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The dihedral genus of a knot

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Let $K \subset S^3$ be a Fox p–colored knot and assume K bounds a locally flat surface $S \subset B^4$ over which the given p–coloring extends. This coloring of S induces a dihedral branched cover $X \to S^4$. Its branching set is a closed surface embedded in S^4 locally flatly away from one singularity whose link is K. When S is homotopy ribbon and X a definite four-manifold, a condition relating the signature of X and the Murasugi signature of K guarantees that S in fact realizes the four-genus of K. We exhibit an infinite family of knots K_m with this property, each with a Fox 3–colored surface of minimal genus m. As a consequence, we classify the signatures of manifolds X which arise as dihedral covers of S^4 in the above sense.

57M12, 57M25, 57Q60

1 Introduction

The slice-ribbon conjecture of Fox [7] asks whether every smoothly slice knot in S^3 bounds a ribbon disk in the four-ball. The analogous question can be asked in the topological category, namely: does every topologically slice knot bound a locally flat homotopy-ribbon disk in B^4? Recall that a properly embedded surface with boundary $F' \subset B^4$ is *homotopy ribbon* if the fundamental group of its complement is generated by meridians of $\partial F'$ in S^3. Ribbon disks are easily seen to be homotopy ribbon whereas homotopy-ribbon disks need not be smooth.

For knots of higher genus, the generalized topological slice-ribbon conjecture asks whether the topological four-genus of a knot is always realized by a homotopy-ribbon surface in B^4. When a knot K admits Fox p–colorings, we approach this problem by studying locally flat, oriented surfaces $F' \subset B^4$ with $\partial F' = K$ over which some p–coloring of K extends, in the sense defined in Section 2.1. The minimal genus of such a surface, when one exists, we call the *p–dihedral genus of K*.

When K is slice and p square-free, it is classically known that the colored surface F' for K can always be chosen to be a disk. This is essentially a consequence of a result

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of Casson and Gordon [6, Lemma 3]; a detailed explanation can be found in work of Geske, Kjuchukova and Shaneson [9, Lemma 9]. Put differently, p–dihedral genus and classical four-genus coincide for slice knots. Furthermore, the topological slice-ribbon conjecture is true for p–colorable slice knots if and only if the minimal p–dihedral genus for these knots can always be realized by homotopy-ribbon surfaces. With this in mind, given a square-free integer p and a p–colorable knot K, we ask:

Question 1 Is the (topological) four-genus of K equal to its (topological) p–dihedral genus?

Question 2 Is the p–dihedral genus of K realized by a homotopy-ribbon surface?

When both of these questions are answered in the affirmative for a knot K with respect to some integer p, it follows that the topological four-genus and homotopy-ribbon genus of K are equal; that is, the generalized topological slice-ribbon conjecture holds for K. If K is not slice, requiring that it satisfy Questions 1 and 2 is a priori a stronger condition than satisfying the generalized slice-ribbon conjecture; however, the advantage of this point of view is that dihedral genus can be studied using dihedral branched covers.

Specifically, our approach is the following. Start with a branched cover of $f'\colon X' \to B^4$ branched along a locally flat properly embedded surface F' with $\partial F' = K$; that is, F' is a properly embedded topological submanifold of B^4. We now construct a new cover $f\colon X \to S^4$ by taking the cones of $\partial X'$, S^3 and the map f. The branching set of f is a surface F embedded in S^4 locally flatly except for one singular point whose link is K. Depending on the knot K and the map f, it may be the case that this construction yields a total space X that is again a manifold. In general, X has one singular (nonmanifold) point $\mathfrak{z}$, the preimage of the singularity on F. The link of $\mathfrak{z}$ is the cover $f_|$ of S^3 branched along K. We will consider the signature of X whether it is a manifold or has an isolated singularity. In the latter case, by the signature of X we mean the Novikov signature of the manifold with boundary obtained by deleting an open neighborhood of $\mathfrak{z}$ in X.

When $f\colon X \to S^4$ in the above construction is a p–fold *irregular dihedral cover* (see the definition on page 1945), an invariant of (p–colored) knots, Ξ_p, is extracted from this construction. This invariant is our main tool. In a general setting, Ξ_p can be thought of as a defect term in the formula for the signature of a branched cover, resulting from the fact that the branching set is not locally flat. Put differently, the presence of a cone singularity K on the branching set causes the signature of the cover to deviate from the smooth case by a term denoted by $\Xi_p(K,\rho)$. This term depends only on the isotopy class of the knot K and its Fox p–coloring ρ, but not on the locally flat part of the branching set.

Given a dihedral cover $f\colon X\to S^4$ whose branching set is orientable with one singularity, we in fact have

$$\Xi_p(K,\rho)=-\sigma(X), \tag{1}$$

by Kjuchukova [13, Theorem 1.4], when X is a manifold, and

$$\Xi_p(K,\rho)=-\sigma(X',\partial X') \tag{2}$$

when X has a singularity, by Geske, Kjuchukova and Shaneson [9, Theorem 7]. In the latter formula, $\partial X'$ is the dihedral cover of K induced by $f_|$, and $\sigma(\cdot,\cdot)$ denotes the Novikov signature of a manifold with boundary. Of course, the first formula for $\Xi_p(K,\rho)$ in terms of X is a special case of the second, since the signature of a manifold is unchanged by deleting an open neighborhood of a point.

Unless explicitly stated otherwise, we will only consider orientable branching sets. Thus, we take the above signature equation to be the definition of $\Xi_p(K,\rho)$. In (5) we recall an explicit formula [13] for Ξ_p which does not rely on constructing the cover X. We also note that $\Xi_p(K,\rho)$ can be computed algorithmically from a colored diagram of K; see Cahn and Kjuchukova [4]. We often suppress notation and write $\Xi_p(K)$ when the choice of coloring is clear, or when a knot admits a unique p–coloring (up to permuting the colors). Thus, for a two-bridge knot K, we will write simply $\Xi_p(K)$. The main result of this paper, Theorem 1, obtains a certain genus bound for K from $\Xi_p(K)$.

As implied by the above, the signature defect $\Xi_p(K)$ is defined for a knot K which arises as the only singularity on the branching set (not necessarily orientable) of an irregular dihedral cover [9]. A knot K is called *p–admissible over S^4*, or simply *p–admissible*, if there exists a p–fold dihedral cover $f\colon X\to S^4$ whose branching set is embedded and locally flat except for one singularity whose link is K. If, in addition, the covering space X is a topological manifold, K is called *strongly p–admissible*. The distinguishing property of *strongly* p–admissible knots[1] is that their dihedral covers are S^3. Admissibility of knots is studied by Kjuchukova and Orr in [14].

[1] Like the invariant Ξ_p, the notion of (strong) p–admissibility of a knot may depend on the choice of coloring. We do not dwell on this presently since all examples in this paper are two-bridge knots and their colorings are unique.

In Section 2, we put side by side the relevant notions of knot four-genus, recall several definitions, and state our main results, Theorems 1, 2 and 3. In Theorem 1, we give a lower bound on the homotopy-ribbon p–dihedral genus of a colored knot K in terms of the invariant $\Xi_p(K)$. We also give a sufficient condition for when this bound is sharp.

In Theorems 2 and 3, we construct, for any integer $m \geq 0$, infinite families of knots for which the 3–dihedral genus and the topological four-genus are both equal to m. The basis of this construction are the knots K_m pictured in Figure 1. The various four-genera of these knots are computed with the help of Theorem 1. In particular, for these knots, the lower bound on genus obtained via branched covers is exact and the generalized topological slice-ribbon conjecture is seen to hold. The proofs of Theorems 1, 2 and 3 are given in Section 4.

The technique we apply is the following. Given a strongly p–admissible knot K, one can evaluate $\Xi_p(K)$ by realizing K as the only singularity on the branch surface of a dihedral cover of S^4. Each of the knots K_m arises as the only singularity on the branching set of a 3–fold dihedral cover

$$f_m\colon \#^{2m+1}\,\overline{\mathbb{CP}}^2 \to S^4.$$

The branching set of f_m is the boundary union of the cone on K_m with the surface F'_m realizing the four-genus of K_m. We construct these covering maps explicitly using singular triplane diagrams, a technique introduced by Cahn and Kjuchukova in [3]. Equivalently, we construct a family of covers $\#^{2m+1}\,\mathbb{CP}^2 \to S^4$, again with oriented, connected branching sets, with the mirror images of the knots K_m as singularities. This construction appears in Section 3. As a corollary of this construction, we realize all odd integers as values of Ξ_3. In Theorem 5, we prove that the range of values of Ξ_3 on strongly admissible knots is precisely the set of odd integers.

We work in the topological category, except where explicitly stated otherwise. Throughout, F denotes a closed, connected, oriented surface, and F' a connected, oriented surface with boundary. D_p denotes the dihedral group of order $2p$, and p is always assumed odd.

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2 Dihedral four-genus and the main theorems

2.1 Some old and new notions of knot genus

We study the interplay between the following notions of four-genus for a Fox p–colorable knot $K \subset S^3$. Classically, the *smooth* (*resp. topological*) *four-genus* is the minimum genus of a smooth (resp. locally flat) embedded orientable surface in B^4 with boundary K. The *smooth* (*resp. topological*) *p–dihedral genus* of a p–colored knot K is, informally, the minimum genus of such a surface F' in B^4 over which the p–coloring of K extends. Precisely, a given p–coloring ρ of K extends over F' if there exists a homomorphism $\overline{\rho}$ which makes the following diagram commute (where i_* is the map induced by inclusion):

$$\begin{array}{ccc} \pi_1(S^3 - K) & \xrightarrow{\ i_*\ } & \pi_1(B^4 - F') \\ \big\downarrow \rho & \swarrow_{\overline{\rho}} & \\ D_p & & \end{array}$$

The p–dihedral genus above is defined for a knot K with a fixed coloring ρ, and hence we denote it by $g_p(K,\rho)$ in the topological case. We define the p–dihedral genus of a p–colorable knot K to be the minimum p–dihedral genus of K over all p–colorings ρ of K, and denote this by $g_p(K)$ in the topological case. Note that not every p–colored knot K admits a surface F' as above. In [14], we determine a necessary and sufficient condition for the existence of a connected oriented surface that fits into this diagram. When there is no surface over which a given coloring ρ of K extends, we define $g_p(K,\rho)$ to be infinite, and similarly for the refined notions of dihedral genus defined below.

The *ribbon genus* of K is the minimum genus of a smooth embedded orientable surface F' in B^4 with boundary K such that F' has only local minima and saddles with respect to the radial height function on B^4. The *smooth* (*topological*) *homotopy-ribbon genus* of a knot K is the minimum genus of a smooth (locally flat) embedded orientable surface F' in B^4 with boundary K such that $i_*\colon \pi_1(S^3-K) \twoheadrightarrow \pi_1(B^4-F')$, that is, inclusion of the boundary into the surface complement induces a surjection on fundamental groups. Finally, given a p–colorable or p–colored knot, its *ribbon p–dihedral genus* or *smooth* (*topological*) *homotopy-ribbon p–dihedral genus* are defined in the obvious way. Observe that all notions of dihedral genus refer to surfaces embedded in the four-ball, even though "four" is not among the multitude of qualifiers we inevitably use.

As a straightforward consequence of the definitions, the following inequalities hold among the *smooth* four-genera of a knot:

$$\begin{array}{ccccc}
\text{four-genus} & \leq & \text{hom. ribbon genus} & \leq & \text{ribbon genus} \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\
p\text{–dihedral genus} & \leq & p\text{–dihedral hom. ribbon genus} & \leq & p\text{–dihedral ribbon genus}
\end{array}$$

Excluding the last column, the inequalities make sense and hold in the topological category too.

2.2 The main theorems

Denote by $g_4(K)$ the topological 4–genus of a knot K, and by $\mathfrak{g}_p(K,\rho)$ the topological homotopy-ribbon p–dihedral genus of a knot K with coloring ρ. Again, the minimum such genus over all colorings ρ of K is $\mathfrak{g}_p(K)$. Let $\sigma(K)$ be the (Murasugi) signature of the knot K. We relate $\mathfrak{g}_p(K,\rho)$, $\Xi_p(K,\rho)$ and $\sigma(K)$. Here, $\Xi_p(K,\rho)$ denotes the invariant discussed in Section 1; it is reviewed in more detail in this section and, in particular, we recall that it can be computed using (5).

Theorem 1 (A) *Let K be a p–admissible knot with p–coloring ρ and denote by M the irregular dihedral cover of K determined by ρ. Then*

$$\mathfrak{g}_p(K,\rho) \geq \frac{|\Xi_p(K,\rho)| - \operatorname{rk} H_1(M;\mathbb{Z})}{p-1} - \frac{1}{2}. \tag{3}$$

(B) *Let K be a p–admissible knot and $F' \subset B^4$ a locally flat homotopy-ribbon oriented surface for K over which a given p–coloring ρ of K extends. Denote by $c(K)$ the cone on K, viewed as embedded in $D^4 = c(S^3)$. If the associated singular dihedral cover of S^4 branched along $F' \cup_K c(K)$ is a definite manifold, then the inequality (3) is sharp. In particular, F' realizes the dihedral genus $\mathfrak{g}_p(K,\rho)$ of K. If, in addition, the equality*

$$|\sigma(K)| = \frac{2|\Xi_p(K,\rho)|}{p-1} - 1$$

holds, then the topological four-genus and the topological homotopy-ribbon p–dihedral genus of K coincide and equal $\frac{1}{2}|\sigma(K)|$, so the generalized topological slice-ribbon conjecture holds for K.

Remark If K has multiple p–colorings, denote by $\min_p(K)$ the minimum value of

$$|\Xi_p(K,\rho)| - \operatorname{rk} H_1(M;\mathbb{Z})$$

over all such colorings of K. Theorem 1 implies

$$\mathfrak{g}_p(K) \geq \frac{\min_p(K)}{p-1} - \frac{1}{2}. \tag{4}$$

Theorem 2 *For every integer $m \geq 0$, there exists a knot K_m and corresponding 3–coloring ρ_m such that*

$$g_4(K_m) = \mathfrak{g}_3(K_m) = \tfrac{1}{2}|\Xi_3(K_m, \rho_m)| - \tfrac{1}{2} = m.$$

That is, the inequality (4) is sharp for these knots and computes their 3–dihedral genus as well as their topological four-genus. The generalized slice-ribbon conjecture holds for these knots.

Theorem 3 *For any integer $m \geq 0$, there exist infinite families of knots whose 3–dihedral genus and topological four-genus are both equal to m.*

2.3 Singular dihedral covers of S^4 and the invariant Ξ_p

In this section, we revisit the definition of a singular branched cover, and dihedral covers in particular. We also review the context in which the invariant Ξ_p arises, as well as a couple of techniques for its calculation.

Definition Let Y be a manifold and $B \subset Y$ a codimension-two submanifold with the property that there exists a surjection $\varphi\colon \pi_1(Y - B) \twoheadrightarrow D_p$. Denote by $\mathring{X}$ the covering space of $Y - B$ corresponding to the conjugacy class of subgroups $\varphi^{-1}(\mathbb{Z}/2\mathbb{Z})$ in $\pi_1(Y - B)$, where $\mathbb{Z}/2\mathbb{Z} \subset D_p$ is any reflection subgroup. The completion of $\mathring{X}$ to a branched cover $f\colon X \to Y$ is called the *irregular dihedral p–fold cover* of Y branched along B.

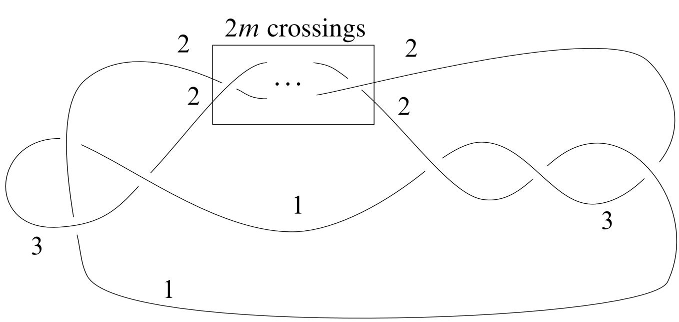

Figure 1: The knot K_m, where $m \geq 0$, and its 3–coloring. We have $K_0 = 6_1$, $K_1 = 8_{11}$, $K_2 = 10_{21}$ and $K_3 = 12a723$.

The manifolds whose irregular dihedral covers we will consider are S^3, B^4 and S^4. The Ξ_p invariant was originally defined in the more general context of a dihedral cover of an arbitrary four-manifold Y with a singularly embedded branching set [13].

Recall the following construction from Section 1. Let F' be a surface with connected boundary K, properly embedded in B^4 and locally flat. Given a branched cover of manifolds with boundary $f'\colon X' \to B^4$, one constructs a *singular* branched cover of S^4 by coning off $\partial X'$, ∂B^4 and the map f'. The resulting covering map, $f\colon X \to S^4$, has total space $X := X' \cup_{\partial X'} c(\partial X')$, where $c(\partial X')$ denotes the cone on $\partial X'$. The branching set is a closed surface $F := F' \cup_K c(K)$ embedded in S^4 with a singularity (the cone point) whose link is K. The space X obtained in this way is a manifold if and only if $\partial X' \cong S^3$.

Denote by $\sigma(X', \partial X')$ the Novikov signature of the manifold with boundary given as a cover of B^4 branched along F'. When $\partial X' = S^3$, denote by $\sigma(X)$ the signature of the manifold X. In this case, we have $\sigma(X) = \sigma(X', \partial X')$.

Given $f'\colon X' \to B^4$ as before with f' an (irregular) dihedral covering map, we always assume that the associated homomorphism $\rho\colon \pi_1(S^3 - K) \to D_p$ is surjective or, equivalently, that $\partial X'$ is connected. In this case, assuming that F' is orientable, $\Xi_p(K, \rho) = -\sigma(X', \partial X')$ by [9, Theorem 7]. In particular, when X is a manifold, this equation reduces to the earlier result $\Xi_p(K, \rho) = -\sigma(X)$ [13, Theorem 1.4].

Below, we recall two formulas for $\Xi_p(K, \rho)$ from [13]. Equation (5) allows $\Xi_p(K, \rho)$ to be computed in terms of K and its coloring using [4] and [2]. Equation (6) expresses $\Xi_p(K, \rho)$ in terms of a singular branched cover of S^4 in the more general case where the branching set is a possibly nonorientable surface.

Refocusing for a moment on the case where the dihedral branched cover X of S^4 is a manifold, we note that there exist many infinite families of knots $K \subset S^3$ whose irregular dihedral covers are homeomorphic to S^3. For example, this is a property shared by all p–colorable two-bridge knots (a well-known fact recalled in the proof of [13, Lemma 3.3]). By definition, if a p–admissible knot K has S^3 as its dihedral cover, then it is in fact strongly p–admissible. We are then able study invariants of K using four-dimensional techniques such as trisections. Criteria for admissibility of singularities are discussed in more detail in [3], where we also use the invariant $\Xi_p(K)$ to give a homotopy-ribbon obstruction for strongly p–admissible knots K. A generalization of this ribbon obstruction to all p–admissible knots appears in [9].

We conclude this section by reviewing the formula for computing the invariant Ξ_p given in [13]. Let p be an odd integer and K a p–admissible knot. Let V be a Seifert surface for K and V° the interior of V. Denote by $\beta \subset V^\circ$ a mod p characteristic knot[2] for K, as defined in [5]. Also denote by L_V the symmetrized linking form for V and by σ_{ζ^i} the Tristram–Levine ζ^i–signature, where ζ is a primitive p^{th} root of unity. Finally, let $W(K,\beta)$ be the cobordism constructed in [5] between the p–fold cyclic cover of S^3 branched along β and the p–fold dihedral cover of S^3 branched along K and determined by ρ. We briefly describe the manifold $W(K,\beta)$. Let Σ be the p–fold cyclic branched cover of β and let $\Sigma_p(\beta)\times[0,1]\to S^3\times[0,1]$ be the induced cyclic cover branched along $\beta\times[0,1]$. Letting $\mathbb{Z}/2\mathbb{Z}$ act on an appropriate subset of $\Sigma_p(\beta)\times\{0\}$, one obtains $W(K,\beta)$ as a quotient of $\Sigma_p(\beta)\times[0,1]$ by this action. One boundary of this quotient, namely $\Sigma_p(\beta)\times\{1\}$, is clearly the p–fold cyclic cover of β. The other boundary component, that is, the image of $\Sigma_p(\beta)\times\{0\}$ under the $\mathbb{Z}/2\mathbb{Z}$ action, is the dihedral cover of α as shown in [5, Proposition 1.1]. By [13, Theorem 1.4],

$$\Xi_p(K,\rho) = \frac{p^2-1}{6p} L_V(\beta,\beta) + \sigma(W(K,\beta)) + \sum_{i=1}^{p-1} \sigma_{\zeta^i}(\beta). \tag{5}$$

The Novikov signature $\sigma(W(K,\beta))$ can be computed in terms of linking numbers in the dihedral cover of K [13, Proposition 2.5]. Thus, the above formula allows $\Xi_p(K)$ to be evaluated directly from a p–colored diagram of K, without direct reference to a four-dimensional construction. An explicit algorithm for performing this computation is outlined in [4]. Note also that when a knot K is realized as the only singularity on an embedded surface $F \subset S^4$ and moreover this surface is presented by a Fox p–colored singular triplane diagram, [3] gives a method for computing $\Xi_p(K)$ from this data, via the signature of the associated cover of S^4. This technique is reviewed and applied in Section 3 below.

We also review the context in which (1) and (2) arise, allowing us to relate $\Xi_p(K,\rho)$ to the signature of a singular branched cover X of S^4. Consider an irregular dihedral cover $f\colon X\to S^4$ whose branching set F is an embedded surface, *not necessarily orientable*, locally flat away from one singularity $z\in F$ of type K. The induced coloring of F is, as always, an extension of ρ. Once again we denote by X' the

[2]Precisely, if K admits multiple p–colorings, one must work with a characteristic knot corresponding to the coloring in question. The sense in which a characteristic knot determines a coloring is laid out in [5, Proposition 1.1]. The examples we construct always admit a unique p–coloring, up to permuting the colors, and therefore a unique equivalence class of mod p characteristic knots.

dihedral cover of B^4 branched along the complement in F of a neighborhood of the singular point z. Note also that X' is obtained by deleting from X a small open neighborhood of $f^{-1}(z)$. We have

$$\Xi_p(K) = -\tfrac{1}{4}(p-1)e(F) - \sigma(X', \partial X'), \tag{6}$$

where $e(F)$ denotes the self-intersection number of F. This is a special case of the signature formula for dihedral branched covers over an arbitrary base[3] given in [9, Theorem 7]. Note that, when F is orientable and X a manifold, (6) reduces to (1), that is, $\Xi_p(K) = -\sigma(X)$. In this case, the Ξ_p invariant of a singularity can be understood entirely in terms of the signature of the branched cover and, in particular, can be computed using four-manifold techniques. We further note that it is possible to realize *all* connected sums $\#^n \mathbb{CP}^2$ as 3–fold dihedral covers of S^4 with one knot singularity on a connected, embedded branching set, if one allows the branching set to be nonorientable [1]. By contrast, we see in Theorem 5 that orientability of the branching set, together with a single singular point, implies that the signature of such a cover is odd.

3 Knots with equal topological and dihedral genera

In this section we construct families of knots for which the topological, ribbon and 3–dihedral genus are equal. We use trisections of four-manifolds [8], triplane diagrams [15], and singular triplane diagrams [3], all of which we review informally for the reader's convenience.

Given a smooth, oriented, 4–manifold X, a $(g; k_1, k_2, k_3)$–trisection of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ into three 4–handlebodies with boundary such that

- $X_i \cong \natural^{k_i}(B^3 \times S^1)$,
- $X_1 \cap X_2 \cap X_3 \cong \Sigma_g$ is a closed, oriented surface of genus g,
- $Y_{ij} = \partial(X_i \cup X_j) \cong \#^{k_l}(S^2 \times S^1)$, where $i, j, l \in \{1, 2, 3\}$ are distinct,
- $\Sigma_g \subset Y_{ij}$ is a Heegaard surface for Y_{ij}.

Every embedded surface $F \subset S^4$ can be described combinatorially by a $(b; c_1, c_2, c_3)$–*triplane diagram* [15]. This is a set of three b–strand trivial tangles (A, B, C) such that

[3] The reference [9] is written in the language of intersection homology. In the case of a singular branched cover $f\colon X \to S^4$, this is equivalent to the Novikov signature $\sigma(X', \partial X')$ since X has only an isolated singularity.

each boundary union of tangles $A \cup \overline{B}$, $B \cup \overline{C}$ and $C \cup \overline{A}$ is a c_i–component unlink, for $i = 1, 2, 3$ respectively. Here $\overline{T}$ denotes the mirror image of T. To obtain F from (A, B, C), one views each of $A \cup \overline{B}$, $B \cup \overline{C}$ and $C \cup \overline{A}$ as unlinks in bridge position in the spokes Y_{12}, Y_{23} and Y_{31} of the standard genus-0 trisection of S^4, glues c_i disks to the components of each of these unlinks, and pushes these disks into the X_i to obtain an embedded surface.

We introduce *singular triplane diagrams* and their colorings in [3]. A $(b; 1, c_2, c_3)$ singular triplane diagram is a triple of b–strand trivial tangles (A, B, C). As above, $B \cup \overline{C}$ and $C \cup \overline{A}$ are c_2– and c_3–component unlinks. $A \cup \overline{B}$ is a knot K. To build a surface with one singularity of type K, one again views each of $A \cup \overline{B}$, $B \cup \overline{C}$ and $C \cup \overline{A}$ in bridge position in the three spokes Y_{12}, Y_{23} and Y_{31} of the standard genus-0 trisection of S^4 and glues c_2 and c_3 disks to the components of each of the two unlinks. Rather than glue disks to $A \cup \overline{B}$, one attaches the cone on K. Note that by interchanging the order of the tangles A and B, one obtains a surface with singularity $\overline{K}$, the mirror of K.

A *p–colored singular triplane diagram* is a singular triplane diagram together with an assignment of values in $\{1, 2, \dots, p\}$ to the arcs of the diagram such that on each tangle, the assignment is a Fox p–coloring and such that the colors along the endpoints of each tangle agree. Such a coloring induces a coloring on the corresponding singular surface.

We use 3–colored singular triplane diagrams to construct a family of 3–fold dihedral covers of S^4 which realize the knots K_m given in Figure 1 as singularities on the branching sets. This construction allows us to compute the values of $\Xi_3(K_m)$ using the induced trisections of the corresponding branched cover. As a corollary, we obtain Theorem 5, which establishes the range of the invariant Ξ_3.

Proposition 4 *Each knot K_m in Figure 1 arises as the only singularity on a 3–fold dihedral branched cover $f_m\colon \#^{2m+1}\,\overline{\mathbb{CP}}^2 \to S^4$ whose branching set F_m is an oriented surface of genus m, embedded smoothly in S^4 away from the one singular point. Equivalently, each knot $\overline{K}_m$ arises as the only singularity on a 3–fold dihedral branched cover $\overline{f}_m\colon \#^{2m+1}\,\mathbb{CP}^2 \to S^4$, also with an embedded oriented branching set of genus m.*

Remark By deleting a small neighborhood of the singularity on the branching set in S^4, one obtains an oriented, 3–colored surface in $F'_m \subset B^4$ with $\partial F'_m = K_m$. In Section 4, we prove that the genus of F'_m is minimal, that is, equal to $g_4(K_m)$. Moreover, by construction, each surface F'_m is ribbon.

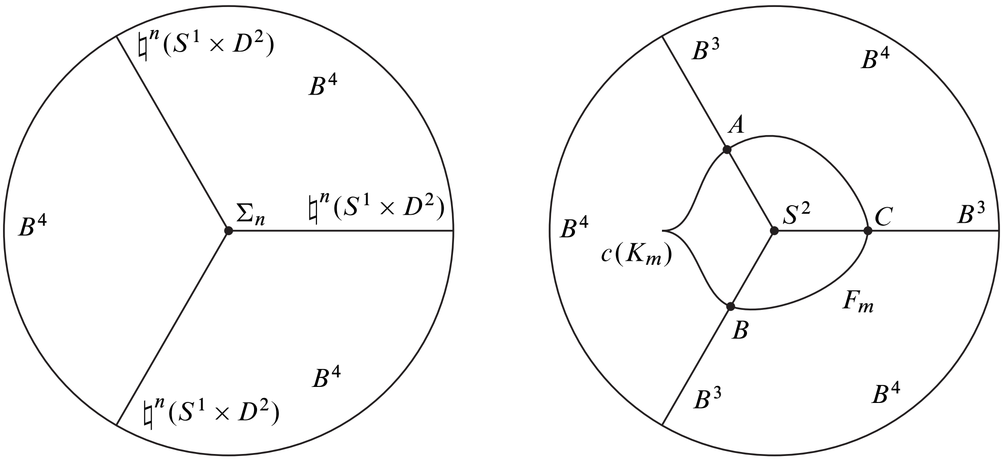

Figure 2: An $(n; 0, 0, 0)$–trisection of $\#^n\,\overline{\mathbb{CP}}^2$, obtained as a branched cover of S^4 over a trisected surface F_m with one singularity K_m.

Proof of Proposition 4 We will construct the surface F_m and will give its Fox coloring using a colored (singular) triplane diagram. From this information, we will produce a trisection of the dihedral cover of S^4 determined by this coloring. We will identify this cover as $\#^n\,\overline{\mathbb{CP}}^2$, where $n = 2m + 1$.

The colored triplane diagram (A_n, B_n, C_n) for F_m, where $m = \frac{1}{2}(n-1)$, is shown in Figure 3. We write the value $i \in \{1, 2, 3\}$ next to an arc of a tangle or knot if the homotopy class of the meridian of that arc is mapped to the reflection in D_3 fixing i.

The union $A_n \cup \overline{B}_n$ is the knot $\overline{K}_m$, while $B_n \cup \overline{C}_n$ and $C_n \cup \overline{A}_n$ are each 2–component unlinks; see Figure 4 for a verification when $n = 3$. A triplane diagram with b bridges and c_i components in each link diagram has Euler characteristic $c_1 + c_2 + c_3 - b$; hence, the surface F_m with singularity $\overline{K}_m$ has Euler characteristic $3 - n$ and genus $m = \frac{1}{2}(n-1)$ since F_m is connected and orientable, and since the tangles A_n, B_n and C_n have $b = n + 2$ bridges.

The fact that F_m is orientable requires a careful check. Consider the cell structure on F_m corresponding to its triplane structure. To show that F_m is orientable, we show that it is possible to coherently orient the faces of this cell structure so that each edge (a bridge in one of the three tangles A_n, B_n or C_n) inherits two different orientations from the two faces adjacent to it. This is shown in Figure 4 in the case $m = 1$ (or $n = 3$).

An Euler characteristic computation shows that the 3–fold dihedral branched cover of the bridge sphere S^2, branched along the $2(n+2)$ endpoints of the bridges, is a

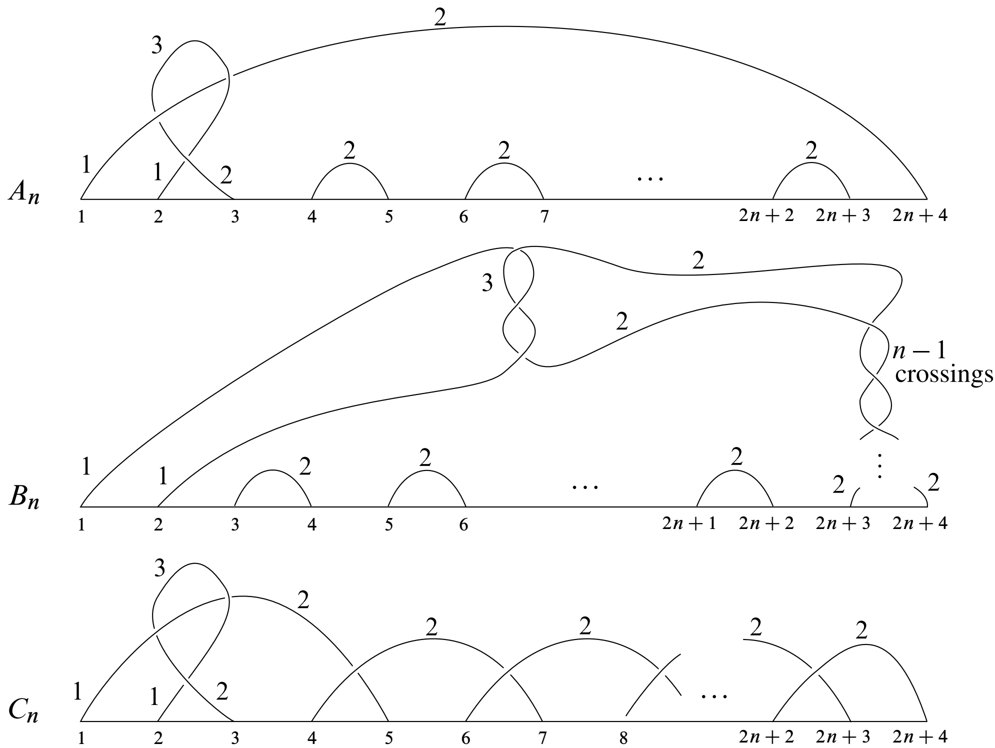

Figure 3: A colored triplane diagram corresponding to a branched covering $\#^n\,\overline{\mathbb{CP}}^2 \to S^4$, in the case where n is odd. The numbers $\{1,2,3\}$ along the arcs describe the coloring. There is one singularity K_m on the branching set, where $m = \frac{1}{2}(n-1)$. By reversing the roles of A_n and B_n, one obtains a branched covering $\#^n\,\mathbb{CP}^2 \to S^4$ with singularity $\overline{K}_m$.

surface Σ_n of genus n. We now show the 3–colored triplane diagram (A_n, B_n, C_n) gives rise to a genus-n trisection of $\#^n\,\overline{\mathbb{CP}}^2$ with central surface Σ_n, following a method explained in [3]. The branching set F_m is orientable and has one singularity of type K_m, so it will follow from (6) that $\Xi_3(K_m) = -\sigma\big(\#^n\,\overline{\mathbb{CP}}^2\big) = n$.

If a properly embedded b–strand tangle $(T, \partial T) \subset (B^3, S^2)$ with arcs $t_1, t_2, \dots, t_b$ is trivial, then by definition there exists a collection of disjoint arcs $d_1, d_2, \dots, d_b$ in S^2 such that the boundary unions $t_i \cup d_i$ bound a collection of disjoint disks in B_3. We refer to the d_i as *disk bottoms*. The existence of such a collection of disks is equivalent to the arcs of T being simultaneously isotopic to a collection of disjoint arcs (the d_i) in S^2.

To determine the trisection diagram, we must first find the disk bottoms for the three tangles A_n, B_n and C_n, then lift them from the bridge sphere S^2 to its irregular

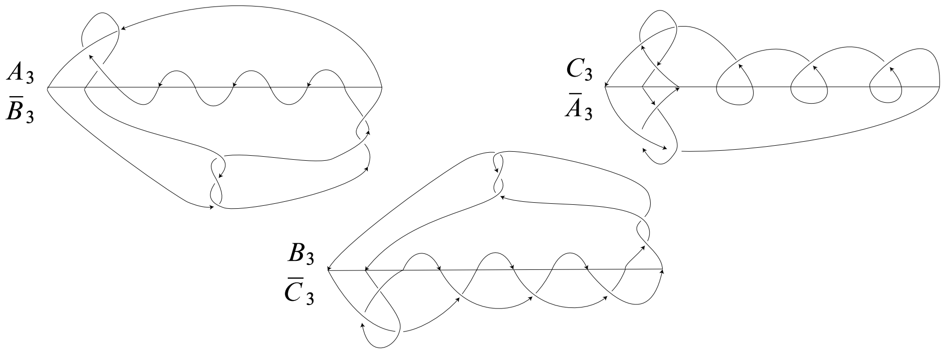

Figure 4: The links $A_3 \cup \overline{B}_3$, $B_3 \cup \overline{C}_3$ and $C_3 \cup \overline{A}_3$. Note that $A_3 \cup \overline{B}_3$ is the knot $\overline{K}_1$.

dihedral cover Σ_n. The curves in the trisection diagram are formed by certain lifts of these disk bottoms; we identify these lifts later.

The disk bottoms for each tangle A_n, B_n and C_n are depicted in Figure 5, in the case $n = 3$. In Figure 7, we draw just three of the disk bottoms for each of A_n (blue), B_n (red) and C_n (green) on the same copy of S^2.

In the next step of the proof, we use a construction of the irregular 3–fold dihedral cover $\Sigma_n \to S^2$, branched along $2(n+2)$ points in S^2, due to Hilden [11]. We review this construction now; the reader should refer to Figure 6 for an example in the case $n = 3$. In this construction, the meridians of two branch points map to the transposition $(2\,3)$ (equivalently, are colored "1"), and the meridians of the remaining $2n+2$ branch points map to the transposition $(1\,3)$ (equivalently, are colored "2"). One first constructs the 6–fold *regular* dihedral cover $R_n \to S^2$ branched along $2(n+2)$ points determined by this coloring. The resulting surface has genus $3n+1$. The 3–fold irregular dihedral cover Σ_n is obtained from this regular one by an involution, namely $180°$ rotation about the vertical axis.

Next, we lift the disk bottoms from the bridge sphere to Σ_n, where Σ_n is constructed as above. Each disk bottom has three lifts to Σ_n, two of which fit together to form a closed curve. Not all of these closed curves are necessarily essential curves on Σ_n; see [3] for further examples. However, we may choose $n-2$ disk bottoms for each tangle (A_n, B_n, C_n) whose lifts are essential. These lifts are shown in Figure 8, again in the case $n = 3$.

The resulting curves form a trisection diagram for $\#^n \overline{\mathbb{CP}}^2$. Moreover, the standard trisection of S^4, branched along F_m, lifts to an $(n; 0, 0, 0)$–trisection of $\#^n \overline{\mathbb{CP}}^2$. This

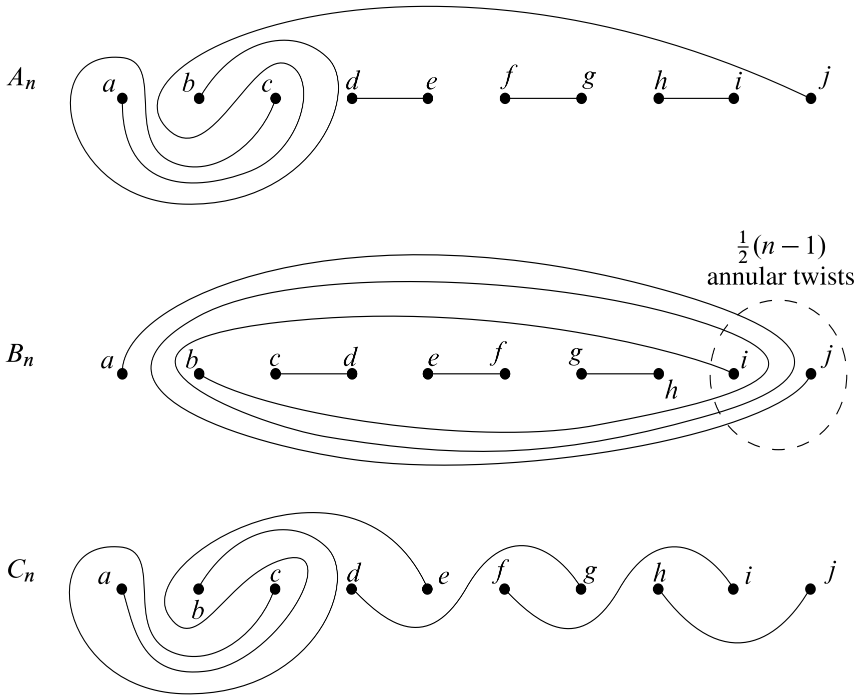

Figure 5: Disk bottoms for the triplane diagram (A_n, B_n, C_n) when $n = 3$.

can be found by analyzing the lifts of the three pieces of the trisection of (S^4, F_m); for details see [3, Theorem 8]. □

We use the above construction to establish the range of the invariant Ξ_3.

Theorem 5 *Let n be an integer. There exists a* ***strongly*** *3–admissible singularity K_n and a 3–coloring ρ_n of K_n such that $\Xi_3(K_n, \rho_n) = n$ if and only if $n \in 2\mathbb{Z} + 1$.*

Remark The proof of Theorem 5 is slightly more general than what the theorem statement requires. That is, we establish that $\Xi_p(K, \rho)$ is odd whenever $p \equiv 3 \bmod 4$. Realizability of all odd integers by Ξ_p is open for $p \neq 3$.

Proof of Theorem 5 We have given a construction realizing each of the knots K_m as the only singularity on a branched cover $\#^{2m+1}\,\overline{\mathbb{CP}}^2 \to S^4$ whose branching set is oriented. By (6), it follows that $\Xi_3(K_m) = -\sigma\big(\#^{2m+1}\,\overline{\mathbb{CP}}^2\big) = 2m+1$, where $m \geq 0$. Note also that $\Xi_p(\overline{K}_m) = -\Xi_p(K_m)$, as proved in [3], where $\overline{K}$ denotes the mirror image of K. Of course, K is (strongly) p–admissible if and only if $\overline{K}$ is. This

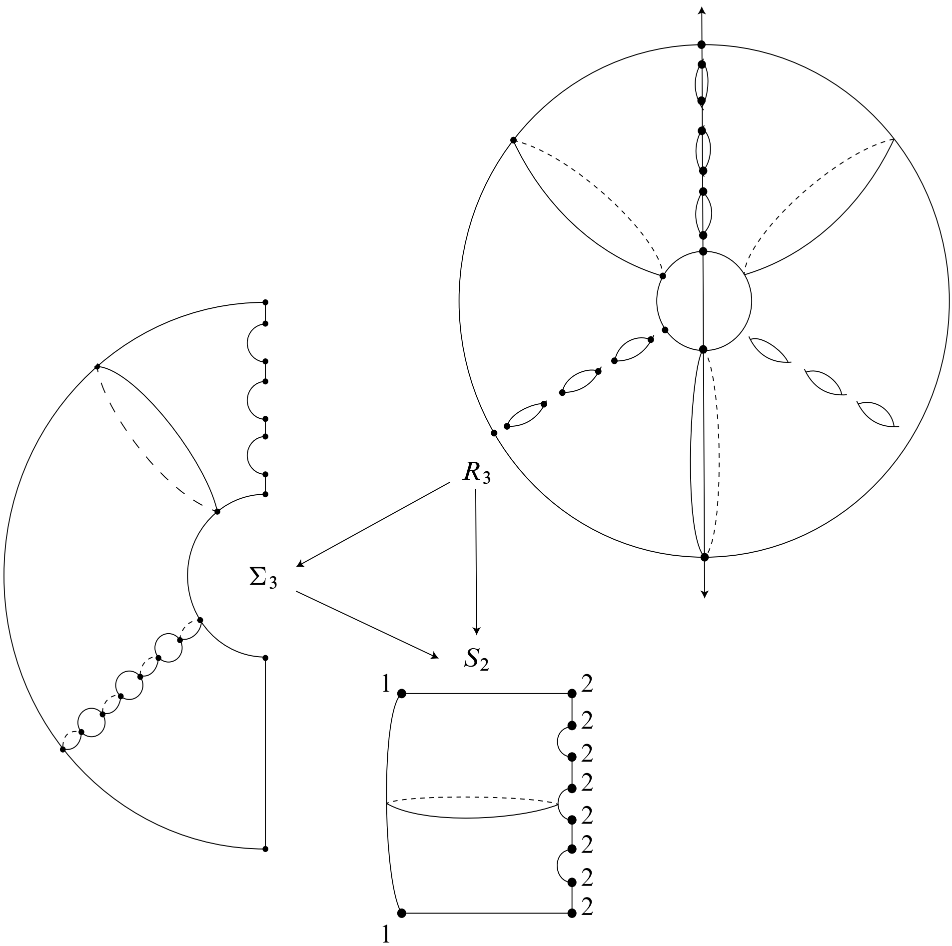

Figure 6: A 6–fold regular dihedral cover R_n of S^2 branched along $2(n+2)$ points, with $n=3$; the irregular cover Σ_n is the quotient of R_n by $180°$ rotation about the vertical axis.

proves that all odd integers are contained in the range of the invariant Ξ_3 on strongly 3–admissible knots.

Conversely, we will verify that for any p–coloring ρ of any strongly p–admissible singularity K, the integer $\Xi_p(K,\rho)$ is odd. It suffices to assume that $p \equiv 3 \bmod 4$. We use (5). Since p is odd, $p^2 \equiv 1 \bmod 4$, so $(p^2-1)/(6p)\, L_V(\beta,\beta)$ is even. It follows from [13, Equation 2.20] that, if $p \equiv 3 \bmod 4$, the rank of $H_2(W(K,\beta);\mathbb{Z})$ is odd, and hence so is the signature. Lastly, each σ_{ζ^i} is an even integer. It follows that $\Xi_p(K)$ is odd. □

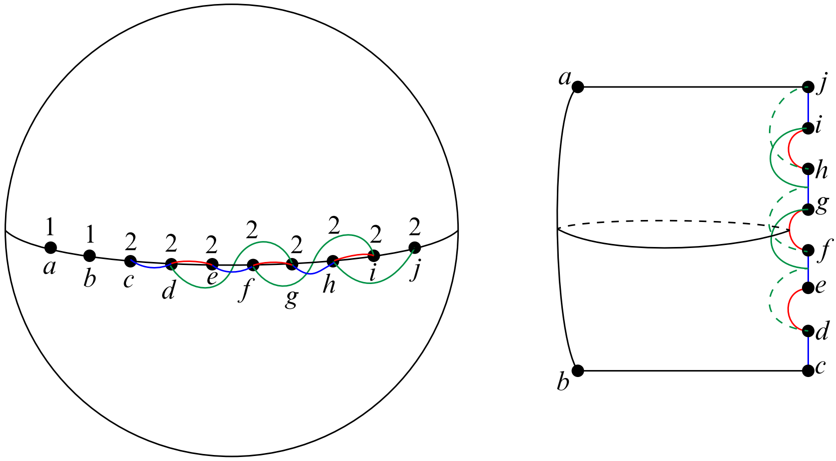

Figure 7: Disk bottoms for the triplane diagram (A_n, B_n, C_n) when $n = 3$, drawn on the bridge sphere.

Remark The knot K_m has bridge number 2, showing that two-bridge knots realize the full range of Ξ_p when $p = 3$. This answers a question posed in [12]. It is not known whether the full range of Ξ_p is realized by two-bridge knots when $p \neq 3$. It would be of interest to establish that it is "sufficient" to consider two-bridge knots when constructing singular dihedral covers of four-manifolds since p–admissibility is particularly easy to detect for two-bridge singularities [14].

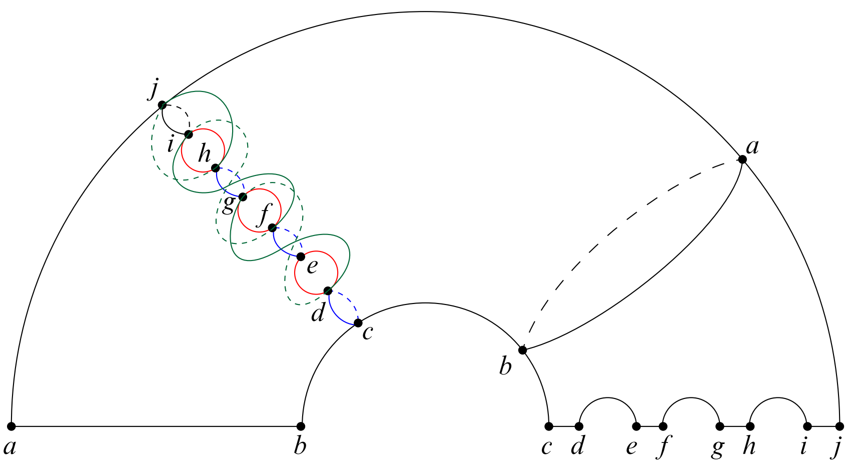

Figure 8: Lifts of disk bottoms to the 3–fold irregular dihedral cover of S^2, for the triplane diagram (A_n, B_n, C_n), when $n = 3$.

4 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 (A) Given a p–admissible knot K with p–coloring ρ, we wish to prove the inequality

$$\mathfrak{g}_p(K,\rho) \geq \frac{|\Xi_p(K,\rho)| - \operatorname{rk} H_1(M)}{p-1} - \frac{1}{2},$$

where M denotes the dihedral cover of S^3 branched along K corresponding to ρ. Throughout, all homology groups are with $\mathbb{Z}$ coefficients.

For K a p–admissible knot, by the definition of homotopy-ribbon dihedral genus, there exists a topologically locally flat orientable homotopy-ribbon surface F' for K such that the genus of F' equals $\mathfrak{g}_p(K,\rho)$. (If ρ does not extend over any locally flat, orientable, homotopy-ribbon surface, $\mathfrak{g}_p(K,\rho) = \infty$ and the inequality is trivial.) Recall that, since F' is orientable, $|\Xi_p(K,\rho)| = \sigma(X', \partial X')$; the right-hand side denotes the Novikov signature of X' as a manifold with boundary. We will find an upper bound for $|\sigma(X', \partial X')|$ in terms of the Euler characteristic of $X = X' \cup c(\partial X')$.

Let

$$\overline{\rho}\colon \pi_1(B^4 - F') \to D_p$$

be the homomorphism which extends the coloring $\rho\colon \pi_1(S^3 - K) \to D_p$ and induces the cover $X' \to B^4$ branched over F'. Let $\widehat{M}$ be the unbranched irregular dihedral cover of $S^3 - K$ corresponding to ρ, and M the induced branched cover. Denote by $F \subset S^4$ the singular surface which is the boundary union of F' and the cone on K, so that X is the dihedral cover of S^4 with branching set F.

We will show that X is simply connected. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\ \mathsf{i}_*\ } & \pi_1(X') \\
\uparrow{\scriptstyle \mathsf{i}_{M*}} & & \uparrow{\scriptstyle \mathsf{i}_{X*}} \\
\pi_1(\widehat{M}) & \xrightarrow{\ j_*\ } & \pi_1(\widehat{X}') \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\
\pi_1(S^3 - K) & \xrightarrow{\ i_*\ } & \pi_1(B^4 - F') \\
\downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \overline{\rho}} & \\
D_p & &
\end{array}$$

All maps in the diagram are either induced by inclusions or by covering maps. Clearly p_* and q_* are injective, as they are induced by covering maps, and i_{M*} and i_{X*}

are surjective, as they are induced by inclusions of unbranched covering spaces into their branched counterparts. The homomorphisms ρ and $\overline{\rho}$ are surjective by definition. Finally, since F' is a homotopy-ribbon surface for K, the homomorphism i_* is surjective.

We now show that j_* is surjective as well. Consider an element $\gamma \in \pi_1(\widehat{X}')$. Since i_* is surjective, there exists an element $\delta \in \pi_1(S^3 - K)$ such that $i_*(\delta) = q_*(\gamma)$. We have that $\overline{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2\mathbb{Z} \subset D_p$, the reflection subgroup which determines the cover $\widehat{X}'$ of $B^4 - F'$. By commutativity of the lower triangle, $\rho(\delta) = \overline{\rho} \circ q_*(\gamma) \in \mathbb{Z}/2$, so $\delta \in \operatorname{im} p_*$. Take $\widetilde{\delta} \in \pi_1(\widehat{M})$ such that $p_*(\widetilde{\delta}) = \delta$. Consider $q_* \circ j_*(\widetilde{\delta})$, which by commutativity is equal to $i_* \circ p_*(\widetilde{\delta})$. Now $q_* \circ j_*(\widetilde{\delta}) = i_*(\delta) = q_*(\gamma)$. By injectivity of q_*, we have $j_*(\widetilde{\delta}) = \gamma$, so j_* is indeed surjective.

Observe that, since j_* and i_{X*} are both surjective, $\mathsf{i}_*\colon \pi_1(M) \to \pi_1(X')$ is surjective as well.

By the Seifert–van Kampen theorem, we have $\pi_1(X) \simeq \pi_1(c(M)) *_{\pi_1(M)} \pi_1(X')$. The cone $c(M)$ is contractible, so $\pi_1(c(M))$ is trivial. Hence

$$\pi_1(c(M)) *_{\pi_1(M)} \pi_1(X') \simeq \pi_1(X')/\operatorname{im} \mathsf{i}_*(\pi_1(M)).$$

The quotient $\pi_1(X')/\operatorname{im} \mathsf{i}_*$ is trivial since i_* is surjective. Hence X is simply connected and, in particular, $H_1(X) = 0$. We also know that $H_0(X) = 1$ since X is path-connected.

Next, consider

$$\chi(X) = \operatorname{rk} H_4(X) - \operatorname{rk} H_3(X) + \operatorname{rk} H_2(X) + 1.$$

Since $X = X' \cup c(\partial X')$,

$$H_n(X) = H_n(X'/\partial X') = \widetilde{H}_n(X', \partial X').$$

By Lefschetz duality and universal coefficients, we see that

$$\operatorname{rk} H_3(X) = \operatorname{rk} H_3(X', \partial X') = \operatorname{rk} H^1(X') = \operatorname{rk} H_1(X'),$$

and

$$\operatorname{rk} H_4(X) = \operatorname{rk} H_4(X', \partial X') = \operatorname{rk} H^0(X') = \operatorname{rk} H_0(X') = 1,$$

and hence

$$\chi(X) = 2 + \operatorname{rk} H_2(X) - \operatorname{rk} H_1(X').$$

By [13, Equation 1.1] we have

(7) $$\chi(X) = 2p - \tfrac{1}{2}(p-1)\chi(F) - \tfrac{1}{2}(p-1).$$

Recall that the Novikov signature $\sigma(X, \partial X')$ is the signature of the intersection form defined on the image of the map $\mathrm{i}_*\colon H_2(X') \to H_2(X', \partial X')$ induced by inclusion. Thus,

$$|\Xi_p(K,\rho)| = \sigma(X, \partial X') \le \mathrm{rk}\,\mathrm{im}\,\mathrm{i}_* \le \mathrm{rk}\, H_2(X', \partial X') = \mathrm{rk}\, H_2(X). \tag{8}$$

The result follows by combining this inequality with the two formulas for $\chi(X)$ above. We substitute

$$\chi(F) = 2 - 2g(F)$$

into (7), and

$$\mathrm{rk}\, H_2(X) = \chi(X) - 2 + \mathrm{rk}\, H_1(X')$$

into (8), which gives

$$|\Xi_p(K,\rho)| \le 2p - \tfrac{1}{2}(p-1)(2-2g(F)) - \tfrac{1}{2}(p-1) - 2 + \mathrm{rk}\, H_1(X').$$

Simplifying, we obtain

$$g(F) \ge \frac{|\Xi_p(K,\rho)| - \mathrm{rk}\, H_1(X')}{p-1} - \frac{1}{2}.$$

Finally, since the inclusion $\mathrm{i}\colon M \to X'$ induces a surjection on fundamental groups, we also know that $\mathrm{rk}\, H_1(M) \ge \mathrm{rk}\, H_1(X')$. Hence

$$g(F) \ge \frac{|\Xi_p(K,\rho)| - \mathrm{rk}\, H_1(M)}{p-1} - \frac{1}{2}.$$

(B) Let K be a p–admissible knot with respect to a coloring ρ and let $F' \subset B^4$ be a homotopy-ribbon, locally flat oriented surface with boundary K such that ρ extends over F'. Denote by $F \subset S^4$ the surface with singularity K obtained as a boundary union of F' and the cone on K, and denote by X the dihedral cover of S^4 determined by the induced coloring of F. We assume that X is a definite manifold. In particular, K is in fact strongly admissible with respect to the coloring ρ, and the corresponding branched cover M of S^3 along K is again S^3. We then wish to show that the inequality (3) is sharp:

$$\mathfrak{g}_p(K,\rho) = \frac{|\Xi_p(K,\rho)|}{p-1} - \frac{1}{2}.$$

Precisely, we will show that the right-hand side of this equation equals the genus of F'. That is, F' will be seen to realize the lower bound from (A) on the dihedral genus $\mathfrak{g}_p(K,\rho)$ of K.

Since X is a definite manifold, $\operatorname{rk} H_2(X) = |\sigma(X)|$. By the proof of (A), X is simply connected; by Poincaré duality we have $\chi(X) = 2 + \operatorname{rk} H_2(X)$ and hence

$$|\Xi_p(K,\rho)| = |\sigma(X)| = \chi(X) - 2.$$

On the other hand, denoting by $g(F)$ the genus of F, by (7) we have

$$\chi(X) = 2p - \tfrac{1}{2}(p-1)(2-2g(F)) - \tfrac{1}{2}(p-1).$$

Putting these two equations together, we conclude that

$$g(F) = \frac{|\Xi_p(K,\rho)|}{p-1} - \frac{1}{2}.$$

By assumption, the coloring ρ extends over F', so

$$g(F') \geq \mathfrak{g}_p(K,\rho) \geq \frac{|\Xi_p(K,\rho)|}{p-1} - \frac{1}{2}.$$

Thus, F' realizes the p–dihedral genus of K.

In the second part of the theorem, we assume in addition that

$$|\sigma(K)| = \frac{2|\Xi_p(K,\rho)|}{p-1} - 1,$$

where $\sigma(K)$ is the signature of the knot K. We wish to show that the topological four-genus and the topological homotopy-ribbon p–dihedral genus of K are both equal to $\frac{1}{2}|\sigma(K)|$.

The additional assumption here can be rewritten as $|\sigma(K)| = 2\mathfrak{g}_p(K,\rho)$. Murasugi's signature bound [16, Theorem 9.1] states that $g_4(K) \geq \frac{1}{2}|\sigma(K)|$. Thus, we have $g_4(K) \geq \mathfrak{g}_p(K,\rho)$. But $g_4(K) \leq \mathfrak{g}_p(K) \leq \mathfrak{g}_p(K,\rho)$ in general, so $g_4(K) = \mathfrak{g}_p(K)$. □

Proof of Theorem 2 Our aim is to show that the equalities

$$g_4(K_m) = \mathfrak{g}_3(K_m) = \tfrac{1}{2}|\Xi_3(K_m,\rho_m)| - \tfrac{1}{2} = m$$

hold for the 3–colored knots K_m introduced in the previous section. In particular, it will follow that the generalized topological slice-ribbon conjecture holds for these knots.

By Theorem 1(B), it suffices to show that

(1) each K_m is the boundary of a homotopy-ribbon surface F'_m such that $\mathfrak{g}_3(K) = g(F'_m)$, and

(2) the signature $\sigma(K_m)$ satisfies the equality
$$|\sigma(K)| = \frac{2|\Xi_p(K,\rho)|}{p-1} - 1$$
for $p=3$.

We first address (1). Surfaces F'_m realizing the lower bound on dihedral homotopy-ribbon genus for the knots K_m are constructed in the proof of Proposition 4: we have shown $g(F'_m) = m$ and $|\Xi_3(K_m)| = 2m+1$, so
$$\frac{|\Xi_p(K_m)|}{p-1} - \frac{1}{2} = m.$$

We note that, since the knots K_m are two-bridge, each of them has a unique 3–coloring (up to permuting the colors), so there is no distinction between $\mathfrak{g}_p(K_m, \rho_m)$ and $\mathfrak{g}_p(K_m)$. By construction, the surface $F'_m \subset B^4$ obtained by deleting a small neighborhood of the singularity K_m is ribbon since $A_m \cup \overline{B}_m$ only bounds the cone on K_m, while the unlinks $B_m \cup \overline{C}_m$ and $C_m \cup \overline{A}_m$ bound standard unknotted disks in B^4.

We now address (2). We will compute the signature $\sigma(K_m)$, and show it is equal to $2m = 2|\Xi_p(K)|/(p-1) - 1$.

The signature of K can be computed using the Goeritz matrix $G(K)$, the matrix of a quadratic form associated to a knot diagram via a checkerboard coloring, and hence a (not necessarily orientable) spanning surface; this technique was introduced by Gordon and Litherland [10]. The advantage of this technique is that the dimension of the Goeritz matrix associated to a projection of a knot may be much smaller than the dimension of the corresponding Seifert matrix; indeed, the dimension of $G(K_m)$ is 4 for all m.

Gordon and Litherland proved that the signature of a knot is equal the signature of the Goeritz matrix of a diagram of the knot plus a certain correction term: $\sigma(K) = \sigma(G(K)) - \mu$. We start by computing the Goeritz matrix $G(K_m)$ and its signature.

One first computes the *unreduced* Goeritz matrix. To do this, one chooses a checkerboard coloring of the knot diagram, and labels the "white" regions $X_1, \dots, X_k$. Such a labeling for the K_m is shown in Figure 9. The entries g_{ij} of the unreduced Goeritz matrix are computed as follows:
$$g_{ij} = \begin{cases} -\sum \eta(c) & \text{for } i \neq j \text{ and } c \text{ a double point incident to } X_i \text{ and } X_j, \\ -\sum_{s\in\{1,\dots,k\}\setminus\{i\}} g_{is} & \text{for } i = j. \end{cases}$$
The signs $\eta(c)$ are computed as in Figure 10; shaded areas correspond to "black" regions of the checkerboard coloring.

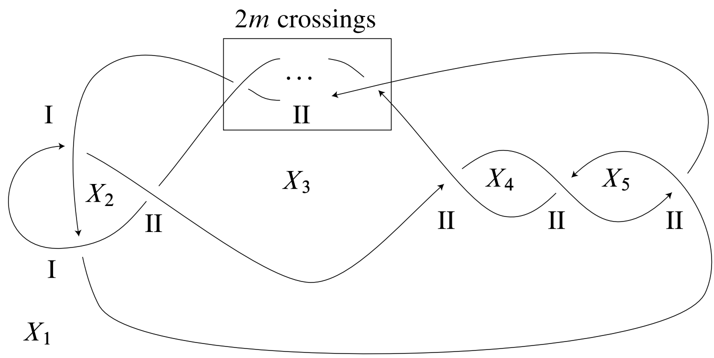

Figure 9: The "white" regions of a checkerboard coloring of K_m, labeled $X_1, X_2, \dots, X_5$.

The unreduced Goeritz matrix of K_m is

$$G'(K_m) = \begin{pmatrix} -2m-3 & -2 & -2m & 0 & -1 \\ -2 & -3 & -1 & 0 & 0 \\ -2m & -1 & -2m-2 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 \\ -1 & 0 & 0 & -1 & -2 \end{pmatrix}.$$

The Goeritz matrix $G(K_m)$ is obtained by deleting the first row and column of $G'(K_m)$. The characteristic polynomial of this matrix is

$$p_{G(K_m)}(\lambda) = (\lambda+3)(\lambda(\lambda+3)^2 + 2(\lambda+1)(\lambda+3)m + 3).$$

Hence $\lambda = -3$ is an eigenvalue. In addition, since $m \geq 0$, it is straightforward to verify that any root of the cubic factor must be negative (if λ is nonnegative, the cubic, as written above, is a sum of three nonnegative terms). Hence, $\sigma(G(K_m)) = -4$.

The correction term $\mu(K)$ in Gordon and Litherland's formula for $\sigma(K)$ is computed as follows. Each crossing c of K can be classified as type I or type II, as shown in Figure 10. Let $\mu(K) = \sum_c \eta(c)$, where the sum is taken over all type II crossings.

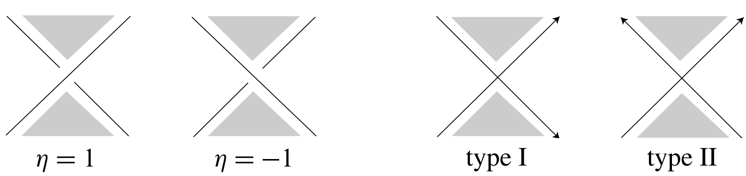

Figure 10: Incidence numbers η and type I and II crossings.

The knot K_m has $4+2m$ type II crossings, each of negative sign; see Figure 9. Hence $\sigma(K_m) = -4 + (4+2m) = 2m$. □

Proof of Theorem 3 Given any nonnegative integer m, our goal is to construct an infinite family of knots whose 3–dihedral and topological 4–genus are both equal to m. Let K_m denote the knot given in Theorem 2 whose 3–dihedral and topological 4–genus equal m. We will prove that, given a nontrivial ribbon knot γ, the knot $K_m \# \gamma$ has the desired property. The theorem follows by taking repeated connect sums of K_m with γ.

Let γ denote any ribbon knot and let $D \subset B^4$ be a ribbon disk with $\partial D = \gamma$. The knot $K_m \# \gamma$ has 3–dihedral genus and topological four-genus equal to m, as we now show. It is clear that the smooth and topological four-genera of $K_m \# \gamma$ are both equal to m since the knot is smoothly concordant to K_m. Next, note that the given 3–coloring ρ_m of K_m induces a 3–coloring ρ_γ of $K_m \# \gamma$ which restricts trivially to γ. Moreover, since ρ_m extends over F'_m, ρ_γ extends over the ribbon surface $F'_m \natural D$, where $\natural$ denotes boundary connected sum. Therefore, the ribbon 3–dihedral genus of $K_m \# \gamma$ is at most m. Since g_4 is a lower bound for the topological 3–dihedral genus, which in turn is a lower bound for the ribbon 3–dihedral genus, it follows that these genera are equal, as claimed. □

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