

Computing ribbon obstructions for colored knots

by

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Abstract. Kjuchukova's Ξ_p invariant gives a ribbon obstruction for Fox p -colored knots. The invariant is derived from dihedral branched covers of 4-manifolds, and can be used to calculate the signatures of these covers when singularities on the branching sets are present. In this note, we give an algorithm for evaluating Ξ_p from a colored knot diagram, and compute a couple of examples.

1. Introduction. A knot $\alpha \subset S^3$ is *slice* if it bounds an embedded disk in B^4 ; *ribbon* if it bounds such a disk which has only local minima and saddle points with respect to the radial height function on B^4 ; and *homotopy ribbon* if α bounds a disk $D \subset B^4$ such that the inclusion-induced map $\pi_1(S^3 - \alpha) \rightarrow \pi_1(B^4 - D)$ is surjective. Every ribbon knot is homotopy ribbon, and every homotopy ribbon knot is slice. The notions of slice and homotopy ribbon make sense in both the smooth and topological categories; the associated disk in B^4 is assumed to be smoothly or locally flatly embedded, respectively. Ribbonness makes sense only in the smooth category. Fox's Slice Ribbon Conjecture asserts that every smoothly slice knot is ribbon [5]. The analogous assertion in the topological category would be that every topologically slice knot is homotopy ribbon.

Now suppose α is a Fox p -colored knot in $S^3 = \partial X$, where X is an oriented topological 4-manifold. Let $\rho : \pi_1(S^3 - \alpha) \rightarrow D_p$ denote the p -coloring of α . Kjuchukova's invariant $\Xi_p(\alpha, \rho)$ is defined for any colored knot α whose p -coloring extends over some locally flat, embedded, oriented surface $F \subset X$ with $\partial F = \alpha$ [6, 7]. When ρ extends over a homotopy-ribbon disk for α , the value $\Xi_p(\alpha, \rho)$ falls in a bounded range. In particular, when the p -fold dihedral branched cover of S^3 along α is a rational homology sphere,

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$|\Xi_p(\alpha, \rho)| \leq (p-1)/2$; in the general case an additional term appears in this inequality [3, 6]. In particular, when $p = 3$, $|\Xi_3(\alpha, \rho)| = 1$. Hence, Ξ_p provides a means for testing potential counterexamples to the Slice Ribbon Conjecture. This is one motivation to develop tools for the evaluation of this invariant, as one of the terms in the formula for $\Xi_p(\alpha, \rho)$ derived in [6, 7] is difficult to compute using data about the knot and its p -coloring. In addition to the purely knot-theoretic interest of Ξ_p , this procedure for evaluating the invariant also allows us to compute signatures of dihedral covers of 4-manifolds with singular branching sets. Indeed, the invariant $\Xi_p(\alpha, \rho)$ was originally defined as the contribution to the signature of a dihedral cover $f : Y^4 \rightarrow X^4$ whose branching set is embedded in X with a singularity whose link is α .

In this note, we lay out an algorithm for computing the ribbon obstruction $\Xi_p(\alpha, \rho)$ from a diagram of α . We then evaluate the value of $\Xi_p(\alpha, \rho)$ in several examples. We focus on the case where $p = 3$ for ease of exposition, but the procedure presented generalizes to all odd p . That is, the computational algorithm given applies to all Fox p -colored knots for which the corresponding branched cover of S^3 along α is a rational homology sphere.

Our main result, Theorem 3.1, allows us to evaluate $\Xi_p(\alpha, \rho)$ from a colored diagram of α using a formula for $\Xi_p(\alpha, \rho)$ proved in [7] (see (3.1)) as well as the algorithm developed in [2] for computing linking numbers of rationally null-homologous knots in dihedral covers of S^3 . We apply this theorem in Section 4.

D_p denotes the dihedral group of order $2p$. We assume $p > 1$ is an odd integer.

2. The invariant Ξ_p . The integer Ξ_p is an invariant of a knot α together with a choice of Fox p -coloring $\rho : \pi_1(S^3 - \alpha) \rightarrow D_p$. Rather than write $\Xi_p(\alpha, \rho)$, we often write $\Xi_p(\alpha)$, as the choice of coloring is often understood or, up to equivalence of colorings, unique.

The invariant $\Xi_p(\alpha)$ arises in the following context. Let X be a closed, oriented 4-manifold. Suppose that $B \subset X$ is a surface, embedded locally flatly away from one singular point whose link is α . Given a surjection $\varphi : \pi_1(X - B) \twoheadrightarrow D_p$, we consider the induced *irregular dihedral p -fold cover* Y of X branched along B . This cover is characterized by the fundamental group of its unbranched counterpart, which is isomorphic to $\varphi^{-1}(\mathbb{Z}/2\mathbb{Z})$ for a choice of reflection subgroup in D_p .

In the above setting, the invariant $\Xi_p(\alpha)$ should be viewed as the contribution to $\sigma(Y)$, the signature of the covering space, resulting from the presence of a singularity, α , on the branching set. For this reason, we refer to Ξ_p as the *signature defect* associated to α . Precisely, $\sigma(Y)$, $\sigma(X)$, and Ξ_p

are related as follows (see [7, Theorem 1.4(2)]):

$$\sigma(Y) = p\sigma(X) - \frac{p-1}{4}e(B) - \Xi_p(\alpha),$$

where $e(B)$ denotes the self-intersection number of the branching set. The fact that Ξ_p gives a ribbon obstruction when Y is a manifold is proven in [3]. This obstruction is generalized in [6] to a larger class of knots, namely all colored knots which bound colored surfaces in some 4-manifold with S^3 boundary. Knots with this property are called *p-admissible*. Theorem 3.1 applies to all 3-admissible knots whose irregular 3-fold dihedral covers are rational homology spheres.

DEFINITION 2.1. Let $\alpha \subset S^3$ be a knot and V a Seifert surface for α with Seifert form A_V . Let $L_V = A_V + A_V^T$ be the corresponding symmetrized form. Let $\beta \subset V^\circ$ be an embedded curve representing a primitive class in $H_1(V; \mathbb{Z})$. If $L_V(\beta, \omega) = A_V(\beta, \omega) + A_V(\omega, \beta) \equiv 0 \pmod{p}$ for all embedded curves ω in V , we say that β is a *mod p characteristic knot* for α .

Characteristic knots are key for computing the signature defect $\Xi_p(\alpha)$. The existence of a p -fold irregular dihedral cover of S^3 branched along α is equivalent to the existence of a mod p characteristic knot for α [4]; the role of a characteristic knot is discussed further in Section 5.1.

3. Overview of the algorithm

The signature defect arising from a singularity. Our combinatorial procedure for computing $\Xi_p(\alpha)$ relies on the formula given in [7, Theorem 1.3], which we now recall. Let α be a p -admissible knot, $\rho : \pi_1(S^3 - \alpha) \rightarrow D_p$ a surjective homomorphism and V a Seifert surface for α . Cappell and Shaneson [4, Proposition 1.1] showed, using the HNN presentation for $\pi_1(S^3 - \alpha)$, that the homomorphism ρ can be expressed using linking numbers of curves in $S^3 - V$ with a characteristic knot for α (see Definition 2.1) contained in the interior of V . Let L_V denote the symmetrized Seifert form of V , β the characteristic knot, ζ a primitive p th root of unity, and σ_{ζ^i} the Tristram–Levine σ_{ζ^i} signature [8, 9]. We have

$$(3.1) \quad \Xi_p(\alpha) = \frac{p^2 - 1}{6p} L_V(\beta, \beta) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta) + \sigma(W(\alpha, \beta)).$$

The first two terms in the above expression for $\Xi_p(\alpha)$ are easily calculated. The third term, $\sigma(W(\alpha, \beta))$, denotes the signature of a 4-manifold $W(\alpha, \beta)$ constructed by Cappell and Shaneson [4]. That is, $W(\alpha, \beta)$ is a cobordism between the p -fold irregular dihedral cover of S^3 branched along α and the p -fold cyclic cover of S^3 branched along β . We recall this construction in Section 5. Computing the signature of the manifold $W(\alpha, \beta)$ in terms of α

is the main result of this paper, Theorem 3.1, and for practical purposes this is equivalent to computing $\Xi_p(\alpha)$. The intersection matrix of $W(\alpha, \beta)$ can be expressed in terms of linking numbers of certain curves in the irregular p -fold dihedral cover M_α of S^3 branched along α . We choose an orientation of β , and let β_r and β_l denote its right and left push-offs in V . Let $V - \beta$ denote the surface with three boundary components α , β_r , and β_l , obtained from V by removing an annular neighborhood of β . The curves whose linking numbers appear in the computation of $\sigma(W(\alpha, \beta))$ are lifts to M_α of a basis $\mathcal{B}_V := \{\beta_r, \beta_l, \omega_1, \dots, \omega_{2g-2}\}$ for $H_1(V - \beta; \mathbb{Z})$. Linking numbers in M_α are computed using the algorithm given in [2].

We condense all the above information in a labeled link diagram of α , β and the ω_j , so that the signature defect can be computed algorithmically. The resulting algorithm is the content of Theorem 3.1.

Set $p = 3$ for the remainder of this section.

First, the arcs of α are labeled “1”, “2”, and “3”, to indicate their coloring by the transpositions (23), (13), and (12).

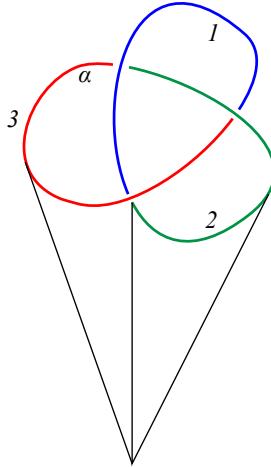


Fig. 1. A cell structure on S^3 determined by the cone on α

The irregular dihedral cover M_α corresponding to ρ is equipped with a cell structure determined by the cone on α . We review the key aspects of this cell structure here (see, e.g., [2] for more details). Equip S^3 with a cell structure that has one 3-cell e^3 , the complement of the cone on α . The “walls” of the cone on α are 2-cells, and so on. The cell structure on M_α is the lift of this cell structure on S^3 . The 3-cell e^3 has three preimages in M_α : e_1^3 , e_2^3 , and e_3^3 . These 3-cells are labeled so that the standard meridians of the arcs of α act on the subscripts according to the coloring $\rho : \pi_1(S^3 - \alpha) \twoheadrightarrow D_3$. See Figure 2.

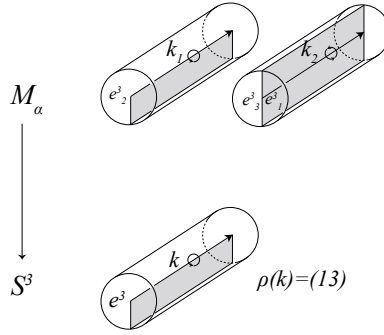


Fig. 2. Bottom: a tubular neighborhood of k , an arc in a diagram of α . Top: the two lifts of this neighborhood, and the action of a meridian of k on the three 3-cells, e^3_1, e^3_2, e^3_3 , of M_α . The shaded rectangles represent 2-cells. Put together, these local pictures determine a cell structure on the 3-fold irregular dihedral cover of S^3 branched along α .

We also choose a designated “zeroth” arc of each component of the link diagram (later, all arcs in the link diagram will be numbered). Let ω_i^j denote the lift of ω_i such that the lift of its zeroth arc lies in the j th 3-cell e^3_j for $j = 1, 2, 3$. The lifts β_r^j and β_l^j are defined analogously.

An *anchor path* for a curve $\omega \subset V - \beta$ is a properly embedded path δ in $V - \beta$ from a point q on the zeroth arc of α to a point r on the zeroth arc of ω . Suppose δ crosses under the arcs $a_1, \dots, a_k$ in the diagram of α , in that order, when traversing from q to r . The *monodromy* of the anchor path δ is the product $\sigma_k \dots \sigma_2 \sigma_1$, where $\rho(a_i) = \sigma_i$ is the permutation associated to the arc a_i of α .

Now we introduce notation in the statement of Theorem 3.1. Assume that M_α , the 3-fold dihedral cover of S^3 branched along α corresponding to ρ , is a rational homology sphere. Let $\mathcal{B}_V = \{\omega_i\}_{i=1}^{2g-2} \cup \{\beta_r, \beta_l\}$ be any basis for $H_1(V - \beta; \mathbb{Z})$ consisting of embedded curves in a Seifert surface V for α , where β is a mod 3 characteristic knot for α , determining ρ . Let δ_i be an anchor path for ω_i , and let γ_r and γ_l be anchor paths for the right and left push-offs of β in V . Let $\mu_{\delta_i}, \mu_{\gamma_r}, \mu_{\gamma_l} \in D_3$ be their monodromies. Let $c_0 \in \{1, 2, 3\}$ be the color of the zeroth arc of α .

Let $\mathcal{B} = \{A_s - B_s\}_{s=1}^{2g-1}$ be the set containing the following $2g - 1$ differences of curves in M_α :

$$\begin{aligned} \omega_i^j - \omega_i^k, \quad & \text{where } \{j, k\} = \{1, 2, 3\} - \{\mu_{\delta_i}(c_0)\}, \quad \text{and} \\ \beta^j - \beta^k, \quad & \text{where } j = \mu_{\gamma_r}(c_0) \text{ and } \{k\} = \{1, 2, 3\} - \{\mu_{\gamma_r}(c_0), \mu_{\gamma_l}(c_0)\}. \end{aligned}$$

Given a simple curve c on the surface $V - \beta$ we denote its positive and negative push-offs by $c^\pm$. We define a $(2g - 1) \times (2g - 1)$ matrix $M = (m_{r,s})_{r,s=1}^{2g-1}$ as follows. Let $A_r - B_r, A_s - B_s \in \mathcal{B}$ and set

$$m_{r,s} := \text{lk}(A_r - B_r, A_s^+ - B_s^-),$$

where “lk” denotes the linking number in M_α . Our main theorem states that the signature of $-M$ is the signature of the cobordism $W(\alpha, \beta)$ constructed by Cappell and Shaneson [4].

MAIN THEOREM 3.1. *Let α be a knot and $\rho : \pi_1(S^3 - \alpha) \twoheadrightarrow D_3$ a surjective homomorphism given by a mod 3 characteristic knot β . Assume that M_α , the 3-fold dihedral cover of S^3 branched along α and determined by ρ , is a rational homology sphere. Let $W(\alpha, \beta)$ be the cobordism constructed in [4] between M_α and the p -fold cyclic cover of S^3 branched along β . Let $M = (m_{r,s})$ be the matrix defined above. Then the signature of the 4-manifold $W(\alpha, \beta)$ is*

$$\sigma(W(\alpha, \beta)) = -\sigma(M).$$

In particular, $\sigma(M)$ is independent of the choices of anchor paths δ_i , γ_r , and γ_l , and the matrix M can be used to compute the invariant $\Xi_3(\alpha)$ associated to ρ , using equation (3.1). Moreover, when M_α is an integer homology sphere, $-M$ represents the intersection form of $W(\alpha, \beta)$.

As remarked earlier, this theorem allows us to perform the non-trivial step in computing the invariant $\Xi_3(\alpha)$, since the other two terms in the formula (3.1) for $\Xi_3(\alpha)$ are determined by the Seifert forms for α and β , and are thus algorithmically computable from diagrams of these knots.

In Section 4, we illustrate how to apply Theorem 3.1 to compute the signature defect associated to a singularity. We use two knots whose dihedral 3-fold covers are homeomorphic to S^3 . Our first example is the knot 6_1 ; this is the 3-admissible knot of smallest crossing number. We use this example to illustrate a characteristic knot, anchor paths, and the associated monodromies. Our second example, the knot 8_{11} , is a 3-admissible knot whose Seifert surface has higher genus, so the additional curves ω_i and their anchor paths come into play. In Section 5, we prove Theorem 3.1 and discuss its generalization to all odd p .

4. Computing the signature defect

Overview of the procedure. First we outline the steps for computing the signature defect $\Xi_p(\alpha)$, and, in particular, the work needed to pass from the geometric formula in [7] to a computation involving only diagrammatic information. We then carry out these steps in examples.

- (1) Fix a diagram and a Seifert surface V for α .
- (2) Find a characteristic knot for α . That is, compute the mod p nullspace of the symmetrized Seifert form for V . Fix a primitive curve, β , in this nullspace. Choose an orientation for β .

- (3) Choose a basis $\mathcal{B}_V = \{\omega_i\}_{i=1}^{2g-2} \cup \{\beta_r, \beta_l\}$ for $H_1(V - \beta; \mathbb{Z})$, where g is the genus of V and β_r, β_l denote the right and left push-offs of β in V .
- (4) Using Theorem 3.1, identify the curves in the 3-fold dihedral cover of S^3 branched along α whose linking numbers contribute to the computation of Ξ_p .
- (5) Compute the linking numbers of these curves using the algorithm in [2]. Evaluate $\sigma(W(\alpha, \beta))$.

EXAMPLE 4.1. In this example, we show $\Xi_3(6_1) = 1$ using Theorem 3.1. The 3-coloring and the Seifert surface V we use are pictured in Figure 3. Also remark that, since 6_1 is a ribbon knot whose 3-fold dihedral cover is S^3 , we can also conclude that $|\Xi_3(6_1)| = 1$ by [3].

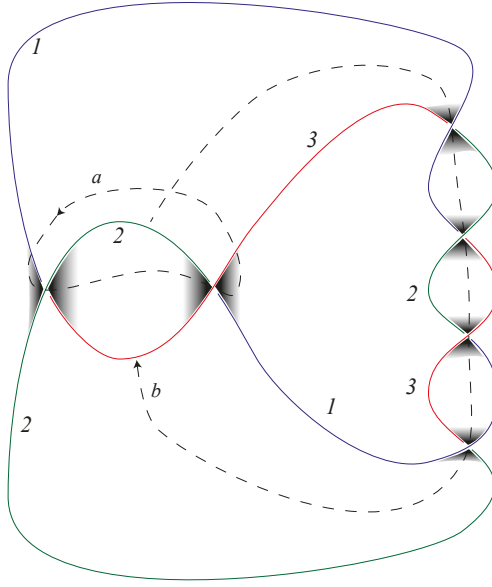


Fig. 3. The knot 6_1 , a Seifert surface V , and a basis $\{a, b\}$ for $H_1(V; \mathbb{Z})$

We begin by finding a mod 3 characteristic knot β for this 3-colored diagram. We compute the matrix of the symmetrized linking form with respect to the basis $\{a, b\}$:

$$[L_V] = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}.$$

Recall that a characteristic knot β is one that satisfies $L_V(\beta, \omega) \equiv 0 \pmod{3}$ for all $\omega \in H_1(V; \mathbb{Z})$. We check that

$$\begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{3}.$$

Hence an embedded representative of the class $a - b$ is a mod 3 characteristic knot. Moreover, since 6_1 is a two-bridge knot, it has a single 3-coloring, up to equivalence, and therefore a single equivalence class of mod 3 characteristic knots. Since V has genus 1, our basis $\mathcal{B}_V$ consists only of β_r and β_l . An embedded curve β , together with a choice of anchor paths γ_r and γ_l , is shown in Figure 4.

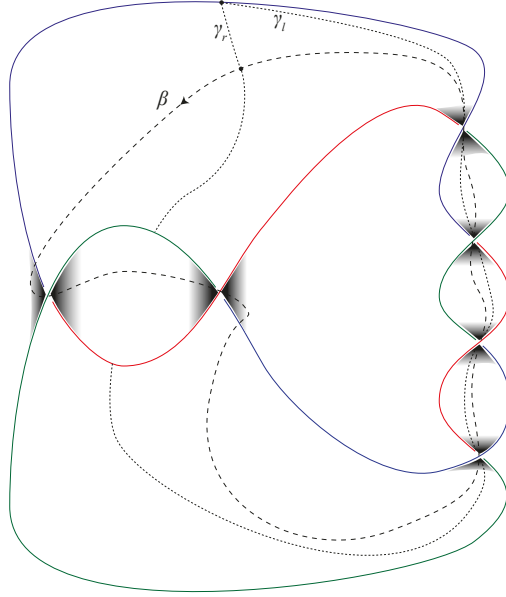


Fig. 4. The knot 6_1 , together with a characteristic knot β and the corresponding arcs γ_r and γ_l

We use the algorithm in [2] to compute linking numbers of the lifts β_r^j and β_l^k , $j, k = 1, 2, 3$. The details of this computation are given in the Appendix 5.3. The jk -entry of the matrix

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

is the linking number of β_r^j and β_l^k . As β_r and β_l are parallel curves in V , we may also view the jk -entry as the linking number of β^j with β^k , in which case the diagonal entries may be interpreted as self-linking numbers.

Next we compute the monodromies of γ_r and γ_l , in order to determine which of the above linking numbers appear in the formula for $\Xi_3(6_1)$:

$$\mu_{\gamma_r} = \text{Id}, \quad \mu_{\gamma_l} = (23)(13)(12)(23)(12)(13) = (123).$$

The zeroth arc of α is colored $c_0 = 1$. Hence $\mu_{\gamma_r}(c_0) = 1$ and $\mu_{\gamma_l}(c_0) = 3$. One can see this in the link diagram in Figure 5; the underlined number $\underline{i}$ on an arc of γ_r or γ_l indicates that the relevant lift of that arc sits in the 3-cell e_i^3 . These cells change from one arc of γ_r or γ_l to the next according to μ_{γ_r} and μ_{γ_l} .

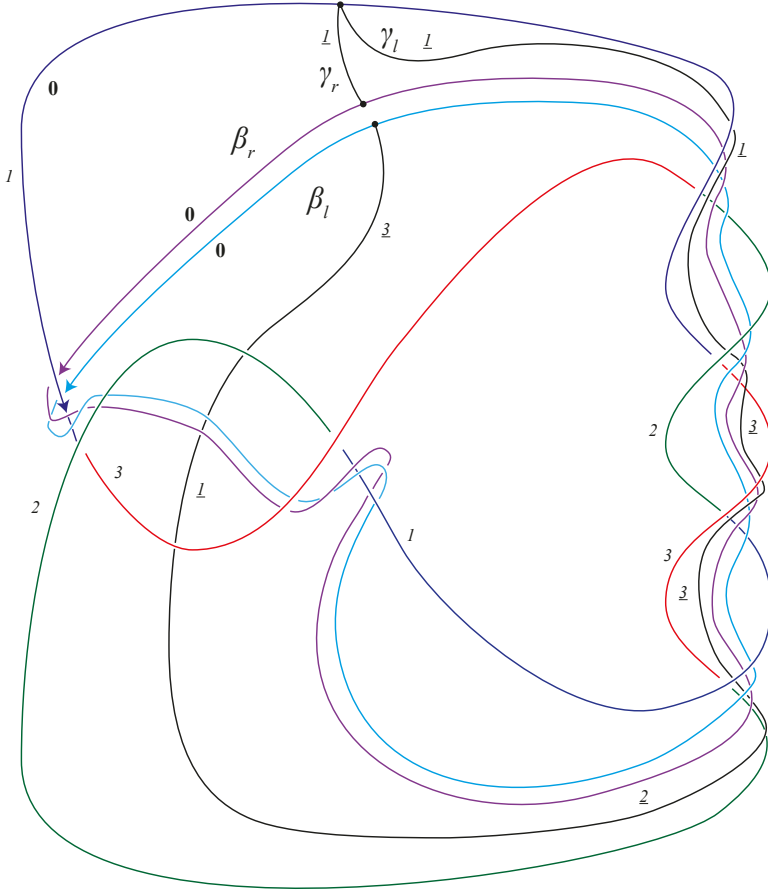


Fig. 5. The link diagram for 6_1 , together with the push-offs β_r and β_l of the characteristic knot, and the anchor paths γ_r and γ_l . The arcs of α are marked with their colors 1, 2, or 3. An $\underline{i}$ on an arc of γ_r or γ_l indicates that the lift of that arc sits in the 3-cell e_i^3 .

By Theorem 3.1, the signature $\sigma(W(\alpha, \beta))$ is the signature of the 1×1 matrix whose entry is the linking of $\beta^1 - \beta^2$ with an appropriate push-off of itself. Using the linking numbers given in the matrix above, we see that $\text{lk}(\beta^1 - \beta^2, \beta^{1,+} - \beta^{2,-}) = -1$. Hence $\sigma(W(\alpha, \beta)) = -\sigma((-1)) = 1$. Since β is an unknot with zero self-linking, the other terms in the formula for $\Xi_3(6_1)$ vanish and we conclude that $\Xi_3(6_1) = 1$.

We verify that

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \pmod{3},$$

so the curve β is a characteristic knot. As in Example 4.1, any mod 3 characteristic knot for 8_{11} determines its unique 3-coloring.

A basis $\mathcal{B}_V$ for $H_1(V - \beta; \mathbb{Z})$ is $\{A, B, \beta_r, \beta_l\}$. We again use the algorithm in [2] to find all linking numbers of lifts of curves in $\mathcal{B}_V$. These linking numbers are displayed in Table 1. We use superscript $\pm$ to denote positive and negative push-offs of curves on V , as well as their lifts.

Table 1. Linking numbers of lifts of the curves $\{A, B, \beta\}$ and their push-offs on V

	A^+	B^+	β^+
A	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$
B	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$
β	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix}$

Table 2. The matrix M of linking numbers of curves in $\mathcal{B}$. The intersection form of $W(\alpha, \beta)$ is given by the matrix $-M$.

	$A^{2,+} - A^{3,-}$	$B^{2,+} - B^{3,-}$	$\beta^{1,+} - \beta^{2,-}$
$A^2 - A^3$	-2	-1	-2
$B^2 - B^3$	-1	-2	-2
$\beta^1 - \beta^2$	-2	-2	-3

The matrix $M = \begin{pmatrix} -2 & -1 & -2 \\ -1 & -2 & -2 \\ -2 & -2 & -3 \end{pmatrix}$ of Theorem 3.1, shown in Table 2, and obtained from Table 1 by the relation $\text{lk}(x^+, y) = \text{lk}(x, y^-)$, has signature -3 . Since β is an unknot and $L_V(\beta, \beta) = 0$, we have $\Xi_3(8_{11}) = \sigma(W(\alpha, \beta)) = -\sigma(M) = 3$.

5. The Cappell–Shaneson construction and proof of the Main Theorem. Before proving Theorem 3.1 we review the Cappell–Shaneson

construction of M_α , the irregular dihedral cover of S^3 branched along α , and a cobordism $W(\alpha, \beta)$ between M_α and M_β , the p -fold cyclic cover of S^3 along β , where β is a mod p characteristic knot for α . We again focus on the case $p = 3$, but our combinatorial procedure can be generalized to all odd p , as discussed in Section 5.3.

5.1. The Cappell–Shaneson construction of the irregular dihedral cover. In [4], Cappell and Shaneson construct the irregular p -fold dihedral cover of S^3 branched along α from the p -fold cyclic cover of S^3 branched along a characteristic knot β . In this subsection, we give an informal overview of their construction. Precise details will be provided later, as needed. Roughly speaking, one begins with the p -fold cyclic cover M_β , and considers the lifts to this cover of a Seifert surface for α , $V \supset \beta$. Remove from M_β a neighborhood J of the union of the preimages of V to obtain a 3-manifold with boundary ∂J . Now identify points on that boundary via an involution $\bar{h}$, defined below. The resulting closed manifold M_α is the p -fold irregular dihedral cover of S^3 branched along α . The surface $S := \bar{h}(\partial J)$ is embedded in M_α , and has one boundary component α_0 , the index-1 lift of α . The index-2 lift of α is an embedded curve on S . In order to compute the signature $\sigma(W(\alpha, \beta))$ in Theorem 3.1, we must compute a matrix of linking numbers of certain elements of $H_1(S; \mathbb{Z})$, namely, a basis for the kernel of the map $i_* : H_1(S; \mathbb{Z}) \rightarrow H_1(\bar{h}(J); \mathbb{Z})$.

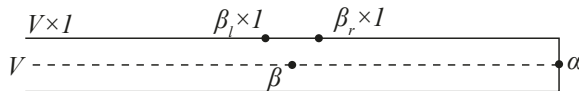
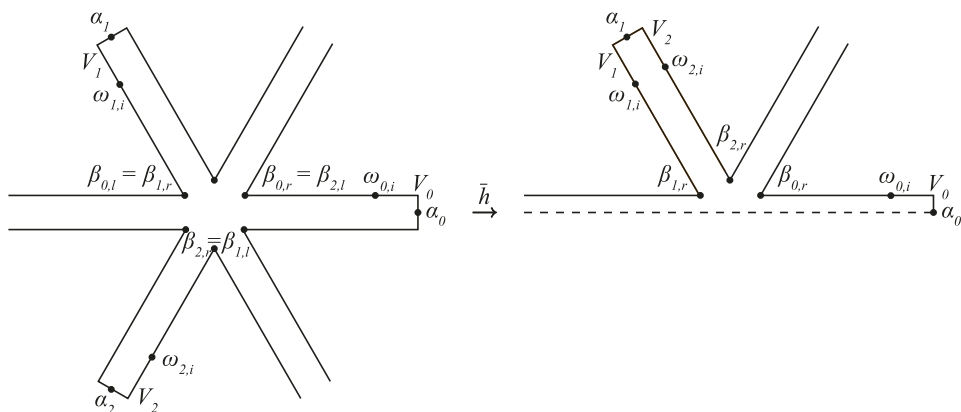


Fig. 7. V is a Seifert surface for α . The needed push-offs of β in $V \times [-1, 1]$ are shown above.

Now we set $p = 3$, describe the construction in detail, and introduce the necessary notation. Let $f : M_\beta \rightarrow S^3$ be a 3-fold cyclic covering map branched along β . By the construction sketched above, we know that M_α can be obtained from M_β as follows. Let $J = f^{-1}(V \times [-1, 1]) \subset M_\beta$. Let $h : V \times [-1, 1] \rightarrow V \times [-1, 1]$ be given by $h(x, t) = h(x, -t)$. Let $\bar{h} : \partial J \rightarrow \partial J$ be the lift of h to M_β restricted to ∂J ; in the schematic Figure 8, $\bar{h}$ is reflection about the horizontal line. Cappell and Shaneson show that M_α is homeomorphic to $(M_\beta - \mathring{J}) / \{\bar{h}(x) \sim x, x \in \partial J\}$, and that the mapping cone $W(\alpha, \beta)$ of $\bar{h}$ is a cobordism from the 3-fold cyclic cover M_β to the irregular 3-fold dihedral cover M_α .

Let $V - \beta$ denote the surface V cut along β , which we obtain by removing a thin annulus between the right and left push-offs β_r and β_l of β in V (note that β is oriented). The surface S above can be obtained by gluing


 Fig. 8. The image of J under the involution $\bar{h}$

together three copies of $V - \beta$ as follows. There are three lifts of $(V - \beta) \times 1$ in M_β , which we label V_0 , V_1 , and V_2 , according to the action of the deck transformation group. Let α_0 , α_1 , and α_2 denote the corresponding lifts of α . Each V_i contains lifts of the curves $\beta_r \times 1$ and $\beta_l \times 1$, and we denote these by $\beta_{i,r}$ and $\beta_{i,l}$. See Figures 7 and 8.

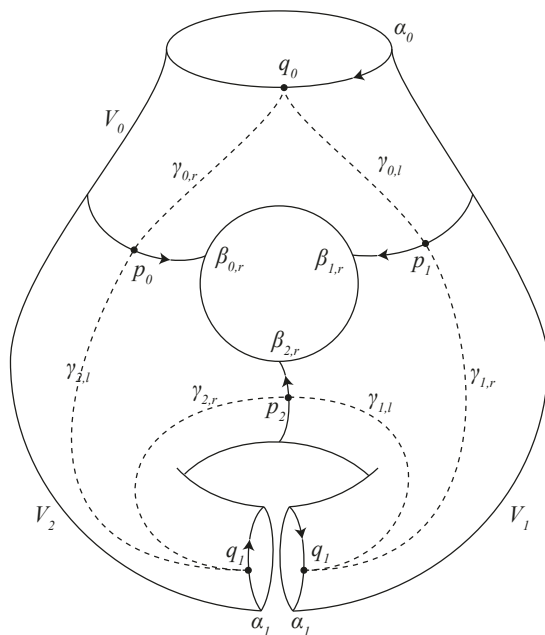


Fig. 9. The surface S , together with the index-1 and index-2 branch curves α_0 and α_1 , the three lifts of β , and the three lifts of γ_r and γ_l . The two copies of α_1 are identified according to the orientations pictured above.

From Figure 8, we can read off the boundaries of the surfaces V_i :

$$\partial V_0 = \alpha_0 + \beta_{0,r} - \beta_{0,l}, \quad \partial V_1 = \alpha_1 + \beta_{1,r} - \beta_{1,l}, \quad \partial V_2 = \alpha_2 + \beta_{2,r} - \beta_{2,l}.$$

Now we construct S by gluing together V_0 , V_1 , and V_2 using the following identifications: $\beta_{0,l}$ is identified with $\beta_{1,r}$; $\beta_{1,l}$ with $\beta_{2,r}$; and $\beta_{2,l}$ with $\beta_{0,r}$. In addition, α_1 and α_2 are identified. The index-1 and index-2 branch curves are $\alpha_0 = \partial S$ and α_1 respectively. Note that $\beta_{0,r}$ and $\beta_{1,r}$ are homologous in S , as they cobound V_0 together with α_0 . The surface S , constructed using these identifications, is pictured in Figure 9, in the case where V has genus 1 and each V_i is a pair of pants. This is in fact the case in our first example, where α is the knot 6_1 . In general the genus of V_i is 1 less than the genus of V .

5.2. Proof of Theorem 3.1. Let $W = W(\alpha, \beta)$ denote the Cappell–Shaneson cobordism, described above, between M_α , the p -fold dihedral branched cover of α , and M_β , the p -fold cyclic branched cover of the characteristic knot β . We will show that $\sigma(W) = -\sigma(M)$, where M is the matrix of linking numbers between the set $\mathcal{B}$ of curves in M_α from the theorem statement. The argument is standard: identify relative classes in $H_2(W, M_\alpha; \mathbb{Z})$ with curves in $\mathcal{B}$, and then show that the intersection numbers of two 2-dimensional classes equal the linking numbers of the corresponding curves. One remark we make right away is that, to calculate the intersection number, we actually use the linking numbers in $\overline{M}_\alpha$, that is, in M_α with the orientation reversed. This is a matter of convention. We adopted the convention used in [1] which means that M_α considered as a cover of S^3 branched along α has the opposite orientation than the one it inherits as a boundary component of the 4-manifold W . Hence, a minus sign appears in the signature formula, $\sigma(W) = -\sigma(M)$, since M is a matrix of linking numbers in M_α . Using the opposite convention would amount to replacing α by its mirror, in which case the orientation reversal would be unnecessary.

We observe that M_α is a rational homology sphere by assumption and M_β is a rational homology sphere because it is the 3-fold branched cover of S^3 along a knot. Thus, we can identify $H_2(W; \mathbb{Z})$ with its image in any of the relative groups $H_2(W, M_\alpha; \mathbb{Z})$, $H_2(W, M_\beta; \mathbb{Z})$ or $H_2(W, M_\alpha \cup M_\beta; \mathbb{Z})$, since the inclusion map in each of the relevant long exact sequences is injective. We will work with the image $\iota_*(H_2(W; \mathbb{Z})) \subset H_2(W, M_\alpha; \mathbb{Z})$, denoted by H from now on, since we happen to have a basis for $H_2(W, M_\alpha; \mathbb{Z})$ on hand. Indeed, by [7, (2.23)], $H_2(W, M_\alpha; \mathbb{Z})$ is free and isomorphic to $\ker i_*$, where $i_* : H_1(S; \mathbb{Z}) \rightarrow H_1(\bar{h}(J); \mathbb{Z})$.

Corollary 2.4 of [7] describes a basis for $\ker i_* \subset H_1(M_\alpha; \mathbb{Z})$, where each element in this kernel is identified with a relative cycle in $H_2(W, M_\alpha; \mathbb{Z})$ via the exact sequence of the pair. A relative cycle in $H_2(W, M_\alpha; \mathbb{Z})$ lies

in $H = \iota_*(H_2(W; \mathbb{Z}))$ if and only if its boundary can be capped off by an oriented surface in M_α . Given two classes in H , we will compute their intersection number in terms of linking numbers between their boundaries, which we will describe in terms of $\ker i_*$.

The remaining ingredient is to characterize the curves in a basis for $\ker i_*$, which lie in the dihedral cover M_α , using only diagrammatic data about the branching set α in S^3 . The curves in $\ker i_*$ project under the branched covering map $M_\alpha \rightarrow S^3$ to curves in $S^3 - \alpha$ and can be described in terms of a basis for $H_1(V - \beta; \mathbb{Z})$ [7]. Each curve in $H_1(V - \beta; \mathbb{Z})$ is covered by three disjoint circles in M_α as seen in the Cappell–Shaneson construction [4]. We use anchor paths and their monodromies to give a combinatorial description of $\ker i_*$ relying solely on diagrammatic information in S^3 . We conclude that the curves in $\mathcal{B}$ represent a basis for $\ker i_*$ and, equivalently, for $H_2(W, M_\alpha; \mathbb{Z})$.

Let q be a point on α , and let p be a point on β . Let $\{\omega_1, \dots, \omega_{2g-2}\} \cup \{\beta_r, \beta_l\}$ be a basis for $H_1(V - \beta; \mathbb{Z})$, where g is the genus of V . Each curve ω_i in $V - \beta$ has three lifts $\omega_{0,i}$, $\omega_{1,i}$, and $\omega_{2,i}$ to $S \subset M_\alpha$. From Figure 8, it is evident that the differences of curves $\omega_{1,i} - \omega_{2,i}$, together with one of either $\beta_{0,r} - \beta_{2,r}$ or $\beta_{1,r} - \beta_{2,r}$, form a basis $\mathcal{B}$ for $\ker i_*$. We refer the reader to Section 3 for the notation and definitions used in Theorem 3.1, namely the lifts ω_i^j , β_r^j , β_l^j , β^j , and the definitions of the anchor paths γ_r , γ_l , δ_i , and their monodromies.

Now we use anchor paths in a diagram of α to identify which of the three lifts β_r^1 , β_r^2 , and β_r^3 are $\beta_{0,r}$, $\beta_{1,r}$, and $\beta_{2,r}$. The lift $\gamma_{0,r}$ of the anchor path γ_r beginning at q_0 has its initial endpoint in the 3-cell $e_{c_0}^3$, on the index-1 lift α_0 of α . Looking at γ_r in a diagram of α (see, e.g., Figure 5), we see $\gamma_{0,r}$ has its final endpoint on the lift β_r^j of β_r that lies in the 3-cell e_j^3 , where $j = \mu_{\gamma_r(c_0)}$. On the other hand, from Figure 9, we see that the final endpoint of $\gamma_{0,r}$ lies on the lift $\beta_{0,r}$ of β . Hence $\beta_{0,r} = \beta_r^j$ with $j = \mu_{\gamma_r(c_0)}$. Similarly, the lift $\gamma_{0,l}$ of γ_l beginning at q_0 has its initial point in the 3-cell $e_{c_0}^3$, on the index-1 lift α_0 of α , and its endpoint on the lift β_l^k of β_l that lies in the 3-cell e_k^3 , where $k = \mu_{\gamma_l(c_0)}$. We also see from Figure 9 that the endpoint of $\gamma_{0,l}$ lies on the lift $\beta_{1,r}$ (which is identified with $\beta_{0,l}$) of β_r , so $\beta_{1,r} = \beta_l^k$ with $k = \mu_{\gamma_l(c_0)}$. Hence the basis element $\beta_{0,r} - \beta_{2,r}$ of $\ker i_*$ is

$$\beta_r^j - \beta_l^k, \quad \text{where } j = \mu_{\gamma_r(c_0)} \text{ and } \{k\} = \{1, 2, 3\} - \{\mu_{\gamma_r(c_0)}, \mu_{\gamma_l(c_0)}\},$$

as stated in Theorem 3.1. Note that we may instead use $\beta^j - \beta^k$, because β_r and β_l are simply push-offs of β in V .

Next we need to identify the basis element $\omega_{1,i} - \omega_{2,i}$ using diagrammatic information. The lift $\delta_{i,0}$ of δ_i beginning at q_0 has its initial endpoint in the 3-cell $e_{c_0}^3$, on the index-1 lift α_0 of α . Looking at δ_i in a diagram of α , we see that $\delta_{i,0}$ has its final endpoint on the lift ω_i^j where $j = \mu_{\delta_i(c_0)}$. On the other

hand, from Figure 10 we see that the final endpoint of $\delta_{0,i}$ lies on the lift $\omega_{0,i}$ of ω_i . Hence $\omega_{0,i} = \omega_i^j$ where $j = \mu_{\delta_i}(c_0)$. Therefore the basis element $\omega_{1,i} - \omega_{2,i}$ is, up to sign,

$$\omega_i^j - \omega_i^k, \text{ where } \{j, k\} = \{1, 2, 3\} - \{\mu_{\delta_i}(c_0)\},$$

as stated in Theorem 3.1.

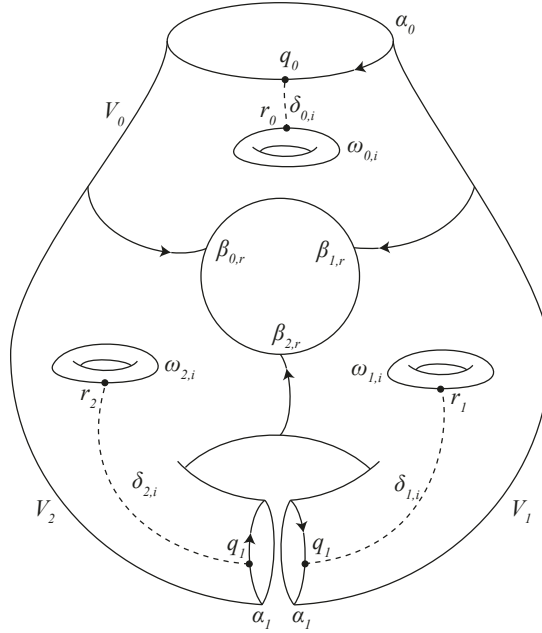


Fig. 10. Lifts to the surface S of an anchor path for ω_i . The two copies of α_1 are identified according to the orientations pictured above.

So far, we have seen that the curves in $\mathcal{B}$ can be identified with a basis for $H_2(W, M_\alpha; \mathbb{Z}) \supset H$; now we wish to calculate the intersection numbers of classes in H . Given any $A_r - B_r \in \mathcal{B}$, we have $A_r - B_r = \partial C_r$ where $[C_r] \in H_2(W, M_\alpha; \mathbb{Z})$ can be represented by a properly embedded cylinder [7, Corollary 2.4].

First suppose M_α is an integer homology sphere. In this case there exist Seifert surfaces $\Sigma_r = \partial A_r$ and $\Sigma'_r = \partial B_r$. The classes represented by the closed surfaces

$$\hat{C}_r := C_r \cup \Sigma_r \cup \Sigma'_r$$

form a basis for H . Note that $A_r - B_r$ and $A_s - B_s$ do not, in general, bound disjoint cycles in $H_2(W, M_\alpha; \mathbb{Z})$, but $A_r - B_r$ and $A_s^+ - B_s^-$ bound disjoint cylinders for any choice of r, s . This is because $A_r - B_r$ bounds a cylinder contained entirely in $\hat{h}(J)$, in the notation of Section 5.1. To give an example

of such a cylinder, set $A_r - B_r = \omega_{1,i} - \omega_{2,i}$, pictured as a pair of points in Figure 8. In this figure, the cylinder $S^1 \times I$ would be depicted as a line segment, properly embedded in $\tilde{h}(J)$, connecting $\omega_{1,i}$ to $\omega_{2,i}$. The same holds for all differences of curves in $\mathcal{B}$. On the other hand, the push-off $A_s^+ - B_s^-$ bounds a cylinder which can be made disjoint from all of $\tilde{h}(J)$. Therefore, the intersection number of $[\hat{C}_r]$ and $[\hat{C}_s]$ is equal to the linking number of $A_r - B_r$ with $A_s^+ - B_s^-$ in $\overline{M}_\alpha$, where again, we reverse the orientation on M_α for the reason previously explained.

Let $M = (m_{r,s})$ be the matrix with entries

$$m_{r,s} = \text{lk}(A_r - B_r, A_s^+ - B_s^-)$$

where $A_r - B_r, A_s - B_s$ run through the elements of $\mathcal{B}$. If M_α is an integer homology sphere, then $\mathcal{B}$ is in fact a basis for H , and the matrix $-M$ represents the intersection form of W . In particular, $\sigma(W) = -\sigma(M)$ as claimed. If M_α is only a rational homology sphere, then we can express a basis $\mathcal{B}'$ for H in terms of the C_r . Given two elements $\sum_i a_i C_{r_i}, \sum_j b_j C_{s_j} \in H$, their intersection number is

$$-\sum_{i,j} a_i b_j \text{lk}(A_{r_i} - B_{r_i}, A_{s_j}^+ - B_{s_j}^-) = -\sum_{i,j} a_i b_j m_{r_i, s_j}.$$

As H and $H_2(W, M_\alpha)$ have the same rank, the intersection matrix with respect to $\mathcal{B}'$ and $-M$ are congruent over $\mathbb{Q}$, so they have the same signature. ■

5.3. Computation for other values of p . We briefly explain how these methods generalize to an algorithm for computing $\Xi_p(\alpha)$ for all odd p . The schematic in Figure 8 allows one to read off a basis for $\ker i_*$ without the hypothesis $p = 3$. (Of course, in the general case, one uses the p -fold cyclic cover of $V \times [-1, 1]$ branched along β , rather than the 3-fold cover.) Anchor paths can be used in the same fashion to distinguish between the p lifts of β and the p lifts of each ω_i in the diagram for $\alpha = \partial V$. The remaining task is the computation of the linking numbers of these lifts in the p -fold dihedral cover of S^3 branched along α . This algorithm carried out in [2] in the case $p = 3$ generalizes to any odd p .

Appendix: Calculating linking numbers in branched covers

A.1. Making lists. We explain how the algorithm in [2] is used to compute the linking numbers that appear in both examples. The input for the algorithm is a labeled link diagram, which is recorded by means of several lists analogous to the Gauss code. One component of the link diagram is the knot α . In order to simplify the combinatorics, we only include in our diagram two of $\{\beta, \omega_1, \dots, \omega_{2g-2}\}$, or one of these curves together with its push-off in V , at any given time. Call these two curves g and h . Because β is a mod 3 characteristic knot, any curve in $V - \beta$ lifts to three closed loops [4]. Thus

for each pair of curves in $\mathcal{B}_V$, we compute nine linking numbers of their lifts, organized in a symmetric 3×3 matrix.

The following set-up allows us to compute the intersection number of any lift of h with a 2-chain whose boundary is any given lift of g . For the details on how this 2-chain is constructed see [2].

(1) The arcs of α in the diagram $\alpha \cup g$ are labeled $0, 1, \dots, m-1$, where m is the number of self-crossings of α plus the number of crossings of α under g . Each arc of α is colored 1, 2 or 3, according to the given homomorphism $\rho : \pi_1(S^3 - \alpha) \twoheadrightarrow D_3$.

(2) The arcs of g in the diagram $\alpha \cup g$ are labeled $0, 1, \dots, n-1$, where n is the number of self-crossings of g plus the number of crossings of g under α .

(3) Now we add h to the above numbered diagram $\alpha \cup g$ without changing the numbering of any existing arcs. The arcs of h are labeled $0, 1, \dots, o-1$, where o is the number of crossings of h under α plus the number of crossings of h under g . In this article, h never has self-crossings.

A.2. Lists. We provide the input used in the computation of our Example 4.1, the knot 6_1 . The first four lists needed are associated to the knot α . The remaining six lists are associated to the two curves g and h described above. The first list, $f = (f(i))_i$, records each number $f(i)$ assigned to the over-arc which meets the head of arc i of α . The second, $\epsilon = (\epsilon(i))_i$, denotes the local writhe number at the head of arc i . Next, $t = (t(i))_i$ denotes the *type* of crossing at the head of arc i ; that is, we set $t(i) = p$ if the over-arc at the head of arc i is an arc of g , and we set $t(i) = k$ if the over-arc at the head of arc i is another arc of α . Recall that the i th arc of α may be a union of smaller arcs, separated by over-crossings by arcs of h . Due to our numbering system, the over-crossing at the end of an arc of α will never be an arc of h . The fourth list, $c = (c(i))_i$, enumerates the colors on the consecutive arcs of α .

Numbering, signs, crossing types, and colors for α :

$$\begin{aligned} f &= (1, 8, 0, 7, 10, 5, 3, 2, 4, 6, 6, 4), \\ \epsilon &= (-, +, -, -, -, -, +, +, +, -, +, -), \\ t &= (p, k, k, p, k, p, p, k, p, k, p, k), \\ c &= (1, 1, 3, 2, 2, 1, 1, 1, 2, 2, 3, 3). \end{aligned}$$

The remaining lists are the over-crossing numbers, signs, and crossing types for the other two components, g and h , of the link diagram.

Numbering, signs, and crossing types for $g = \beta_I$:

$$\begin{aligned} &(0, 8, 2, 6, 6, 10, 4, 0), \\ &(-, +, -, +, -, +, -, +), \\ &(k, k, k, k, k, k, k, k). \end{aligned}$$

Numbering, signs, and crossing types for $h = \beta_r$:

$$\begin{aligned} &(0, 1, 8, 2, 3, 6, 4, 6, 10, 6, 4, 0), \\ &(-, -, +, -, -, +, +, -, +, +, -, +), \\ &(k, p, k, k, p, k, p, k, k, p, k, k). \end{aligned}$$

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