

Smoothly Bounded Domains Covering Compact Manifolds

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ABSTRACT. We show that if a bounded domain in complex Euclidean space with $C^{1,1}$ boundary covers a compact manifold, then the domain is biholomorphic to the unit ball. In particular, this shows that complex hyperbolic spaces are the only Hermitian symmetric spaces biholomorphic to bounded domains with $C^{1,1}$ boundary.

1. INTRODUCTION

Given a domain $\Omega \subset \mathbb{C}^d$ let $\text{Aut}(\Omega)$ denote the biholomorphism group of Ω . When Ω is bounded, H. Cartan proved that $\text{Aut}(\Omega)$ is a Lie group (with possibly infinitely many connected components) and acts properly on Ω .

We say that a domain $\Omega \subset \mathbb{C}^d$ *covers a compact manifold* if there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ such that Γ acts freely, properly discontinuously, and co-compactly on Ω . The simplest example of a domain which covers a compact manifold is the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$. In this case, $\text{Aut}(\mathbb{B}_d)$ is isomorphic to the matrix group $\text{PU}(1, d)$, and any co-compact torsion-free lattice $\Gamma \leq \text{Aut}(\mathbb{B}_d)$ acts freely, properly discontinuously, and co-compactly on $\mathbb{B}_d$.

In this paper we prove that, up to biholomorphism, the unit ball is the only domain covering a compact manifold with $C^{1,1}$ boundary.

Theorem 1.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain which covers a compact manifold. If $\partial\Omega$ is $C^{1,1}$, then Ω is biholomorphic to the unit ball.*

A bounded domain $\Omega \subset \mathbb{C}^d$ is called *symmetric* if $\text{Aut}(\Omega)$ is a semisimple Lie group which acts transitively on Ω . A theorem of Borel [2] says that every bounded symmetric domain covers a compact manifold, and so we have the following corollary.

Corollary 1.2. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded symmetric domain and $\partial\Omega$ is $C^{1,1}$. Then, Ω is biholomorphic to the unit ball.*

Theorem 1.1 extends a classical result of Rosay and Wong from the 1970s.

Theorem 1.3 (Rosay [15], Wong [22]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain which covers a compact manifold. If $\partial\Omega$ is C^2 , then Ω is biholomorphic to the unit ball.*

Remark 1.4. Wong proved Theorem 1.3 for strongly pseudoconvex domains, and Rosay extended the result to any bounded domain with C^2 boundary.

Over the last forty years many different proofs of Theorem 1.3 have been found, but they all rely on essentially the same idea: every bounded domain $\Omega \subset \mathbb{C}^d$ with C^2 boundary has at least one strongly pseudoconvex boundary point, and the interior complex geometry of Ω near a strongly pseudoconvex boundary point is close to the interior complex geometry of the unit ball. Then, since Ω covers a compact manifold, the interior complex geometry of Ω is everywhere close to the interior complex geometry of the unit ball. Then, a limiting argument shows that Ω is biholomorphic to the ball.

One way to make this precise is to consider the Bergman metric g on Ω . This is a $\text{Aut}(\Omega)$ -invariant Kähler metric on Ω and, since the boundary is C^2 , also complete by a result of Ohsawa [14]. Kim-Yu [9] proved that the holomorphic sectional curvature of g is limited to $-4/(d+1)$ at ξ_0 (see also [10]). Since $\text{Aut}(\Omega)$ acts co-compactly on Ω , for any point $z \in \Omega$ there exists a sequence $\varphi_n \in \text{Aut}(\Omega)$ such that $\varphi_n(z) \rightarrow \xi_0$. Then, by the invariance of g , the holomorphic curvature at z equals $-4/(d+1)$. Since z was arbitrary, (Ω, g) has constant holomorphic curvature and hence, by a theorem of Q. K. Lu [12], Ω is biholomorphic to the ball. (For more details, see Section 5 in [9].)

In the $C^{1,1}$ case it is no longer possible to simply localize around a strongly pseudoconvex point, which makes the argument more complicated.

1.1. A conjecture. Recently, we generalized Theorem 1.3 in a different direction by only assuming that the domain covers a finite volume manifold.

Theorem 1.5 (Z. [24]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^2 boundary and $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group acting freely on Ω . If $\Gamma \backslash \Omega$ has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then Ω is biholomorphic to the unit ball.*

Based on this it seems natural to ask if Theorem 1.1 can also be extended to the finite volume case.

Conjecture 1.6. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with $C^{1,1}$ boundary and $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group acting freely on Ω . If $\Gamma \backslash \Omega$ has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then Ω is biholomorphic to the unit ball.*

1.2. Outline of the proof of Theorem 1.1. Our proof in the $C^{1,1}$ case is substantially different than the standard proofs in the C^2 case and requires both local and global arguments.

Fix a bounded domain $\Omega \subset \mathbb{C}^d$ with $C^{1,1}$ boundary and a discrete group $\Gamma \leq \text{Aut}(\Omega)$ such that Γ acts freely, properly discontinuously, and co-compactly on Ω .

Step 1: For $\alpha > 0$ define

$$\mathcal{P}_\alpha = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \text{Re}(z_1) > \alpha \sum_{j=2}^d |z_j|^2 \right\}.$$

Notice that $\mathcal{P}_\alpha$ is biholomorphic to the unit ball.

We use a rescaling argument to show that Ω is biholomorphic to a domain $D \subset \mathbb{C}^d$ where the following hold:

- (1) There exist $0 < \beta < \alpha$ such that $\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$.
- (2) $\text{Aut}(D)$ contains the one-parameter subgroup

$$u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d).$$

In particular, $\text{Aut}(\Omega) \cong \text{Aut}(D)$ is non-discrete.

Step 2: Next, we use a theorem of Frankel and Nadel to deduce that Ω is a bounded symmetric domain.

Theorem 1.7 (Frankel, Nadel [7, 13]). *Suppose M is a compact complex manifold with $c_1(M) < 0$ and $\widetilde{M}$ is the universal cover of M . If $\text{Aut}(\widetilde{M})$ is non-discrete, then $\widetilde{M}$ is biholomorphic to either a bounded symmetric domain, or a non-trivial product $D_1 \times D_2$ where D_1 is a bounded symmetric domain and $\text{Aut}(D_2)$ is discrete.*

Remark 1.8.

- (1) Nadel [13] proved Theorem 1.7 when $d = 2$, and then Frankel [7] extended the result to all dimensions.
- (2) Theorem 1.7 is one of several rigidity results which consider manifolds whose universal cover has a non-discrete isometry group (see [4, 6, 11]).
- (3) In our setting, the quotient $\Gamma \backslash \Omega$ will be aspherical, and in this special case an alternative proof of Theorem 1.7 can be found in [6].

If $M := \Gamma \backslash \Omega$, then $c_1(M) < 0$ (see the discussion on [7, pg. 286]). Further, the domain Ω is simply connected (see Proposition 3.3), and hence is the universal cover of M . Thus, by Step 1 and Theorem 1.7, we see that Ω is either symmetric or biholomorphic a product $D_1 \times D_2$ where D_1 is symmetric and D_2 has discrete automorphism group. We will use the geometry of the rescaled domain from Step 1 to show that it is impossible for Ω to be biholomorphic to such a product. Thus, Ω is a bounded symmetric domain.

Step 3: To finish the proof of Theorem 1.1 we exploit the geometry of the rescaled domain D and the theory of bounded symmetric domains. Let $\Omega_{HC} \subset \mathbb{C}^d$ be the image of the Harish-Chandra embedding of Ω . Then, by Step 1, there exists a biholomorphism $F : D \rightarrow \Omega_{HC}$.

To show that Ω is biholomorphic to the ball, we introduce the holomorphic function

$$J : \mathbb{D} \rightarrow \mathbb{C}$$

$$J(\lambda) = \det \left(F' \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right) \right),$$

where $F'(z)$ is the Jacobian matrix of F . This function measures the volume contraction/expansion of F along the linear slice

$$\mathbb{C} \cdot e_1 \cap D = \{(z, 0, \dots, 0) \in \mathbb{C}^d : \operatorname{Re}(z) > 0\}$$

of D . Since F is a biholomorphism, J is nowhere zero.

We will estimate J using the “change of variable formula” for the Bergman kernels on D and Ω_{HC} :

$$\kappa_D(z, w) = \kappa_{\Omega_{HC}}(F(z), F(w)) \det(F'(z)) \overline{\det(F'(w))}.$$

Combining this with a formula for the Bergman kernel on Ω_{HC} from [5], we show the following: if Ω is not biholomorphic to the ball, then J extends continuously to $\partial \mathbb{D}$ and $J|_{\partial \mathbb{D}} \equiv 0$. But then the maximal principle would imply that $J \equiv 0$, which is impossible. Thus, Ω is biholomorphic to the unit ball. A key part in this step is showing that

$$\lambda \in \mathbb{D} \rightarrow F \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right) \in \Omega_{HC}$$

parameterizes the diagonal of a maximal polydisk in Ω_{HC} .

2. PRELIMINARIES

2.1. Notation. For $z_0 \in \mathbb{C}^d$ and $r > 0$ let

$$\mathbb{B}_d(z_0; r) = \{z \in \mathbb{C}^d : \|z - z_0\| < r\}.$$

Also let $e_1, \dots, e_d$ denote the standard basis of $\mathbb{C}^d$.

2.2. The Kobayashi metric. Given a domain $\Omega \subset \mathbb{C}^d$, the (infinitesimal) Kobayashi metric is the pseudo-Finsler metric

$$k_\Omega(x; v) = \inf \{ |\xi| : f \in \operatorname{Hol}(\Delta, \Omega), f(0) = x, d(f)_0(\xi) = v \}.$$

By a result of Royden [16, Proposition 3] the Kobayashi metric is an upper semi-continuous function on $\Omega \times \mathbb{C}^d$. In particular, if $\sigma : [a, b] \rightarrow \Omega$ is an absolutely continuous curve (as a map $[a, b] \rightarrow \mathbb{C}^d$), then the function

$$t \in [a, b] \rightarrow k_{\Omega}(\sigma(t); \sigma'(t))$$

is integrable, and we can define the *length* of σ to be

$$\ell_{\Omega}(\sigma) = \int_a^b k_{\Omega}(\sigma(t); \sigma'(t)) dt.$$

One can then define the *Kobayashi pseudo-distance* to be

$$K_{\Omega}(x, y) = \inf \{ \ell_{\Omega}(\sigma) : \sigma : [a, b] \rightarrow \Omega \text{ is absolutely continuous,} \\ \text{with } \sigma(a) = x, \text{ and } \sigma(b) = y \}.$$

This definition is equivalent to the standard definition of K_{Ω} via analytic chains (see [20, Theorem 3.1]).

We will use the following property of the Kobayashi metric (which is immediate from the definition).

Observation 2.1. Suppose $\Omega_1 \subset \mathbb{C}^{d_1}$, $\Omega_2 \subset \mathbb{C}^{d_2}$ are domains. If $f : \Omega_1 \rightarrow \Omega_2$ is holomorphic, then

$$K_{\Omega_2}(f(p), f(q)) \leq K_{\Omega_1}(p, q) \quad \text{and} \quad k_{\Omega_2}(f(p); d(f)_p(v)) \leq k_{\Omega_1}(p; v)$$

for all $p, q \in \Omega_1$ and $v \in \mathbb{C}^d$.

We will also consider the following special class of maps of the disk into a domain.

Definition 2.2. Suppose that $\Omega \subset \mathbb{C}^d$ is a domain. A holomorphic map $\varphi : \mathbb{D} \rightarrow \Omega$ is called a *complex geodesic* if $K_{\Omega}(\varphi(z), \varphi(w)) = K_{\mathbb{D}}(z, w)$ for all $z, w \in \mathbb{D}$.

2.3. The Bergman kernel and basic properties. Let μ denote the Lebesgue measure on $\mathbb{C}^d$. Then, for a domain $\Omega \subset \mathbb{C}^d$ let $\mathcal{H}^2(\Omega)$ be the Hilbert space of holomorphic functions $f : \Omega \rightarrow \mathbb{C}$ with $\int_{\Omega} |f|^2 d\mu < +\infty$. If $\{\phi_j : j \in J\}$ is an orthonormal basis of $\mathcal{H}^2(\Omega)$, then the function

$$\kappa_{\Omega} : \Omega \times \Omega \rightarrow \mathbb{C} \\ \kappa_{\Omega}(z, w) = \sum_{j \in J} \phi_j(z) \overline{\phi_j(w)}$$

is called the *Bergman kernel* of Ω .

We now recall two important properties of the Bergman kernel, proofs of both can be found in [8, Chapter 12].

Proposition 2.3 (Monotonicity). *If $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^d$ are domains, then we have that $\kappa_{\Omega_2}(z, z) \leq \kappa_{\Omega_1}(z, z)$ for all $z \in \Omega_1$.*

Proposition 2.4 (Change of variable formula). *If $\Omega_1, \Omega_2 \subset \mathbb{C}^d$ are domains and $F : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism, then*

$$\kappa_{\Omega_1}(z, w) = \kappa_{\Omega_2}(F(z), F(w)) \det(F'(z)) \overline{\det(F'(w))}$$

for all $z, w \in \Omega_1$.

We will also use the following well-known calculation.

Observation 2.5. Suppose $\alpha > 0$ and

$$\mathcal{P}_\alpha := \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Re}(z_1) > \alpha \sum_{j=2}^d |z_j|^2 \right\}.$$

Then, there exists $C_\alpha > 0$ such that

$$\kappa_{\mathcal{P}_\alpha}((\lambda, 0, \dots, 0), (\lambda, 0, \dots, 0)) = C_\alpha \operatorname{Re}(\lambda)^{-(d+1)}$$

for all $(\lambda, 0, \dots, 0) \in \mathcal{P}_\alpha$.

Since the proof is short we include it.

Proof. Let

$$C_\alpha := \kappa_{\Omega_1}((1, 0, \dots, 0), (1, 0, \dots, 0)),$$

and consider the automorphisms $a_t, u_t \in \operatorname{Aut}(\mathcal{P}_\alpha)$ given by

$$a_t(z_1, \dots, z_d) = (e^t z_1, e^{t/2} z_2, \dots, e^{t/2} z_d)$$

and

$$u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d).$$

Then,

$$(\lambda, 0, \dots, 0) = u_{\operatorname{Im}(\lambda)} a_{\log \operatorname{Re}(\lambda)}(1, 0, \dots, 0),$$

and so Proposition 2.4 implies that

$$\kappa_{\mathcal{P}_\alpha}((\lambda, 0, \dots, 0), (\lambda, 0, \dots, 0)) = C_\alpha \operatorname{Re}(\lambda)^{-(d+1)}.$$

□

2.4. A higher-dimensional variant of Hurwitz's theorem. We will use the following higher-dimensional variant of Hurwitz's theorem.

Theorem 2.6 (Deng-Guan-Zhang [3, Theorem 2.2]). Suppose $D \subset \mathbb{C}^d$ is a bounded domain and $x \in D$. Let $f_n : D \rightarrow \mathbb{C}^d$ be a sequence of injective holomorphic maps such that $f_n(x) = 0$ for all n and f_n converges locally uniformly to a map $f : D \rightarrow \mathbb{C}^d$. If there exists $\varepsilon > 0$ such that $\mathbb{B}_d(0; \varepsilon) \subset f_n(D)$ for all n , then f is injective.

3. DOMAINS WITH CO-COMPACT AUTOMORPHISM GROUPS

In this subsection we prove two basic facts about domains whose automorphism group acts co-compactly (i.e., there is a compact subset whose translates by the automorphism group cover the domain). Both are probably well known.

Proposition 3.1. If $\Omega \subset \mathbb{C}^d$ is a bounded domain and $\text{Aut}(\Omega)$ acts co-compactly on Ω , then (Ω, K_Ω) is a proper metric space. Hence, Ω is pseudoconvex.

Remark 3.2.

- (1) Recall that a metric space is called *proper* if bounded sets are relatively compact. Proper metric spaces are clearly Cauchy complete, and so the “hence” part of Proposition 3.1 follows from a result of Wu [23, Theorem F].
- (2) Siegel [19] proved that if a bounded domain covers a compact manifold, then the domain is pseudoconvex (see [18, Section 2.1] for an exposition).

Proposition 3.3. If $\Omega \subset \mathbb{C}^d$ is a bounded domain, $\text{Aut}(\Omega)$ acts co-compactly on Ω , and $\partial\Omega$ is C^1 , then for every $m \geq 1$ the m^{th} homotopy group $\pi_m(\Omega)$ is trivial. In particular, Ω is simply connected.

Remark 3.4. The proof of Proposition 3.3 is a simple modification of the proof of the Lemma on pg. 256 in [22] which in [22] is attributed to R. Greene.

3.1. Proof of Proposition 3.1. Before proceeding, we recall some terminology. If (X, d) is a metric space, $[a, b] \subset \mathbb{R}$, and $\sigma : [a, b] \rightarrow X$ is continuous, then we define the *length* of σ to be

$$\ell(\sigma) = \sup \left\{ \sum_{j=1}^N d(\sigma(t_j), \sigma(t_{j+1})) : N \geq 1, a \leq t_1 < t_2 < \cdots < t_N \leq b \right\}.$$

Then, (X, d) is called a *length space* if

$$d(x, y) = \inf \{ \ell(\sigma) : \sigma : [0, 1] \rightarrow X \text{ continuous with } \sigma(0) = x, \sigma(1) = y \}$$

for every $x, y \in X$.

We will use the following version of the Hopf-Rinow Theorem (for a proof, see, e.g., [1, Chapter I, Theorem 2.2]).

Theorem 3.5 (Hopf–Rinow). Suppose (X, d) is a locally compact length metric space. Then, the following are equivalent:

- (1) (X, d) is a proper metric space.
- (2) (X, d) is Cauchy complete.

We will also use the following lemma.

Lemma 3.6. Suppose (X, d) is a locally compact metric space and there exists a compact set $K \subset X$ such that $X = \text{Isom}(X, d) \cdot K$. Then, (X, d) is Cauchy complete.

Proof. We first claim that there exists $\delta > 0$ such that for any $x \in X$ the set

$$\overline{B}_X(x; \delta) := \{y \in X : d(x, y) \leq \delta\}$$

is compact. Since (X, d) is locally compact, for any $k \in K$ there exists $\delta_k > 0$ such that $\overline{B}_X(k; \delta_k)$ is compact. Then, since

$$K \subset \cup_{k \in K} \{y \in X : d(k, y) < \delta_k/2\},$$

there exist $k_1, \dots, k_N \in K$ such that

$$K \subset \cup_{i=1}^N \{y \in X : d(k_i, y) < \delta_{k_i}/2\}.$$

Then, if $\delta := \min\{\delta_{k_i}/2 : i = 1, \dots, N\}$, we see that $\overline{B}_X(x; \delta)$ is compact for any $x \in K$. Since $X = \text{Isom}(X, d) \cdot K$, then $\overline{B}_X(x; \delta)$ is compact for any $x \in X$.

Now suppose that $(x_n)_{n \geq 1}$ is a Cauchy sequence in (X, d) . Then, there exists $N > 0$ such that $d(x_n, x_N) < \delta$ for $n \geq N$. Thus, $\{x_n : n \geq N\} \subset \overline{B}_X(x_N; \delta)$. But then there exists a subsequence $(x_{n_k})_{k \geq 1}$ which converges. Thus, (X, d) is Cauchy complete. $\square$

Proof of Proposition 3.1. By construction, (Ω, K_Ω) is a length metric space. Further, since Ω is bounded, it is easy to show that (Ω, K_Ω) is locally compact.

By Lemma 3.6 the metric space (Ω, K_Ω) is Cauchy complete. Thus, by Theorem 3.5, K_Ω is a proper metric on Ω .

Since (Ω, K_Ω) is Cauchy complete, a result of Royden [16, Corollary pg. 136] says Ω is taut. Then, Ω is pseudoconvex by a result of Wu [23, Theorem F]. $\square$

3.2. Proof of Proposition 3.3. Without loss of generality, we can suppose that $0 \in \Omega$. Then, by rotating and scaling we can assume $1 = \max\{\|z\| : z \in \partial\Omega\}$ and $e_1 \in \partial\Omega$. Then, define the function

$$\begin{aligned} f : \overline{\Omega} &\rightarrow \mathbb{C} \\ f(z_1, \dots, z_d) &= e^{z_1 - 1}. \end{aligned}$$

Then, $|f(z)| \leq 1$ for all $z \in \overline{\Omega}$ with equality if and only if $z = e_1$.

Now fix a sequence $(p_n)_{n \geq 1}$ in Ω with $e_1 = \lim_{n \rightarrow \infty} p_n$. Since $\text{Aut}(\Omega)$ acts co-compactly on Ω , there exist sequences $(\varphi_n)_{n \geq 1}$ in $\text{Aut}(\Omega)$ and $(k_n)_{n \geq 1}$ in Ω such that the following hold:

- (1) $p_n = \varphi_n(k_n)$.
- (2) $\{k_n : n \geq 0\}$ is relatively compact in Ω .

By Montel's theorem we can pass to a subsequence so that φ_n converges locally uniformly to a holomorphic map $\varphi_\infty : \Omega \rightarrow \overline{\Omega}$. By passing to another subsequence, we can also assume that $k_n \rightarrow k \in \Omega$. Then,

$$\varphi_\infty(k) = \lim_{n \rightarrow \infty} \varphi_n(k_n) = \lim_{n \rightarrow \infty} p_n = e_1.$$

We now consider $g = f \circ \varphi_\infty$; then, $|g(z)| \leq 1$ for all $z \in \Omega$ and $|g(k)| = 1$. Thus, by the maximal principle, $g \equiv 1$, and so $\varphi_\infty \equiv e_1$.

Since $\partial\Omega$ is C^1 , there exists a neighborhood $\mathcal{O}$ of e_1 such that $\Omega \cap \mathcal{O}$ is contractible. Now fix a continuous map $\sigma : \mathbb{S}^m \rightarrow \Omega$. Since φ_n converges locally uniformly to $\varphi_\infty \equiv e_1$ and $\sigma(\mathbb{S}^m)$ is compact, there exists some $n \geq 0$ such that $(\varphi_n \circ \sigma)(\mathbb{S}^m) \subset \mathcal{O} \cap \Omega$. Thus, $(\varphi_n \circ \sigma)$ is homotopically trivial, and so σ is homotopically trivial. Since σ was arbitrary, $\pi_m(\Omega) = 1$.

4. RESCALING

As before, for $\alpha > 0$ define

$$\mathcal{P}_\alpha = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Re}(z_1) > \alpha \sum_{j=2}^d |z_j|^2 \right\}.$$

Proposition 4.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with $C^{1,1}$ boundary. If $\operatorname{Aut}(\Omega)$ acts co-compactly on Ω , then Ω is biholomorphic to a domain $D \subset \mathbb{C}^d$ where the following hold:*

- (1) $\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$ for some $0 < \beta < \alpha$.
- (2) $\operatorname{Aut}(D)$ contains the one-parameter subgroup

$$u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d).$$

4.1. Rescaling Euclidean balls. Before proving Proposition 4.1 we describe a rescaling procedure.

We begin by recalling the definition of the local Hausdorff topology on the set of all convex domains in $\mathbb{C}^d$. First, define the *Hausdorff distance* between two non-empty compact sets $A, B \subset \mathbb{C}^d$ by

$$d_H(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} \|a - b\|, \max_{b \in B} \min_{a \in A} \|b - a\| \right\}.$$

To obtain a topology on the set of all non-empty convex domains in $\mathbb{C}^d$, we consider the *local Hausdorff pseudo-distances* defined by

$$d_H^{(R)}(A, B) = d_H \left(A \cap \overline{\mathbb{B}_d(0; R)}, B \cap \overline{\mathbb{B}_d(0; R)} \right), \quad R > 0.$$

Then, a sequence of convex domains Ω_n converges to a convex domain Ω if there exists some $R_0 \geq 0$ such that $\lim_{n \rightarrow \infty} d_H^{(R)}(\overline{\Omega_n}, \overline{\Omega}) = 0$ for all $R \geq R_0$.

The Kobayashi distance is continuous with respect to this topology (see, e.g., [26, Theorem 4.1]).

Theorem 4.2. *Suppose $\Omega_n \subset \mathbb{C}^d$ is a sequence of convex domains and $\Omega = \lim_{n \rightarrow \infty} \Omega_n$ in the local Hausdorff topology. Assume the Kobayashi metric is non-degenerate on Ω and each Ω_n . Then,*

$$K_\Omega(p, q) = \lim_{n \rightarrow \infty} K_{\Omega_n}(p, q)$$

for all $p, q \in \Omega$. Moreover, the convergence is uniform on compact subsets of $\Omega \times \Omega$.

Remark 4.3. Under the hypothesis of Theorem 4.2: if $K \subset \Omega$ is a compact set, then $K \subset \Omega_n$ for n sufficiently large (see [26, Lemma 4.4]). Thus, $K_{\Omega_n}(p, q)$ is well defined for n sufficiently large (which depends on p, q).

We end this discussion with the following example.

Example 4.4. Fix $r > 0$, a sequence $r_n > 0$ converging to 0, and the sequence of linear maps

$$\Lambda_n(z_1, \dots, z_d) = \left(\frac{1}{r_n} z_1, \frac{1}{\sqrt{r_n}} z_2, \dots, \frac{1}{\sqrt{r_n}} z_d \right).$$

Then, $\mathcal{P}_{1/(2r)} = \lim_{n \rightarrow \infty} \Lambda_n \mathbb{B}_d(re_1; r)$ in the local Hausdorff topology.

4.2. The proof of Proposition 4.1. The rest of the section is devoted to the proof of the Proposition. Suppose then Ω is a bounded domain with $C^{1,1}$ boundary and there exists a compact set $K \subset \Omega$ such that $\text{Aut}(\Omega) \cdot K = \Omega$.

Lemma 4.5. *After applying an affine transformation, we can assume that*

$$\mathbb{B}_d(re_1; r) \subset \Omega \subset \mathbb{B}_d(e_1; 1)$$

for some $r \in (0, 1)$.

Proof. By translating we can assume that $e_1 \in \Omega$. Then, we pick $\xi_0 \in \partial\Omega$ such that $\|\xi_0 - e_1\| = \max\{\|\xi - e_1\| : \xi \in \partial\Omega\}$. By rotating and scaling Ω about e_1 , we can assume that $\xi_0 = 0$. Then, $\Omega \subset \mathbb{B}_d(e_1; 1)$.

For $\xi \in \partial\Omega$, let $n_\Omega(\xi)$ be the inward pointing normal unit vector at ξ . Since $\partial\Omega$ is $C^{1,1}$, there is some $r > 0$ such that $\mathbb{B}_d(\xi + rn_\Omega(\xi); r) \subset \Omega$ for every $\xi \in \partial\Omega$. Then, since $\Omega \subset \mathbb{B}_d(e_1; 1)$, we have $n_\Omega(0) = e_1$ and so $\mathbb{B}_d(re_1; r) \subset \Omega$. $\square$

Fix a sequence $r_n \in (0, r)$ converging to 0. Then, pick $\varphi_n \in \text{Aut}(\Omega)$ and $k_n \in K$ such that $\varphi_n(k_n) = r_n e_1$. Then, consider the dilations

$$\Lambda_n(z_1, \dots, z_d) = \left(\frac{1}{r_n} z_1, \frac{1}{\sqrt{r_n}} z_2, \dots, \frac{1}{\sqrt{r_n}} z_d \right).$$

Let $\Omega_n := \Lambda_n \Omega$ and $F_n := \Lambda_n \varphi_n : \Omega \rightarrow \Omega_n$. Then,

$$\Lambda_n \mathbb{B}_d(re_1; r) \subset \Omega_n \subset \Lambda_n \mathbb{B}_d(e_1; 1).$$

Further, by Example 4.4,

$$\mathcal{P}_\alpha = \lim_{n \rightarrow \infty} \Lambda_n \mathbb{B}_d(re_1; r) \text{ where } \alpha := \frac{1}{2r}$$

and

$$\mathcal{P}_\beta = \lim_{n \rightarrow \infty} \Lambda_n \mathbb{B}_d(e_1; 1) \text{ where } \beta := \frac{1}{2}$$

in the local Hausdorff topology.

Lemma 4.6. *After passing to a subsequence, F_n converges to a holomorphic embedding $F : \Omega \rightarrow \mathbb{C}^d$. Moreover, if $D = F(\Omega)$, then $\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$.*

Proof. By construction $F_n(k_n) = e_1$ and Observation 2.1 implies that

$$K_{\Omega_n}(z, w) \leq K_{\Lambda_n \mathbb{B}_d(e_1; 1)}(z, w)$$

for all $z, w \in \Omega_n$. Theorem 4.2 implies that $K_{\mathcal{P}_\beta} = \lim_{n \rightarrow \infty} K_{\Lambda_n \mathbb{B}_d(e_1; 1)}$ locally uniformly. Thus, using the Arzelà-Ascoli theorem, we can pass to a subsequence where F_n converges locally uniformly to a holomorphic map $F : \Omega \rightarrow \mathbb{C}^d$.

Let $D = F(\Omega)$. Since

$$\Lambda_n \mathbb{B}_d(re_1; r) \subset F_n(\Omega) \subset \Lambda_n \mathbb{B}_d(e_1; 1)$$

for every n we see that

$$(4.1) \quad \overline{\mathcal{P}_\alpha} \subset D \subset \overline{\mathcal{P}_\beta}.$$

Next, we use Theorem 2.6 to show that F is injective. Since

$$\mathcal{P}_\alpha = \lim_{n \rightarrow \infty} \Lambda_n \mathbb{B}_d(re_1; r)$$

in the local Hausdorff topology and $\Lambda_n \mathbb{B}_d(re_1; r) \subset \Omega_n$ for every $n \geq 0$, there exists $\varepsilon > 0$ such that $\mathbb{B}_d(e_1; \varepsilon) \subset F_n(\Omega)$ for every $n \geq 0$. By passing to a subsequence we can suppose that $k_n \rightarrow k \in K$. Then, consider the maps $G_n(z) = F_n(z) - F_n(k)$. Since

$$\lim_{n \rightarrow \infty} F_n(k) = \lim_{n \rightarrow \infty} F_n(k_n) = F(k) = e_1,$$

G_n converges locally uniformly to $F - e_1$. Further, by passing to another subsequence we can suppose that $\|e_1 - F_n(k)\| < \varepsilon/2$ for every $n \geq 0$. Then, for every $n \geq 0$, the map G_n is injective, $G_n(k) = 0$, and $\mathbb{B}_d(0; \varepsilon/2) \subset G_n(\Omega)$. Thus, F is injective by Theorem 2.6, so F is an embedding.

Now, since F is an embedding, D is an open set, so Equation (4.1) becomes

$$\mathcal{P}_\alpha = \text{int}(\overline{\mathcal{P}_\alpha}) \subset D \subset \text{int}(\overline{\mathcal{P}_\beta}) = \mathcal{P}_\beta.$$

This completes the proof. $\square$

Showing that $\text{Aut}(D)$ contains a one-parameter subgroup requires some preliminary lemmas.

Lemma 4.7. *Suppose $(z_n)_{n \geq 1}$ is a sequence, $z_n \in \Omega_n$ for every n , $\lim_{n \rightarrow \infty} z_n = z$, and $\liminf_{n \rightarrow \infty} K_{\Omega_n}(e_1, z_n) < +\infty$. Then, $z \in D$.*

Proof. Fix $z_0 \in \Omega$ and let $R = \max_{k \in K} K_\Omega(z_0, k)$. Then, pick $n_j \rightarrow \infty$ such that

$$M := \lim_{j \rightarrow \infty} K_{\Omega_{n_j}}(e_1, z_{n_j}) < +\infty.$$

Since $F_n(k_n) = e_1$ and $k_n \in K$, for each $j \geq 0$, there exists

$$w_j \in \overline{B_\Omega(z_0; R + M)}$$

such that $F_{n_j}(w_j) = z_{n_j}$. By Proposition 3.1, K_Ω is a proper metric on Ω . Thus, we can pass to a subsequence such that $w_j \rightarrow w \in \Omega$. Since $F_n \rightarrow F$ locally uniformly, we then have

$$F(w) = \lim_{j \rightarrow \infty} F_{n_j}(w_j) = \lim_{j \rightarrow \infty} z_j = z.$$

Thus, $z \in F(\Omega) = D$. $\square$

Before proceeding we recall some standard notations. First, let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^d$, that is, $\langle z, w \rangle = \overline{w}^t z$ for all $z, w \in \mathbb{C}^d$. Then, given a C^1 function $f : \mathbb{C}^d \rightarrow \mathbb{R}$, let ∇f denote the gradient of f , that is,

$$\lim_{h \rightarrow 0} \frac{f(z + hv) - f(z)}{h} = \text{Re} \langle \nabla f(z), v \rangle$$

for all $z, v \in \mathbb{C}^d$. The next lemma essentially says that the distance to the boundary in the tangential direction is much larger than the distance to the boundary in the normal direction.

Lemma 4.8. *For every $m > 0$, there exists $\delta_m > 0$ such that the following hold: if $z_0 \in \Omega \cap \mathbb{B}_d(0; \delta_m)$, $T > 0$, and*

$$\{z_0 + x e_1 : -T < x < T\} \subset \Omega,$$

then

$$\{z_0 + (x + iy)e_1 : -T/2 \leq x \leq T/2, -mT \leq y \leq mT\} \subset \Omega.$$

Proof. Fix a C^1 defining function $\rho : \mathbb{C}^d \rightarrow \mathbb{R}$ of Ω ; that is,

$$\Omega = \{z \in \mathbb{C}^d : \rho(z) < 0\}$$

and $\nabla \rho(z) \neq 0$ in a neighborhood of $\partial\Omega$. Since $0 \in \partial\Omega$ and $\Omega \subset \mathbb{B}_d(e_1; 1)$, we must have $\nabla \rho(0) = -e_1$.

Since $\{0\} = \{z \in \overline{\Omega} : \operatorname{Re} \langle z, e_1 \rangle = 0\}$, there exists $\delta_m > 0$ such that if $z \in \Omega$ and $\operatorname{Re} \langle z, e_1 \rangle < 2\delta_m$, then

$$\|\nabla \rho(z) - (-e_1)\| < \frac{1}{2m+1}.$$

Now fix $z_0 \in \Omega \cap \mathbb{B}_d(0; \delta_m)$ and $T > 0$ so that $\{z_0 + xe_1 : -T < x < T\} \subset \Omega$. Since $\Omega \subset \mathbb{B}_d(e_1; 1)$, we have

$$0 \leq \operatorname{Re} \langle z_0 - Te_1, e_1 \rangle = \operatorname{Re} \langle z_0, e_1 \rangle - T < \delta_m - T,$$

and so $T \leq \delta_m$. Thus, $\operatorname{Re} \langle z_0 + xe_1, e_1 \rangle < 2\delta_m$ when $-T < x < T$.

Now fix $-T/2 \leq x \leq T/2$. Then,

$$\begin{aligned} \rho(z_0 + xe_1) &= \rho(z_0 - Te_1) + \int_{-T}^x \operatorname{Re} \langle \nabla \rho(z_0 + te_1), e_1 \rangle dt \\ &\leq \rho(z_0 - Te_1) + \int_{-T}^x \left(-1 + \frac{1}{2m+1}\right) dt \\ &\leq 0 - \frac{2m}{2m+1}(x+T) \leq -\frac{m}{2m+1}T. \end{aligned}$$

Let

$$y^+ := \min\{y \geq 0 : z_0 + (x + iy)e_1 \in \partial\Omega\}.$$

Notice that, if $y \in [0, y^+)$, then $z_0 + (x + iy)e_1 \in \Omega$ and

$$\operatorname{Re} \langle z_0 + (x + iy)e_1, e_1 \rangle = \operatorname{Re} \langle z_0 + xe_1, e_1 \rangle < 2\delta_m.$$

Thus,

$$\begin{aligned} 0 &= \rho(z_0 + (x + iy^+)e_1) \\ &= \rho(z_0 + xe_1) + \int_0^{y^+} \operatorname{Re} \langle \nabla \rho(z_0 + (x + iy)e_1), ie_1 \rangle dy \\ &\leq -\frac{m}{2m+1}T + \int_0^{y^+} \frac{1}{2m+1} dy = \frac{1}{2m+1}(y^+ - mT). \end{aligned}$$

Hence, $y^+ \geq mT$. Next, define

$$y^- = \max\{y \leq 0 : z_0 + (x + iy)e_1 \in \partial\Omega\}.$$

Then, a similar argument shows that $y^- \leq -mT$, and so

$$\{z_0 + (x + iy)e_1 : -mT \leq y \leq mT\} \subset \Omega.$$

Since $-T/2 \leq x \leq T/2$ was arbitrary, this completes the proof of the lemma. $\square$

Lemma 4.9. *Aut(D) contains the one-parameter subgroup*

$$u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d).$$

Proof. It is enough to fix $w_0 \in D$ and $t \in \mathbb{R}$; then, we show $w_0 + ite_1 \in D$.

Since the sequence F_n converges locally uniformly to F , there exist $\varepsilon > 0$ and $N \geq 0$ such that $\mathbb{B}_d(w_0; \varepsilon) \subset F_n(\Omega) = \Lambda_n(\Omega)$ for all $n \geq N$.

Define $w_n := \Lambda_n^{-1}w_0$. Then,

$$(4.2) \quad \{w_n + xe_1 : -r_n\varepsilon < x < r_n\varepsilon\} \subset \Lambda_n^{-1}\mathbb{B}_d(w_0; \varepsilon) \subset \Omega$$

when $n \geq N$.

Fix $m \in \mathbb{N}$ such that $|t| < m\varepsilon$. Let $\delta_m > 0$ be the associated constant from Lemma 4.8. Since $r_n \rightarrow 0$, we have $\lim_{n \rightarrow \infty} w_n = 0$. Thus, by increasing N we can assume $w_n \in \mathbb{B}_d(0; \delta_m)$ when $n \geq N$. Then, by Equation (4.2) and Lemma 4.8,

$$\{w_n + (x + iy)e_1 : -r_n\varepsilon/2 < x < r_n\varepsilon/2, -mr_n\varepsilon < y < mr_n\varepsilon\} \subset \Omega$$

when $n \geq N$. Thus,

$$S := \{w_0 + (x + iy)e_1 : -\varepsilon/2 < x < \varepsilon/2, -m\varepsilon < y < m\varepsilon\} \subset \Lambda_n\Omega.$$

Then, when $n \geq N$,

$$K_{\Lambda_n\Omega}(w_0, w_0 + ite_1) \leq K_S(w_0, w_0 + ite_1),$$

and so

$$\sup_{n \geq N} K_{\Lambda_n\Omega}(w_0, w_0 + ite_1) < +\infty.$$

Hence, Lemma 4.7 implies that $w_0 + ite_1 \in D$. $\square$

5. THE GEOMETRY OF THE RESCALED DOMAIN

For the rest of this section, suppose that $D \subset \mathbb{C}^d$ is a domain where $P_\alpha \subset D \subset P_\beta$ for some $0 < \beta < \alpha$.

Define

$$\mathcal{H}_D := D \cap \mathbb{C} \cdot e_1 = \{(z, 0, \dots, 0) \in \mathbb{C}^d : \operatorname{Re}(z) > 0\}$$

and $\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

Observation 5.1. If $z, w \in \mathcal{H}$, then

$$K_D((z, 0, \dots, 0), (w, 0, \dots, 0)) = K_{\mathcal{H}}(z, w).$$

Proof. The inclusion map

$$\begin{aligned} \iota : \mathcal{H} &\rightarrow D \\ \iota(z) &= (z, 0, \dots, 0) \end{aligned}$$

implies that

$$K_D((z, 0, \dots, 0), (w, 0, \dots, 0)) \leq K_{\mathcal{H}}(z, w).$$

for all $z, w \in \mathcal{H}$. Since

$$D \subset \mathcal{P}_\beta \subset \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > 0\},$$

the projection map

$$\begin{aligned} \pi : D &\rightarrow \mathcal{H} \\ \pi(z_1, \dots, z_d) &= z_1 \end{aligned}$$

implies that

$$K_{\mathcal{H}}(z, w) \leq K_D((z, 0, \dots, 0), (w, 0, \dots, 0))$$

for all $z, w \in \mathcal{H}$. □

Observation 5.1 implies that $\mathcal{H}_D$ can be parametrized to be a complex geodesic (see Definition 2.2). The next observation proves that, up to parametrization, this is the only complex geodesic joining two points in $\mathcal{H}_D$.

Proposition 5.2. Suppose $p, q \in \mathcal{H}_D$ are distinct and $\varphi : \mathbb{D} \rightarrow D$ is a complex geodesic with $p, q \in \varphi(\mathbb{D})$. Then, there exists $\phi \in \operatorname{Aut}(\mathbb{D})$ such that

$$(\varphi \circ \phi)(\lambda) = \left(\frac{1 + \lambda}{1 - \lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$. In particular, $\varphi(\mathbb{D}) = \mathcal{H}_D$.

Proof. By hypothesis, $p = (p_1, 0, \dots, 0)$ and $q = (q_1, 0, \dots, 0)$ for some $p_1, q_1 \in \mathcal{H}$. Let $f : \mathcal{H} \rightarrow \mathbb{D}$ be a biholomorphism, and consider the map $\hat{\varphi} := \varphi \circ f : \mathcal{H} \rightarrow D$. Let $\hat{\varphi}_1, \dots, \hat{\varphi}_d$ denote the coordinate functions of $\hat{\varphi}$. Since

$$D \subset \mathcal{P}_\beta \subset \{(z_1, \dots, z_d) \in \mathbb{C}^d : \operatorname{Re}(z_1) > 0\},$$

we have $\hat{\varphi}_1(\mathcal{H}) \subset \mathcal{H}$. Further, by Observation 5.1, if $\hat{\varphi}(\lambda_1) = p$ and $\hat{\varphi}(\lambda_2) = q$, then

$$\begin{aligned} K_{\mathcal{H}}(\lambda_1, \lambda_2) &= K_{\mathbb{D}}(f(\lambda_1), f(\lambda_2)) = K_D(\varphi(f(\lambda_1)), \varphi(f(\lambda_2))) = K_D(p, q) \\ &= K_{\mathcal{H}}(p_1, q_1) = K_{\mathcal{H}}(\hat{\varphi}_1(\lambda_1), \hat{\varphi}_1(\lambda_2)). \end{aligned}$$

Thus, by the Schwarz lemma, $\hat{\varphi}_1$ is a biholomorphism of $\mathcal{H}$. Then, by replacing f with $f \circ \hat{\varphi}_1^{-1}$, we can assume that $\hat{\varphi}_1 = \text{id}$.

We claim that $\hat{\varphi}_j \equiv 0$ for $2 \leq j \leq d$. Fix $t \in \mathbb{R}$; then, since $D \subset \mathcal{P}_{\beta}$ we have

$$\limsup_{\lambda \rightarrow it} \sum_{j=2}^d \left| \hat{\varphi}_j(\lambda) \right|^2 \leq \limsup_{\lambda \rightarrow it} \text{Re}(\hat{\varphi}_1(\lambda)) = \limsup_{\lambda \rightarrow it} \text{Re}(\lambda) = 0.$$

Thus,

$$\lim_{\lambda \rightarrow it} \hat{\varphi}_j(\lambda) = 0$$

for $2 \leq j \leq d$, so $\hat{\varphi}_j$ extends continuously to $\mathcal{H} \cup i\mathbb{R}$ with $\hat{\varphi}_j|_{i\mathbb{R}} \equiv 0$. Thus, by the Schwarz reflection principle, $\hat{\varphi}_j$ extends holomorphically to all of $\mathbb{C}$. But then, since $\hat{\varphi}_j|_{i\mathbb{R}} \equiv 0$, we have $\hat{\varphi}_j \equiv 0$.

Thus,

$$\varphi(\lambda) = (\varphi_1(\lambda), 0, \dots, 0)$$

where $\varphi_1 : \mathbb{D} \rightarrow \mathcal{H}$ is a biholomorphism. Finally, define $\phi \in \text{Aut}(\mathbb{D})$ by

$$\phi(\lambda) = \varphi_1^{-1} \left(\frac{1 + \lambda}{1 - \lambda} \right).$$

Then,

$$(\varphi \circ \phi)(\lambda) = \left(\frac{1 + \lambda}{1 - \lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$. □

Proposition 5.3. *Suppose $(z_n)_{n \geq 1}, (w_n)_{n \geq 1}$ are sequences in D with*

$$\lim_{n \rightarrow \infty} z_n = \xi \in (i\mathbb{R}) \times \{(0, \dots, 0)\} = \overline{\mathcal{H}}_D \cap \partial D$$

and $\limsup_{n \rightarrow \infty} K_D(w_n, z_n) < +\infty$; then, $\lim_{n \rightarrow \infty} w_n = \xi$.

Proof. Notice that $K_{\mathcal{P}_{\beta}}(z, w) \leq K_D(z, w)$ for all $z, w \in D$ and

$$(i\mathbb{R}) \times \{(0, \dots, 0)\} \subset \partial D \cap \partial \mathcal{P}_{\beta}.$$

Thus, this proposition follows immediately from the well-understood geometry of $(\mathcal{P}_{\beta}, K_{\mathcal{P}_{\beta}})$; it is a standard model of complex hyperbolic d -space.

For the reader's convenience we provide a complete argument. Consider the complex affine hyperplane

$$H = \xi + \text{Span}_{\mathbb{C}}\{e_2, \dots, e_d\}.$$

Then, $H \cap \mathcal{P}_\beta = \emptyset$ and $\xi \in H$. By standard estimates for the Kobayashi distance on a convex domain (see, e.g., [25, Lemma 4.2]),

$$K_{\mathcal{P}_\beta}(z, w) \geq \frac{1}{2} \log \frac{d_{\text{Euc}}(w, H)}{d_{\text{Euc}}(z, H)}$$

for all $z, w \in \mathcal{P}_\beta$. Thus, we must have $\lim_{n \rightarrow \infty} d_{\text{Euc}}(w_n, H) = 0$. Then, since $H \cap \overline{\mathcal{P}_\beta} = \{\xi\}$, we have $\lim_{n \rightarrow \infty} w_n = \xi$. $\square$

6. THE DOMAIN IS SYMMETRIC

In this section we prove the following.

Proposition 6.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain which covers a compact manifold. If $\partial\Omega$ is $C^{1,1}$, then Ω is a bounded symmetric domain.*

Before starting the proof, we recall the following notation.

Definition 6.2. Given a domain $\Omega \subset \mathbb{C}^d$, let $\text{Aut}_0(\Omega)$ denote the connected component of the identity in $\text{Aut}(\Omega)$.

Proof of Proposition 6.1. Proposition 4.1 implies that $\text{Aut}(\Omega)$ is non-discrete, and Proposition 3.3 implies that Ω is simply connected. Hence, by Theorem 1.7 either Ω is a bounded symmetric domain, or Ω is biholomorphic to $D_1 \times D_2$ where D_1 is a bounded symmetric domain and $\text{Aut}(D_2)$ is an infinite discrete group.

We assume the second possibility and derive a contradiction.

By Proposition 4.1, there exists a biholomorphism $F : D_1 \times D_2 \rightarrow D$ where $D \subset \mathbb{C}^d$ is a domain such that $\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$ for some $\alpha > \beta > 0$, and $\text{Aut}(D)$ contains the one-parameter subgroup $u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d)$.

Define

$$G := F \circ (\{\text{id}\} \times \text{Aut}(D_2)) \circ F^{-1} \leq \text{Aut}(D).$$

Then, by assumption, G is an infinite discrete subgroup of $\text{Aut}(D)$, and G commutes with

$$\text{Aut}_0(D) = F \circ (\text{Aut}(D_1) \times \{\text{id}\}) \circ F^{-1}.$$

We will obtain a contradiction by establishing the following.

Claim: G is a finite group.

Since $\text{Aut}_0(D_1 \times D_2) = \text{Aut}_0(D_1) \times \{\text{id}\}$, for any $z = (z_1, z_2) \in D_1 \times D_2$ we have

$$\text{Aut}_0(D_1 \times D_2) \cdot z = D_1 \times \{z_2\}.$$

In particular, the orbit $\text{Aut}_0(D_1 \times D_2) \cdot z$ is a complex analytic variety in $D_1 \times D_2$. Thus, for any $w \in D$, the orbit $\text{Aut}_0(D) \cdot w$ is a complex analytic variety in D . Further, since $\text{Aut}(D)$ contains the one-parameter group

$$u_t(z_1, \dots, z_d) = (z_1 + it, z_2, \dots, z_d),$$

for any $w_0 \in D$ and $w \in \text{Aut}_0(D) \cdot w_0$, the tangent space $T_w(\text{Aut}_0(D) \cdot w_0)$ of $\text{Aut}_0(D) \cdot w_0$ at w contains ie_1 . Thus, since $\text{Aut}_0(D) \cdot w_0$ is a complex analytic variety,

$$(6.1) \quad \mathbb{C} \cdot e_1 \subset T_w(\text{Aut}_0(D) \cdot w_0).$$

As before, let $\mathcal{H}_D := \{(z, 0, \dots, 0) : \text{Re}(z) > 0\}$. Then, by Equation (6.1),

$$\mathcal{H}_D \subset \text{Aut}_0(D) \cdot e_1.$$

Thus, for each $z \in \mathcal{H}_D$, there exists $\phi_z \in \text{Aut}_0(D)$ such that $\phi_z(e_1) = z$.

Now suppose $g \in G$. Then, for $z \in \mathcal{H}_D$ we have

$$K_D(z, g(z)) = K_D(\phi_z(e_1), g\phi_z(e_1)) = K_D(\phi_z(e_1), \phi_z g(e_1)) = K_D(e_1, g(e_1)),$$

since G commutes with $\text{Aut}_0(D)$. Hence,

$$(6.2) \quad \sup_{z \in \mathcal{H}_D} K_D(z, g(z)) = K_D(e_1, g(e_1)) < +\infty.$$

Let $\mathcal{H} := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$, and define the map

$$\begin{aligned} \psi : \mathcal{H} &\rightarrow D \\ \psi(\lambda) &= g(\lambda, 0, \dots, 0). \end{aligned}$$

Then, let $\psi_1, \dots, \psi_d$ denote the coordinate functions of ψ . By Proposition 5.3 and Equation (6.2), if $t \in \mathbb{R}$, then

$$\lim_{\lambda \rightarrow it} \psi(\lambda) = (it, 0, \dots, 0).$$

Thus, by applying the Schwarz reflection principle to each ψ_j , we can extend ψ to a map $\mathbb{C} \rightarrow \mathbb{C}^d$ such that

$$\psi(it) = (it, 0, \dots, 0)$$

for $t \in \mathbb{R}$. But then, by the identity theorem for holomorphic functions, we have $\psi(\lambda) = (\lambda, 0, \dots, 0)$ for all $\lambda \in \mathbb{C}$. In particular, $g(e_1) = e_1$.

Since $g \in G$ was arbitrary we see that $G \cdot e_1 = e_1$. Since D is biholomorphic to a bounded domain, $\text{Aut}(D)$ acts properly on D , and hence G must be compact. Since G is also discrete, we see that G is finite. Thus, we have a contradiction. $\square$

7. POLYDISKS IN BOUNDED SYMMETRIC DOMAINS

In this section we recall some facts about polydisks in bounded symmetric domains.

Definition 7.1. Suppose Ω is a bounded symmetric domain. The *real rank* of Ω is the largest integer r such that there exists a holomorphic isometric embedding $f : (\mathbb{D}^r, K_{\mathbb{D}^r}) \rightarrow (\Omega, K_{\Omega})$.

From the characterization of bounded symmetric domains, every bounded symmetric domain has real rank at least one. Moreover, the real rank is one if and only if the symmetric domain is biholomorphic to the unit ball.

The next result says that there are many isometric embeddings of polydisks (see [21, pg. 280]).

Theorem 7.2 (Polydisk Theorem). Suppose Ω is a bounded symmetric domain with real rank r . If $z_1, z_2 \in \Omega$, then there exist a holomorphic isometric embedding $f : (\mathbb{D}^r, K_{\mathbb{D}^r}) \rightarrow (\Omega, K_{\Omega})$ whose image contains z_1, z_2 .

For any bounded symmetric domain $\Omega \subset \mathbb{C}^d$, Harish-Chandra constructed an embedding $F : \Omega \hookrightarrow \mathbb{C}^d$ whose image is convex and bounded (see [17, Chapter II, Section 4]). Further, there exists a norm $\|\cdot\|_{HC}$ on $\mathbb{C}^d$ such that

$$F(\Omega) = \{z \in \mathbb{C}^d : \|z\|_{HC} < 1\}.$$

We will use the following terminology.

Definition 7.3. A bounded symmetric domain $\Omega \subset \mathbb{C}^d$ is in *standard form* if it coincides with the image of its Harish-Chandra embedding.

We now recall the following well-known description of the Bergman kernel on a bounded symmetric domain (see, e.g., [5] or [17, Chapter II, Section 5]).

Theorem 7.4. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded symmetric domain in standard form with real rank r . Assume $\Phi : (\mathbb{D}^r, K_{\mathbb{D}^r}) \rightarrow (\Omega, K_{\Omega})$ is a holomorphic isometric embedding with $\Phi(0) = 0$. Then, there exist constants $p, C > 0$ such that

$$\kappa_{\Omega}(\Phi(z), \Phi(z)) = C \left(\prod_{j=1}^r (1 - |z_j|^2) \right)^{-p}$$

for all $z \in \mathbb{D}^r$. Moreover, $p \geq (d + r)/r$.

Here are precise references for the proof of Theorem 7.4: by the discussion on pages 76 and 77 in [5], there exist constants $p, C > 0$ such that

$$\kappa_{\Omega}(\Phi(z), \Phi(z)) = C \left(\prod_{j=1}^r (1 - |z_j|^2) \right)^{-p}$$

for all $z \in \mathbb{D}^r$. The lower bound on p follows from Equations (1.9) and (3.3) in [5].

7.1. Complex geodesics in polydisks. We will use the following observations about complex geodesics in polydisks.

Lemma 7.5. *Suppose $z = (z_1, \dots, z_r) \in \mathbb{D}^r$ and $|z_a| \neq |z_b|$ for some $1 \leq a, b \leq r$. Then, there exist two complex geodesics $\varphi_1, \varphi_2 : \mathbb{D} \rightarrow \mathbb{D}^r$ whose images contain z and 0, but $\varphi_1(\mathbb{D}) \neq \varphi_2(\mathbb{D})$.*

Proof. By permuting the coordinates we can assume that

$$0 \leq |z_1| \leq |z_2| \leq \dots \leq |z_r|.$$

Since $|z_1| < |z_r|$ there exist two holomorphic functions $f_1, f_2 : \mathbb{D} \rightarrow \mathbb{D}$ such that $f_1(0) = f_2(0) = 0$, $f_1(z_r) = f_2(z_r) = z_1$, and $f_1 \neq f_2$. For $2 \leq j \leq r-1$, select $\omega_j \in \overline{\mathbb{D}}$ such that $\omega_j z_r = z_j$. Then, for $j = 1, 2$, define the map

$$\begin{aligned} \varphi_j : \mathbb{D} &\rightarrow \mathbb{D}^r \\ \varphi_j(\lambda) &= (f_j(\lambda), \omega_2 \lambda, \dots, \omega_{r-1} \lambda, \lambda). \end{aligned}$$

Since each φ_j is holomorphic, we have

$$K_{\mathbb{D}^r}(\varphi_j(\lambda_1), \varphi_j(\lambda_2)) \leq K_{\mathbb{D}}(\lambda_1, \lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$. Further, by projecting onto the last component, we have

$$K_{\mathbb{D}^r}(\varphi_j(\lambda_1), \varphi_j(\lambda_2)) \geq K_{\mathbb{D}}(\lambda_1, \lambda_2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$. Thus, $\varphi_1, \varphi_2 : \mathbb{D} \rightarrow \mathbb{D}^r$ are both complex geodesics with $\varphi_1(z_r) = \varphi_2(z_r) = z$. Finally, since $f_1 \neq f_2$, we have $\varphi_1(\mathbb{D}) \neq \varphi_2(\mathbb{D})$. $\square$

Lemma 7.6. *Suppose $z = (z_1, \dots, z_r) \in \mathbb{D}^r - \{0\}$ and*

$$|z_1| = \dots = |z_r|.$$

If $\varphi : \mathbb{D} \rightarrow \mathbb{D}^r$ is a complex geodesic with $\varphi(0) = 0$ and $\varphi(\lambda_0) = z$, then $|\lambda_0| = |z_1|$ and

$$\varphi(\lambda) = \left(\frac{z_1}{\lambda_0} \lambda, \dots, \frac{z_r}{\lambda_0} \lambda \right)$$

for all $\lambda \in \mathbb{D}$.

Proof. Since

$$K_{\mathbb{D}}(0, \lambda_0) = K_{\mathbb{D}^r}(0, z) = \max_{1 \leq j \leq r} K_{\mathbb{D}}(0, z_j) = K_{\mathbb{D}}(0, z_1),$$

we must have $|\lambda_0| = |z_1|$. Then, applying the Schwarz lemma to each component function of φ shows that

$$\varphi(\lambda) = \left(\frac{z_1}{\lambda_0} \lambda, \dots, \frac{z_r}{\lambda_0} \lambda \right)$$

for all $\lambda \in \mathbb{D}$. $\square$

8. BOUNDED SYMMETRIC DOMAINS WITH SMOOTH BOUNDARIES

Theorem 8.1. *Suppose Ω is a bounded symmetric domain with $C^{1,1}$ boundary. Then, Ω is biholomorphic to the unit ball.*

The rest of the section is devoted to the proof of the Theorem. Suppose then Ω is a bounded symmetric domain with $C^{1,1}$ boundary. Let r denote the real rank of Ω .

Let Ω_{HC} denote the image of the Harish-Chandra embedding of Ω . By Proposition 4.1, there exists a biholomorphism $F : D \rightarrow \Omega_{HC}$ where $D \subset \mathbb{C}^d$ is a domain such that $\mathcal{P}_\alpha \subset D \subset \mathcal{P}_\beta$ for some $\alpha > \beta > 0$. By post-composing F with an element of $\text{Aut}(\Omega_{HC})$ we may assume that $F(e_1) = 0$.

Lemma 8.2. *A holomorphic isometric embedding $\Phi : (\mathbb{D}^r, K_{\mathbb{D}^r}) \rightarrow (D, K_D)$ exists such that*

$$\Phi(\lambda, \dots, \lambda) = \left(\frac{1 + \lambda}{1 - \lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$.

Proof. As before, define

$$\mathcal{H}_D := D \cap \mathbb{C} \cdot e_1 = \left\{ (z, 0, \dots, 0) \in \mathbb{C}^d : \text{Re}(z) > 0 \right\}.$$

Fix $w_0 \in \mathcal{H}_D \setminus \{e_1\}$. By Theorem 7.2 there exists a holomorphic isometric embedding $\Phi_0 : (\mathbb{D}^r, K_{\mathbb{D}^r}) \rightarrow (D, K_D)$ with $e_1, w_0 \in \Phi_0(\mathbb{D}^r)$. By pre-composing Φ_0 with an element of $\text{Aut}(\mathbb{D}^r) \geq \text{Aut}(\mathbb{D}) \times \dots \times \text{Aut}(\mathbb{D})$, we may assume that

$$\Phi_0(0) = e_1 \quad \text{and} \quad \Phi_0(t_1, \dots, t_r) = w_0$$

for some real numbers $t_1, \dots, t_r \in [0, 1)$.

By the “in particular” part of Proposition 5.2, every complex geodesic in D containing e_1, w_0 has image $\mathcal{H}_D$. Thus, by Lemma 7.5, we must have $t_1 = \dots = t_r$. Then, by Lemma 7.6,

$$\mathcal{H}_D = \{\Phi_0(\lambda, \dots, \lambda) : \lambda \in \mathbb{D}\}.$$

Then, by the first part of Proposition 5.2, there exists $\phi \in \text{Aut}(\mathbb{D})$ such that

$$\Phi_0(\phi(\lambda), \dots, \phi(\lambda)) = \left(\frac{1 + \lambda}{1 - \lambda}, 0, \dots, 0 \right)$$

for all $\lambda \in \mathbb{D}$. Finally, the map

$$\Phi(z_1, \dots, z_r) := \Phi_0(\phi(z_1), \dots, \phi(z_r))$$

has all of the desired properties. □

Next, consider the function

$$J : \mathbb{D} \rightarrow \mathbb{C}$$

$$J(\lambda) = \det \left(F' \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right) \right) = \det(F'(\Phi(\lambda, \dots, \lambda)))$$

where $F'(z)$ is the complex Jacobian matrix of F . Since F is a biholomorphism, J is nowhere zero. We will show that Ω_{HC} is biholomorphic to the ball by estimating the boundary values of J . To that end, define $\Phi_{HC} := F \circ \Phi$. Notice that $\Phi_{HC}(0) = F(e_1) = 0$.

Let κ_D and $\kappa_{\Omega_{HC}}$ be the Bergman kernels of D and Ω_{HC} , respectively. We will use the notation that

$$\kappa_D(z) := \kappa_D(z, z) \quad \text{and} \quad \kappa_{\Omega_{HC}}(w) := \kappa_{\Omega_{HC}}(w, w)$$

for $z \in D$ and $w \in \Omega_{HC}$. Then, by Proposition 2.4,

$$(8.1) \quad |J(\lambda)|^2 = \frac{\kappa_D(\Phi(\lambda, \dots, \lambda))}{\kappa_{\Omega_{HC}}(\Phi_{HC}(\lambda, \dots, \lambda))} = \frac{\kappa_D\left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0\right)}{\kappa_{\Omega_{HC}}(\Phi_{HC}(\lambda, \dots, \lambda))}$$

for all $\lambda \in \mathbb{D}$.

Lemma 8.3. *There exist constants $0 < a < b$ such that*

$$a \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(d+1)} \leq \kappa_D \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right) \leq b \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(d+1)}$$

for all $\lambda \in \mathbb{D}$.

Proof. Since $\Omega \subset \mathcal{P}_\beta$, Proposition 2.3 and Observation 2.5 imply there exists a constant $C_\beta > 0$ such that

$$C_\beta (\operatorname{Re}(z))^{-(d+1)} = K_{\mathcal{P}_\beta}(z, 0, \dots, 0) \leq K_D(z, 0, \dots, 0)$$

for all $z \in \mathcal{H}$. Further,

$$\frac{1-|\lambda|}{|1-\lambda|^2} \leq \operatorname{Re} \left(\frac{1+\lambda}{1-\lambda} \right) \leq 2 \frac{1-|\lambda|}{|1-\lambda|^2}$$

for all $\lambda \in \mathbb{D}$. Combining these two estimates provides the lower bound

$$C_\beta \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(d+1)} \leq \kappa_D \left(\frac{1+\lambda}{1-\lambda}, 0, \dots, 0 \right).$$

The same argument with $\mathcal{P}_\alpha \subset \Omega$ yields the upper bound. □

Lemma 8.4. Ω is biholomorphic to the unit ball.

Proof. By Equation (8.1), Theorem 7.4, and Lemma 8.3, there exist constants $C > 0$ and $p \geq (d+r)/r$ such that

$$\begin{aligned} |J(\lambda)|^2 &\leq C \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(d+1)} (1-|\lambda|^2)^{rp} \leq C \left(\frac{1-|\lambda|}{|1-\lambda|^2} \right)^{-(d+1)} (1-|\lambda|^2)^{d+r} \\ &\leq C |1-\lambda|^{2(d+1)} (1+|\lambda|)^{d+r} (1-|\lambda|)^{r-1} \end{aligned}$$

for all $\lambda \in \mathbb{D}$.

Thus, if $r > 1$, then J extends continuously to $\partial \mathbb{D}$ and $J|_{\partial \mathbb{D}} \equiv 0$. Then, by the maximal principle, $J \equiv 0$. But this contradicts the fact that J is nowhere vanishing. Thus, we must have $r = 1$, and hence Ω is biholomorphic to the unit ball. $\square$

9. PROOF OF THEOREM 1.1

Theorem 1.1 is an immediate consequence of Proposition 6.1 and Theorem 8.1.

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