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A LOWER BOUND FOR THE KÄHLER-EINSTEIN DISTANCE FROM THE DIEDERICH-FORNÆSS INDEX

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Abstract. In this paper we establish a lower bound for the distance induced by the Kähler-Einstein metric on pseudoconvex domains with positive hyperconvexity index (e.g. positive Diederich-Fornæss index). A key step is proving an analog of the Hopf lemma for Riemannian manifolds with Ricci curvature bounded from below.

1. Introduction

Every bounded pseudoconvex domain $\Omega \subset \mathbb{C}^m$ has a unique complete Kähler-Einstein metric, denoted by g_{KE}, with Ricci curvature $-(2m-1)$. This was constructed by Cheng and Yau [4] when Ω has $\mathcal{C}^2$ boundary and by Mok and Yau [9] in general.

Let dist_{KE} be the distance induced by g_{KE}. Since g_{KE} is complete, if we fix $z_0 \in \Omega$, then

$$\lim_{z \to \partial\Omega} \operatorname{dist}_{KE}(z, z_0) = \infty. \tag{1.1}$$

In this note we consider quantitative versions of Equation (1.1). In particular, it is natural to ask for lower bounds on $\operatorname{dist}_{KE}(z, z_0)$ in terms of the Euclidean distance to the boundary

$$\delta_\Omega(z) := \min\{\|w - z\| : w \in \partial\Omega\}.$$

Mok and Yau proved for every $z_0 \in \Omega$ there exist $C_1, C_2 > 0$ such that

$$\operatorname{dist}_{KE}(z, z_0) \geq -C_1 + C_2 \log\log \frac{1}{\delta_\Omega(z)}$$

for all $z \in \Omega$, see [9, pg. 47]. Further, by considering the case of a punctured disk, this lower bound is the best possible for general pseudoconvex domains.

However, for certain classes of bounded pseudoconvex domains, there are much better lower bounds. For instance, if Ω is convex, then for any $z_0 \in \Omega$ there exist $C_1, C_2 > 0$ such that

$$\operatorname{dist}_{KE}(z, z_0) \geq -C_1 + C_2 \log \frac{1}{\delta_\Omega(z)} \tag{1.2}$$

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for all $z \in \Omega$, see [7]. In this note, we show that Estimate (1.2) holds for a large class of domains - those with positive hyperconvexity index.

First we recall the well studied Diederich-Fornæss index. Suppose $\Omega \subset \mathbb{C}^m$ is a bounded pseudoconvex domain. A number $\tau \in (0,1)$ is a called a *Diederich-Fornæss exponent of* Ω if there exist $C > 1$ and a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ such that

$$\frac{1}{C}\delta_\Omega(z)^\tau \leq -\psi(z) \leq C\delta_\Omega(z)^\tau$$

for all $z \in \Omega$. Then the *Diederich-Fornæss index of* Ω is defined to be

$$\eta(\Omega) := \sup\{\tau : \tau \text{ is a Diederich-Fornæss exponent of } \Omega\}.$$

It is known that $\eta(\Omega) > 0$ for many domains. For instance, Diederich-Fornæss [5] proved that $\eta(\Omega) > 0$ when $\partial\Omega$ is $\mathcal{C}^2$. Later, Harrington [8] generalized this result and proved that $\eta(\Omega) > 0$ when $\partial\Omega$ is Lipschitz.

The hyperconvexity index, introduced by Chen [3], is a similar quantity associated to a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^m$. In particular, a number $\tau \in (0,1)$ is a called a *hyperconvexity exponent of* Ω if there exist $C > 1$ and a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ such that

$$-\psi(z) \leq C\delta_\Omega(z)^\tau$$

for all $z \in \Omega$. Then the *hyperconvexity index of* Ω is defined to be

$$\alpha(\Omega) := \sup\{\tau : \tau \text{ is a hyperconvexity exponent of } \Omega\}.$$

By definition $\alpha(\Omega) \geq \eta(\Omega)$. Further, it is sometimes easier to verify that the hyperconvexity index is positive, see [3, Appendix].

For domains with positive hyperconvexity index we will establish the following lower bound for dist_{KE}.

Theorem 1.1. *Suppose $\Omega \subset \mathbb{C}^m$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$. If $z_0 \in \Omega$ and $\epsilon > 0$, then there exists some $C = C(z_0, \epsilon) \leq 0$ such that*

$$\mathrm{dist}_{KE}(z, z_0) \geq C + \left(\frac{\alpha(\Omega)}{2m-1} - \epsilon\right)\log\frac{1}{\delta_\Omega(z)}$$

for all $z \in \Omega$.

Remark 1.2. In this note we have normalized the Kähler-Einstein metric to have Ricci curvature equal to $-(2m-1)$. If we instead normalized so that the Ricci curvature equals $-(2m-1)\lambda$ we would obtain the lower bound

$$C + \frac{1}{\sqrt{\lambda}}\left(\frac{\alpha(\Omega)}{2m-1} - \epsilon\right)\log\frac{1}{\delta_\Omega(z)}.$$

Theorem 1.1 is a consequence of the following more general result which shows that Estimate (1.2) holds for any complete Kähler metric with Ricci curvature bounded from below.

Theorem 1.3. *Suppose $\Omega \subset \mathbb{C}^m$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$, g is a complete Kähler metric on Ω with $\mathrm{Ric}_g \geq -(2m-1)$, and dist_g is the distance associated to g. If $z_0 \in \Omega$ and $\epsilon > 0$, then there exists some $C = C(z_0, \epsilon) \leq 0$ such that*

$$\mathrm{dist}_g(z_0, z) \geq C + \left(\frac{\alpha(\Omega)}{2m-1} - \epsilon\right)\log\frac{1}{\delta_\Omega(z)}$$

for all $z \in \Omega$.

1.1. **Lower bounds on the Bergman metric.** It is conjectured that the Bergman distance on a bounded pseudoconvex domain with $\mathcal{C}^2$ boundary also satisfies Estimate (1.2). In this direction, the best general result is due to Błocki [1] who extended work of Diederich-Ohsawa [6] and established a lower bound of the form

$$C_1 + C_2 \frac{1}{\log\log(1/\delta_\Omega(z))} \log \frac{1}{\delta_\Omega(z)}$$

for the Bergman distance on a bounded pseudoconvex domain with $\mathcal{C}^2$ boundary.

Notice that Theorem 1.3 implies the conjectured lower bound for the Bergman distance under the additional assumption that the Ricci curvature of the Bergman metric is bounded from below.

2. A Hopf lemma for Riemannian manifolds

The standard proof of the Hopf lemma implies the following estimate:

Proposition 2.1 (Hopf Lemma). *If $D \subset \mathbb{R}^m$ is a bounded domain with $\mathcal{C}^2$ boundary and $\varphi : D \to (-\infty, 0)$ is subharmonic, then there exists $C > 0$ such that*

$$\varphi(x) \leq -C\delta_D(x)$$

for all $x \in D$.

We will prove a variant of (this version of) the Hopf Lemma for Riemannian manifolds with Ricci curvature bounded below.

Given a complete Riemannian manifold (X, g), let dist_g denote the distance induced by g, let ∇_g denote the gradient, and let Δ_g denote the Laplace-Beltrami operator on X. A function $\varphi : X \to \mathbb{R}$ is *subharmonic* if $\Delta_g \varphi \geq 0$ in the sense of distributions. Also, for $x \in X$ and $r > 0$ define

$$B_g(x, r) = \{y \in X : \operatorname{dist}_g(x, y) < r\}.$$

Proposition 2.2. *Suppose that (X, g) is a complete m-dimensional Riemannian manifold with $\operatorname{Ric}(g) \geq -(m-1)$. If $x_0 \in X$, $\epsilon > 0$, and $\varphi : X \to (-\infty, 0)$ is subharmonic, then there exists $C > 0$ such that*

$$\varphi(x) \leq -C \exp\Big(-(m-1+\epsilon)\operatorname{dist}_g(x, x_0)\Big)$$

for all $x \in X$.

Remark 2.3. In this section we are working in the category of Riemannian manifolds with real dimension m, which explains the lower bound on Ricci curvature of $-(m-1)$, as compared to the lower bound of $-(2m-1)$ in the statement of Theorem 1.3.

We require one lemma.

Lemma 2.4. *Suppose that (X, g) is a complete m-dimensional Riemannian manifold with $\operatorname{Ric}(g) \geq -(m-1)$. Then for every $x_0 \in X$ and $\epsilon > 0$, there exists $r_0 > 0$ such that the function*

$$\Phi(x) = \exp\Big(-(m-1+\epsilon)\operatorname{dist}_g(x, x_0)\Big)$$

is subharmonic on $X \setminus B_g(x_0, r_0)$.

This lemma is probably well known. The proof below is based on an argument of Calabi [2]. The lemma can also be deduced from the non-smooth version of the "Global Laplacian Theorem" stated in [12, Theorem 3.6]. In the special case when the function $x \to \mathrm{dist}_g(x, x_0)$ is smooth on $X \setminus \{x_0\}$ (e.g. X is simply connected and has no conjugate points), the lemma follows from the Laplacian comparison theorem as stated in [10, Lemma 7.1.9].

Proof. It is enough to show that there exists some $r_0 > 0$ such that $\Delta_g \Phi \geq 0$ *in the barrier sense* on $X \setminus B_g(x_0, r_0)$, that is: for every $q \in X \setminus B_g(x_0, r_0)$ and $\delta > 0$ there exists a $\mathcal{C}^2$ function $f : \mathcal{O} \to \mathbb{R}$ defined on a neighborhood $\mathcal{O}$ of q such that:

(1) $f(q) = \Phi(q)$,
(2) $f \leq \Phi$ on $\mathcal{O}$, and
(3) $\Delta_g f \geq -\delta$ on $\mathcal{O}$

(see the discussion in [12, Section 3]).

Fix $r_0 > 1$ sufficiently large. Then fix $q \in X \setminus B_g(x_0, r_0)$ and let $\sigma : [0, T] \to X$ be a unit speed geodesic joining x_0 to q. Then consider the function $r_q(x) = \mathrm{dist}_g(x, \sigma(1)) + 1$ (notice that $T > r_0 > 1$). By the proof of [10, Lemma 7.1.9], q is not in the cut locus of $\sigma(1)$. In particular, there exists a neighborhood $\mathcal{O}$ of q such that r_q is $\mathcal{C}^\infty$ and

$$\|\nabla_g r_q\| \equiv 1$$

on $\mathcal{O}$, see [11, Proposition III.4.8]. By shrinking $\mathcal{O}$ we can assume that $\mathcal{O} \subset X \setminus B_g(x_0, r_0)$. Then, by the smooth Laplacian comparison theorem

$$\begin{aligned}\Delta_g r_q(x) &\leq (m-1)\coth\left(r_q(x) - 1\right) = (m-1)\coth \mathrm{dist}_g(x, \sigma(1)) \\ &\leq (m-1)\coth(r_0 - 1)\end{aligned}$$

on $\mathcal{O}$, see [10, Lemma 7.1.9]. Next consider the function $f : \mathcal{O} \to [0, \infty)$ defined by

$$f(x) = \exp\Big(-(m-1+\epsilon) r_q(x) \Big).$$

Then $f(q) = \Phi(q)$, $f \leq \Phi$ on $\mathcal{O}$, and

$$\begin{aligned}\Delta_g f(x) &= f(x)\left((m-1+\epsilon)^2 \left\|\nabla_g r_q\right\|^2 - (m-1+\epsilon)\Delta_g r_q(x)\right) \\ &\geq f(x)\left((m-1+\epsilon)^2 - (m-1+\epsilon)(m-1)\coth(r_0 - 1)\right).\end{aligned}$$

So for $r_0 > 0$ sufficiently large (which only depends on ϵ), $\Delta_g f \geq 0$ on $\mathcal{O}$.

Hence $\Delta_g \Phi(x) \geq 0$ in the barrier sense on $X \setminus B_g(x_0, r_0)$. □

Proof of Proposition 2.2. Fix $r_0 > 0$ such that

$$x \to \exp\Big(-(m-1+\epsilon)\, \mathrm{dist}_g(x, x_0)\Big)$$

is subharmonic on $X \setminus B_g(x_0, r_0)$. Since $\varphi < 0$, there exists $C > 0$ such that

$$\varphi(x) \leq -C \exp\Big(-(m-1+\epsilon)\, \mathrm{dist}_g(x, x_0)\Big)$$

for all $x \in B_g(x_0, r_0)$. Then consider

$$f(x) = \varphi(x) + C \exp\Big(-(m-1+\epsilon)\, \mathrm{dist}_g(x, x_0)\Big).$$

Then f is subharmonic on $X \setminus B_g(x_0, r_0)$. Fix $R > r_0$ and let

$$A_R = B_g(x_0, R) \setminus B_g(x_0, r_0)$$

Then $f(x) \leq 0$ on $\partial B_g(x_0, r_0)$ and

$$f(x) \leq C \exp\Big(-(m-1+\epsilon)R \Big)$$

on $\partial B_g(x_0, R)$. So by the maximum principle

$$f(x) \leq C \exp\Big(-(m-1+\epsilon)R \Big)$$

on A_R. Then sending $R \to \infty$ shows that

$$f(x) \leq 0$$

on $X \setminus B_g(x_0, r_0)$. So

$$\varphi(x) \leq -C \exp\Big(-(m-1+\epsilon)\operatorname{dist}_g(x, x_0) \Big)$$

for all $x \in X$. □

3. Proof of Theorem 1.3

Suppose $\Omega \subset \mathbb{C}^m$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$, g is a complete Kähler metric on Ω with $\operatorname{Ric}_g \geq -(2m-1)$, $z_0 \in \Omega$, and $\epsilon > 0$.

Fix $\epsilon_1 > 0$ and a hyperconvexity exponent $\tau \in (0,1)$ such that

$$\frac{\tau}{2m-1+\epsilon_1} \geq \frac{\alpha(\Omega)}{2m-1} - \epsilon.$$

Then there exist $a > 1$ and a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ such that

$$-\psi(z) \leq a\delta_\Omega(z)^\tau$$

for all $z \in \Omega$.

Since ψ is plurisubharmonic and g is Kähler, ψ is subharmonic on (Ω, g). So by Proposition 2.2 there exists $C_0 > 0$ such that

$$\psi(z) \leq -C_0 \exp\Big(-(2m-1+\epsilon_1)\operatorname{dist}_g(x, x_0) \Big)$$

for all $z \in \Omega$. Then

$$-a\delta_\Omega(z)^\tau \leq -C_0 \exp\Big(-(2m-1+\epsilon_1)\operatorname{dist}_g(x, x_0) \Big)$$

and so there exists $C_1 \in \mathbb{R}$ such that

$$C_1 + \left(\frac{\tau}{2m-1+\epsilon_1} \right) \log \frac{1}{\delta_\Omega(z)} \leq \operatorname{dist}_g(z, z_0)$$

for all $z \in \Omega$. Since the set $\{z \in \Omega : \delta_\Omega(z) \geq 1\}$ is compact and

$$\frac{\tau}{2m-1+\epsilon_1} \geq \frac{\alpha(\Omega)}{2m-1} - \epsilon,$$

there exists $C \in \mathbb{R}$ such that

$$C + \Big(\frac{\alpha(\Omega)}{2m-1} - \epsilon \Big) \log \frac{1}{\delta_\Omega(z)} \leq \operatorname{dist}_g(z, z_0)$$

for all $z \in \Omega$.

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References

[1] Zbigniew Błocki, *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2613–2625, DOI 10.1090/S0002-9947-05-03738-4. MR2139520

[2] E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry*, Duke Math. J. **25** (1958), 45–56. MR92069

[3] Bo-Yong Chen, *Bergman kernel and hyperconvexity index*, Anal. PDE **10** (2017), no. 6, 1429–1454, DOI 10.2140/apde.2017.10.1429. MR3678493

[4] Shiu Yuen Cheng and Shing Tung Yau, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33** (1980), no. 4, 507–544, DOI 10.1002/cpa.3160330404. MR575736

[5] Klas Diederich and John Erik Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), no. 2, 129–141, DOI 10.1007/BF01390105. MR437806

[6] Klas Diederich and Takeo Ohsawa, *An estimate for the Bergman distance on pseudoconvex domains*, Ann. of Math. (2) **141** (1995), no. 1, 181–190, DOI 10.2307/2118631. MR1314035

[7] Sidney Frankel, *Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 183–208, DOI 10.1090/pspum/052.2/1128543. MR1128543

[8] Phillip S. Harrington, *The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries*, Math. Res. Lett. **15** (2008), no. 3, 485–490, DOI 10.4310/MRL.2008.v15.n3.a8. MR2407225

[9] Ngaiming Mok and Shing-Tung Yau, *Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*, The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), Proc. Sympos. Pure Math., vol. 39, Amer. Math. Soc., Providence, RI, 1983, pp. 41–59. MR720056

[10] Peter Petersen, *Riemannian geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. MR3469435

[11] Takashi Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author. MR1390760

[12] Guofang Wei, *Manifolds with a lower Ricci curvature bound*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 203–227, DOI 10.4310/SDG.2006.v11.n1.a7. MR2408267

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