{x}_j)\right\| &\leq K_{\omega^*}\|e_j(t)\|.
\end{aligned}$$

Theorem 1: Given that $V^*(x)$ is the solution of HJIE (10). Let Assumptions 1–3 hold. Then, $\mu^*(\hat{x}_j)$ in (12) can force system (1) with the worst disturbance $\nu^*(\hat{x}_j)$ in (13) to be stable in the sense of UUB if the triggering condition is set as

$$\|e_j(t)\|^2 \leq \frac{\left(1 - \eta^2\right)\lambda_{\min}(Q)}{2\left(K_{u^*}^2 + \gamma^2 K_{\omega^*}^2\right)}\|x(t)\|^2 \triangleq \bar{e}_T(t) \tag{15}$$

where $\gamma > 0$ is given in (4), $0 < \eta < 1$, and K_{u^*} and K_{ω^*} are Lipschitz constants given in Assumption 3, and $\bar{e}_T(t)$ is the event-triggering threshold.

Proof: Since $V^*(x)$ is the solution of HJIE (10), we can find $V^*(x) \geq 0$ (see [1]). Thus, we take $V^*(x)$ as the Lyapunov function candidate. Differentiating $V^*(x)$ along with the solution of $\dot{x} = f(x) + g(x)\mu^*(\hat{x}_j) + k(x)\nu^*(\hat{x}_j)$, it follows:

$$\begin{aligned}
\dot{V}^*(x) &= \left(V_x^*\right)^{\mathsf{T}}\left(f(x) + g(x)\mu^*(\hat{x}_j) + k(x)\nu^*(\hat{x}_j)\right) \\
&= \left(V_x^*\right)^{\mathsf{T}}\left(f(x) + g(x)u^*(x) + k(x)\omega^*(x)\right) \\
&\quad + \left(V_x^*\right)^{\mathsf{T}} g(x)\left(\mu^*(\hat{x}_j) - u^*(x)\right) \\
&\quad + \left(V_x^*\right)^{\mathsf{T}} k(x)\left(\nu^*(\hat{x}_j) - \omega^*(x)\right).
\end{aligned} \tag{16}$$

According to (6)–(9), there holds

$$\begin{cases}
\left(V_x^*\right)^{\mathsf{T}}(f(x) + g(x)u^*(x) + k(x)\omega^*(x)) \\
\qquad = \gamma^2\|\omega^*(x)\|^2 + \rho V^*(x) - Q(x) - \mathcal{R}(u^*(x)) \\
\left(V_x^*\right)^{\mathsf{T}} g(x) = -2\beta\left(\tanh^{-1}((u^*(x) - \ell_b)/\beta)\right)^{\mathsf{T}} \\
\left(V_x^*\right)^{\mathsf{T}} k(x) = 2\gamma^2(\omega^*(x))^{\mathsf{T}}.
\end{cases} \tag{17}$$

Then, using (17), we have (16) further developed as

$$\begin{aligned}\dot{V}^*(x) = &\ \underbrace{2\beta\Big(\tanh^{-1}\big((u^*(x)-\ell_b)/\beta\big)\Big)^{\mathsf{T}}\big(u^*(x)-\mu^*(\hat{x}_j)\big)}_{\Lambda_1}\\ &+\underbrace{2\gamma^2\big(\omega^*(x)\big)^{\mathsf{T}}\big(\nu^*(\hat{x}_j)-\omega^*(x)\big)}_{\Lambda_2}+\gamma^2\big\|\omega^*(x)\big\|^2\\ &+\rho V^*(x)-Q(x)-\mathcal{R}\big(u^*(x)\big). \end{aligned}\tag{18}$$

Applying Young's inequality $2c^{\mathsf{T}}d \le \|c\|^2+\|d\|^2$ (*note: c* and *d* are vectors having proper dimensions and the same below) to Λ_1 in (18) and using Assumption 3 as well as the second equation of (17), we find

$$\begin{aligned}\Lambda_1 &\le \Big\|\beta\tanh^{-1}\big((u^*(x)-\ell_b)/\beta\big)\Big\|^2+\big\|u^*(x)-\mu^*(\hat{x}_j)\big\|^2\\ &\le \frac{1}{4}\Big\|\big(V_x^*\big)^{\mathsf{T}}g(x)\Big\|^2+K_{u^*}^2\|e_j(t)\|^2.\end{aligned}$$

Likewise, using the above-mentioned Young's inequality and Assumption 3, we have

$$\begin{aligned}\Lambda_2 &\le \gamma^2\big\|\omega^*(x)\big\|^2+\gamma^2\big\|\nu^*(\hat{x}_j)-\omega^*(x)\big\|^2\\ &\le \gamma^2\big\|\omega^*(x)\big\|^2+\gamma^2K_{\omega^*}^2\|e_j(t)\|^2.\end{aligned}$$

Thus, (18) yields

$$\begin{aligned}\dot{V}^*(x) \le &-Q(x)-\mathcal{R}\big(u^*(x)\big)\\ &+\Big(K_{u^*}^2+\gamma^2K_{\omega^*}^2\Big)\|e_j(t)\|^2+\frac{1}{4}\Big\|\big(V_x^*\big)^{\mathsf{T}}g(x)\Big\|^2\\ &+2\gamma^2\big\|\omega^*(x)\big\|^2+\rho V^*(x).\end{aligned}\tag{19}$$

As stated in [2], $V^*(x)$ is continuously differentiable on Ω. Thus, both $V^*(x)$ and V_x^* are bounded on Ω. To facilitate later discussion, we denote $\max\{\|V^*(x)\|, \|V_x^*\|\} \le \delta_{V^*}$ with $\delta_{V^*} > 0$ being the constant. Then, using Assumption 2 and noticing the facts that $Q(x) = x^{\mathsf{T}}Qx \ge \lambda_{\min}(Q)\|x\|^2$ and $-\mathcal{R}(u^*(x)) \le 0$, we have (19) developed as

$$\begin{aligned}\dot{V}^*(x) \le &-\frac{(1+\eta^2)}{2}\lambda_{\min}(Q)\|x\|^2\\ &-\frac{(1-\eta^2)}{2}\lambda_{\min}(Q)\|x\|^2\\ &+\Big(K_{u^*}^2+\gamma^2K_{\omega^*}^2\Big)\|e_j(t)\|^2+a_0\end{aligned}\tag{20}$$

where

$$a_0 = (1/2)k_M^2\delta_{V^*}^2/\gamma^2+(1/4)g_M^2\delta_{V^*}^2+\rho\delta_{V^*}.$$

Letting (15) hold, we derive from (20) that

$$\dot{V}^*(x) \le -\frac{(1+\eta^2)}{2}\lambda_{\min}(Q)\|x\|^2+a_0.$$

Thus, $\dot{V}^*(x) < 0$ holds if $x \notin \Omega_x$ with Ω_x being defined as

$$\Omega_x = \left\{x : \|x\| \le \sqrt{\frac{2a_0}{(1+\eta^2)\lambda_{\min}(Q)}}\right\}.$$

Using the Lyapunov theorem extension [28], we obtain UUB of x with its ultimate bound being $\sqrt{2a_0/((1+\eta^2)\lambda_{\min}(Q))}$. That is, system (1) with the worst disturbance $\nu^*(\hat{x}_j)$ in (13) is guaranteed to be stable in the sense of UUB under the optimal control $\mu^*(\hat{x}_j)$ in (12). ■

Remark 3: Theorem 1 presents a triggering condition [that is, (15)] for designing the event-driven H_∞-constrained controller. As stated in Section I, the proposed triggering condition must have a non-negative event-triggering threshold. Apparently, the triggering threshold $\bar{e}_T(t)$ in (15) is non-negative under the condition $\eta \in (0, 1)$. In other words, the selection of the prescribed level of disturbance attenuation [i.e., γ in (4)] does not affect whether $\bar{e}_T(t)$ is positive or not. This characteristic is different from [19]–[26], which need to properly choose the prescribed level of disturbance attenuation [i.e., γ in (4)] to make the triggering threshold non-negative.

Next, we show that the Zeno behavior is excluded. According to [36, Theor. III.1], the Zeno behavior can be obviated if the minimal intersample time $\min_j\{T_j\}$ (*note:* $T_j = t_{j+1} - t_j$) is positive. Now, we prove $\min_j\{T_j\} > 0$ with the triggering condition (15). To facilitate later discussion, we introduce another assumption derived from Assumptions 1–3.

Assumption 4: There exist positive constants l_1, l_2, and c_0 such that, for all $x, \hat{x}_j \in \Omega$

$$\big\|f(x)+g(x)\mu^*(\hat{x}_j)+k(x)\nu^*(\hat{x}_j)\big\| \le l_1\|x\|+l_2\|e_j(t)\|+c_0.$$

Theorem 2: Consider system (1) with the optimal event-driven control $\mu^*(\hat{x}_j)$ in (12) and the worst disturbance $\nu^*(\hat{x}_j)$ in (13). Suppose that Assumption 4 holds and the triggering condition is set as (15). Then, there holds $\min_j\{T_j\} > 0$, where $T_j = t_{j+1} - t_j$, $j \in \{0, 1, 2, \ldots\}$.

Proof: See Appendix A. ■

C. ACL for Solving the Event-Driven HJIE

Within the framework of ACL, the event-driven HJIE (14) will be solved in this section through a single critic network. According to [37], $V^*(x)$ can be restated via a neural network over Ω as

$$V^*(x) = W_c^{\mathsf{T}}\sigma_c(x)+\varepsilon_c(x)\tag{21}$$

where $W_c \in \mathbb{R}^{\tilde{n}_c}$ is the ideal weight vector to be determined, $\tilde{n}_c \in \mathbb{R}$ is the number of neurons, $\sigma_c(x) = [\sigma_{c1}(x), \sigma_{c2}(x), \ldots, \sigma_{c\tilde{n}_c}(x)]^{\mathsf{T}} \in \mathbb{R}^{\tilde{n}_c}$ is the activation function vector with $\sigma_{c1}(x), \sigma_{c2}(x), \ldots, \sigma_{c\tilde{n}_c}(x)$ being linearly independent (*note:* $\sigma_{ci}(x) \in C^1(\Omega)$ and $\sigma_{ci}(0) = 0$, $i = 1, 2, \ldots, \tilde{n}_c$), and $\varepsilon(x) \in \mathbb{R}$ is the approximation error.

The derivative of $V^*(x)$ in (21) at the sampled state $\hat{x}_j$ is

$$V_{\hat{x}_j}^* = \frac{\partial V^*(x)}{\partial x}\bigg|_{x=\hat{x}_j} = \nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)W_c+\nabla\varepsilon_c(\hat{x}_j)\tag{22}$$

where $\nabla\mathcal{C}(\hat{x}_j) = (\partial\mathcal{C}(x)/\partial x)|_{x=\hat{x}_j}$ with $\mathcal{C}(\cdot) = \sigma_c(\cdot)$ or $\varepsilon_c(\cdot)$.

Using (22), we can rewrite the optimal event-driven control $\mu^*(\hat{x}_j)$ in (12) as (*note:* $t \in [t_j, t_{j+1})$)

$$\mu^*(\hat{x}_j) = -\beta\tanh\big(\mathcal{A}_1(\hat{x}_j)\big)+\varepsilon_{\mu^*}(\hat{x}_j)+\ell_b\tag{23}$$

where

$$\mathcal{A}_1(\hat{x}_j) = \frac{1}{2\beta}g^{\mathsf{T}}(\hat{x}_j)\nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)W_c$$

$$\varepsilon_{\mu^*}(\hat{x}_j) = -\frac{1}{2}\big(I_m - \mathscr{C}(h(\hat{x}_j))\big)g^{\mathsf{T}}(\hat{x}_j)\nabla\varepsilon_c(\hat{x}_j)$$

with I_m being the $m \times m$ identity matrix and $\mathscr{C}(h(\hat{x}_j)) = \mathrm{diag}\{\tanh^2(h_1(\hat{x}_j)), \tanh^2(h_2(\hat{x}_j)), \ldots, \tanh^2(h_m(\hat{x}_j))\}$. Here, $h(\hat{x}_j) = [h_1(\hat{x}_j), h_2(\hat{x}_j), \ldots, h_m(\hat{x}_j)]^{\mathsf{T}} \in \mathbb{R}^m$ with $h(\hat{x}_j) \in \mathbb{R}^m$ being chosen between $(1/(2\beta))g^{\mathsf{T}}(\hat{x}_j)\nabla V^*(\hat{x}_j)$ and $\mathcal{A}_1(\hat{x}_j)$.

Likewise, utilizing (22), we can restate the worst event-driven disturbance $\nu^*(\hat{x}_j)$ in (13) as (*note:* $t \in [t_j, t_{j+1})$)

$$\nu^*(\hat{x}_j) = \frac{1}{2\gamma^2}k^{\mathsf{T}}(\hat{x}_j)\nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)W_c + \varepsilon_{\nu^*}(\hat{x}_j) \tag{24}$$

where $\varepsilon_{\nu^*}(\hat{x}_j) = k^{\mathsf{T}}(\hat{x}_j)\nabla\varepsilon_c(\hat{x}_j)/(2\gamma^2)$.

Because W_c is to be determined (or rather, unavailable), we cannot implement $\mu^*(\hat{x}_j)$ in (23) and $\nu^*(\hat{x}_j)$ in (24). To cope with this problem, we consider the approximate value of $V^*(x)$, denoted by $\hat{V}^*(x)$, which is regarded as the output of the critic network, that is

$$\hat{V}(x) = \hat{W}_c^{\mathsf{T}}\sigma_c(x) \tag{25}$$

with $\hat{W}_c$ being the weight vector applied to estimate the ideal/unavailable weight vector W_c.

Using (25) and taking the same procedure of deriving $\mu^*(\hat{x}_j)$ in (23), we obtain the estimated value of $\mu^*(\hat{x}_j)$ as

$$\hat{\mu}(\hat{x}_j) = -\beta\tanh\big(\mathcal{A}_2(\hat{x}_j)\big) + \ell_b, \quad t \in [t_j, t_{j+1}) \tag{26}$$

where

$$\mathcal{A}_2(\hat{x}_j) = \frac{1}{2\beta}g^{\mathsf{T}}(\hat{x}_j)\nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)\hat{W}_c.$$

Similarly, we have the estimated value of $\nu^*(\hat{x}_j)$ in (24) formulated as

$$\hat{\nu}(\hat{x}_j) = \frac{1}{2\gamma^2}k^{\mathsf{T}}(\hat{x}_j)\nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)\hat{W}_c, \quad t \in [t_j, t_{j+1}). \tag{27}$$

On the other hand, using (8) and (9), we derive from (6) that

$$H\big(x, V_x^*, u^*(x), \omega^*(x)\big) = 0. \tag{28}$$

Replacing V_x^*, $u^*(x)$, and $\omega^*(x)$ in (28) with $\hat{V}_x$, $\hat{\mu}(\hat{x}_j)$, and $\hat{\nu}(\hat{x}_j)$, respectively, we are able to present the approximate Hamiltonian as (*note:* $t \in [t_j, t_{j+1})$)

$$\begin{aligned}\hat{H}\Big(x, \hat{V}_x, \hat{\mu}(\hat{x}_j), \hat{\nu}(\hat{x}_j)\Big) &= \hat{W}_c^{\mathsf{T}}\nabla\sigma_c(x)\big(f(x) + g(x)\hat{\mu}(\hat{x}_j) + k(x)\hat{\nu}(\hat{x}_j)\big)\\ &\quad - \rho\hat{W}_c^{\mathsf{T}}\sigma_c(x) + Q(x) + \mathcal{R}\big(\hat{\mu}(\hat{x}_j)\big) - \gamma^2\big\|\hat{\nu}(\hat{x}_j)\big\|^2.\end{aligned}$$

Then, like [19], we describe the gap between $\hat{H}(x, \hat{V}_x, \hat{\mu}(\hat{x}_j), \hat{\nu}(\hat{x}_j))$ and $H(x, V_x^*, u^*(x), \omega^*(x))$ through an error function

$$\begin{aligned}e_c &= \hat{H}\Big(x, \hat{V}_x, \hat{\mu}(\hat{x}_j), \hat{\nu}(\hat{x}_j)\Big) - H\big(x, V_x^*, u^*(x), \omega^*(x)\big)\\ &= \hat{W}_c^{\mathsf{T}}\phi + Q(x) + \mathcal{R}\big(\hat{\mu}(\hat{x}_j)\big) - \gamma^2\big\|\hat{\nu}(\hat{x}_j)\big\|^2\end{aligned} \tag{29}$$

where

$$\phi = \nabla\sigma_c(x)\big(f(x) + g(x)\hat{\mu}(\hat{x}_j) + k(x)\hat{\nu}(\hat{x}_j)\big) - \rho\sigma_c(x).$$

To make $e_c \to 0$, a feasible way is to tune $\hat{W}_c$ in (29). Generally, the tuning rule for $\hat{W}_c$ is obtained by applying the gradient descent method to the goal function $(1/2)e_c^{\mathsf{T}}e_c$. For purposes of improving the efficiency in using the historical state data and obviating the difficulty in checking the PE condition, we introduce a novel objective function as follows:

$$E = \frac{(1/2)e_c^{\mathsf{T}}e_c}{\big(1 + \phi^{\mathsf{T}}\phi\big)^2} + \sum_{p=1}^{\mathcal{N}_0}\frac{(1/2)e_{c_{(p)}}^{\mathsf{T}}e_{c_{(p)}}}{\Big(1 + \phi_{(p)}^{\mathsf{T}}\phi_{(p)}\Big)^2} \tag{30}$$

where $p \in \{1, 2, \ldots, \mathcal{N}_0\}$ (*note:* $\mathcal{N}_0 \geq \tilde{n}_c$ with $\tilde{n}_c$ denoting the number of neurons used in the critic network) is the index applied to mark the historical state data $x(t_p)$ ($t_p \in [t_j, t_{j+1})$), $e_{c_{(p)}} = e_c(x(t_p))$, and $\phi_{(p)} = \phi(x(t_p))$, that is

$$\begin{aligned}e_{c_{(p)}} &= \hat{W}_c^{\mathsf{T}}\phi_{(p)} + Q(x(t_p)) + \mathcal{R}\big(\hat{\mu}(\hat{x}_j)\big) - \gamma^2\big\|\hat{\nu}(\hat{x}_j)\big\|^2\\ \phi_{(p)} &= \nabla\sigma_c(x(t_p))\big(f(x(t_p)) + g(x(t_p))\hat{\mu}(\hat{x}_j)\\ &\qquad + k(x(t_p))\hat{\nu}(\hat{x}_j)\big) - \rho\sigma_c(x(t_p)).\end{aligned}$$

Here, $(1 + \phi^{\mathsf{T}}\phi)^{-2}$ and $(1 + \phi_{(p)}^{\mathsf{T}}\phi_{(p)})^{-2}$ are the normalization terms. We apply the gradient descent method to E in (30) and then obtain that $\hat{W}_c$ is updated via (*note:* $t \in [t_j, t_{j+1})$)

$$\begin{aligned}\dot{\hat{W}}_c &= -l_c\frac{\partial E}{\partial\hat{W}_c}\\ &= -l_c\frac{\phi e_c}{\big(1 + \phi^{\mathsf{T}}\phi\big)^2} - \sum_{p=1}^{\mathcal{N}_0}l_c\frac{\phi_{(p)}e_{c_{(p)}}}{\Big(1 + \phi_{(p)}^{\mathsf{T}}\phi_{(p)}\Big)^2}\end{aligned} \tag{31}$$

with $l_c > 0$ being the design parameter.

We denote

$$\varphi = \phi\Big/\Big(1 + \phi^{\mathsf{T}}\phi\Big) \quad \text{and} \quad \varphi_{(p)} = \phi_{(p)}\Big/\Big(1 + \phi_{(p)}^{\mathsf{T}}\phi_{(p)}\Big)$$

and define the weight estimation error as $\tilde{W}_c = W_c - \hat{W}_c$. Then, from (31), we have

$$\begin{aligned}\dot{\tilde{W}}_c &= -l_c\left(\varphi\varphi^{\mathsf{T}} + \sum_{p=1}^{\mathcal{N}_0}\varphi_{(p)}\varphi_{(p)}^{\mathsf{T}}\right)\tilde{W}_c\\ &\quad + \frac{l_c\varphi\varepsilon_H}{1 + \phi^{\mathsf{T}}\phi} + \sum_{p=1}^{\mathcal{N}_0}\frac{l_c\varphi_{(p)}\varepsilon_{H_{(p)}}}{1 + \phi_{(p)}^{\mathsf{T}}\phi_{(p)}}, \quad t \in [t_j, t_{j+1})\end{aligned} \tag{32}$$

where $\varepsilon_H = -\nabla\varepsilon_c^{\mathsf{T}}(x)(f(x) + g(x)\hat{\mu}(\hat{x}_j) + k(x)\hat{\nu}(\hat{x}_j)) + \rho\varepsilon_c(x)$ and $\varepsilon_{H_{(p)}} = -\nabla\varepsilon_c^{\mathsf{T}}(x(t_p))(f(x(t_p)) + g(x(t_p))\hat{\mu}(\hat{x}_j) + k(x(t_p))\hat{\nu}(\hat{x}_j)) + \rho\varepsilon_c(x(t_p))$ are residual errors (*note:* due to the process of deriving ε_H and $\varepsilon_{H_{(p)}}$ similar to [38], here we omit it in order to avoid redundancy).

Remark 4: The expression (31) indicates that the historical states $x(t_1), x(t_2), \ldots, x(t_{\mathcal{N}_0})$ and the concurrent state $x(t)$ are utilized to tune the weight vector $\hat{W}_c$. Chowdhary [39] coined this characteristic concurrent learning. Recently, due to experience replay sharing nearly the same feature as concurrent learning, they are regarded as synonyms (see [40]–[42]). Similar to [40], the tuning rule (31) can force $\tilde{W}_c$ to converge to a small neighborhood of zero without requiring the PE condition only if

$$\operatorname{rank}\mathcal{E} = \tilde{n}_c \tag{33}$$

where $\mathcal{E}$ is the set consisting of $\mathcal{N}_0$ historical states, that is

$$\mathcal{E} = \big\{\sigma_c(x(t_1)), \sigma_c(x(t_2)), \ldots, \sigma_c(x(t_{\tilde{n}_c})), \ldots, \sigma_c(x(t_{\mathcal{N}_0}))\big\}.$$

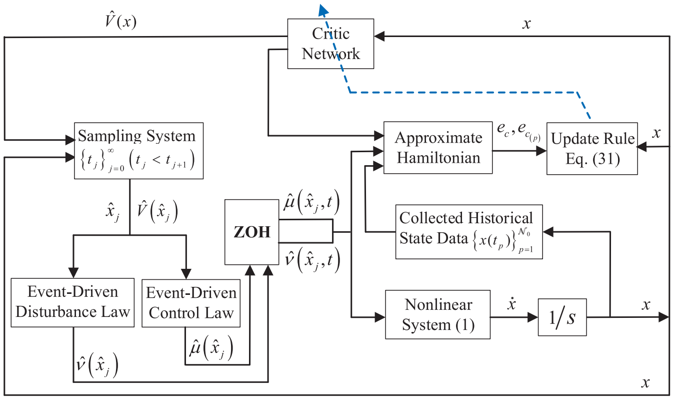

Fig. 1. Block diagram of the present event-driven H_∞-constrained control strategy (*note:* "ZOH" means "zero-order hold").

Apparently, (33) holds only when a sufficiently large number of historical state data (i.e., $\mathcal{N}_0$) is collected.

To elaborate the present event-driven H_∞-constrained control strategy, we provide a block diagram as displayed in Fig. 1.

D. Stability Analysis of the Closed-Loop System

Before proceeding further, we present an assumption sharing the same feature as it utilized in [43]–[46].

Assumption 5: For every $x \in \Omega$, $\|\nabla\sigma_c(x)\| \le b_{\sigma_c}$ with $b_{\sigma_c} > 0$ the known constant. Meanwhile, there have $\|\varepsilon_{\mu^*}(x)\| \le b_{\varepsilon_{\mu^*}}$, $\|\varepsilon_{\nu^*}(x)\| \le b_{\varepsilon_{\nu^*}}$, and $\|\varepsilon_H\| \le b_{\varepsilon_H}$, where $b_{\varepsilon_{\mu^*}} > 0$, $b_{\varepsilon_{\nu^*}} > 0$, and $b_{\varepsilon_H} > 0$ are known constants.

Theorem 3: Consider system (1) with the event-driven control (26) and the event-driven disturbance (27). Suppose that Assumptions 1–5 and the condition (33) hold. Meanwhile, let the initial control policy for system (1) be admissible and let the tuning rule for $\hat{W}_c$ be constructed as (31). Then, the closed-loop system (1) and the weight estimation error $\tilde{W}_c$ are stable in the sense of UUB as long as the triggering condition is set as (15) and the following inequality holds:

$$l_c\lambda_{\min}\big(\Phi(\varphi,\varphi_{(p)})\big) - 2k_M^2 b_{\sigma_c}^2/\gamma^2 > 0 \tag{34}$$

where

$$\Phi(\varphi,\varphi_{(p)}) = \varphi\varphi^{\mathsf{T}} + \sum_{p=1}^{\mathcal{N}_0}\varphi_{(p)}\varphi_{(p)}^{\mathsf{T}}. \tag{35}$$

Proof: See Appendix B. ■

IV. Simulation Study

We consider the CT nonlinear system with a disturbance described by the equations as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_1 + x_2\\ -0.5(x_1+x_2) + 0.5x_2\sin^2(x_1)\end{bmatrix} + \begin{bmatrix}0\\ \sin(x_1)\end{bmatrix}u + \begin{bmatrix}1\\ 0\end{bmatrix}\omega \tag{36}$$

where $x = [x_1, x_2]^{\mathsf{T}} \in \mathbb{R}^2$ with $x_0 = x(0) = [0.5, -0.5]^{\mathsf{T}}$, $u \in \mathcal{U} = \{u \in \mathbb{R} : -2 \le u \le 3\}$ (i.e., $u_{\min} = -2$ and $u_{\max} = 3$), and $\omega \in \mathbb{R}$. From (36), we can see that $g(x) = [0, \sin(x_1)]^{\mathsf{T}}$, which implies $g(0) = 0$. Meanwhile, for every $x \in \mathbb{R}^2$, $g(x) = [0, \sin(x_1)]^{\mathsf{T}}$ (*note:* $\|\sin(x_1)\| \le 1$ for $x_1 \in \mathbb{R}$) and $k(x) = [1, 0]^{\mathsf{T}}$ are bounded. Thus, Assumption 2 is satisfied.

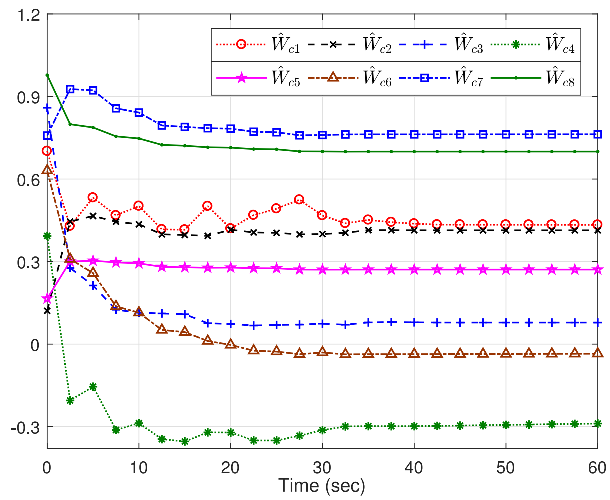

Fig. 2. Convergence of $\hat{W}_c = [\hat{W}_{c1}, \hat{W}_{c2}, \ldots, \hat{W}_{c8}]^{\mathsf{T}}$.

Based on (2), if letting Q be the 2×2 identity matrix, then we can write the fictitious output for system (36) as

$$\|z\|^2 = x_1^2 + x_2^2 + \mathcal{R}(u)$$

where [*note:* according to (3), $\beta = 2.5$ and $b = 0.5$]

$$\begin{aligned}\mathcal{R}(u) &= 2\int_b^u \beta\tanh^{-1}((\tau - b)/\beta)d\tau\\ &= 2\beta(u-b)\tanh^{-1}((u-b)/\beta)\\ &\quad + \beta^2\ln\Big(1-(u-b)^2/\beta^2\Big).\end{aligned}$$

The discount factor is presented as $\rho = 0.85$. It is desirable to solve the event-driven H_∞-constrained control problem corresponding to system (36) with $\gamma = 1$ [or rather, to solve the event-driven HJIE (14)]. To this end, we first set parameters utilized in the triggering condition (15) as follows: $\eta = \sqrt{2}/2$, $K_{u^*} = 2.5$, and $K_{\omega^*} = 2.5$. Then, we use the critic network (25) to acquire the approximate solution of (14). Specifically, we choose $\sigma_c(x)$ in (25) to be (*note:* $\tilde{n}_c = 8$)

$$\sigma_c(x) = \Big[x_1^2, x_2^2, x_1x_2, x_1^4, x_2^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3\Big]^{\mathsf{T}}$$

and write $\hat{W}_c$ in (25) as $\hat{W}_c = [\hat{W}_{c1}, \hat{W}_{c2}, \ldots, \hat{W}_{c8}]^{\mathsf{T}}$. To make the initial control policy for system (36) admissible, we present the initial weight vector as $\hat{W}_c^{\text{initial}} = [0.701, 0.121, 0.859, 0.393, 0.166, 0.631, 0.758, 0.978]^{\mathsf{T}}$ (*note:* because there is no general way to obtain the initial admissible control policy, here we derive $\hat{W}_c^{\text{initial}}$ via trial and error). Meanwhile, we set the parameters in (31) as $l_c = 0.9$ and $\mathcal{N}_0 = 10$ in order to update the weight vector $\hat{W}_c$.

We carry out the simulation via the MATLAB (2017a) software package and obtain simulation results shown in Figs. 2–8. Fig. 2 presents the convergence of the weight vector $\hat{W}_c$. As displayed in Fig. 2, $\hat{W}_c$ is convergent after the first 45 s with its converged value $\hat{W}^{\text{converged}} =$

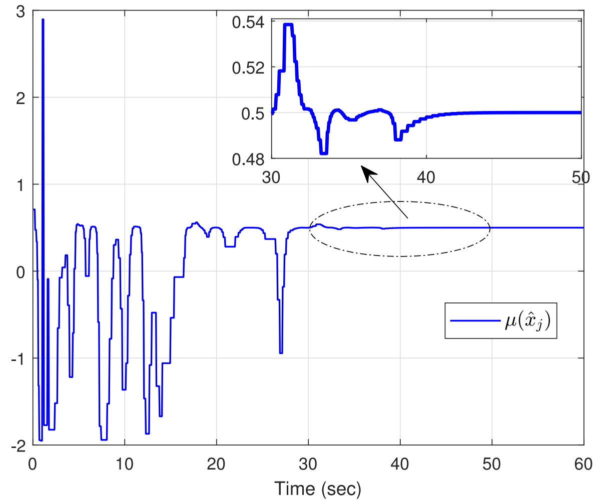

Fig. 3. Approximate optimal event-driven control $\mu(\hat{x}_j)$.

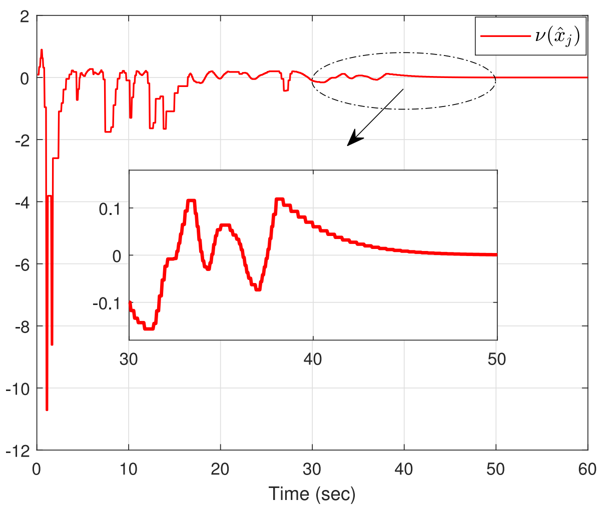

Fig. 4. Approximate worst event-driven disturbance $\nu(\hat{x}_j)$.

$[0.4335, 0.4136, 0.0784, -0.2886, 0.2710, -0.0348, 0.7627, 0.6999]^{\mathsf{T}}$. Figs. 3 and 4 illustrate the approximate optimal event-driven control $\mu(\hat{x}_j)$ and the approximate worst event-driven disturbance $\nu(\hat{x}_j)$, respectively. From Fig. 3, we can see that $\mu(\hat{x}_j)$ does not override the asymmetric input bounds. Thus, the difficulty of asymmetric control constraints is overcome. It also can be observed from Fig. 3 that $\mu(\hat{x}_j)$ does not converge to zero (actually, $\mu(\hat{x}_j)$ converges to 0.5). This feature is in accordance with Remark 1 2). Fig. 5(a) depicts the trajectories of system (36), that is, $x_1(t)$ and $x_2(t)$, and Fig. 5(b) shows that the norm of the error function (i.e., $\|e_j(t)\|$) and the square root of the event-triggering threshold (i.e., $\sqrt{\bar{e}_T(t)}$). According to Fig. 5(a) and (b), we can find both $\|e_j(t)\|$ and $\sqrt{\bar{e}_T(t)}$ approximate zero when the states $x_1(t)$ and $x_2(t)$ go to zero. Fig. 6 describes the intersample time T_j (*note:* $T_j = t_{j+1} - t_j$). Clearly, $\min_j\{T_j\} = 0.1 > 0$, which validates Theorem 2. Thus, the Zeno behavior is excluded.

Inserting the weight vector $\hat{W}^{\text{converged}}$ into (26), we can derive the approximate optimal event-driven control. Let the

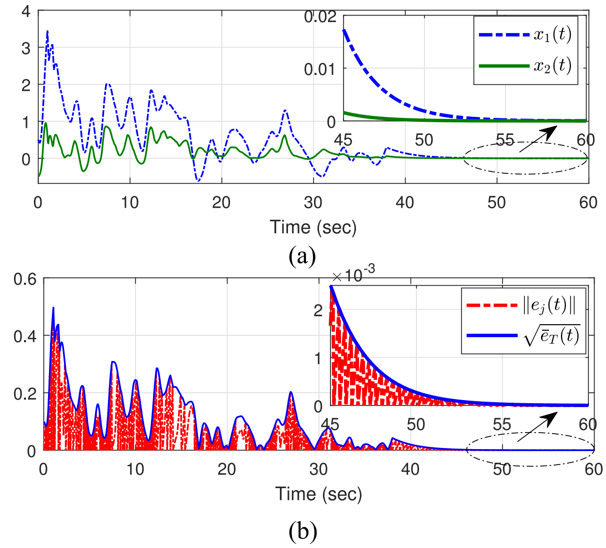

Fig. 5. (a) Trajectories of system (36) under $\mu(\hat{x}_j)$ and $\nu(\hat{x}_j)$, that is, $x_1(t)$ and $x_2(t)$. (b) Norm of the error function (i.e., $\|e_j(t)\|$) and the square root of the event-triggering threshold (i.e., $\sqrt{\bar{e}_T(t)}$).

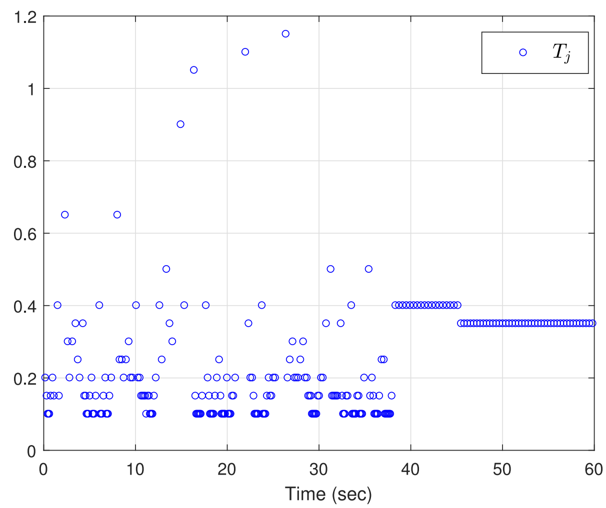

Fig. 6. Intersample time T_j (*note:* $T_j = t_{j+1} - t_j$).

disturbance signal be given in the form

$$\omega(t) = \begin{cases} 12r_1 e^{-0.25(t-t_0)} \cos(t - t_0), & t \geq t_0 \\ 0, & t < t_0. \end{cases} \tag{37}$$

with r_1 being randomly selected within the interval [0, 1] and $t_0 = 5$ s. Then, when considering that system (36) is at rest and suffering from the disturbance $\omega(t)$ given in (37), we obtain the closed-loop system's states $x_1(t)$ and $x_2(t)$ shown in Fig. 7(a). Meanwhile, the event-driven control $\mu(\hat{x}_j)$ for the closed-loop system is described as Fig. 7(b). Let the ratio of the disturbance attenuation be defined as

$$\gamma_d = \left(\frac{\int_t^\infty e^{-\rho(\tau-t)}\big(x^{\mathsf{T}}(\tau)Qx(\tau) + \mathcal{R}(u(\tau))\big)d\tau}{\int_t^\infty e^{-\rho(\tau-t)}\|\omega(\tau)\|^2 d\tau} \right)^{1/2}.$$

Then, the evolution of disturbance attenuation γ_d is displayed as Fig. 8. That is, after the first 20 s, γ_d converges to 0.4438, that is, $\gamma_d^{\text{converged}} = 0.4438$ ($< \gamma = 1$).

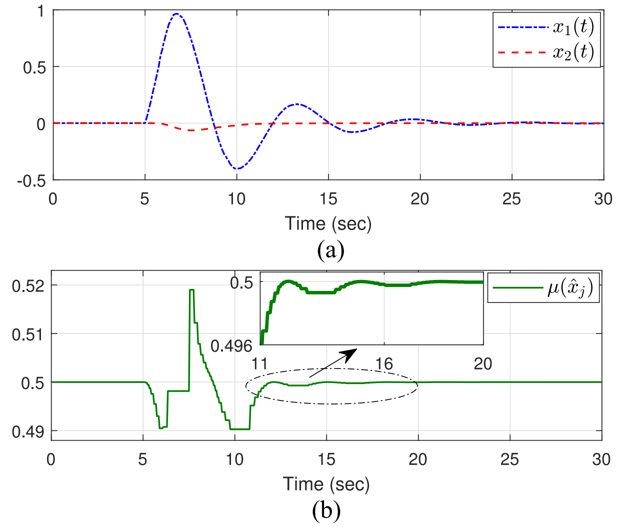

Fig. 7. (a) States $x_1(t)$ and $x_2(t)$ under the disturbance $\omega(t)$ given in (37). (b) Event-driven control $\mu(\hat{x}_j)$ for the closed-loop system.

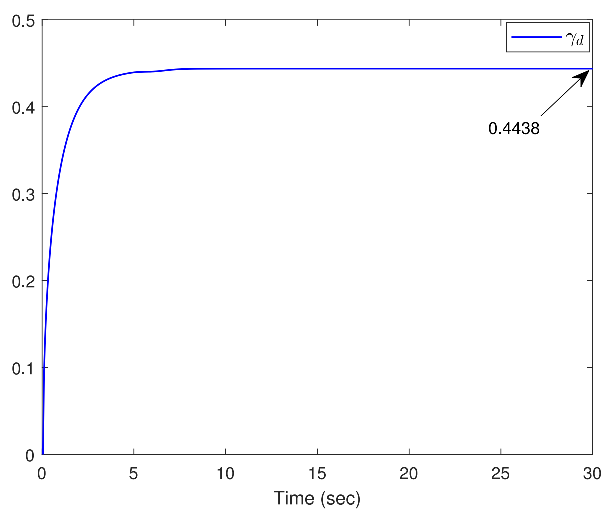

Fig. 8. Evolution of the disturbance attenuation γ_d.

Therefore, the prescribed L_2-gain performance level γ can be achieved under the obtained event-driven control policy $\mu(\hat{x}_j)$.

To elaborate that the computational load is reduced, we make comparisons between the present event-driven ACL and the ADP approach developed in [30] (see Table I). From Table I, we can see that the implementation of the event-driven ACL uses only 257 sampling states, while the implementation of the ADP method proposed in [30] needs 1200 sampling states. Thus, the present event-driven ACL has a higher efficiency in using sampling state data. Actually, using Table I and making some calculations, we find that the computational load is reduced up to 75.01%. This verifies that the computational load is remarkably decreased when using the present event-driven ACL. Furthermore, in order to illustrate that tuning both the control policy and the disturbance policy in the event-driven mechanism has better performance than only updating the control policy in the event-driven mechanism, we provide comparisons in Table II. According to

TABLE I
COMPARISON OF THE COMPUTATIONAL LOAD BETWEEN THE PRESENT EVENT-DRIVEN ACL AND THE ADP APPROACH DEVELOPED IN [30]

Method		ADP Used in [30]	Event-Driven ACL
Sampling States		1200	257
Number of Additions and Multiplications at Each Sampling State for Control Policy and Disturbance Policy	Controller Update	43	43
	Disturbance Update	41	41
	Triggering Condition	0	14
Total Number of Computation		100800	25186

TABLE II
COMPUTATIONAL LOADS OF ACL WITH TIME-DRIVEN DISTURBANCE AND ACL WITH EVENT-DRIVEN DISTURBANCE (*Note:* THE TWO CASES USE THE EVENT-DRIVEN CONTROL POLICY)

Method	Total Number of Computation (Including Number of Additions and Multiplications for Control and Disturbance at Each Sampling State)
ACL With Event-Driven Control and *Time-Driven* Disturbance	63336
ACL With Event-Driven Control and *Event-Driven* Disturbance	25186

Table II, after performing some calculations, we find that the computational load is reduced up to 60.23% when using ACL with both the event-driven control and the event-driven disturbance.

V. CONCLUSION

An event-driven H_∞ control strategy has been developed for CT nonlinear systems with asymmetric input constraints. The proposed H_∞ control scheme uses historical and instantaneous state data simultaneously to update both the control policy and the disturbance in an event-driven mechanism. Advantages of such an H_∞ control strategy lie in that it not only relaxes the PE condition but also brings down the computational load. A precondition of implementing the present event-driven H_∞ control strategy is that the control matrix of system (1) should satisfy $g(0) = 0$ (see Assumption 2). This condition excludes those systems with the control matrix $g(0) \neq 0$, which is a limitation. As stated in Remark 1 2), the purpose of presenting the condition $g(0) = 0$ is to make the equilibrium point to zero. Thus, in order to remove the condition $g(0) = 0$, we can consider the case that the equilibrium point is nonzero. On the other hand, it is often unable to acquire the information of controlled systems in real-world applications, let alone the knowledge of its control matrix. Therefore, our consecutive study tends to design event-driven H_∞ controllers for unknown nonlinear systems having nonzero equilibrium points as well as suffering from asymmetric input constraints.

Appendix A
Proof of Theorem 2

Based on Assumption 4, we have that system (1) with $\mu^*(\hat{x}_j)$ in (12) and $\nu^*(\hat{x}_j)$ in (13) yields

$$\|\dot{x}\| = \left\|f(x) + g(x)\mu^*(\hat{x}_j) + k(x)\nu^*(\hat{x}_j)\right\| \le l_1\|x\| + l_2\|e_j(t)\| + c_0. \tag{38}$$

Since (11) yields $\dot{x}(t) = -\dot{e}_j(t)$ and $x(t) = \hat{x}_j - e_j(t)$ for $t \in [t_j, t_{j+1})$, we can obtain from (38) that (*note:* $t \in [t_j, t_{j+1})$)

$$\|\dot{e}_j(t)\| \le (l_1 + l_2)\|e_j(t)\| + l_1\left\|\hat{x}_j\right\| + c_0. \tag{39}$$

Using the fact that $e_j(t_j) = 0$ ($j \in \{0, 1, 2, \ldots\}$) and the comparison lemma [47, Lemma 3.4], we have the solution of (39) satisfied (*note:* $t \in [t_j, t_{j+1})$)

$$\|e_j(t)\| \le \frac{l_1\left\|\hat{x}_j\right\| + c_0}{l_1 + l_2}\left(e^{(l_1+l_2)(t-t_j)} - 1\right). \tag{40}$$

Clearly, the next triggering instant t_{j+1} is released only when the square of the right-hand side of (40) violates (or is larger than) the event-triggering threshold $\bar{e}_T(t)$ given in (15). Therefore, there holds

$$\frac{l_1\left\|\hat{x}_j\right\| + c_0}{l_1 + l_2}\left(e^{(l_1+l_2)(t_{j+1}-t_j)} - 1\right) > \sqrt{\bar{e}_T(t_{j+1})} = \sqrt{\frac{(1-\eta^2)\lambda_{\min}(Q)}{2\left(K_{u^*}^2 + \gamma^2 K_{\omega^*}^2\right)}\|x(t_{j+1})\|^2}. \tag{41}$$

According to Theorem 1, the state x is stable in the sense of UUB. Thus, we can conclude that $\|x(t_{j+1})\| \ne 0$ at the triggering instant t_{j+1}. In other words, $\sqrt{\bar{e}_T(t_{j+1})} > 0$. Then, (41) yields

$$T_j = t_{j+1} - t_j > \frac{1}{l_1 + l_2}\ln\left(1 + \pi_j\right) \tag{42}$$

where

$$\pi_j = \frac{l_1 + l_2}{l_1\left\|\hat{x}_j\right\| + c_0}\sqrt{\bar{e}_T(t_{j+1})} > 0.$$

Taking the minimum values over both sides of (42), we thus obtain

$$\min_j\{T_j\} > \frac{1}{l_1 + l_2}\ln\left(1 + \min_j\left\{\pi_j\right\}\right) > 0.$$

This completes the proof.

Appendix B
Proof of Theorem 3

Noticing that the closed-loop system (1) contains the states $x(t)$ and $\hat{x}_j$ as well as $\tilde{W}_c$ (*note:* $\hat{W}_c = W_c - \tilde{W}_c$), we let the Lyapunov function candidate be

$$\mathcal{L}(t) = V^*(\hat{x}_j) + \underbrace{V^*(x(t))}_{\mathcal{L}_1(t)} + \underbrace{(1/2)\tilde{W}_c^{\mathsf{T}}\tilde{W}_c}_{\mathcal{L}_2(t)}. \tag{43}$$

According to the characteristic of (43), we partition the discussion of stability into two aspects.

Situation I: Let events not be triggered, that is, $t \in [t_j, t_{j+1})$, $j \in \{0, 1, 2, \ldots\}$. Then, it follows that $\dot{V}^*(\hat{x}_j) = 0$. Differentiating $\mathcal{L}_1(t)$ in (43) along with the trajectory generated from the dynamical system $\dot{x} = f(x) + g(x)\hat{\mu}(\hat{x}_j) + k(x)\hat{\nu}(\hat{x}_j)$, we find

$$\begin{aligned}\dot{\mathcal{L}}_1(t) &= \left(\nabla V^*(x)\right)^{\mathsf{T}}\left(f(x) + g(x)\hat{\mu}(\hat{x}_j) + k(x)\hat{\nu}(\hat{x}_j)\right)\\ &= \left(\nabla V^*(x)\right)^{\mathsf{T}}\left(f(x) + g(x)u^*(x) + k(x)\omega^*(x)\right)\\ &\quad + \left(\nabla V^*(x)\right)^{\mathsf{T}}g(x)\left(\hat{\mu}(\hat{x}_j) - u^*(x)\right)\\ &\quad + \left(\nabla V^*(x)\right)^{\mathsf{T}}k(x)\left(\hat{\nu}(\hat{x}_j) - \omega^*(x)\right).\end{aligned} \tag{44}$$

Inserting (17) into (44), we have

$$\begin{aligned}\dot{\mathcal{L}}_1(t) &= \underbrace{2\beta\left(\tanh^{-1}\left((u^*(x) - \ell_b)/\beta\right)\right)^{\mathsf{T}}\left(u^*(x) - \hat{\mu}(\hat{x}_j)\right)}_{\Theta_1}\\ &\quad + \underbrace{2\gamma^2\left(\omega^*(x)\right)^{\mathsf{T}}\left(\hat{\nu}(\hat{x}_j) - \omega^*(x)\right)}_{\Theta_2} + \rho V^*(x)\\ &\quad + \gamma^2\left\|\omega^*(x)\right\|^2 - Q(x) - \mathcal{R}\left(u^*(x)\right).\end{aligned} \tag{45}$$

By using Young's inequality (i.e., $2c^{\mathsf{T}}d \le \|c\|^2 + \|d\|^2$), we have Θ_1 and Θ_2 in (45) satisfied

$$\begin{aligned}\Theta_1 &\le \left\|\beta\tanh^{-1}\left((u^*(x) - \ell_b)/\beta\right)\right\|^2 + \left\|u^*(x) - \hat{\mu}(\hat{x}_j)\right\|^2\\ &= \frac{1}{4}\left\|\left(V_x^*\right)^{\mathsf{T}}g(x)\right\|^2 + \left\|u^*(x) - \hat{\mu}(\hat{x}_j)\right\|^2\\ \Theta_2 &\le \gamma^2\left\|\omega^*(x)\right\|^2 + \gamma^2\left\|\hat{\nu}(\hat{x}_j) - \omega^*(x)\right\|^2.\end{aligned}$$

Thus, (45) yields

$$\begin{aligned}\dot{\mathcal{L}}_1(t) &\le -Q(x) - \mathcal{R}\left(u^*(x)\right) + \rho V^*(x)\\ &\quad + 2\gamma^2\left\|\omega^*(x)\right\|^2 + \frac{1}{4}\left\|\left(V_x^*\right)^{\mathsf{T}}g(x)\right\|^2\\ &\quad + \underbrace{\left\|u^*(x) - \hat{\mu}(\hat{x}_j)\right\|^2}_{\Sigma_1} + \gamma^2\underbrace{\left\|\omega^*(x) - \hat{\nu}(\hat{x}_j)\right\|^2}_{\Sigma_2}.\end{aligned} \tag{46}$$

Letting Young's inequality $\|c + d\|^2 \le 2\|c\|^2 + 2\|d\|^2$ be applied to Σ_1 in (46) and using (23) and (26), as well as Assumption 3, we obtain

$$\begin{aligned}\Sigma_1 &= \left\|\left(u^*(x) - \mu^*(\hat{x}_j)\right) + \left(\mu^*(\hat{x}_j) - \hat{\mu}(\hat{x}_j)\right)\right\|^2\\ &\le 2\left\|\mu^*(\hat{x}_j) - \hat{\mu}(\hat{x}_j)\right\|^2 + 2\left\|u^*(x) - \mu^*(\hat{x}_j)\right\|^2\\ &\le 4\beta^2\left\|\tanh\left(\mathcal{A}_2(\hat{x}_j)\right) - \tanh\left(\mathcal{A}_1(\hat{x}_j)\right)\right\|^2\\ &\quad + 4\left\|\varepsilon_{\mu^*}(\hat{x}_j)\right\|^2 + 2K_{u^*}^2\|e_j(t)\|^2\\ &\le 8\beta^2\left(\left\|\tanh\left(\mathcal{A}_1(\hat{x}_j)\right)\right\|^2 + \left\|\tanh\left(\mathcal{A}_2(\hat{x}_j)\right)\right\|^2\right)\\ &\quad + 4\left\|\varepsilon_{\mu^*}(\hat{x}_j)\right\|^2 + 2K_{u^*}^2\|e_j(t)\|^2.\end{aligned} \tag{47}$$

Denoting $\mathcal{A}_\iota(\hat{x}_j) = [\mathcal{A}_{\iota 1}(\hat{x}_j), \mathcal{A}_{\iota 2}(x), \ldots, \mathcal{A}_{\iota m}(\hat{x}_j)]^{\mathsf{T}} \in \mathbb{R}^m$ (*note:* $\iota = 1, 2$) and noting that $|\tanh(y)| \le 1$ for all $y \in \mathbb{R}$, we find

$$\left\|\tanh\left(\mathcal{A}_\iota(\hat{x}_j)\right)\right\|^2 = \sum_{i=1}^{m}\tanh^2(\mathcal{A}_{\iota i}(x) \le m, \quad \iota = 1, 2.$$

Therefore, according to (47) and Assumption 5, there holds

$$\Sigma_1 \le 2K_{u^*}^2\|e_j(t)\|^2 + 16\beta^2 m + 4b_{\varepsilon_{\mu^*}}^2. \tag{48}$$

Likewise, calculating Σ_2 in (46), we have

$$\begin{aligned}\Sigma_2 &= \left\|\left(\omega^*(x) - \nu^*(\hat{x}_j)\right) + \left(\nu^*(\hat{x}_j) - \hat{\nu}(\hat{x}_j)\right)\right\|^2 \\ &\le 2\left\|\omega^*(\hat{x}_j) - \nu^*(\hat{x}_j)\right\|^2 + 2\left\|\nu^*(\hat{x}_j) - \hat{\nu}(\hat{x}_j)\right\|^2 \\ &\le 2\left\|\frac{k^{\mathsf{T}}(\hat{x}_j)}{2\gamma^2}\nabla\sigma_c^{\mathsf{T}}(\hat{x}_j)\tilde{W}_c + \varepsilon_{\nu^*}(\hat{x}_j)\right\|^2 \\ &\quad + 2K_{\omega^*}^2\|e_j(t)\|^2 \\ &\le 2K_{\omega^*}^2\|e_j(t)\|^2 + \left(k_M^2 b_{\sigma_c}^2/\gamma^4\right)\left\|\tilde{W}_c\right\|^2 + 4b_{\varepsilon_{\nu^*}}^2. \quad (49)\end{aligned}$$

Based on the facts that $\max\{\|V^*(x)\|, \|V_x^*\|\} \le \delta_{V^*}$ (*note:* see the proof of Theorem 1) and $-\mathcal{R}(u^*(x)) \le 0$, as well as (48) and (49), we obtain from (46) that

$$\begin{aligned}\dot{\mathcal{L}}_1(t) \le &-Q(x) + 2\left(K_{u^*}^2 + \gamma^2 K_{\omega^*}^2\right)\|e_j(t)\|^2 \\ &+ \left(k_M^2 b_{\sigma_c}^2/\gamma^2\right)\left\|\tilde{W}_c\right\|^2 + d_0 \quad (50)\end{aligned}$$

where

$$d_0 = 16\beta^2 m + 4\gamma^2 b_{\varepsilon_{\nu^*}}^2 + 4b_{\varepsilon_{\mu^*}}^2 + a_0$$

with a_0 being given as in (20).

Differentiating $\mathcal{L}_2(t)$ along the solution of (32), we obtain

$$\begin{aligned}\dot{\mathcal{L}}_2(t) \le &-l_c\tilde{W}_c^{\mathsf{T}}\Phi\left(\varphi, \varphi_{(p)}\right)\tilde{W}_c + l_c\tilde{W}_c^{\mathsf{T}}\frac{\varphi\varepsilon_H}{1+\phi^{\mathsf{T}}\phi} \\ &+ \sum_{p=1}^{\mathcal{N}_0} l_c\tilde{W}_c^{\mathsf{T}}\frac{\varphi_{(p)}\varepsilon_{H_{(p)}}}{1+\phi_{(p)}^{\mathsf{T}}\phi_{(p)}} \quad (51)\end{aligned}$$

with $\Phi(\varphi, \varphi_{(p)})$ being presented as (35).

Using Young's inequality $c^{\mathsf{T}}d \le (1/2)c^{\mathsf{T}}c + (1/2)d^{\mathsf{T}}d$ and noticing that $(1+\phi^{\mathsf{T}}\phi)^{-1} \le 1$, we have the second term on the right-hand side of (51) satisfied

$$\begin{aligned}\frac{l_c\tilde{W}_c^{\mathsf{T}}\varphi\varepsilon_H}{1+\phi^{\mathsf{T}}\phi} &\le \frac{l_c}{1+\phi^{\mathsf{T}}\phi}\left(\frac{1}{2}\tilde{W}_c^{\mathsf{T}}\varphi\varphi^{\mathsf{T}}\tilde{W}_c + \frac{1}{2}\varepsilon_H^{\mathsf{T}}\varepsilon_H\right) \\ &\le \frac{l_c}{2}\tilde{W}_c^{\mathsf{T}}\varphi\varphi^{\mathsf{T}}\tilde{W}_c + \frac{l_c}{2}\varepsilon_H^{\mathsf{T}}\varepsilon_H.\end{aligned}$$

Likewise, observing that $(1+\phi_{(p)}^{\mathsf{T}}\phi_{(p)})^{-1} \le 1$, we have

$$\begin{aligned}\sum_{p=1}^{\mathcal{N}_0}\frac{l_c\tilde{W}_c^{\mathsf{T}}\varphi_{(p)}\varepsilon_{H_{(p)}}}{1+\phi_{(p)}^{\mathsf{T}}\phi_{(p)}} \le &\frac{l_c}{2}\tilde{W}_c^{\mathsf{T}}\left(\sum_{p=1}^{\mathcal{N}_0}\varphi_{(p)}\varphi_{(p)}^{\mathsf{T}}\right)\tilde{W}_c \\ &+ \frac{l_c}{2}\sum_{p=1}^{\mathcal{N}_0}\varepsilon_{H_{(p)}}^{\mathsf{T}}\varepsilon_{H_{(p)}}.\end{aligned}$$

Thus, (51) yields

$$\begin{aligned}\dot{\mathcal{L}}_2(t) &\le -\frac{l_c}{2}\tilde{W}_c^{\mathsf{T}}\Phi\left(\varphi, \varphi_{(p)}\right)\tilde{W}_c + \frac{l_c}{2}\varepsilon_H^{\mathsf{T}}\varepsilon_H \\ &\quad + \frac{l_c}{2}\sum_{p=1}^{\mathcal{N}_0}\varepsilon_{H_{(p)}}^{\mathsf{T}}\varepsilon_{H_{(p)}} \\ &\le -\frac{l_c}{2}\lambda_{\min}\left(\Phi\left(\varphi, \varphi_{(p)}\right)\right)\left\|\tilde{W}_c\right\|^2 \\ &\quad + \frac{l_c}{2}(1+\mathcal{N}_0)b_{\varepsilon_H}^2. \quad (52)\end{aligned}$$

Combining (50) and (52) together with $\dot{V}^*(\hat{x}_j) = 0$ and noticing that $Q(x) = x^{\mathsf{T}}Qx \ge \lambda_{\min}(Q)\|x\|^2$, we have that the derivative of $\mathcal{L}(t)$ in (43) yields

$$\begin{aligned}\dot{\mathcal{L}}(t) \le &-\eta^2\lambda_{\min}(Q)\|x\|^2 - \left(1-\eta^2\right)\lambda_{\min}(Q)\|x\|^2 \\ &+ 2\left(K_{u^*}^2 + \gamma^2 K_{\omega^*}^2\right)\|e_j(t)\|^2 - \frac{1}{2}F_0\left\|\tilde{W}_c\right\|^2 \\ &+ \frac{l_c}{2}(1+\mathcal{N}_0)b_{\varepsilon_H}^2 + d_0\end{aligned}$$

where

$$F_0 = l_c\lambda_{\min}\left(\Phi\left(\varphi, \varphi_{(p)}\right)\right) - 2k_M^2 b_{\sigma_c}^2/\gamma^2.$$

Therefore, letting (15) and (34) hold, we find that $\dot{\mathcal{L}}(t) < 0$ if either $x \notin \bar{\Omega}_x$ or $\tilde{W}_c \notin \Omega_{\tilde{W}_c}$ with $\bar{\Omega}_x$ and $\Omega_{\tilde{W}_c}$ being given as follows:

$$\bar{\Omega}_x = \left\{x : \|x\| \le \frac{1}{\eta}\sqrt{\frac{l_c(1+\mathcal{N}_0)b_{\varepsilon_H}^2/2 + d_0}{\lambda_{\min}(Q)}}\right\} \quad (53)$$

$$\Omega_{\tilde{W}_c} = \left\{\tilde{W}_c : \left\|\tilde{W}_c\right\| \le \sqrt{\frac{l_c(1+\mathcal{N}_0)b_{\varepsilon_H}^2 + 2d_0}{F_0}}\right\}. \quad (54)$$

Using the Lyapunov theorem extension [28], we obtain UUB of x and $\tilde{W}_c$ with their ultimate bounds being the same as the bounds of $\bar{\Omega}_x$ in (53) and $\Omega_{\tilde{W}_c}$ in (54), respectively.

Situation II: Let events be triggered, that is, $t = t_{j+1}$, $j \in \{0, 1, 2, \ldots\}$. Then, the difference of the Lyapunov function candidate $\mathcal{L}(t)$ in (43) should be taken into account. That is

$$\Delta\mathcal{L}(t_j) = V^*(\hat{x}_{j+1}) - V^*(\hat{x}_j) + \Pi$$

where

$$\begin{aligned}\Pi = &V^*\left(x(t_{j+1})\right) - V^*\left(x(t_{j+1}^-)\right) \\ &+ \frac{1}{2}\tilde{W}_c^{\mathsf{T}}\left(t_{j+1}\right)\tilde{W}_c\left(t_{j+1}\right) - \frac{1}{2}\tilde{W}_c^{\mathsf{T}}\left(t_{j+1}^-\right)\tilde{W}_c\left(t_{j+1}^-\right)\end{aligned}$$

and $x(t_{j+1}^-) = \lim_{\epsilon\to 0^+} x(t_{j+1} - \epsilon)$ with $\epsilon \in (0, t_{j+1} - t_j)$.

As proved in Situation I, if either $x \notin \tilde{\Omega}_x$ or $\tilde{W}_c \notin \tilde{\Omega}_{\tilde{W}_c}$, then, for every $t \in [t_j, t_{j+1})$, there holds $d\mathcal{L}(t)/dt < 0$. This implies $d(\mathcal{L}_1(t) + \mathcal{L}_2(t))/dt < 0$ for every $t \in [t_j, t_{j+1})$. [*Note:* $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ are defined as in (43)]. Thus, $\mathcal{L}_1(t) + \mathcal{L}_2(t)$ is strictly monotonically decreasing on $[t_j, t_{j+1})$. Note that for every $\epsilon \in (0, t_{j+1} - t_j)$, there holds $t_{j+1} > t_{j+1} - \epsilon$. Thus, we have

$$\mathcal{L}_1(t_{j+1}) + \mathcal{L}_2(t_{j+1}) < \mathcal{L}_1(t_{j+1} - \epsilon) + \mathcal{L}_2(t_{j+1} - \epsilon). \quad (55)$$

The right limit over both sides of (55) (i.e., $\epsilon \to 0^+$) yields

$$\begin{aligned}\mathcal{L}_1(t_{j+1}) + \mathcal{L}_2(t_{j+1}) &\le \lim_{\epsilon\to 0^+}\left(\mathcal{L}_1(t_{j+1} - \epsilon) + \mathcal{L}_2(t_{j+1} - \epsilon)\right) \\ &= \mathcal{L}_1(t_{j+1}^-) + \mathcal{L}_2(t_{j+1}^-). \quad (56)\end{aligned}$$

Using the definitions of $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ in (43), we rewrite (56) as

$$\begin{aligned}&V^*\left(x(t_{j+1})\right) + \frac{1}{2}\tilde{W}_c^{\mathsf{T}}\left(t_{j+1}\right)\tilde{W}_c\left(t_{j+1}\right) \\ &\quad \le V^*\left(x(t_{j+1}^-)\right) + \frac{1}{2}\tilde{W}_c^{\mathsf{T}}\left(t_{j+1}^-\right)\tilde{W}_c\left(t_{j+1}^-\right).\end{aligned}$$

This proves $\Pi \leq 0$. On the other hand, due to UUB stability of $x(t)$ proved in Situation I, we can conclude that $V_i^*(\hat{x}_{j+1}) \leq V_i^*(\hat{x}_j)$. Accordingly, we have $\Delta\mathcal{L}(t_j) < 0$ when letting either $x \notin \bar{\Omega}_x$ or $\tilde{W}_c \notin \tilde{\Omega}_{\tilde{W}_c}$. According to the Lyapunov theorem extension [28], we obtain UUB of x and $\tilde{W}_c$ with their ultimate bounds being the same as the bounds of $\bar{\Omega}_x$ in (53) and $\Omega_{\tilde{W}_c}$ in (54), respectively. This completes the proof.

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