



2-Segal objects and algebras in spans

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Abstract

We define a category parameterizing Calabi–Yau algebra objects in an infinity category of spans. Using this category, we prove that there are equivalences of infinity categories relating, firstly: 2-Segal simplicial objects in $\mathcal{C}$ to algebra objects in $\text{Span}(\mathcal{C})$; and secondly: 2-Segal cyclic objects in $\mathcal{C}$ to Calabi–Yau algebra objects in $\text{Span}(\mathcal{C})$.

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Introduction

2-Segal objects and associativity

A familiar concept in higher category theory is that of *Segal objects* in an ∞ -category $\mathcal{C}$, that is, simplicial objects $X : \Delta^{\text{op}} \rightarrow \mathcal{C}$ such that the natural map

$$X_n \rightarrow \overbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}^{\times n}.$$

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is an equivalence. Introduced by Rezk in [17], Segal objects show up in a variety of guises, from monoidal ∞ -categories (cf. [12]) to the nerves of 1-categories. Of particular interest is the algebraic content of the Segal condition. Given a Segal set X , the span

$$X_1 \times_{X_0} X_1 \leftarrow X_2 \rightarrow X_1 \quad (1)$$

can be read as a multiplication law, owing to the invertibility of the left hand morphism. Moreover, the Segal condition on higher simplices also expresses the associativity of this multiplication.

The Segal condition on a simplicial set was generalized to the ‘higher-dimensional’ *2-Segal condition* by Dyckerhoff and Kapranov [4] and Gálvez-Carrillo et al. [10] (2-Segal spaces are called *decomposition spaces* in the latter). In a sense, the 2-Segal condition no longer requires that the span (1) define a multiplication, but retains the higher associativity conditions encoded in the higher simplices. More precisely, the 2-Segal condition on a simplicial object $X : \Delta^{\text{op}} \rightarrow \mathcal{C}$ requires that the diagrams

$$\begin{array}{ccc} X_n & \longrightarrow & X_{i,\dots,j} \\ \downarrow & & \downarrow \\ X_{1,\dots,i,j,\dots,n} & \longrightarrow & X_{i,j} \end{array}$$

all be pullback diagrams in $\mathcal{C}$. The 2-Segal condition is indeed a generalization of the Segal condition, insofar as every Segal simplicial object is 2-Segal.

A 2-Segal object X is said to be *unital* if, additionally, the diagrams

$$\begin{array}{ccc} X_{n-1} & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_{i,i+1} \end{array}$$

are pullback in $\mathcal{C}$. Throughout this paper, we adopt the convention that ‘2-Segal object’ refers to a *unital* 2-Segal object in the terminology of [4]. This terminology acquired a post-hoc justification from the recent paper [7], in which it is shown that all 2-Segal objects in the Dyckerhoff-Kapranov sense are unital 2-Segal, obviating the need for the distinction drawn in [4].

The sense in which such structures encode associativity relies on thinking of spans

$$X_1 \times X_1 \times \cdots \times X_1 \longleftarrow X_n \longrightarrow X_1$$

as ‘ n -fold multiplications’, regardless of whether the left-hand morphisms are equivalences. We then compose by concatenating spans and taking a pullback, thinking of the result as a ‘space of compositions’. In this language, the 2-Segal condition says that the space of compositions of n -fold multiplications with m -fold multiplications is precisely the space of $n + m - 1$ -fold multiplications.

There are a number of ways to make this intuitive picture rigorous (see, for example, the relation to Hall algebras presented in [4], and the connection with operads from [18]). The present paper concerns itself with one such perspective, namely, considering the relation between 2-Segal objects in an ∞ -category $\mathcal{C}$ and algebra objects in an ∞ -category $\text{Span}(\mathcal{C})$ whose morphisms are spans in $\mathcal{C}$ (for a more precise construction, see 1.4). Several results in this direction have already appeared in the literature. In the original Dyckerhoff-Kapranov paper [4], monads and algebra objects in $(\infty, 2)$ -categories of spans were constructed from 2-Segal objects. More recently, Penney [16] defined lax algebras in spans coming from simplicial objects, and showed that the associativity of these lax algebras was equivalent to the 2-Segal condition. In this paper, we prove a more general result.

Theorem A *Let $\mathcal{C}$ be an ∞ -category with small limits. There is an equivalence of ∞ -categories*

$$\left\{ \begin{array}{c} \text{Algebra objects} \\ \text{in } \text{Span}(\mathcal{C}) \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal simplicial} \\ \text{objects in } \mathcal{C} \end{array} \right\}.$$

This theorem appears in full detail in the text as Theorem 2.23.

The proof of Theorem 2.23 is by necessity quite technical. However, in broad strokes, the steps of the proof are relatively straightforward. To avoid confusion in the combinatorial material which is to follow, we briefly sketch the elements of the proof here.

A sketch of the main argument

The first necessary ingredient in the proof is left-hand ∞ -category—algebra objects in $\text{Span}(\mathcal{C})$ equipped with the pointwise-Cartesian monoidal structure. The model for this monoidal category used here is a Cartesian fibration

$$\text{Span}_{\Delta}(\mathcal{C}^{\boxtimes}) \rightarrow N(\Delta)$$

as constructed in [4] and recapitulated in Sect. 1.4 of this paper. The definition of $\text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$ proceeds via a pair of adjunctions, so that maps of ∞ -(co)operads

$$\begin{array}{ccc} N(\Delta) & \xrightarrow{\quad} & \text{Span}_{\Delta}(\mathcal{C}^{\boxtimes}) \\ & \searrow & \swarrow \\ & N(\Delta) & \end{array}$$

correspond to functors of ∞ -categories

$$\Theta := \text{Tw}(\Delta) \times_{\Delta} \Delta^{\text{II}} \rightarrow \mathcal{C}$$

satisfying four additional conditions. We define a category $\text{Alg}_{\text{Sp}}(\mathcal{C})$ to be the full ∞ -subcategory of $\text{Fun}(\text{Tw}(\Delta) \times_{\Delta} \Delta^{\text{II}}, \mathcal{C})$ on the functors satisfying these conditions

It is essential to here remark on two features of this definition:

- Firstly, the construction in terms of Cartesian fibrations is quite natural in this setting. However, it does require that we work with coalgebras (as above) rather than algebras in $\text{Span}(\mathcal{C})$. Fortunately, this is a minor inconvenience, as, owing to the duality $\text{Span}(\mathcal{C}) \simeq \text{Span}(\mathcal{C})^{\text{op}}$, algebras and coalgebras in $\text{Span}(\mathcal{C})$ can be identified.
- Secondly, this definition of $\text{Alg}_{\text{Sp}}(\mathcal{C})$ imposes a somewhat unusual functoriality on the result. Whereas the morphisms between algebras in $\text{Span}(\mathcal{C})$ are natural transformations whose components are themselves spans, the morphisms in $\text{Alg}_{\text{Sp}}(\mathcal{C})$ are natural transformations whose components are morphisms in $\mathcal{C}$. The latter morphisms can be viewed as the special case of the former in which one leg of each component span is a degenerate morphism in $\mathcal{C}$. More formally, one might say: “Morphisms in $\text{Alg}_{\text{Sp}}(\mathcal{C})$ are defined to be natural transformations of the corresponding adjoint diagram in $\mathcal{C}$, rather than natural transformations in $\text{Span}(\mathcal{C})$ ”.

With these definitions in place, we note that Θ can be expressed as the nerve of a 1-category, and the proof proceeds by progressively eliminating or reinterpreting the four conditions which describe the functors in $\text{Alg}_{\text{Sp}}(\mathcal{C})$.

FIRST CONDITION: The first of these conditions merely amounts to certain objects being sent to the terminal object of $\mathcal{C}$, and is briefly dealt with in Sect. 2.2 by eliminating those objects from Θ .

SECOND CONDITION: The second condition specifies a set E of morphisms in Θ which must be sent to equivalences in $\mathcal{C}$. By universal property, this means that the ∞ -category of functors satisfying this condition will correspond to functors out of the ∞ -categorical localization of Θ at the morphisms in E . The handling of this ∞ -categorical localization forms the most technical combinatorial argument of the paper.

To effectively deal with this second condition, we first interpret the objects of Θ as rooted planar forests with a linear order on their roots, and a chosen interval in the linearly-ordered set of roots. This pictorial reinterpretation then allows us to observe that the morphisms in E induce bijections on the chosen interval of the roots, and moreover, induce a bijection on the sets of leaf interstices of the chosen trees.

Armed with this interpretation, we define an auxiliary category Δ^* of ordered tuples of elements in Δ . There is then a functor $\mathcal{L} : \Theta \rightarrow \Delta^*$ which, loosely speaking, sends a planar tree to the ordered set of interstices between its leaves.¹ Section 2.3 and Appendix A are devoted to formalizing this picture and proving that the map $\mathcal{L} : \Theta \rightarrow \Delta^*$ is an ∞ -categorical localization at the morphisms in E . The latter is accomplished via direct combinatorics on the slices of the map $\mathcal{L}$.

The category Δ^* thus plays a key role in the proof overall. On the one hand, it is an ∞ -categorical localization of Θ ; on the other, it contains $\Delta^{\text{op}} \subset \Delta^*$ as a full subcategory. This means that we can use Δ^* to interpolate between Θ and Δ^{op} , and thus between algebra objects in $\text{Span}(\mathcal{C})$ and 2-Segal objects in $\mathcal{C}$.

¹ The reader familiar with the work [18] may notice a similarity with Walde’s treatment of dendroidal spaces presenting invertible ∞ -operads. The two constructions involve a very similar intuition, which suggests a deep connection between the 2-Segal conditions and the combinatorics of trees.

Once Δ^* has been displayed as an ∞ -categorical localization, it is a simple matter to translate the remaining two conditions on the elements of $\mathrm{Alg}_{\mathrm{Sp}}(\mathcal{C})$ into conditions on functors $\Delta^* \rightarrow \mathcal{C}$.

THIRD CONDITION: The third condition, when translated to functors $F : \Delta^* \rightarrow \mathcal{C}$, requires that the image $F([m_1], [m_2], \dots, [m_k])$ of a tuple $([m_1], [m_2], \dots, [m_k])$ in Δ^* be canonically displayed as a product $F([m_1]) \times \dots \times F([m_k])$. After some careful unwinding in Sect. 2.4, we find that this precisely characterizes the image of the right Kan extension along $\Delta^{\mathrm{op}} \hookrightarrow \Delta^*$. Consequently, the ∞ -category of functors $\Delta^* \rightarrow \mathcal{C}$ satisfying the third condition is equivalent to the ∞ -category of functors $\Delta^{\mathrm{op}} \rightarrow \mathcal{C}$.

FOURTH CONDITION: The fourth condition is a pullback condition which, when translated through the localization and the restriction to Δ^{op} , is precisely a reformulation of the Segal conditions. This is shown in Sect. 2.4.

Polygons, surfaces, and topological field theories

There is an additional geometric intuition underlying the 2-Segal condition. Fix a standard $n + 1$ -gon P_n , and a simplicial object $X : \Delta^{\mathrm{op}} \rightarrow \mathcal{C}$. The set of vertices V_n of P_n defines a simplicial set Δ^{V_n} . For any triangulation $\mathcal{T}$ of P_n with vertices in P_n , one can define a simplicial subset $\Delta^{\mathcal{T}} \subset \Delta^{V_n}$ whose 2-simplices correspond to the triangles in $\mathcal{T}$. Taking limits of the simplicial object X over the corresponding categories of simplices, the inclusion $\Delta^{\mathcal{T}} \subset \Delta^{V_n}$ yields a morphism

$$X_n \rightarrow \lim_{\Delta/\Delta^{\mathcal{T}}} X_k.$$

By [4, Proposition 2.3.2], the 2-Segal condition is equivalent to the condition that this morphism be an equivalence for every $n \geq 2$ and every such triangulation $\mathcal{T}$ of P_n . Intuitively, this means that the 2-Segal condition allows one to glue together the X_n to get invariants of 2-dimensional simplicial complexes.

The connection of 2-Segal spaces to 2-dimensional geometry can be extended further with recourse to 2-Segal *cyclic* objects, that is cyclic objects in $\mathcal{C}$ whose underlying simplicial objects are 2-Segal. In [5, Section V.2], Dyckerhoff and Kapranov construct invariants $X(S, M)$ of stable marked surface (S, M) with boundary, associated to a 2-Segal cyclic object $X : \Delta^{\mathrm{op}} \rightarrow \mathcal{C}$. For the subset $N \subset M$ of marked points on the boundary of S , this invariant comes equipped with a projection $X(S, M) \rightarrow X_1^{|N|}$. More suggestively, if we label some of these marked points as ‘incoming’ and the rest as ‘outgoing’, we can read the invariant $X(S, M)$ as a span

$$X_1^{|N_{\mathrm{in}}|} \leftarrow X(S, M) \rightarrow X_1^{|N_{\mathrm{out}}|}.$$

Moreover, the $X(S, M)$ come equipped with coherent actions of the mapping class group. It is therefore natural to ask whether the invariants $X(S, M)$ form an open, oriented, ∞ -categorical topological field theory in $\mathrm{Span}(\mathcal{C})$.

Such open, oriented theories have attracted some attention in the literature already. In [2], Costello considers open oriented theories equipped with a set of D-branes and valued in the (dg-)category of chain complexes. He shows that such field theories are equivalent to Calabi–Yau A_∞ categories—a generalization of the Calabi–Yau algebras in chain complexes. A similar classification which has more bearing on the situation detailed above, is that of Lurie:

Theorem [15, Theorem 4.2.11] *Let $\mathcal{C}$ be a symmetric monoidal ∞ -category. The following types of data are equivalent:*

1. *Open oriented topological field theories in $\mathcal{C}$.*
2. *Calabi–Yau algebra objects in $\mathcal{C}$.*

Based on this theorem, the latter half of this paper seeks to relate cyclic 2-Segal objects to Calabi–Yau algebras. Such a relation is realized by:

Theorem B *Let $\mathcal{C}$ be an ∞ -category with small limits. There is an equivalence of ∞ -categories*

$$\left\{ \begin{array}{l} \text{Calabi–Yau} \\ \text{Algebra objects} \\ \text{in } \text{Span}(\mathcal{C}) \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{2-Segal cyclic} \\ \text{objects in } \mathcal{C} \end{array} \right\}.$$

This appears in the text in full detail as Theorem 3.25. As a consequence of Theorem 3.25, we see that 2-Segal cyclic objects in $\mathcal{C}$ are equivalent to open oriented topological field theories in $\text{Span}(\mathcal{C})$.

The proof of Theorem 3.25 proceeds along much the same lines as that of Theorem 2.23. Rather than recapitulate the method here, we briefly mention the changes which pose additional difficulties.

- We need to define a category Ass_{CY} which parameterizes Calabi–Yau algebras. This is done in Sect. 1.2.
- We must add conditions which formally account for the non-degeneracy of the trace in the definition of a Calabi–Yau algebra.
- We need to account for the cyclic symmetries inherent in Calabi–Yau algebras. We thus work with symmetric monoidal categories of spans, and must make use of the combinatorics relating Δ to Fin_* .
- The construction of a category Λ^* (a cyclic analogue of Δ^*) is more difficult, and the pictorial intuition more involved (see, for instance, the difference between Figs. 3, 4 and 5). Similarly, the localization result is commensurately more combinatorially complex.
- One must pass through two adjunctions—a restriction-Kan extension adjunction and a reflective localization—to arrive from Λ^* to Λ^{op} .

Regrettably, these difficulties necessitate a certain amount of repetition in this paper—in particular the inclusion of two localization results. Where possible, the exposition has been streamlined to eliminate some duplication. In some cases, the proofs of results, while not identical, are similar enough that I have chosen to leave the second to the reader, with every confidence that the former will prove a sufficient guide to the argument.

Examples and consequences

Once the correspondence of Theorem 3.25 is established, a wealth of avenues to construct topological field theories open up. A number of examples of interest have already been explored in the literature.

- Per [6], the Waldhausen S-construction also gives rise to many cyclic 2-segal spaces. An interesting special case is discussed in [3,5,6], where various versions of topological Fukaya categories are constructed as invariants $X(S, M)$ associated to 2-Segal objects arising from the Waldhausen S-construction.
- 1-Segal cyclic objects also provide a zoo of interesting examples. As a particular example, consider a morphism $f : A \rightarrow B$ in the ∞ -category of spaces $\mathcal{S}$. The Čech nerve of this morphism is the 1-Segal simplicial space

$$\cdots \quad A \times_B A \times_B A \rightrightarrows A \times_B A \rightrightarrows A$$

which realizes to B . An appropriately chosen circle action on B equips the Čech nerve with a canonical cyclic structure, and similarly, a cyclic structure on the Čech nerve equips its realization with a coherent S^1 -action. Loosely speaking, the surface invariant $X(S, M)$ associated to this cyclic Čech nerve of f is the space of ‘ S^1 -equivariant B -local systems on the circle bundle of a twisted tangent bundle of (S, M) equipped with reduction of structure group to A over the marked points’. When B is $BSL_2(\mathbb{R})$ and A is BU , where U is the subgroup of upper unitriangular matrices, this construction can be related to the higher Teichmüller spaces constructed by Fock and Goncharov in [9].

- Another interesting incarnation of the cyclic Čech nerve construction is its application to a morphism $f : * \rightarrow X$ into a connected space X . In this context, the Čech nerve has the loop space ΩX based at $f(*)$ as its space of 1-simplices, and we expect the resulting surface invariants to relate to string topology.

Theorems 2.23 and 3.25 also bear an interesting relation to another construction in the literature. Following Cisinski and Moerdijk (cf. [1]), Walde defines a notion of a cyclic ∞ -operad in [18], and shows that there are equivalences of ∞ -categories

$$\left\{ \begin{array}{c} \text{invertible cyclic} \\ \infty\text{-operads} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal cyclic} \\ \text{objects in } \mathcal{S} \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \text{invertible} \\ \infty\text{-operads} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{2-Segal simplicial} \\ \text{objects in } \mathcal{S} \end{array} \right\}.$$

Which now has the immediate implication of relating invertible (cyclic) ∞ -operads to (Calabi–Yau) algebras in $\text{Span}(\mathcal{S})$.

There are also a number of possible generalizations of Theorems 2.23 and 3.25. For instance, the cyclic category Λ is one example of a *crossed simplicial group*, a notion defined by Fiedorowicz and Loday [8] and Krasauskas [11]. In [5], invariants analogous to the $X(S, M)$ were constructed for functors $X : \Delta \mathfrak{G}^{\text{op}} \rightarrow \mathcal{C}$ satisfying

the 2-Segal condition, where $\Delta\mathcal{G}$ is a crossed simplicial group. We expect that the relation between open topological field theories in spans and 2-Segal cyclic objects generalizes to this additional structure, which will be the basis for some future work on the subject.

1 The menagerie: notations, conventions, and constructions

This section will be given over to a review of the fundamental definitions and constructions that we will use in the proof of the main results. Along the way, we will also prove relations between these objects, so as to somewhat ameliorate the density of our later arguments.

1.1 Linear and cyclic orders

We first review the combinatorics of categories of ordered and unordered sets. The material here is mostly standard (with the possible exception of the imbrication in Definition 1.5), and the section serves mostly to introduce notation. The section may thus comfortably be skipped by most readers, and used as a reference for later sections.

Remark 1.1 It is worth noting that we follow a general convention: the skeletal version of a category will be denoted by the conventional symbol (e.g. Δ), whereas a blackboard bold symbol will be used for an “enlarged” version (e.g. $\mathbb{\Delta}$ for *all* finite non-empty linearly ordered sets.) While it is not strictly necessary to work with both the skeletal and non-skeletal versions of these categories, a number of constructions (e.g. cyclic duality, Construction 1.11) are more canonical in their non-skeletal form. We therefore discuss both versions of these combinatorial categories, in hopes of clarifying the constructions to come.

Definition 1.2 The *simplex category* Δ has objects the standard linearly ordered sets $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and morphisms the order-preserving maps. The *enlarged simplex category* $\mathbb{\Delta}$ has objects finite non-empty linearly ordered sets, and morphisms order-preserving maps.

The *augmented simplex category* Δ_+ (resp. the *augmented simplex category* $\mathbb{\Delta}_+$) is obtained from Δ (resp. $\mathbb{\Delta}$) by appending an initial object $\emptyset$, which will also sometimes be denoted by $[-1]$.

The *interval category* ∇ is the subcategory of Δ on the objects $[n]$ for $n \geq 1$, the morphisms of which preserve maximal and minimal elements. The *enlarged interval category* $\mathbb{\nabla}$ is the subcategory of $\mathbb{\Delta}$ on those sets of cardinality ≥ 2 , whose morphisms preserve maximal and minimal elements.

The *augmented interval category* ∇_+ (resp. the *augmented extended interval category* $\mathbb{\nabla}_+$) is the subcategory of Δ (resp. $\mathbb{\Delta}$) whose objects have cardinality ≥ 1 and whose morphisms preserve the maximal and minimal elements.

Definition 1.3 The category of the standard finite sets $\underline{n} := \{1, 2, \dots, n\}$ for $n \geq 0$ will be denoted $\mathbf{Fin}$. The category of the standard finite pointed sets $\langle n \rangle := \underline{n} \sqcup \{*\}$ will be denoted $\mathbf{Fin}_*$. The category of all finite sets (resp. the category of all finite

pointed sets) will be denoted by $\mathbb{F}in$ (resp. by $\mathbb{F}in_*$). When convenient, we will denote by $\mathbb{I}$ (resp. by $\mathbb{I}^*$) the opposites of the categories $\mathbb{F}in_*$ (resp. $\mathbb{F}in$). Given a pointed set $S \in \mathbb{F}in_*$, we denote by S° the set $S \setminus \{*\}$, where $*$ denotes the basepoint of S .

We additionally denote by $\mathcal{A}ss$ the *associative operad*, i.e. the category whose objects are objects of $\mathbb{F}in_*$, and whose morphisms $\phi : S \rightarrow T$ are morphisms in $\mathbb{F}in_*$ equipped with a chosen linear order on the fiber $\phi^{-1}(i)$ for each $i \in T^\circ$. Composition is defined by composition in $\mathbb{F}in_*$, together with the lexicographic orders. Note that there is a forgetful functor $\mathcal{A}ss \rightarrow \mathbb{F}in_*$, which equips $N(\mathcal{A}ss)$ with the structure of an ∞ -operad in the sense of [13].

Construction 1.4 (*Linear interstices*) Given a linearly ordered set $S \in \Delta$ we define an *inner interstice* of S to be an ordered pair $(k, k+1) \in S \times S$, where $k+1$ denotes the successor to k . The set of inner interstices of S is, itself, a linearly ordered set, with the order

$$(k, k+1) \leq (j, j+1) \Leftrightarrow k \leq j$$

We will denote the linearly ordered set of inner interstices of S by $\mathbb{I}(S)$. Note that $\mathbb{I}([0]) = \emptyset$.

Given a linearly ordered set $S \in \Delta_+$, let $\hat{S}$ be the set $\{a\} \sqcup S \sqcup \{b\}$, where b is taken to be maximal and a minimal. We define an *outer interstice* of S to be an inner interstice of $\hat{S}$. We will denote the linearly ordered set of outer interstices of S by $\mathbb{O}(S)$. Note that $\mathbb{O}(\emptyset) = \{(a, b)\}$.

We define functors

$$\mathbb{O} : \Delta_+^{\text{op}} \rightarrow \nabla_+; \quad S \mapsto \mathbb{O}(S)$$

and

$$\mathbb{I} : \nabla_+^{\text{op}} \rightarrow \Delta; \quad S \mapsto \mathbb{I}(S)$$

as follows (we will define $\mathbb{O}$ explicitly, the definition of $\mathbb{I}$ is similar). Given a morphism $f : S \rightarrow T$ in Δ_+ , we define a morphism $\mathbb{O}(f) : \mathbb{O}(S) \rightarrow \mathbb{O}(T)$ by setting

$$\mathbb{O}(f)(j, j+1) = \begin{cases} (k, k+1) & f(k) \leq j \leq j+1 \leq f(k+1) \\ (a, a+1) & j \leq f(k) \quad \forall k \in S \\ (b-1, b) & j \geq f(k) \quad \forall k \in S. \end{cases}$$

Pictorially, we can represent the morphism $\mathbb{O}(f)$ as a forest as in Fig. 1, thinking leaves $j \in \mathbb{O}(T)$ as being attached to the root $k \in \mathbb{O}(S)$ if $\mathbb{O}(f)(j) = k$.

Note that the functors $\mathbb{I}$ and $\mathbb{O}$ define an equivalence of categories. Since Δ_+ (resp. ∇_+) is the skeletal version of Δ_+ (resp. ∇_+), all isomorphisms in these categories are identities, we see that we get an induced *isomorphism* of categories

$$\mathbb{O} : \Delta_+^{\text{op}} \xrightarrow{\cong} \nabla_+ : \mathbb{I}$$

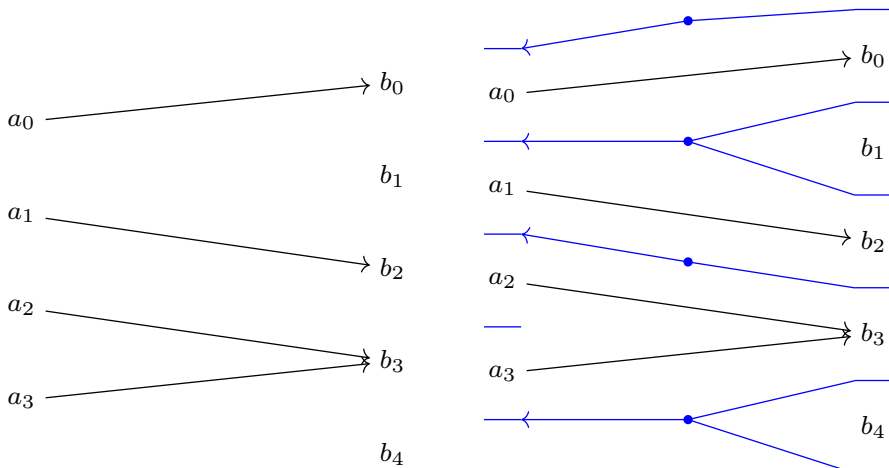


Fig. 1 Left: a morphism f of linearly ordered sets. Right: the morphism $\mathbb{O}(f)$, visualized as a forest (blue)

Moreover, we can define a functor $\nabla_+ \rightarrow \text{Fin}_*$ by

$$S \mapsto (S \amalg \{*\}) / \max(S) \sim \min(s) \sim *$$

We then find that the induced functor

$$\Delta_+^{\text{op}} \hookrightarrow \Delta_+^{\text{op}} \xrightarrow{O} \nabla_+ \rightarrow \text{Fin}_*$$

is precisely the functor $\text{cut} : \Delta^{\text{op}} \rightarrow \text{Fin}_*$ defined in [13, 4.1.2.9].

The next definition, the *imbrication*, is of particular import in the main argument. Loosely, the imbrication is the dual monoidal structure to the ordinal sum, as expressed on ∇_+ . It will be used in Sect. 2.3 to give a precise description of the category Δ^* , which interpolates between the indexing category for algebra objects in spans, and the category Δ^{op} .

Definition 1.5 Given two linearly ordered sets $S, T \in \Delta_+$ define the *ordinal sum* $S \oplus T$ to be the set $S \amalg T$ equipped with the linear order defined by the orders on S and T and the proscription that for all $s \in S$ and $t \in T$, $s \leq t$. The ordinal sum defines a monoidal structure on Δ_+ .

Given two linearly ordered sets $S, T \in \nabla_+$, with b the maximum of S and a the minimum of T , define the *imbrication* $S \star T$ to be the linearly ordered set $(S \oplus T)_{/a \sim b}$ (note that since a is the successor to b in $S \oplus T$, there is a canonical linear order on $S \star T$ compatible with the quotient map).

Lemma 1.6 *The functor $\mathbb{O}$ is a monoidal functor sending the ordinal sum to the imbrication.*

Remark 1.7 It is worth noting here that some of these notions appear in [10] (using different notation). The duality between Δ_+ and ∇_+ is the content of [10, Lemma 6.2], and the imbrication appears in Sect. 2.4 as the *amalgamated ordinal sum*.

Definition 1.8 A *cyclic order* on a finite set S is a transitive $\mathbb{Z}$ -action on S . Equivalently, this is simply transitive action of $\mathbb{Z}/|S|$ on S .

Definition 1.9 Given a cyclic set S , and a collection $\{[n_i]\}_{i \in S}$ of objects in Δ_+ , we define a cyclic set $\bigcup^S([n_i])$ as follows. The underlying set is $\coprod_{i \in S} [n_i]$, and the cyclic order is given by the $\mathbb{Z}/n$ -action (where $n := \sum_{i \in S} (n_i + 1)$) that sends

$$j \in [n_i] \mapsto \begin{cases} j + 1 & j < n_i \\ 0 \in [n_{i+1}] & j = n_i \end{cases}$$

where $i + 1$ denotes the successor of i in the cyclic order on S . We call this order on $\coprod_{i \in S} [n_i]$ the *lexicographic (cyclic) order*.

Definition 1.10 A *morphism of cyclically ordered sets* $S \rightarrow T$ consists of a map of sets $\phi : S \rightarrow T$, and a linear order on each fiber such that the lexicographic cyclic order on S agrees with the predefined cyclic order on S .

The *cyclic category* has as its objects the standard cyclicly ordered sets $\langle n \rangle$ for $n \geq 0$, and as its morphisms the maps of finite sets respecting the cyclic order. The *enlarged cyclic category* $\mathbb{A}$ has as its objects all finite, non-empty, cyclicly ordered sets, and as its morphisms the maps which respect the cyclic order.

Construction 1.11 (*Cyclic duality*) In analogy to the construction of the linear interstice functors, we define a duality

$$\mathbb{D} : \mathbb{A}^{\text{op}} \rightarrow \mathbb{A}$$

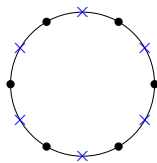
on the cyclic category. Let $S \in \mathbb{A}$ be a cyclicly ordered set. We define a *cyclic interstice* of S to be an ordered pair $(a, a + 1) \in S \times S$, where $a + 1$ denotes the successor of a under the cyclic order. We denote the set of cyclic interstices of S by $\mathbb{D}(S)$. The set $\mathbb{D}(S)$ inherits a canonical cyclic order from S , which can be visualized as in Fig. 2. The functor $\mathbb{D}$ is specified on morphisms by an analogue of Construction 1.4, namely, for $f : S \rightarrow T$ in $\mathbb{A}$ and $(j, j + 1) \in \mathbb{D}(T)$, we set

$$\mathbb{D}(f)(j, j + 1) := (k, k + 1)$$

where $(k, k + 1)$ is the unique interstice of S such that the linearly ordered set $(f(k), f(k) + 1, f(k) + 2, \dots, f(k + 1))$ contains $(j, j + 1)$ in order.

This functor is an equivalence of categories. Since $\mathbb{A}$ is the skeletal version of $\mathbb{A}$, $\mathbb{D}$ descends to an equivalence $D : \mathbb{A}^{\text{op}} \rightarrow \mathbb{A}$

Fig. 2 A cyclic set with its cyclic order visualized via an embedding into the oriented circle (black), together with its set of cyclic interstices (blue crosses)



Construction 1.12 (*Cyclic closures*) We define a functor $\mathbb{K} : \Delta \rightarrow \mathbb{A}$ in the following way. Given a linearly ordered set S of cardinality $n + 1$, there is a unique order-preserving bijection $\phi : S \rightarrow [n]$. We define a bijection

$$S \rightarrow r(n); \quad j \mapsto \exp\left(\frac{2\pi i \phi(j)}{n+1}\right)$$

to the n th roots of unity in S^1 . The orientation on S^1 then yields a canonical cyclic order on S . Passing to skeletal versions yields the well-known functor $\kappa : \Delta \rightarrow \Lambda$.

Via the equivalences $\mathbb{O}$ and $\mathbb{D}$ we can then define a functor $\mathbb{C} : \nabla \rightarrow \Lambda$ such that the diagram

$$\begin{array}{ccc} \Delta^{\text{op}} & \xrightarrow{\mathbb{O}} & \nabla \\ \mathbb{K} \downarrow & & \downarrow \mathbb{C} \\ \Lambda^{\text{op}} & \xrightarrow{\mathbb{D}} & \Lambda \end{array}$$

commutes up to natural isomorphism. The functor $\mathbb{C}$ admits the following explicit description on objects. Let $S \in \nabla$ with maximal element b and minimal element a . Then $\mathbb{C}(S)$ can be identified with with quotient of $\mathbb{K}(S)$ by the identification $a \sim b$. Once again, we have that $\mathbb{C}$ descends to a functor $C : \nabla \rightarrow \Lambda$.

Lemma 1.13 *Let $S \in \mathbb{A}$, a set $\{[n_i]\}_{i \in S}$ of elements in Δ_+ , and a compatible linear order $\phi : \mathbb{K}([m]) \cong S$, there is a canonical isomorphism*

$$\mathbb{K}\left(\bigoplus_{i \in [m]} [n_{\phi(i)}]\right) \cong \bigcup^S [n_i]$$

which acts as the identity on underlying sets.

Proof We compare the $\mathbb{Z}/n$ -actions. When $j \in \bigoplus_{i \in [m]} [n_i]$ is not maximal, the successor function for the ordinal sum agrees with the $\mathbb{Z}/n$ -action on $\bigcup^S [n_i]$. If j is maximal, we have that the action on the left sends j to $0 \in n_{\phi(0)}$, which agrees with the definition of the cyclic order on the right. $\square$

1.2 Calabi–Yau algebras

We now turn our attention to Calabi–Yau algebras. After briefly recalling the main definitions, we construct a category $\mathcal{A}ss_{CY}$ such that maps from $\mathcal{A}ss_{CY}$ into a symmetric monoidal ∞ -category satisfying non-degeneracy conditions are the same as Calabi–Yau algebras. We then use $\mathcal{A}ss_{CY}$ to provide a definition of the ∞ -category of Calabi–Yau algebras in a specified symmetric monoidal ∞ -category. The proof that $\mathcal{A}ss_{CY}$ does, in fact, parameterize the desired algebraic structures is somewhat long and combinatorial, and forms the greater part of this section.

Throughout the section, we take $\mathcal{C}^{\otimes} \rightarrow \mathbb{F}n_*$ to be a symmetric monoidal ∞ -category with monoidal unit $\mathbb{1}$ and tensor product $\otimes$.

Construction 1.14 There is a functor $B : \mathbb{A} \rightarrow \mathcal{A}ss$ defined as follows. On objects, send each $S \in \mathbb{A}$ to $S \sqcup \{*\}$, forgetting the cyclic order. On morphisms, send $f : S \rightarrow T$ to its underlying map of sets. Define a linear order on the fibers of f by choosing embeddings of S and T into S^1 compatible with the cyclic order, and representing f as a commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\tilde{f}} & S^1 \\ \alpha \uparrow & & \uparrow \beta \\ S & \xrightarrow{f} & T \end{array}$$

where $\tilde{f}$ is monotone of degree 1. For $i \in T$, the preimage of $\beta(i)$ under $\tilde{f}$ is an interval, and $\beta(f^{-1}(i)) \subset \tilde{f}^{-1}(\beta(i))$. The orientation of S^1 induces an orientation of $\tilde{f}^{-1}(\beta(i))$, and hence a linear order on $f^{-1}(i)$.

Definition 1.15 The *cyclic bar object* of an algebra object $X : \mathcal{A}ss \rightarrow \mathcal{C}^{\otimes}$ is the composition $B^*(X)$. A *cyclic trace* on X is a natural transformation η from $B^*(X)$ to the constant cyclic object on $\mathbb{1} \in \mathcal{C}$. We call a pair (X, η) consisting of an algebra object in $\mathcal{C}^{\otimes}$ and a cyclic trace a *trace algebra*.

Remark 1.16 A natural transformation to a constant cyclic object may be modeled as a functor from the category $\mathbb{A}_{\diamond}$ obtained from $\mathbb{A}$ by formally adjoining a terminal object. We denote the terminal object of $\mathbb{A}_{\diamond}$ by $\diamond$.

Definition 1.17 A morphism $\gamma : X \otimes X \rightarrow \mathbb{1}$ in $\mathcal{C}$ is called *non-degenerate* if there exists a morphism $\eta : \mathbb{1} \rightarrow X \otimes X$ such that

- The composite

$$X \xrightarrow{\sim} X \otimes \mathbb{1} \xrightarrow{\eta \otimes \text{id}_{\mathbb{1}}} X \otimes X \otimes X \xrightarrow{\text{id}_{\mathbb{1}} \otimes \gamma} \mathbb{1} \otimes X \xrightarrow{\sim} X$$

is homotopic to the identity.

- The composite

$$X \xrightarrow{\sim} \mathbb{1} \otimes X \xrightarrow{\text{id}_{\mathbb{1}} \otimes \eta} X \otimes X \otimes X \xrightarrow{\gamma \otimes \text{id}_{\mathbb{1}}} X \otimes \mathbb{1} \xrightarrow{\sim} X$$

is homotopic to the identity.

Definition 1.18 Let (X, η) be a trace algebra in $\mathcal{C}$, and let $\eta_2 : X \otimes X \rightarrow \mathbb{1}$ be the map induced by $\langle 2 \rangle \rightarrow \diamond$ in $\mathbb{A}_\diamond$ under η . We call (X, η) a *Calabi–Yau algebra* in $\mathcal{C}$ if η_2 is non-degenerate.

Remark 1.19 The definition above is precisely that of [15, Example 4.2.8]. When Hochschild homology is defined, the map $\eta : B^*(X) \rightarrow \mathbb{1}$ is equivalently an S^1 -equivariant trace

$$\int_{S^1} X \rightarrow \mathbb{1}.$$

Definition 1.20 Let $\mathcal{Ass}_{CY}$ be the category with

- Objects $\text{ob}(\mathcal{Ass}) \sqcup \{\diamond\}$.
- Morphisms between $S, T \in \mathcal{Ass}$

$$\text{Hom}_{\mathcal{Ass}_{CY}}(S, T) := \text{Hom}_{\mathcal{Ass}}(S, T).$$

- For $S \in \mathcal{Ass}$,

$$\text{Hom}_{\mathcal{Ass}_{CY}}(\diamond, S) := \emptyset$$

and a morphism $S \rightarrow \diamond$ is a choice of a subset $T \subset S^\circ$ and a cyclic order on T .

- For $S, T \in \mathcal{Ass}$, and morphisms $\phi : S \rightarrow T$ and $\psi : T \rightarrow \diamond$, the composite $\psi \circ \phi$ is given by the induced cyclic order

Note that $\mathcal{Ass}_{CY}$ comes equipped with a functor $\mathcal{Ass}_{CY} \rightarrow \mathbb{F}\text{in}_*$ sending $\diamond \mapsto \langle 1 \rangle$.

Construction 1.21 Let $\mathbb{A} \rightarrow \mathbb{A}_\diamond$ and $\mathcal{Ass} \rightarrow \mathcal{Ass}_{CY}$ be the inclusions. Define a functor $F : \mathbb{A}_\diamond \rightarrow \mathcal{Ass}_{CY}$ by setting $F = B$ on $\mathbb{A} \subset \mathbb{A}_\diamond$, and sending $\diamond \mapsto \diamond$. By definition, the diagram

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{A}_\diamond \\ B \downarrow & & \downarrow F \\ \mathcal{Ass} & \longrightarrow & \mathcal{Ass}_{CY} \end{array} \quad (2)$$

commutes.

Definition 1.22 We take $\mathfrak{P}$ to be the categorical pattern of [13, Proposition 2.1.4.6]. In the following proof, we will freely make reference to this proposition, and Appendix B from the same.

Lemma 1.23 *The diagram*

$$\begin{array}{ccc} N(\mathbb{A}) & \longrightarrow & N(\mathbb{A}_\diamond) \\ B \downarrow & & \downarrow F \\ N(\mathcal{Ass}) & \longrightarrow & N(\mathcal{Ass}_{CY}) \end{array}$$

induces an $\mathfrak{P}$ -anodyne morphism of ∞ -categories

$$\theta : N(\mathcal{A}ss) \coprod_{N(\mathbb{A})} N(\mathbb{A}_\diamond) \rightarrow N(\mathcal{A}ss_{CY})$$

over $\mathbb{F}n_*$, where the non-degenerate marked simplices are precisely the inert morphisms of $\mathcal{A}ss$.

Proof An n -simplex of $N(\mathcal{A}ss) \coprod_{N(\mathbb{A})} N(\mathbb{A}_\diamond)$ is an equivalence class in $N(\mathcal{A}ss) \sqcup N(\mathbb{A}_\diamond)$ under the relation that

$$\underbrace{(S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_n)}_{\in N(\mathcal{A}ss)_n} \sim \underbrace{(T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n)}_{\in N(\mathbb{A})_n}$$

if and only if

$$B(T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n) = (S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_n).$$

In particular, θ is injective, and a bijection on 0-simplices.

We proceed by induction. For ease of notation, we set $Q = N(\mathcal{A}ss) \coprod_{N(\mathbb{A})} N(\mathbb{A}_\diamond)$.

1. Suppose $f : S \rightarrow \diamond$ is a 1-simplex not contained in the image of θ . Then S is determined by $T \subsetneq S^\circ$ and a cyclic order on S . Adding a basepoint to T to get $T_f \in \mathcal{A}ss_{CY}$ we get a factorization of f as

$$\begin{array}{ccc} & T_f & \\ \beta \nearrow & & \searrow \alpha \\ S & \xrightarrow{f} & \diamond \end{array}$$

in $\mathcal{A}ss_{CY}$. Taking such a 2-simplex σ_f for every such f , we can form the pushout

$$\begin{array}{ccc} \coprod_{\{f\}} (\Delta_1^2)^b & \longrightarrow & Q_0 \\ \downarrow & & \downarrow \\ \coprod_{\{f\}} (\Delta^2)^b & \longrightarrow & Q_1 \end{array}$$

The morphism on the left is of type (C_1) from [13, B.1.1], so we get a factorization

$$\begin{array}{ccccc} & \theta & & & \\ & \curvearrowright & & & \\ Q_0 & \xrightarrow{\tau_1} & Q_1 & \xrightarrow{\theta_1} & N(\mathcal{A}ss_{CY}) \end{array}$$

where τ_1 is $\mathfrak{P}$ -anodyne, and θ_1 is bijective on 1-simplices.

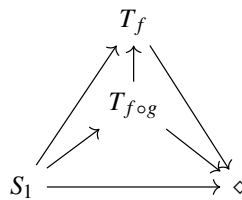
2. Now suppose that $\sigma : \Delta^2 \rightarrow \mathcal{A}ss_{CY}$ is a 2-simplex not in the image of θ_1 . Then σ must be given by a sequence

$$S_1 \xrightarrow{g} S_2 \xrightarrow{f} \diamond$$

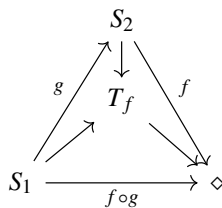
(if σ does not contain $\diamond$, it is the image of a simplex in $\mathcal{A}ss$, if it contains two copies of $\diamond$, it is degenerate). Consequently, we get two 2-simplices, $\sigma_{f \circ g}$ and σ_g in the image of θ_1 . Moreover, g restricts to a morphism

$$g : T_{f \circ g} \rightarrow T_f,$$

and we get a 2-simplex $S_1 \rightarrow S_2 \rightarrow T_f$. We then note that the Λ_1^3 horn



can be filled to a 2-simplex $S_1 \rightarrow T_f \rightarrow \diamond$ via a horn of type (C_1) . Finally, we get a Λ_2^3 -horn



of type (C_1) . This gives us a factorization of θ as $Q_0 \xrightarrow{\tau_2} Q_2 \xrightarrow{\theta_2} N(\mathcal{A}ss_{CY})$ where τ_1 is $\mathfrak{P}$ -anodyne and θ_2 is bijective on simplices of dimension ≤ 2 .

3. Now suppose inductively that we have obtained a factorization through $\theta_{n-1} : Q_n \rightarrow N(\mathcal{A}ss_{CY})$ such that
- θ_{n-1} is bijective on k -simplices for $k \leq n-1$.
 - The image of θ_{n-1} contains all n -simplices of the form

$$S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_{n-1} \rightarrow \diamond$$

where $S_{n-1} \rightarrow \diamond$ is a 1-simplex in the image of $\Lambda_\diamond$.

Suppose given an n -simplex σ not in the image of θ_{n-1} . Then, by similar reasoning to that above, σ must be of the form

$$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \rightarrow S_{n-1} \xrightarrow{\phi_n} \diamond$$

with $S_{n+1} \rightarrow \diamond$ not in the image of $\Lambda_\diamond$. Define $\psi_k := \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_{n-k}$, we then get n -simplices in the image of θ_{n-1}

$$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \rightarrow \widehat{S_k} \rightarrow S_{n-1} \rightarrow T_{\phi_n} \rightarrow \diamond$$

and an n -simplex in the image of θ_{n-1}

$$S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_{n-1} \rightarrow T_f.$$

These n n -simplices form a Λ_n^{n+1} -horn in $N(\text{Ass}_{\mathcal{C}\mathcal{Y}})$ which, once again, can be filled by a pushout of type (C_1) .

We therefore get a factorization

$$Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow N(\text{Ass}_{\mathcal{C}\mathcal{Y}})$$

which exhausts $N(\text{Ass}_{\mathcal{C}\mathcal{Y}})$. Each morphism in this sequence is $\mathfrak{P}$ -anodyne, and so the transfinite composition $Q_0 \rightarrow N(\text{Ass}_{\mathcal{C}\mathcal{Y}})$ is $\mathfrak{P}$ -anodyne. $\square$

Corollary 1.24 *The ∞ -category of trace algebras in $\mathcal{C}$ is equivalent to the full subcategory of $\text{Map}_{\mathbb{H}\mathbb{N}_*}^\sharp(N(\text{Ass}_{\mathcal{C}\mathcal{Y}}), \mathcal{C}^\otimes)$ sending $\diamond$ to $\mathbb{1}$.*

Definition 1.25 We define the ∞ -category of Calabi–Yau algebras in $\mathcal{C}$ to be the full subcategory of $\text{Map}_{\mathbb{H}\mathbb{N}_*}^\sharp(N(\text{Ass}_{\mathcal{C}\mathcal{Y}}), \mathcal{C}^\otimes)$ on those objects which

1. send $\diamond$ to $\mathbb{1}$, and
2. send the morphism $\langle 2 \rangle \rightarrow \diamond$ in $\mathbb{A}_\diamond$ to a non-degenerate morphism $X \times X \rightarrow \mathbb{1}$.

1.3 Cartesian monoidal structures

We now briefly recapitulate Cartesian monoidal structures. This is necessitated by the fact that, throughout this paper, we will model (symmetric) monoidal structures by Cartesian fibrations, rather than the coCartesian fibrations used in [13]. Indeed, this section amounts, more or less, to a dualized recapitulation of the corresponding sections of [13].

The Cartesian fibrations modeling these monoidal structures will be defined via adjunctions, as laid out below. Throughout this section, $\mathcal{C}$ will denote an ∞ -category which admits finite products.

Definition 1.26 The category $\Delta^\sqcup$ has as its objects pairs $([n], \{i, j\})$, where $[n] \in \Delta$ and $i \leq j$ are elements in $[n]$. The morphisms $([n], \{i, j\}) \rightarrow ([m], \{k, \ell\})$ consist of

a morphism $\phi : [n] \rightarrow [m]$ such that $\phi(i) \leq k \leq \ell \leq \phi(j)$. We will, in general, think of $\{i, j\}$ as an interval inside $[n]$, and denote by $\{i \leq j\}$ the linearly ordered set

$$\{i \leq j\} := \{i, i+1, \dots, j\} \subset [n].$$

The category $\mathbb{F}\mathbf{in}_*^{\sqcup}$ has as its objects pairs (S, T) where $S \in \mathbb{F}\mathbf{in}_*$ and $T \subset S^\circ$. A morphism $(S, T) \rightarrow (P, Q)$ consists of a morphism $\phi : S \rightarrow P$ in $\mathbb{F}\mathbf{in}_*$ such that $\phi(T) \subset Q$. We will sometimes denote by $\mathbb{F}^{\sqcup}$ the category $(\mathbb{F}\mathbf{in}_*^{\sqcup})^{\text{op}}$.

Remark 1.27 We can provide an alternate characterization of $\Delta^{\sqcup}$ and $\mathbb{F}\mathbf{in}_*^{\sqcup}$. The functor $\Delta^{\sqcup} \rightarrow \Delta$ is the coCartesian fibration defined as a Grothendieck construction of the functors

$$\Delta \rightarrow \text{Cat}; \quad [n] \mapsto I_{[n]}^{\text{op}}.$$

The functor $\mathbb{F}\mathbf{in}_*^{\sqcup} \rightarrow \mathbb{F}\mathbf{in}$ is the Cartesian fibration defined as a Grothendieck construction of the (contravariant) power set functor

$$\mathbb{F}\mathbf{in}_*^{\text{op}} \rightarrow \text{Cat}; \quad S \mapsto \mathcal{P}(S^\circ).$$

Note that, as in [4, Remark 10.3.2], these constructions relate to the constructions $\Delta^\times \rightarrow \Delta$ and $\Gamma^\times \rightarrow \mathbb{F}\mathbf{in}_*$ from [12, Proposition 1.2.8] and [13, Proposition 2.4.1.5] respectively. In particular, the functor $\Gamma^\times \rightarrow \mathbb{F}\mathbf{in}_*$ is the Cartesian fibration arising as the Grothendieck construction of

$$\mathbb{F}\mathbf{in}^* \rightarrow \text{Cat}; \quad S \mapsto \mathcal{P}(S^\circ)^{\text{op}}.$$

For an ∞ -category $\mathcal{D}$ with enough colimits, the functor $\mathbb{F}\mathbf{in}_*^{\sqcup} \rightarrow \mathbb{F}\mathbf{in}_*$ can therefore be used to construct a coCartesian fibration $\mathcal{D}^{\sqcup} \rightarrow \mathbb{F}\mathbf{in}_*$ modeling the coCartesian symmetric monoidal structure on $\mathcal{D}$.

Construction 1.28 The functor $\text{cut} : \Delta \rightarrow \mathbb{F}\mathbf{in}_*^{\text{op}}$ yields a functor $\Delta^{\sqcup} \rightarrow (\mathbb{F}\mathbf{in}_*^{\sqcup})^{\text{op}}$. To see this, we first note that for $\{i, j\} \subset [n]$ in $\Delta^{\sqcup}$, we have $\mathbb{O}(\{i \leq j\}) \subset \mathbb{O}([n])$. On objects we therefore define $\{i, j\} \subset [n] \mapsto (\mathbb{O}([n]), \mathbb{O}(\{i \leq j\}))$

Given a morphism $f : ([n], \{i, j\}) \rightarrow ([m], \{k, \ell\})$ in $\Delta^{\sqcup}$, we get a morphism $\mathbb{O}(f) : \mathbb{O}([m]) \rightarrow \mathbb{O}([n])$. Moreover, the condition that $f(i) \leq k \leq \ell \leq f(j)$ ensures that $\mathbb{O}(f)(\mathbb{O}(\{k \leq \ell\})) \subset \mathbb{O}(\{i \leq j\})$.

Construction 1.29 (*Cartesian monoidal structures*) Given an ∞ -category $\mathcal{C}$ with finite products, we can associate two Cartesian fibrations to $\mathcal{C}$ as follows.

We define a functor of ∞ -categories $\overline{\mathcal{C}^{\boxtimes}} \rightarrow \Delta$ via the universal property

$$\text{Hom}_\Delta(K, \overline{\mathcal{C}^{\boxtimes}}) \cong \text{Hom}_{\text{Set}_\Delta}(K \times_\Delta \Delta^{\sqcup}, \mathcal{C}).$$

Similarly, we define a functor $\overline{\mathcal{C}^\times} \rightarrow \mathbb{F}$ via the universal property

$$\text{Hom}_\mathbb{F}(K, \overline{\mathcal{C}^\times}) \cong \text{Hom}_{\text{Set}_\Delta}(K \times_\mathbb{F} \mathbb{F}^{\sqcup}, \mathcal{C}).$$

Both of these are Cartesian fibrations by dint of [14, 3.2.2.13].

We now let $\mathcal{C}^{\boxtimes} \subset \overline{\mathcal{C}^{\boxtimes}}$ be the full subcategory on those objects $G : I_{[n]}^{\text{op}} \rightarrow \mathcal{C}$ for which G displays $G(\{i \leq j\})$ as a product over $G(\{k \leq k+1\})$ for $i \leq k < j$.

Similarly, we let $\mathcal{C}^{\times} \subset \overline{\mathcal{C}^{\times}}$ be the full subcategory on those objects $G : \mathcal{P}(S^{\circ})^{\text{op}} \rightarrow \mathcal{C}$ for which G displays $G(S)$ as a product over $G(i)$ for $i \in S$.

With these definitions in place, we can now recall the main results characterizing them.

Proposition 1.30 *The functor $\mathcal{C}^{\boxtimes} \rightarrow \Delta$ is a Cartesian fibration exhibiting the Cartesian monoidal structure on $\mathcal{C}$.*

Proof This is [4, Prop. 10.3.8]. $\square$

Proposition 1.31 *The functor $\mathcal{C}^{\times} \rightarrow \mathbb{I}$ is a Cartesian fibration exhibiting the Cartesian symmetric monoidal structure on $\mathcal{C}$.*

Proof The proof of this statement is, *mutatis mutandis*, the same as the proof of [13, Proposition 2.4.1.5]. $\square$

There is a further feature of interest to the presentation of monoidal structures by Cartesian fibrations. Given a Cartesian fibration $p : \mathcal{D}^{\boxtimes} \rightarrow \Gamma$ representing a monoidal structure on $\mathcal{D} := \mathcal{D}_{(1)}^{\boxtimes}$, the opposite map $p^{\text{op}} : (\mathcal{D}^{\boxtimes})^{\text{op}} \rightarrow \Gamma^{\text{op}}$ is the coCartesian fibration representing the induced monoidal structure on $\mathcal{D}^{\text{op}}$. The Cartesian fibration p thus gives us direct access to coalgebras in $\mathcal{D}$:

Definition 1.32 Given an ∞ -operad $\mathcal{O}^{\boxtimes} \rightarrow \Gamma^{\text{op}}$ and a Cartesian fibration $\mathcal{D}^{\boxtimes} \rightarrow \Gamma$ exhibiting a symmetric monoidal structure on $\mathcal{D}$, a $\mathcal{O}^{\boxtimes}$ -coalgebra in $\mathcal{D}$ is a functor

$$A : (\mathcal{O}^{\boxtimes})^{\text{op}} \rightarrow \mathcal{D}^{\boxtimes}$$

over Γ such that (the opposite of) every inert morphism f in $\mathcal{O}^{\boxtimes}$ is sent to a Cartesian morphism in $\mathcal{D}$.

The ∞ -category of $\mathcal{O}^{\boxtimes}$ -coalgebras in $\mathcal{D}$ is the full subcategory $\text{CoAlg}_{\mathcal{O}}(\mathcal{D}^{\boxtimes}) := \text{Fun}_{\Gamma}((\mathcal{O}^{\boxtimes})^{\text{op}}, \mathcal{D}^{\boxtimes})$ on the $\mathcal{O}^{\boxtimes}$ -coalgebras.

In our exploration of ∞ -categories of spans, we will freely make use of the relation between algebras and coalgebras induced by the duality on spans, as described in the lemma below.

Lemma 1.33 *Let $p : \mathcal{D}^{\boxtimes} \rightarrow \Gamma$ be a Cartesian fibration exhibiting a model structure $\boxtimes$ on $\mathcal{D}$, and suppose that we have a duality $\mathcal{D} \simeq \mathcal{D}^{\text{op}}$ so that $p^{\text{op}} : (\mathcal{D}^{\boxtimes})^{\text{op}} \rightarrow \Gamma^{\text{op}}$ exhibits a monoidal structure $\boxtimes$ on $\mathcal{D}$. Then for every ∞ -operad $\mathcal{O}^{\boxtimes}$, there is an equivalence*

$$\text{CoAlg}_{\mathcal{O}}(\mathcal{D}^{\boxtimes}) \simeq \text{Alg}_{\mathcal{O}}(\mathcal{D}^{\boxtimes}).$$

Proof Follows directly from unwinding the definitions. $\square$

1.4 ∞ -Categories of Spans

We will briefly recall here the requisite constructions and definitions for ∞ -categories of spans. For a fuller exposition, see [4, Chapter 10]. Throughout this section, we will assume that $\mathcal{C}$ is an ∞ -category with small limits.

Definition 1.34 Let S be a linearly ordered set. We define I_S to be the poset of non-empty sub-intervals $\{i \leq j\} \subset S$.

Let Δ^n be the standard n -simplex. We define the *spine* $\mathcal{J}^n \subset \Delta^n$ to be

$$\mathcal{J}^n := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

Construction 1.35 (*Categories of spans*) We define the functor $\mathrm{Tw} : \Delta \rightarrow \mathrm{Set}_\Delta$ by

$$[n] \mapsto N(I_{[n]})^{\mathrm{op}}.$$

By left Kan extension along the Yoneda embedding and restriction, we get an adjunction, which we will also denote by

$$\mathrm{Tw} : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Set}_\Delta : \overline{\mathrm{Span}}. \quad (3)$$

For an ∞ -category $\mathcal{D}$, the simplicial set $\mathrm{Tw}(\mathcal{D})$ is an ∞ -category, which we will call the *twisted arrow ∞ -category* of $\mathcal{D}$. Note that $\mathrm{Tw}(\mathcal{D})$ comes with a canonical projection $\eta_{\mathcal{D}} : \mathrm{Tw}(\mathcal{D}) \rightarrow \mathcal{D}$. If $\mathcal{D}$ is the nerve of a 1-category D , $\mathrm{Tw}(\mathcal{D})$ can be identified with the nerve of the 1-category $\mathrm{Tw}(D)$ whose objects are morphisms $f : a \rightarrow b$ in D and whose morphisms $f \rightarrow g$ are commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \uparrow \\ c & \xrightarrow{g} & d \end{array}$$

in D , i.e. factorizations $f = h \circ g \circ \ell$.

Given $X \in \mathrm{Set}_\Delta$, we can extend the adjunction 3 to an adjunction

$$\mathrm{Tw}_X : (\mathrm{Set}_\Delta)_{/X} \leftrightarrow (\mathrm{Set}_\Delta)_{/X} : \overline{\mathrm{Span}}_X$$

by setting $\mathrm{Tw}_X(S \rightarrow X)$ to be the composite

$$\mathrm{Tw}(S) \rightarrow \mathrm{Tw}(X) \xrightarrow{\eta_X} X$$

and by setting $\overline{\text{Span}}_X(S \rightarrow X)$ to be the left-hand column of the pullback

$$\begin{array}{ccc} \overline{\text{Span}}_X(S) & \longrightarrow & \overline{\text{Span}}(S) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \overline{\text{Span}}(X) \end{array}$$

in Set_Δ .

Let $p : S \rightarrow X$ be a map of simplicial sets. We call an n -simplex in $\overline{\text{Span}}_X(S)$ represented by a map $\sigma : \text{Tw}(\Delta^n) \rightarrow S$ a *Segal simplex* if, for every $\Delta^k \subset \Delta^n$, the composite diagram

$$\{0, k\} \star \text{Tw}(\mathcal{J}^k) \subset \text{Tw}(\Delta^k) \subset \text{Tw}(\Delta^n) \xrightarrow{\sigma} S$$

is a p -limit diagram. We denote by $\text{Span}_X(S) \subset \overline{\text{Span}}_X(S)$ the simplicial subset consisting of the Segal simplices.

The use of these relative constructions, for our purposes, is that they define Cartesian fibrations exhibiting the pointwise Cartesian model structures.

Proposition 1.36 [4, 10.2.31] *Let $p : \mathcal{C}^\otimes \rightarrow N(\Delta)$ be a Cartesian fibration exhibiting a monoidal structure on $\mathcal{C}_{[1]}^\otimes$ such that p admits relative pullbacks. Then $\text{Span}_\Delta(\mathcal{C}^\otimes) \rightarrow N(\Delta)$ is a Cartesian fibration exhibiting a monoidal structure on $\text{Span}_*(\mathcal{C}_{[1]}^\otimes)$.*

Corollary 1.37 *Let $p : \mathcal{C}^\otimes \rightarrow N(\Gamma)$ be a Cartesian fibration exhibiting a symmetric monoidal structure on $\mathcal{C}_{(1)}^\otimes$ such that p admits relative pullbacks. Then $\text{Span}_\Gamma(\mathcal{C}^\otimes) \rightarrow N(\Gamma)$ is a Cartesian fibration exhibiting a symmetric monoidal structure on $\text{Span}_*(\mathcal{C}^\otimes)$.*

Corollary 1.38 *Let $\mathcal{C}$ be an ∞ -category that admits small limits. Then the functors*

$$\begin{aligned} \text{Span}_\Delta(\mathcal{C}^\boxtimes) &\rightarrow N(\Delta) \\ \text{Span}_{\mathbb{I}}(\mathcal{C}^\times) &\rightarrow N(\mathbb{I}) \end{aligned}$$

are Cartesian fibrations exhibiting a monoidal or a symmetric monoidal structure on $\text{Span}_(\mathcal{C})$ respectively, called the pointwise Cartesian (symmetric) monoidal structure.*

Remark 1.39 Tracing through the definition, it is easy to see that, for any ∞ -category $\mathcal{C}$ which admits small limits, there is a duality $\text{Span}(\mathcal{C}) \cong \text{Span}(\mathcal{C})^{\text{op}}$. Under this duality, the opposites of the Cartesian fibrations of Corollary 1.38 are coCartesian fibrations which classify the same pointwise Cartesian (symmetric) monoidal structure on $\text{Span}(\mathcal{C})$. Consequently, Lemma 1.33 allows us to identify algebras in $\text{Span}(\mathcal{C})$ and coalgebras in $\text{Span}(\mathcal{C})$. A more general account of this phenomenon (in $(\infty, 2)$ -categories of bispans) is given in [4, Remarks 11.1.16 and 11.1.17]. We will use this identification freely in the sequel, working with coalgebra objects in $\text{Span}(\mathcal{C})$.

2 Algebras in spans

We now begin the proof of the first of our main results: Theorem 2.23, which shows the equivalence between associative algebras in $\text{Span}(\mathcal{C})$ and 2-Segal simplicial objects in $\mathcal{C}$. As a matter of course, we will use the identification of Remark 1.39, and work with coalgebras in $\text{Span}(\mathcal{C})$. This is primarily a device to simplify our computations, allowing us to work entirely within the framework of Cartesian fibrations developed above.

Before embarking on the proof, let us briefly expand on the general course we will follow. For the remainder of the section, we fix an ∞ -category $\mathcal{C}$ with small limits. By construction, a functor

$$\overline{G} : \Delta \rightarrow \text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$$

over Δ corresponds to a functor

$$G : \text{Tw}(\Delta) \times_{\Delta} \Delta^{\sqcup} \rightarrow \mathcal{C}$$

satisfying certain conditions. Throughout this section, we will fix the notation $\Theta := \text{Tw}(\Delta) \times_{\Delta} \Delta^{\sqcup}$. We can identify the objects of Θ with pairs consisting of a morphism $f : [n] \rightarrow [m]$ in Δ together with an interval $\{i, j\} \subset [n]$. We will sometimes denote an object by the pair $(f, \{i, j\})$. Morphisms in Θ will be represented as diagrams

$$\begin{array}{ccc} \{i, j\} & \subseteq & [n] \xrightarrow{f} [m] \\ & & \downarrow g \quad \uparrow \overline{g} \\ \{i', j'\} & \subseteq & [n'] \xrightarrow{f'} [m'] \end{array}$$

in Δ .

As a first step, we identify sufficient and necessary conditions for the adjoint map $G : \Theta \rightarrow \mathcal{C}$ to define an algebra object $\overline{G} : \Delta \rightarrow \text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$. The first of these conditions allows us to instead consider functors $\Omega \rightarrow \mathcal{C}$, where Ω is the full subcategory of Θ on the objects such that the interval $\{i, j\} \subset [n]$ is non-degenerate. We then define $\text{Alg}_{\text{Sp}}(\mathcal{C})$ to be the full subcategory of $\text{Fun}(\Omega, \mathcal{C})$ satisfying these conditions.

The equivalence between $\text{Alg}_{\text{Sp}}(\mathcal{C})$ and 2-Seg($\mathcal{C}$) is achieved by defining an auxiliary category Δ^* and a full subcategory $\text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C}) \subset \text{Fun}(\Delta^*, \mathcal{C})$ which fits into a sequence of equivalences

$$\text{Alg}_{\text{Sp}}(\mathcal{C}) \xleftarrow[\simeq]{\mathcal{L}^*} \text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C}) \xleftarrow[\simeq]{} 2\text{-Seg}(\mathcal{C}).$$

The latter of these equivalences descends from a restriction-Kan extension adjunction, and is quite straightforward to prove. The bulk of the combinatorial work in the proof of the main theorem is thus devoted to defining the functor $\mathcal{L} : \Omega \rightarrow \Delta^*$, and showing that it is an ∞ -categorical localization at the necessary set of morphisms. This is

accomplished by direct computation, using a lemma from [18]. To streamline the presentation, we defer the proof of the localization to Appendix A.

2.1 Conditions on functors

The first step of our proof will be to identify conditions on a functor $G : \Theta \rightarrow \mathcal{C}$ sufficient and necessary for G to define a coalgebra object $\overline{G} : \Delta \rightarrow \text{Span}_\Delta(\mathcal{C}^{\boxtimes})$. We begin with the conditions necessary for the adjoint map to send simplices to Segal simplices.

To this end, suppose we are given a functor $G : \Theta \rightarrow \mathcal{C}$ which corresponds to a functor

$$\tilde{G} : \text{Tw}(\Delta) \rightarrow \mathcal{C}^{\boxtimes}$$

over Δ .

Proposition 2.1 *The functor G defines a functor $\overline{G} : \Delta \rightarrow \text{Span}_\Delta(\mathcal{C}^{\boxtimes})$ if and only if, for every simplex $[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} [n_k]$ in Δ and every interval $\{i, j\} \subset [n_0]$, the corresponding diagram*

$$\begin{array}{ccccccc} & & G(\psi_k, \{i, j\}) & & & & \\ & \swarrow & & \searrow & & & \\ G(\phi_1, \{i, j\}) & & \cdots & & G(\phi_k, \psi_{k-1}(i, j)) & & \\ \swarrow \quad \searrow & & \swarrow \quad \searrow & & \swarrow \quad \searrow & & \\ G([n_0], \{i, j\}) & G([n_1], \psi_1(i, j)) & \cdots & G([n_{k-1}], \psi_{k-1}(i, j)) & G([n_k], \psi_k(i, j)) \end{array} \quad (4)$$

is a limit diagram in $\mathcal{C}$. Here $\psi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$, and $\psi(i, j)$ denotes the interval $\{\psi(i), \psi(j)\}$.

Proof By definition, G defines a functor

$$\overline{G} : \Delta \rightarrow \text{Span}_\Delta(\mathcal{C}^{\boxtimes})$$

if and only if every restriction of $\tilde{G}$ to $\text{Tw}(\Delta^n) \subset \text{Tw}(\Delta)$ is a Segal simplex in $\mathcal{C}^{\boxtimes}$.

Let $\Delta^k \hookrightarrow \Delta$ be the simplex

$$[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_k} [n_k].$$

Then by [4, Lemma 10.2.13], there is a functor

$$H : \left(\Delta^1 \times \text{Tw}(\Delta^k) \right) \times_\Delta \Delta^1 \rightarrow \mathcal{C}$$

representing a homotopy

$$\tilde{H} : \Delta^1 \times \text{Tw}(\Delta^k) \rightarrow \mathcal{C}^{\boxtimes}.$$

This homotopy has components that are Cartesian morphisms, and the component $\tilde{G}_0 := \tilde{H}|_{\{0\} \times \text{Tw}(\Delta^k)}$ has image contained in $\mathcal{C}_{[n_0]}^{\boxtimes}$. Since this is the case, the condition that $\tilde{G}$ is a p -limit diagram when restricted to the Segal cone is equivalent to the condition that $\tilde{G}_0$ is a limit diagram in $\mathcal{C}_{[n_0]}^{\boxtimes}$ when restricted to the Segal cone. This can be checked componentwise, using one component for each subinterval of $[n_0]$.

Fix one such subinterval, $\{i, j\}$. Then the corresponding Segal cone diagram in $\mathcal{C}$ will be

$$\begin{array}{ccccc}
 & G_0(\phi_n \circ \cdots \circ \phi_1, \{i, j\}) & & & \\
 & \swarrow & & \searrow & \\
 G_0(\phi_1, \{i, j\}) & & \cdots & & G_0(\phi_k, \{i, j\}) \\
 \swarrow \quad \searrow & & \swarrow \quad \searrow & & \swarrow \quad \searrow \\
 G_0([n_0], \{i, j\}) & G_0([n_1], \{i, j\}) & \cdots & G_0([n_{k-1}], \{i, j\}) & G_0([n_k], \{i, j\})
 \end{array}$$

Since the homotopy has Cartesian components, H will restrict to a natural equivalence between this diagram and the diagram (1). Therefore, a simplex is Segal if and only if all such diagrams are limit diagrams. $\square$

2.1.1 Cartesian morphisms and equivalences

The condition above, which corresponds to $\overline{G}$ sending Segal simplices to Segal simplices, does not guarantee that $\overline{G}$ is a coalgebra object. For that to hold, we need the additional condition that inert morphisms are sent to Cartesian morphisms in $\text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$.

Suppose G represents a coalgebra object. Given an inert morphism $\Delta^1 \xrightarrow{\{\phi\}} \Delta$ ($\phi : [n] \rightarrow [m]$), G must send ϕ to a Cartesian morphism in $\text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$. This means that the adjoint map

$$\text{Tw}(\Delta^1) \rightarrow \mathcal{C}^{\boxtimes}$$

is comprised only of Cartesian morphisms. Therefore:

- For the source map $\phi \rightarrow [n]$ in $\text{Tw}(\Delta)$, and for any $\{i, j\} \in [n]$, the induced morphism

$$G(\phi, \{i, j\}) \rightarrow G([n], \{i, j\})$$

is an equivalence.

- For the target map $\phi \rightarrow [m]$ in $\text{Tw}(\Delta)$, and for any $\{i, j\} \in [n]$ The induced morphism

$$G(\phi, \{i, j\}) \rightarrow G([m], \{\phi(i), \phi(j)\})$$

is an equivalence.

We will write $\phi_{i,j} : [i, \dots, j] \rightarrow [n]$ for the inert morphism which includes the interval $[i, \dots, j]$.

Proposition 2.2 *Suppose G represents a coalgebra object. Let $f : [n] \rightarrow [m]$ be a morphism in Δ , viewed as an object in $\text{Tw}(\Delta)$.*

1. *Let $f|_{[i,j]} : [i, \dots, j] \rightarrow [m]$ be the restriction of f to $[i, \dots, j] \subset [n]$. Then the induced morphism*

$$G(f|_{[i,j]}, \{i, j\}) \rightarrow G(f, \{i, j\})$$

is an equivalence.

2. *Let $\tilde{f} : [n] \rightarrow [i, \dots, j] \subset [m]$ be a morphism such that composing with the inert morphism $\phi_{i,j} : [i, \dots, j] \rightarrow [m]$ yields f . Then the induced morphism*

$$G(f, \{i, j\}) \rightarrow G(\tilde{f}, \{i, j\})$$

is an equivalence.

Proof Applying our conclusion from Proposition 2.1, we find that in case (1), the diagram

$$\begin{array}{ccc} & G(f|_{[i,j]}, \{i, j\}) & \\ \swarrow & & \searrow \\ G(\phi_{i,j}, \{i, j\}) & & G(f, \{i, j\}) \\ \searrow & & \swarrow \\ & G(\text{id}_{[n]}, \{i, j\}) & \end{array}$$

must be pullback. Therefore, since $G(\phi_{i,j}, \{i, j\}) \rightarrow G(\text{id}_{[n]}, \{i, j\})$ must be an equivalence, so must $G(f|_{[i,j]}, \{i, j\}) \rightarrow G(f, \{i, j\})$.

Similarly, in case (2), the diagram

$$\begin{array}{ccc} & G(f, \{i, j\}) & \\ \swarrow & & \searrow \\ G(\tilde{f}, \{i, j\}) & & G(\phi_{i,j}, \{i, j\}) \\ \searrow & & \swarrow \\ & G(\text{id}_{[i,\dots,j]}, \{i, j\}) & \end{array}$$

must be pullback. Therefore, since $G(\phi_{i,j}, \{i, j\}) \rightarrow G(\text{id}_{[i,\dots,j]}, \{i, j\})$ must be an equivalence, so must $G(f, \{i, j\}) \rightarrow G(\tilde{f}, \{i, j\})$. $\square$

Lemma 2.3 Suppose G sends the morphisms from Proposition 2.2 to equivalences. Let

$$\mu := \left\{ \begin{array}{ccc} [k] \cong \{i, j\} & \subseteq & [n] \xrightarrow{f} [m] \\ & & \downarrow g \quad \bar{g} \uparrow \\ [k] \cong \{i', j'\} & \subseteq & [n'] \xrightarrow{f'} [m'] \end{array} \right\}$$

be a morphism such that g restricts to an isomorphism $[i, \dots, j] \xrightarrow{\cong} [i', \dots, j']$ and $\bar{g}$ restricts to an isomorphism $[f'(i'), f'(i') + 1, \dots, f'(j')] \xrightarrow{\cong} [f(i), f(i) + 1, \dots, f(j)]$. Then G sends μ to an equivalence.

Proof We first note that, under the given hypotheses, G will send morphisms of the form

$$\nu := \left\{ \begin{array}{ccc} \{0, k\} & \subseteq & [k] \xrightarrow{s} [m] \\ & & \text{id}_{[k]} \downarrow \quad \uparrow h \\ \{0, k\} & \subseteq & [k] \xrightarrow{s'} [m'] \end{array} \right\}$$

to equivalences, where h sends $[s'(i'), s'(i') + 1, \dots, s'(j')]$ isomorphically to $[m]$. This follows from composing

$$\begin{array}{ccc} \{0, k\} & \subseteq & [k] \xrightarrow{s} [m] \\ & & \text{id}_{[k]} \downarrow \quad \uparrow h \\ \{0, k\} & \subseteq & [k] \xrightarrow{s'} [m'] \\ & & \text{id}_{[k]} \downarrow \quad \uparrow \psi \\ \{0, k\} & \subseteq & [k] \xrightarrow{s} [m] \end{array}$$

Where ψ is the inclusion of the interval $[f(0), \dots, f(k)]$. The lower morphism is then one of the morphisms of type (2) from Proposition 2.2 and the two morphisms compose to the identity. So, by 2-out-of-3, ν must be sent to an equivalence.

Now write $[\ell] := [f(i), f(i) + 1, \dots, f(j)]$, and consider the composition

$$\begin{array}{ccc} \{0, k\} & \subseteq & [k] \xrightarrow{s} [\ell] \\ & & \downarrow \phi_{i,j} \quad h \uparrow \\ \{i, j\} & \subseteq & [n] \xrightarrow{f} [m] \\ & & \downarrow g \quad \bar{g} \uparrow \\ [k] \cong \{i', j'\} & \subseteq & [n'] \xrightarrow{f'} [m'] \end{array}$$

where h sends $[\ell]$ isomorphically to itself. The upper morphism is the composite of a morphism of type (1) from Proposition 2.2 and a morphism of the same kind as ν .

Moreover, the composite

$$\begin{array}{ccc} \{0, k\} & \subseteq & [k] \xrightarrow{s} [\ell] \\ & \phi_{i', j'} \downarrow & \uparrow h' \\ [k] \cong \{i', j'\} & \subseteq & [n'] \xrightarrow{f'} [m'] \end{array}$$

is also the composite of a morphism of type (1) from Proposition 2.2 and a morphism of the same kind as v . Therefore, by the 2-out-of-3 property, μ must be sent to an equivalence. $\square$

Definition 2.4 We define E to be the set of all morphisms of the form from Lemma 2.3. Note that E is closed under composition.

Let us briefly unwind what we have shown. We assumed that $G : \Omega \rightarrow \mathcal{C}$ corresponded to a functor $\tilde{G} : \text{Tw}(\Delta) \rightarrow \mathcal{C}^{\boxtimes}$ over Δ . We then found two conditions on G — the limit condition of Proposition 2.1 and the condition that G send the morphisms in E to equivalences — which are necessary and sufficient for $G : \text{Tw}(\Delta) \rightarrow \mathcal{C}^{\boxtimes}$ to define a coalgebra object.

However, not every functor $G : \Omega \rightarrow \mathcal{C}$ defines a functor $\tilde{G} : \text{Tw}(\Delta) \rightarrow \mathcal{C}^{\boxtimes}$ over Δ . We therefore must also require the product diagram condition of Construction 1.29. For reasons which will become clear shortly, we separate out the condition that G send degenerate intervals to the terminal object.

Corollary 2.5 A functor $G : \Theta \rightarrow \mathcal{C}$ defines a coalgebra object in $\text{Span}_{\Delta}(\mathcal{C}^{\boxtimes})$ if and only if

1. G sends degenerate intervals to the terminal object.
2. G sends the morphisms in E to equivalences.
3. G sends $(\{i, j\} \subset [n] \xrightarrow{f} [m])$ together with its projections to sub-intervals to a product diagram.
4. G sends all diagrams of the form (4) to limit diagrams.

With these conditions in place, we can now define our category $\text{Alg}_{\text{Sp}}(\mathcal{C})$ of algebras in $\text{Span}(\mathcal{C})$.

Definition 2.6 We denote by $\text{Alg}_{\text{Sp}}(\mathcal{C})$ the full sub- ∞ -category of $\text{Fun}(\Theta, \mathcal{C})$ on those functors satisfying conditions (1)-(4) from the corollary. We denote by $\text{Fun}^*(\Theta, \mathcal{C})$ the full sub- ∞ -category of functors sending every degenerate interval to a terminal object in $\mathcal{C}$ (i.e., those functors satisfying condition (1) from the corollary).

2.2 Forgetting degenerate intervals

We now begin the process of relating functors $G : \Theta \rightarrow \mathcal{C}$ satisfying conditions (1)-(4) to 2-Segal objects in $\mathcal{C}$. Our first order of business will be to show that we can ‘forget’ the degenerate intervals in Θ without losing any information.

Definition 2.7 Let Ω be the full subcategory of Θ on those objects $\{i, j\} \subset [n] \xrightarrow{f} [m]$ such that the interval $\{i, j\}$ is not degenerate (i.e. $i \neq j$). Pulling back along the inclusion $\Omega \rightarrow \Theta$ induces a functor $S : \text{Fun}^*(\Theta, \mathbb{C}) \rightarrow \text{Fun}(\Omega, \mathbb{C})$.

Our aim will be to show that S is an equivalence of ∞ -categories. To do this, we first prove a general lemma.

Definition 2.8 Given a 1-category D , call an object $d \in D$ *attracting* if, for all $a \in D$,

$$\text{Hom}_D(a, d) \neq \emptyset, \quad \text{and} \quad \text{Hom}_D(d, a) = \emptyset.$$

Lemma 2.9 Let $d \in D$ be an attracting object, denote by $\text{Fun}^*(D, \mathbb{C})$ the full sub- ∞ -category on those functors sending d to the terminal object, and denote by D° the full subcategory on all objects other than d . Then the functor

$$\text{Fun}^*(D, \mathbb{C}) \rightarrow \text{Fun}(D^\circ, \mathbb{C})$$

is an equivalence.

Proof Without loss of generality, we assume that $\mathbb{C}$ has a unique terminal object. when f sends d to the terminal object. Denote by $\mathbb{C}' \subset \mathbb{C}$ the largest subcategory not containing morphisms from the terminal object to any other object, and denote by $\mathbb{C}^\circ$ the full subcategory on non-terminal objects. Then we have an equivalence $\mathbb{C}' \simeq (\mathbb{C}^\circ)^\triangleright$ since the hom-spaces to the terminal object are all contractible. Any simplex in $\text{Fun}^*(D, \mathbb{C})$ factors through $\text{Fun}^*(D, \mathbb{C}')$, so it will suffice to show that

$$\text{Fun}^*(D, (\mathbb{C}^\circ)^\triangleright) \rightarrow (D^\circ, \mathbb{C})$$

is a trivial Kan fibration.

Unwinding the definitions, this amounts to solving the extension problem

$$\begin{array}{ccc} (\partial \Delta^n \times D) \coprod_{\partial \Delta^n \times D^\circ} \Delta^n \times D^\circ & & \\ \downarrow & \searrow f & \\ \Delta^n \times D & \dashrightarrow & (\mathbb{C}^\circ)^\triangleright \end{array}$$

where f sends $\partial \Delta^n \times D$ to the cone point. However, this implies that f factors through $(\Delta^n \times D^\circ)^\triangleright$. Pulling back along $\Delta^n \times D \rightarrow (\Delta^n \times D^\circ)^\triangleright$ then gives the desired extension.

□

Corollary 2.10 The functor $S : \text{Fun}^*(\Theta, \mathbb{C}) \rightarrow \text{Fun}(\Omega, \mathbb{C})$ is an equivalence of ∞ -categories.

Proof We again assume that $\mathcal{C}$ has a unique terminal object. Let Θ^{deg} be the full subcategory on only the degenerate intervals. We can write $\text{Fun}^*(\Theta, \mathcal{C})$ as a pullback in Set_Δ

$$\begin{array}{ccc} \text{Fun}^*(\Theta, \mathcal{C}) & \longrightarrow & \text{Fun}(\Theta, \mathcal{C}) \\ \downarrow & & \downarrow \\ \text{Fun}(\Theta^{deg}, *) & \longrightarrow & \text{Fun}(\Theta^{deg}, \mathcal{C}) \end{array}$$

There is a natural transformation of diagrams to the pullback diagram

$$\begin{array}{ccc} \text{Fun}^*(\Theta \coprod_{\Theta^{deg}} *, \mathcal{C}) & \longrightarrow & \text{Fun}(\Theta, \mathcal{C}) \\ \downarrow & & \downarrow \\ \text{Fun}^*(*, \mathcal{C}) & \longrightarrow & \text{Fun}(\Theta^{deg}, \mathcal{C}) \end{array}$$

Since this natural transformation is an isomorphism on the bottom three objects, the universal property of the pullback gives us an isomorphism $\text{Fun}^*(\Theta, \mathcal{C}) \cong \text{Fun}^*(\Theta \coprod_{\Theta^{deg}} *, \mathcal{C})$. $*$ $\in \Theta \coprod_{\Theta^{deg}} *$ is an attracting object, and so Lemma 2.9 yields the desired result. $\square$

2.3 The localization map

We now come to the meat of the proof: the construction of the category Δ^* and the functor $\mathcal{L} : \Omega \rightarrow \Delta^*$, and the proof the $\mathcal{L}$ is an ∞ -categorical localization at the morphisms in E . There is an underlying graphical intuition which underlies these constructions, which is sketched in Fig. 3.

To start off, we can immediately write down the definition of the category Δ^* which will interpolate between Ω and Δ^{op} .

Definition 2.11 Define a category Δ^* to have objects finite (non-empty) ordered tuples of elements in Δ . The morphisms of Δ^* from $([n_0], \dots, [n_k]) \rightarrow ([m_0], \dots, [m_\ell])$ consist of

1. A morphism $\phi : [\ell] \rightarrow [k]$ in Δ .
2. For each $i \in \{0, 1, \dots, k\}$, with $\phi^{-1}(i) = (j_1, \dots, j_r)$, a morphism

$$f_i : [m_{j_1}] \star [m_{j_2}] \star \dots \star [m_{j_r}] \rightarrow [n_i]$$

in Δ .

Satisfying the conditions that

1. If there is a $p \in \langle \ell \rangle^\circ$ with $r > \max_{j \in \phi^{-1}(i)}(j)$, then f_i hits $n_i \in [n_i]$.
2. If there is a $p \in \langle \ell \rangle^\circ$ with $r < \min_{j \in \phi^{-1}(i)}(j)$, then f_i hits $0 \in [n_i]$.

Remark 2.12 We could equivalently define the morphisms to be

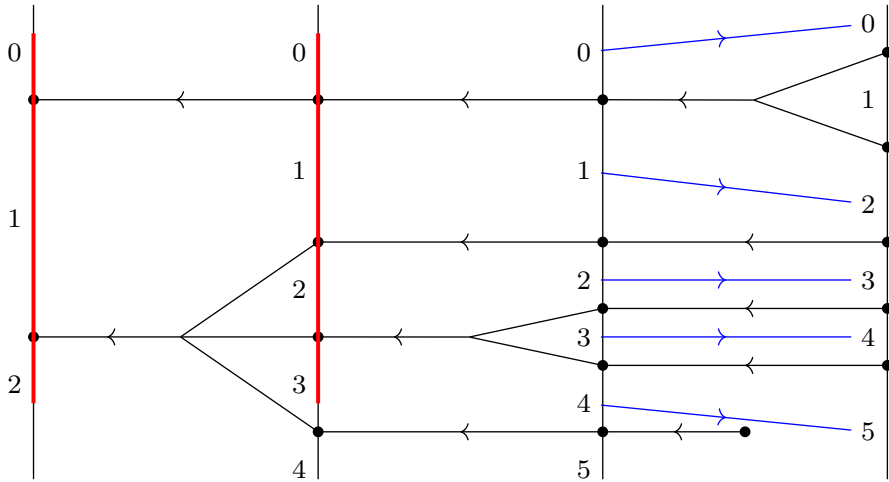


Fig. 3 A pictorial representation of a morphism μ in Ω , viewed as a triple of composable morphisms $[n] \xrightarrow{g} [n'] \xrightarrow{f'} [m'] \xrightarrow{\bar{g}} [m]$ in Δ . The dual forest is drawn in black, the chosen subintervals of $[n]$ and $[n']$ marked in red, and the induced morphism $\mathcal{L}(\mu)$ is drawn in blue. Note that the source of $\mathcal{L}(\mu)$ is the imbrication of the ordered sets $\{f'(i), f'(i) + 1, \dots, f'(i + 1)\}$

1. A morphism $\phi : [\ell] \rightarrow [k]$ in Δ .
2. A morphism

$$f : [m_1] \star [m_2] \star \dots \star [m_\ell] \rightarrow [n_1] \star [n_2] \star \dots \star [n_k]$$

in Δ .

Satisfying the condition that, for any $i \in [k]$ with $\phi^{-1}(i) = (j_1, \dots, j_r)$, the restriction

$$f_i : [m_{j_1}] \star [m_{j_2}] \star \dots \star [m_{j_r}] \rightarrow [n_1] \star [n_2] \star \dots \star [n_k]$$

has image contained in $[n_i]$.

Before defining the localization functor $\mathcal{L} : \Omega \rightarrow \Delta^*$, we will give a construction of a map which will simplify the construction.

Construction 2.13 Let $\phi : ([n], \{i, j\}) \rightarrow ([m], \{k, \ell\})$ be a morphism in $\Delta^{\mathbb{U}}$. and write $\{i \leq j\}$ for the linearly ordered set $\{i, i + 1, \dots, j\}$. Applying $\mathbb{O}$ to ϕ , we obtain a diagram

$$\begin{array}{ccc} \mathbb{O}([m]) & \xrightarrow{\mathbb{O}(\phi)} & \mathbb{O}([n]) \\ \cup & & \cup \\ \mathbb{O}(\{k \leq \ell\}) & & \mathbb{O}(\{i \leq j\}) \\ \cup & & \cup \\ \mathbb{I}(\{k \leq \ell\}) & & \mathbb{I}(\{i \leq j\}) \end{array}$$

Since $\phi(i) \leq k \leq \ell \leq \phi(j)$, we see that for every $a \in \{k \leq \ell\}$, there exists a $b \in \{i \leq j\}$ such that $\phi(b) \leq a \leq a+1 \leq \phi(b+1)$. That is, $\mathbb{O}(\phi)$ descends uniquely to a map

$$\text{res}(\phi) : \mathbb{I}(\{k \leq \ell\}) \rightarrow \mathbb{I}(\{i \leq j\}).$$

Note that we here apply the convention that $\mathbb{I}([0]) = \emptyset$. We therefore obtain a functor

$$\text{res} : \Delta^{\sqcup} \rightarrow \Delta_+^{\text{op}}$$

which sends all non-degenerate intervals into $\Delta \subset \Delta_+$.

Construction 2.14 We now define a functor $\mathcal{L} : \Omega \rightarrow \Delta^*$. On objects it is given by

$$\{i, j\} \subset [n] \xrightarrow{f} [m] \mapsto (\{f(i) \leq f(i+1)\}, \dots, \{f(j-1) \leq f(j)\})$$

where $\{f(k) \leq f(k+1)\} := \{f(k), f(k+1), \dots, f(k+1)\}$ are considered to be ordered via the order on $[m]$. Note that the indexing set of $\mathcal{L}(\{i, j\}, f)$ is precisely $\mathbb{I}(\{i \leq j\})$

On morphisms, $\mathcal{L}$ is more complicated. A morphism in Ω is given by a commutative diagram of the form

$$\mu = \left\{ \begin{array}{ccc} [k] = \{i, j\} & \subseteq & [n] \xrightarrow{f} [m] \\ & \searrow \scriptstyle g \downarrow & \uparrow \scriptstyle \bar{g} \\ [k'] = \{i', j'\} & \subseteq & [n'] \xrightarrow{f'} [m'] \end{array} \right\}$$

where $g(i) \leq i' \leq j' \leq g(j)$. We define $\mathcal{L}(\mu)$ to be a pair (ϕ_f, ψ_f) . We then write $\phi_f := \text{res}(g) : \mathbb{I}(\{i' \leq j'\}) \rightarrow \mathbb{I}(\{i \leq j\})$.

Since the diagram commutes, for each pair $\{p, p+1\} \subset \{i, j\} \subset [n]$, we have that $\bar{g}(f(g(p))) = p$ and $\bar{g}(f(g(p+1))) = p+1$, so that $\bar{g}$ descends to a map of ordered sets

$$\begin{aligned} \bar{g}_p : \{f'(g(p) \leq f'(g(p)+1)) \star \dots \star \{f'(g(p+1)-1) \leq f'(g(p+1))\} \\ \rightarrow \{f(p) \leq f(p+1)\} \end{aligned}$$

It is easy to verify that conditions (1) and (2) from the definition of Δ^* are satisfied by the $\bar{g}_p$. On morphisms, therefore, we define

$$\mathcal{L}(\mu) := \left(\phi(g), \{\bar{g}_p\}_{i \leq p < j} \right).$$

This is functorial via the functoriality of res and the restriction of $\bar{g}$.

2.3.1 Localizing

We now discuss a key element in our argument: the proof $\mathcal{L}$ is an ∞ -categorical localization of Ω at the morphisms in E . Our helpmeet in this endeavor will be the following lemma

Lemma 2.15 [18, Lemma 3.1.1] *Let $L : C \rightarrow D$ be a functor of 1-categories. For each $d \in D$, define the weak fiber $C_d \subset C_{/d}$ to be the full subcategory of the slice on the isomorphisms $L(c) \cong d$. Suppose that, for every $d \in D$, there is a subcategory $B_d \subset C_d$ such that*

- B_d has an initial object b_d and
- the inclusion $N(B_d) \rightarrow N(C)_{/d}$ is cofinal.

Then L is an ∞ -categorical localization of C at the morphisms of the categories B_d .

In our application of Lemma 2.15, we will be interested in the weak fibers of the functor $\mathcal{L} : \Omega \rightarrow \Delta^*$. We first note that, given an object $M = ([m_1], \dots, [m_k]) \in \Delta^*$, the weak fiber Ω_M is non-empty. We can explicitly build an object

$$\{0, k\} \subset [k] \xrightarrow{f_M} [m_1] \star [m_2] \star \cdots \star [m_k] =: [m]$$

in the fiber over M , given by

$$f_M(i) = \begin{cases} 0 \in [m_{i+1}] & i < k \\ m_k \in [m_k] & i = k. \end{cases}$$

Secure in the knowledge that our weak fibers do not fall at the first hurdle, we can define the subcategories necessary for our application of Lemma 2.15.

Definition 2.16 For $M = ([m_1], \dots, [m_k]) \in \Delta^*$, we define a subcategory $\Omega_M^E \subset \Omega_M$ as follows. The objects of Ω_M^E are the same as those of Ω_M , but the morphisms are only those in E . Note that since elements of E are sent to isomorphisms by $\mathcal{L}$, every morphism of E appears in a category Ω_M^E for some $M \in \Delta^*$.

The desired localization result will then follow as a corollary of

Proposition 2.17 *For every object $M \in \Delta^*$*

1. Ω_M^E has an initial object.
2. The inclusion $\Omega_M^E \hookrightarrow \Omega_{/M}$ is cofinal.

Due to its long and combinatorial nature, we will defer the proof of Proposition 2.17 to Appendix A. More precisely, Lemma A.4 shows the first assertion, and Lemma A.5 shows the second.

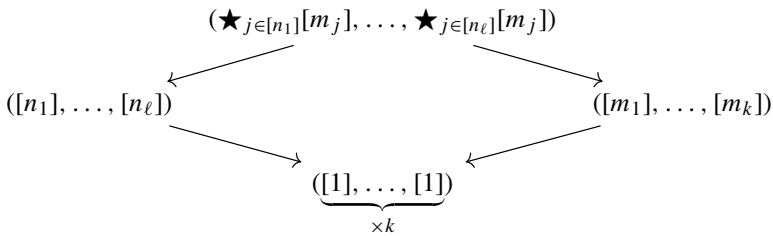
Corollary 2.18 *The functor $\mathcal{L} : \Omega \rightarrow \Delta^*$ is an ∞ -categorical localization at the morphisms in E .*

2.3.2 Algebra conditions

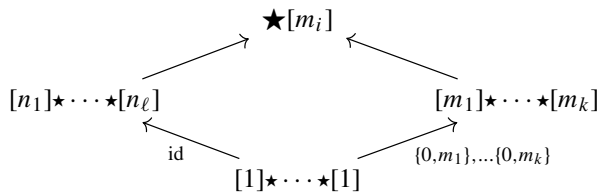
We have shown that $\mathcal{L}$ localizes $\mathcal{Q}$ at the morphisms in E , which necessarily identifies $\text{Alg}_{\text{Sp}}(\mathcal{C})$ with a full subcategory of $\text{Fun}(\Delta^*, \mathcal{C})$. It remains for us to characterize this subcategory — i.e. to translate conditions (3) and (4) into conditions on functors $\Delta^* \rightarrow \mathcal{C}$.

Denote by $\text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C})$ the full sub- ∞ -category of functors f which

(A) send the diagrams

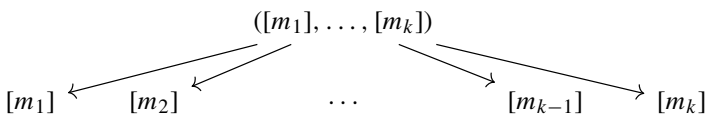


opposite the diagrams



to pullback diagrams and

(B) send the diagrams



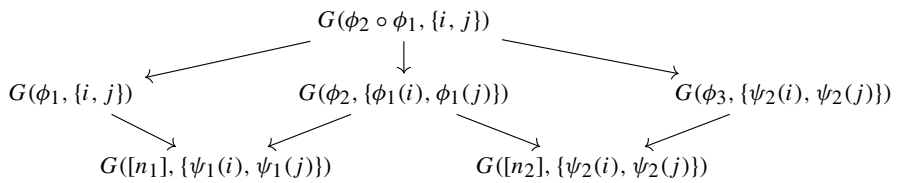
to product diagrams.

Proposition 2.19 *There is an equivalence of ∞ -categories*

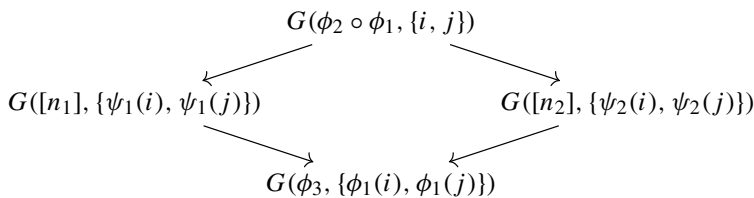
$$\text{Alg}_{\text{Sp}}(\mathcal{C}) \simeq \text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C}).$$

Proof It is clear that condition (B) corresponds to condition (3) from Corollary 2.5.

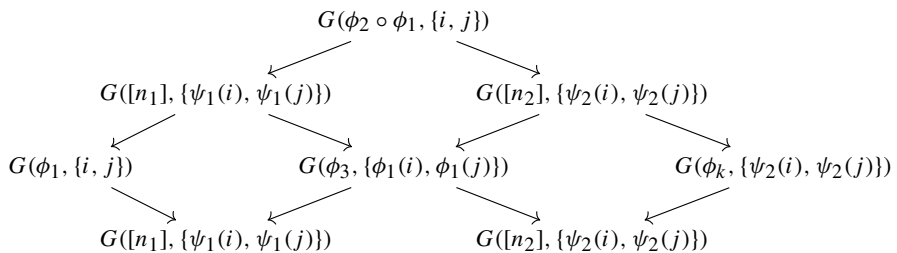
For condition (A), first consider a 3-simplex $[n_0] \xrightarrow{\phi_1} [n_1] \xrightarrow{\phi_2} [n_2] \xrightarrow{\phi_3} [n_3]$ in Δ . The corresponding limit diagram (4) can be written as



However, by (the dual of) [14, Proposition 4.4.2.2], this diagram is a limit if and only if the induced diagram



is pullback. However, combining these two diagrams, we get



By the pasting property for pullback diagrams, we thus see that it is sufficient to require that each of the diagrams corresponding to the sub-2-simplices of our simplex is pullback. Iterating this argument, we find that property (4) of corollary 2.5 is satisfied if and only if it is satisfied on 2-simplices. Since condition (A) is the image of this 2-simplex condition under $\mathcal{L}$, this proves the proposition. $\square$

Lemma 2.20 *A functor $f \in \text{Fun}(\Delta^*, \mathcal{C})$ satisfies condition (A) if and only if it satisfies condition (A) for collections where all but one of the $[m_i]$ are equal to $[1]$.*

Proof This follows from applying the pasting law to diagrams of the form

$$\begin{array}{ccccc}
 ([n_1], \dots, [n_\ell]) & \longrightarrow & ([1], \dots, [1]) \\
 \uparrow & & \uparrow \\
 ([m_1] \star [1]^{*k-1}, [1], \dots, [1]) & \longrightarrow & ([m_1], [1], \dots, [1]) & \longrightarrow & ([1], \dots, [1]) \\
 \uparrow & & \uparrow & & \uparrow \\
 ([m_1] \star [m_2] \star [1]^{*k-2}, [1], \dots, [1]) & \triangleright & ([m_1], [m_2], \dots, [1]) & \triangleright & ([1], \dots, [1], [m_2], [1], \dots, [1]) \\
 \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots
 \end{array}$$

If condition (A) is satisfied for squares where all but one of the $[m_i]$ are equal to $[1]$, then the bottom right square and the bottom rectangle are both pullback. Therefore, the bottom left square is pullback. Since our restricted version of condition (A) also implies that the top left square is pullback, the left-hand rectangle is pullback. Iterating this argument yields the lemma. $\square$

2.4 Extension and restriction

We now come to the final step of the proof: showing an equivalence between $\text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C})$ and $2\text{-Seg}_\Delta(\mathcal{C})$. As mentioned above, this follows from a restriction-Kan extension adjunction.

Considering the full subcategory of Δ^* on the objects $([n])$ for $n \geq 0$ we get

$$\iota : \Delta^{op} \rightarrow \Delta^*.$$

Taking restriction and right Kan extension gives us an adjunction of infinity categories

$$\iota_* : \text{Fun}(\Delta^*, \mathcal{C}) \leftrightarrow \text{Fun}(\Delta^{op}, \mathcal{C}) : \iota_!$$

Denote by $\text{Fun}^\times(\Delta^*, \mathcal{C})$ the full sub- ∞ -category that sends each diagram

$$\begin{array}{ccccccc}
 & & ([m_1], \dots, [m_k]) & & & & \\
 & \swarrow & & \searrow & & \swarrow & \searrow \\
 [m_1] & & [m_2] & & \dots & & [m_{k-1}] & & [m_k]
 \end{array}$$

to a limit diagram.

Proposition 2.21 *The adjunction $\iota_* : \text{Fun}(\Delta^*, \mathcal{C}) \leftrightarrow \text{Fun}(\Delta^{op}, \mathcal{C}) : \iota_!$ descends to an equivalence of ∞ -categories*

$$\text{Fun}^\times(\Delta^*, \mathcal{C}) \simeq \text{Fun}(\Delta^{op}, \mathcal{C}).$$

Proof We compute the overcategory $(\Delta^{op})_{([m_1], \dots, [m_k])}/$. An object in the overcategory will consist of a choice of $i \in \{1, 2, \dots, k\}$ and a morphism $[n] \rightarrow [m_i]$. A morphism

$(i, [n] \rightarrow [m_i]) \rightarrow (j, [\ell] \rightarrow [m_j])$ only exists if $i = j$, and in this case is given by a commutative diagram

$$\begin{array}{ccc} [\ell] & \xrightarrow{\quad} & [n] \\ & \searrow \quad \swarrow & \\ & [m_i] & \end{array}$$

consequently, we find that the induced diagram

$$\begin{array}{ccccccc} & & (\Delta^{op})_{([m_1], \dots, [m_k])}/ & & & & \\ & \nearrow & & \nwarrow & & \nearrow & \\ (\Delta^{op})_{([m_1])}/ & & (\Delta^{op})_{([m_2])}/ & \cdots & (\Delta^{op})_{([m_{k-1}])}/ & & (\Delta^{op})_{([m_k])}/ \end{array}$$

displays $(\Delta^{op})_{([m_1], \dots, [m_k])}/$ as a coproduct, and, hence, for any $f \in \text{Fun}(\Delta^{op}, \mathbb{C})$, the diagram

$$\begin{array}{ccccccc} & & \iota_! f([m_1], \dots, [m_k]) & & & & \\ & \swarrow & & \searrow & & \swarrow & \searrow \\ \iota_! f([m_1]) & & \iota_! f([m_2]) & \cdots & \iota_! f([m_{k-1}]) & & \iota_! f([m_k]) \end{array} \quad (5)$$

displays $\iota_! f([m_1], \dots, [m_k])$ as a product. Consequently, the adjunction descends to an adjunction $\iota_* : \text{Fun}^\times(\Delta^\star, \mathbb{C}) \leftrightarrow \text{Fun}(\Delta^{op}, \mathbb{C}) : \iota_!$.

Since this is a right Kan extension from a full subcategory, the counit is an equivalence. Moreover, the components of the unit are equivalences on the objects of Δ^{op} . However, for every object $([m_1], \dots, [m_k])$, the unit induces a natural transformation of limit diagrams of the form in diagram (5). Therefore, we see that the components of the unit are equivalences for all objects, and thus, the unit is also an equivalence. $\square$

Proposition 2.22 *Denote by $2\text{-Seg}_\Delta(\mathbb{C})$ the full subcategory of $\text{Fun}(\Delta^{op}, \mathbb{C})$ on 2-Segal objects. Then the equivalence of the previous proposition descends to an equivalence of ∞ -categories*

$$\text{Fun}^{alg}(\Delta^\star, \mathbb{C}) \simeq 2\text{-Seg}_\Delta(\mathbb{C}).$$

Proof Let $G \in \text{Fun}^{alg}(\Delta^\star, \mathbb{C})$, and consider the diagram

$$\begin{array}{ccc} [n] & \xleftarrow{\quad} & [n+m-1] \\ \downarrow & & \downarrow \\ \{0, 1\}, \{1, 2\}, \dots, \{n-1, n\} & \xleftarrow{\quad} & ([1], \dots, \underbrace{[m]}^{j^{th}}, \dots, [1]) \end{array}$$

in Δ^* . We can expand this diagram to

$$\begin{array}{ccc}
 [n] & \longleftarrow & [n+m-1] \\
 \downarrow & & \downarrow \\
 (\{0, 1\}, \{1, 2\}, \dots, \{n-1, n\}) & \longleftarrow & ([1], \dots, \overbrace{[m]}^{j^{th}}, \dots, [1]) \\
 \downarrow & & \downarrow \\
 \{j-1, j\} & \longleftarrow & [m]
 \end{array}$$

Since the two vertical morphisms in the lower square are sent to projections onto factors of a product, the lower square is sent to a pullback diagram under G . We therefore see that the exterior square is sent to a pullback if and only if the upper square is sent to a pullback. However, the exterior square is opposite to the diagram

$$\begin{array}{ccc}
 [n] & \longrightarrow & [n+m-1] \\
 \{j-1, j\} \uparrow & & \uparrow \\
 [1] & \xrightarrow{\{0, m\}} & [m]
 \end{array}$$

in Δ , which is precisely the diagram for the 2-Segal conditions when $[m] \neq [0]$, and is the diagram for the unitality condition when $[m] = [0]$. Therefore, we see that $G \in \text{Fun}^\times(\Delta^*, \mathcal{C})$ is in $\text{Fun}^{\text{alg}}(\Delta^*, \mathcal{C})$ if and only if the underlying simplicial object is 2-Segal. $\square$

We can summarize our results in the following theorem.

Theorem 2.23 *There is an equivalence of ∞ -categories*

$$\text{Alg}_{\text{Sp}}(\mathcal{C}) \simeq 2\text{-Seg}_\Delta(\mathcal{C}).$$

3 Calabi–Yau algebras in spans

This final section is devoted to the proof of the second main theorem of the paper—Theorem 3.25—which gives the equivalence between Calabi–Yau (co)algebras in $\text{Span}(\mathcal{C})$ and 2-Segal cyclic objects in $\mathcal{C}$. From a bird’s perspective, the forms of the proofs of Theorems 2.23 and 3.25 are identical. We define an ancillary category Λ^* and a sub-category $\text{Fun}^{\text{alg}}(\Lambda^*, \mathcal{C}) \subset \text{Fun}(\Lambda^*, \mathcal{C})$, and then show two equivalences

$$\text{Alg}_{\text{Sp}}^{\text{CY}}(\mathcal{C}) \xleftarrow[\simeq]{\mathcal{L}^*} \text{Fun}^{\text{alg}}(\Lambda^*, \mathcal{C}) \xrightarrow[\simeq]{} 2\text{-Seg}_\Delta(\mathcal{C})$$

The first of these proceeds, as before, from an ∞ -categorical localization.

However, as soon as one zooms in to the combinatorial details, significant differences between the proofs appear. So as to avoid recapitulating too much material,

we will state lemmata in this section without proof when the corresponding proof in the previous section is sufficiently similar. We will similarly, insofar as it is possible, parallel the structure of Sect. 2 here, to make the analogy more plain.

As before, we let $\mathcal{C}$ be an ∞ -category with small limits, and we set $\Theta := \mathrm{Tw}(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}}) \times_{\mathbb{F}} \mathbb{F}^{\mathrm{II}}$. Note that, analogously to the associative case, the functors

$$\overline{G} : \mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}} \rightarrow \mathrm{Span}_{\mathbb{F}}(\mathcal{C}^{\times})$$

over $\mathbb{F}$ that define Calabi–Yau (co)algebras will correspond to functors

$$G : \Theta \rightarrow \mathcal{C}$$

satisfying additional conditions. We will represent morphisms in Θ diagrammatically as

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f} & T \\ & & \uparrow g & & \downarrow \bar{g} \\ P & \subseteq & S' & \xleftarrow{f'} & T' \end{array}$$

where f , f' , g , and $\bar{g}$ are morphisms in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$ (not $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}}$).

In general, for a morphism $\diamond \xleftarrow{f} T$ in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$, we will denote the two possible subsets of the image of $\diamond$ in $\mathbb{F}\mathrm{n}_{*}$ by $\emptyset$ and $\{1\}$.

3.1 Conditions on functors

We again seek to find conditions on functors $\Theta \rightarrow \mathcal{C}$ which are necessary and sufficient for the adjoint $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}} \rightarrow \mathrm{Span}_{\mathbb{F}}(\mathcal{C}^{\times})$ to exist and define a Calabi–Yau (co)algebra in spans.

To this end, suppose we have a functor

$$G : \Theta \rightarrow \mathcal{C}$$

which corresponds to a functor $\mathrm{Tw}(\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}}) \rightarrow \mathcal{C}^{\times}$ over $\mathbb{F}$. The condition on G which guarantees that we obtain a map $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}} \rightarrow \mathrm{Span}_{\mathbb{F}}(\mathcal{C}^{\times})$ is an analogue of Proposition 2.1.

Proposition 3.1 *The functor G defines a functor $\overline{G} : \mathcal{A}\mathrm{ss}_{\mathrm{CY}}^{\mathrm{op}} \rightarrow \mathrm{Span}_{\mathbb{F}}(\mathcal{C}^{\times})$ if and only if for every simplex $S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} S_2 \rightarrow \cdots \rightarrow S_n$ in $\mathcal{A}\mathrm{ss}_{\mathrm{CY}}$, and every subset $P \subset S_n^{\circ}$ the corresponding diagram*

$$\begin{array}{c}
 G(\psi_{n-1}, P) \\
 \swarrow \quad \searrow \\
 G(\phi_n, P) \quad \dots \quad G(\phi_1, \psi_{n-2}^{-1}(P)) \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 G(S_n, P) \quad G(S_{n-1}, \phi_n^{-1}(P)) \quad \dots \quad G(S_1, \psi_{n-2}^{-1}(P)) \quad G(S_0, \psi_{n-1}^{-1}(P))
 \end{array} \quad (6)$$

is a limit diagram in $\mathcal{C}$, where $\psi_k := \phi_n \circ \phi_{n-1} \cdots \circ \phi_{n-k}$.

Proof This is, *mutatis mutandis*, the same as the proof of Proposition 2.1. Note that if $S_k = \diamond$, then $S_j = \diamond$ for all $j \geq k$. $\square$

Definition 3.2 We define a set of morphisms E in Θ to consist of the following two types of morphisms.

1. Morphisms

$$\begin{array}{ccccc}
 \{1\} & \subseteq & \diamond & \xleftarrow{f} & T \\
 & & \text{id} \uparrow & & \downarrow \bar{g} \\
 \{1\} & \subseteq & \diamond & \xleftarrow{h} & S
 \end{array}$$

such that $\bar{g}|_{f^{-1}(\diamond)} : f^{-1}(\diamond) \rightarrow g^{-1}(\diamond)$ is an isomorphism.

2. Morphisms

$$\begin{array}{ccccc}
 Q & \subseteq & S & \xleftarrow{f} & T \\
 & & g \uparrow & & \downarrow \bar{g} \\
 P & \subseteq & U & \xleftarrow{h} & V
 \end{array}$$

such that $g|_P : P \rightarrow Q$ is an isomorphism, $P = g^{-1}(Q)$, and $\bar{g}|_{f^{-1}(Q)} : f^{-1}(Q) \rightarrow h^{-1}(P)$ is an isomorphism.

Analogously to the associative case, one can then show

Proposition 3.3 A functor $G : \Theta \rightarrow \mathcal{C}$ defines a trace co-algebra in $\text{Span}_F(\mathcal{C}^\times)$ if and only if it satisfies the following conditions:

1. G sends empty subsets to the terminal object.
2. G sends the morphisms in E to equivalences.
3. G sends $P \subset S \leftarrow T$ together with its projections to $\{i\} \subset S \leftarrow T$ for $i \in P$ to a product diagram.
4. G sends all diagrams of the form (6) to limit diagrams.

3.1.1 Non-degeneracy

Unlike in the associative case, we must consider one additional condition on our functors. The difference between Calabi–Yau algebras and trace algebras is precisely that certain morphisms must be non-degenerate. Our goal in this section is to unwind precisely what this non-degeneracy means in terms of the adjoint map $G : \Theta \rightarrow \mathcal{C}$.

To this end, we consider a morphism γ in $\text{Span}_\Gamma(\mathcal{C}^\times)$ represented by

$$X \times X \xleftarrow{(\gamma_1, \gamma_2)} Y \rightarrow *.$$

Lemma 3.4 *The morphism γ is non-degenerate in the sense of Definition 1.17 if and only if γ_1 and γ_2 are equivalences.*

Proof If γ_1 and γ_2 are equivalences, we can define a morphism

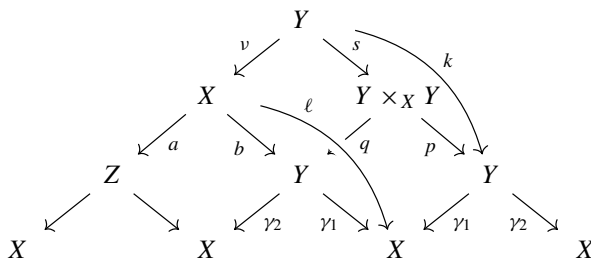
$$* \leftarrow Y \xrightarrow{(\gamma_1, \gamma_2)} X \times X$$

which displays the non-degeneracy of γ .

Now suppose that γ is non-degenerate, and let $\eta := (\eta_1, \eta_2)$ be a morphism

$$* \leftarrow Z \xrightarrow{(\eta_1, \eta_2)} X \times X$$

displaying the non-degeneracy of γ . Then we have the diagram



where every square is pullback. The left hand pullback must define an equivalence in $\text{Span}(\mathcal{C})$, and therefore, the morphism ℓ is an equivalence. We thus see that γ_1 must have a left inverse up to homotopy. Similarly, we see that the morphism k must be an equivalence. By the symmetry of the left-hand pullback square, $q \circ s$ must be an equivalence, and thus, $b \circ v$ is an equivalence. However, v is a pullback of γ_1 along an equivalence, and therefore is homotopic to γ_1 . Therefore, we see that γ_1 has a right inverse up to homotopy, and so, γ_1 is an equivalence. A similar argument shows that γ_2 is an equivalence. $\square$

Construction 3.5 Let $G : \Theta \rightarrow \mathcal{C}$ be a functor representing a trace (co)algebra in $\text{Span}_\Gamma(\mathcal{C}^\times)$. In particular, we have the object

$$Y := G(\{1\} \subset \diamond \leftarrow (2))$$

and the object

$$X_n := G(\{2\} \subset \langle 2 \rangle \xleftarrow{f} \langle n+1 \rangle)$$

where $f(1) = 1$ and $f(i) = 2$ for all $i \neq 1$. Finally, we have the object

$$Z_n := G(\{1\} \subset \diamond \leftarrow \langle n \rangle)$$

By 3.1, we get a pullback diagram

$$\begin{array}{ccc} & Z_n & \\ \swarrow & & \searrow \\ X_n & & Y \\ \searrow & & \swarrow \\ & ((2), \{2\}) & \end{array}$$

By 3.4, we know that the trace is non-degenerate if and only if the bottom right morphism is an equivalence. From the structure of the pullback diagram, we see that this is equivalent to requiring that the morphism $Z_n \rightarrow X_n$ is an equivalence for all n .

We have thus found a final condition, which allows us to guarantee that G is adjoint to a Calabi–Yau algebra. Combining this with the conditions that $G : \Theta \rightarrow \mathcal{C}$ defines a map $\text{Tw}(\text{Ass}_{\mathcal{C}^{\text{Y}}}^{\text{op}}) \rightarrow \mathcal{C}^{\times}$ over Γ , we can summarize the conditions we have obtained in the following corollary

Corollary 3.6 *A functor $G : \Theta \rightarrow \mathcal{C}$ defines a Calabi–Yau co-algebra in $\text{Span}_{\Gamma}(\mathcal{C}^{\times})$ if and only if it satisfies the following conditions:*

1. *G sends empty subsets to the terminal object.*
2. *G sends the morphisms in E to equivalences.*
3. *G sends $P \subset S \leftarrow T$ together with its projections to $\{i\} \subset S \leftarrow T$ for $i \in P$ to a product diagram.*
4. *G sends the morphisms $Z_n \rightarrow X_n$ from 3.5 to equivalences.*
5. *G sends all diagrams of the form (6) to limit diagrams.*

With these conditions in place, we can now define our category of Calabi–Yau algebras in $\text{Span}_{\Gamma}(\mathcal{C}^{\times})$.

Definition 3.7 We define $\text{Alg}_{\text{Sp}}^{\text{CY}}(\mathcal{C})$ to be the full ∞ -subcategory of $\text{Fun}(\Theta, \mathcal{C})$ satisfying the conditions of Corollary 3.6.

Before continuing on to the localization, we again forget the objects corresponding to degenerate (empty) subsets.

Definition 3.8 Let Ω be the full subcategory of Θ on those objects

$$Q \subset S \xleftarrow{f} T$$

such that $Q \neq \emptyset$ and $f : T \rightarrow S$ is not $\text{id}_\diamond$.

Lemma 3.9 *There is an equivalence of ∞ -categories*

$$\text{Fun}^*(\Theta, \mathcal{C}) \simeq \text{Fun}(\Omega, \mathcal{C})$$

Where Fun^* denotes the full subcategory on those functors which send empty subsets to the terminal object of $\mathcal{C}$.

Proof This is, *mutatis mutandis*, the same proof as that of Lemma 2.9. $\square$

3.2 The localization map

The key technical difficulty of the proof, as in the associative case, is the construction of the map $\mathcal{L} : \Omega \rightarrow \Lambda^*$ and the proof that it is an ∞ -categorical localization at the morphisms in E . Each of the facets of the construction and proof is rather more involved than in the associative case. However, there is still a useful underlying graphical intuition, which is suggested by Figs. 4 and 5. Owing to the substantial differences between the localization maps in the associative and Calabi–Yau cases, it is necessary to present the construction here in full.

We begin with the construction of Λ^* . This generalizes the construction of Δ^* in three ways. Firstly, since our morphisms in Ass (viewed, as before, as planar forests) are equipped with a *subset* of the *set* of roots, rather than an *interval* in an *ordered set* of roots, we must index the collections $\{[m_i]\}$ by unordered sets, rather than by ordered sets. Similarly, the product associated with morphisms of such collections is no longer the imbrication, but the ordinal sum. Finally, we must also include objects $\langle n \rangle \in \Lambda$, so as to account for the morphisms $P \rightarrow \diamond$ in Ass_{CY} .

Definition 3.10 Let Λ^* be the category with objects

- finite collections $\{[m_i]\}_{i \in S}$ in Δ indexed by $S \in \mathbb{F}\mathbf{n}$, and
- $\langle n \rangle$ in Λ ,

and morphisms given by:

1. A morphism $\{[m_i]\}_{i \in S} \rightarrow \{[n_j]\}_{j \in T}$ is given by
 - a morphism $\phi : T \rightarrow S$ in $\mathbb{F}\mathbf{n}$, with a chosen linear order on each fiber, and
 - for each $i \in S$, a morphism

$$\bigoplus_{j \in \phi^{-1}(i)} [n_j] \rightarrow [m_i]$$

2. A morphism $\langle n \rangle \rightarrow \{[m_i]\}_{i \in S}$ is given by
 - a cyclic order on S , and
 - a morphism

$$\bigcup^S [m_i] \rightarrow \langle n \rangle$$

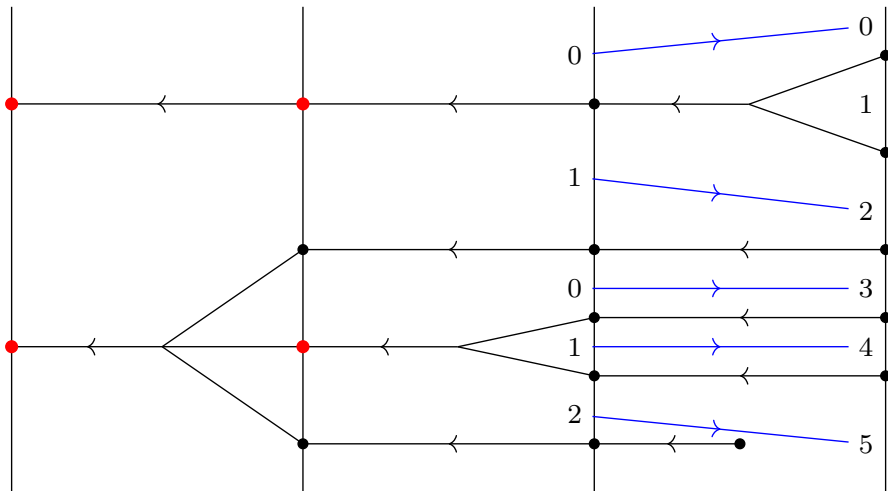


Fig. 4 A pictorial representation of a morphism μ in Ω , considered as a sequence $T \xrightarrow{\bar{g}} V \xrightarrow{h} U \xrightarrow{g} S$ of morphisms in $\mathcal{Ass}$. The chosen subsets $Q \subset S$ and $P \subset U$ are marked in red, and the induced morphism $\mathcal{L}(\mu)$ is drawn in blue. Note that, unlike in the analogous Fig. 3, the source of $\mathcal{L}(\mu)$ is the ordinal sum $\bigoplus_{i \in P} O(f^{-1}(i))$, owing to the presence interstitial trees with roots not in P

in Λ .

3. A morphism $\langle n \rangle \rightarrow \langle m \rangle$ is given by a morphism $\langle m \rangle \rightarrow \langle n \rangle$ in Λ .
4. Empty homsets $\{[m_i]\} \rightarrow \langle n \rangle$.

Composition is defined by taking lexicographic linear and cyclic orders. It is well-defined by Lemma 1.13.

The construction of $\mathcal{L} : \Omega \rightarrow \Lambda^*$ is substantially more complex than in the associative case. However, the underlying intuition is more or less the same: we forget all information about a rooted tree except the interstices between its leaves, together with the induced (linear or cyclic) order.

Construction 3.11 We define a functor $\mathcal{L} : \Omega \rightarrow \Lambda^*$ as follows. Let

$$P \subset S \xleftarrow{f} T$$

be an object in Ω with f a morphism in $\mathcal{Ass}$. We send this object to the collection

$$\left\{ O\left(f^{-1}(i)\right)\right\}_i \in P.$$

Let

$$\{\diamond\} \subset \diamond \xleftarrow{f} S$$

be an object in Ω . Then we send this object to

$$D(f^{-1}(\diamond)) \in \mathcal{A}.$$

To define $\mathcal{L}$ on morphisms, we proceed by cases:

1. Suppose we have a diagram

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f} & T \\ & & \uparrow g & & \downarrow \bar{g} \\ P & \subseteq & U & \xleftarrow{h} & V \end{array}$$

representing a morphism μ in Ω , where all of the objects are in $\mathcal{A}ss \subset \mathcal{A}ss_{CY}$. $\mathcal{L}(\mu)$ will be given by a morphism ϕ_μ in $\mathcal{F}in_*$ and a set of morphisms $\{\psi_i\}_{i \in Q}$ in Δ . The morphism ϕ_μ we take to be the restriction of g to $P \subset U^\circ$. Fixing $i \in Q$, we see that $\bar{g}$ restricts to a morphism $\bar{g}_i : f^{-1}(i) \rightarrow h^{-1}(g^{-1}(i))$ of linearly ordered sets. This can be rewritten as

$$\bar{g}_i : f^{-1}(i) \rightarrow \bigoplus_{j \in g^{-1}(i)} h^{-1}(j)$$

It therefore induces a morphism

$$\star_{j \in \bar{g}^{-1}(i)} O(h^{-1}(i)) \rightarrow O(f^{-1}(i))$$

We then define ψ_i to be the composite

$$\bigoplus_{j \in \bar{g}^{-1}(i) \cap P} O(h^{-1}(i)) \rightarrow \star_{j \in \bar{g}^{-1}(i)} O(h^{-1}(i)) \rightarrow O(f^{-1}(i))$$

See Fig. 4 for a pictorial representation.

2. Suppose we have a diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f} & T \\ & & \uparrow \text{id} & & \downarrow \bar{g} \\ \{1\} & \subseteq & \diamond & \xleftarrow{h} & V \end{array}$$

representing a morphism μ in Ω with $T, U \in \mathcal{A}ss$. Then $\mathcal{L}(\mu)$ will be given by a morphism $\psi : D(h^{-1}(\diamond)) \rightarrow D(f^{-1}(\diamond))$. The morphism $\bar{g}$ restricts to a morphism of cyclically ordered sets

$$\bar{g}_\diamond : f^{-1}(\diamond) \rightarrow h^{-1}(\diamond)$$

we therefore define ψ to be $D(\bar{g}_\diamond)$.

3. Suppose we have a diagram

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f} & T \\ & & \uparrow g & & \downarrow \bar{g} \\ P & \subseteq & U & \xleftarrow{h} & V \end{array}$$

representing a morphism μ in Ω , where all objects except $\diamond$ are in $\mathcal{A}\text{ss}$. The morphism $\mathcal{L}(\mu)$ will be given by a cyclic order on P and a morphism $\psi : \bigcup^S O(f^{-1}(i)) \rightarrow D(f^{-1}(\diamond))$. The cyclic order on P is induced by the cyclic order on $g^{-1}(\diamond) \supset P$. The morphism $\bar{g}$ restricts to a morphism

$$\bar{g}_\diamond : f^{-1}(\diamond) \rightarrow (g \circ h)^{-1}(\diamond)$$

of cyclically ordered sets. Passing through D gives a morphism

$$D(\bar{g}_\diamond) : D((g \circ h)^{-1}(\diamond)) \rightarrow D(f^{-1}(\diamond)).$$

Choosing any linear order on $g^{-1}(\diamond)$ compatible with the cyclic order we can write $D(\bar{g}_\diamond)$ as

$$C \left(O \left(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i) \right) \right) = D \left(K \left(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i) \right) \right) \rightarrow D(f^{-1}(\diamond))$$

We then have the canonical morphism

$$K \left(\bigoplus_{i \in g^{-1}(\diamond)} O(h^{-1}(i)) \right) \rightarrow C \left(\star_{i \in g^{-1}(\diamond)} O(h^{-1}(i)) \right) = C \left(O \left(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i) \right) \right)$$

And so we define ψ to be the composite

$$\begin{array}{ccc} K \left(\bigoplus_{i \in P} O(h^{-1}(i)) \right) & \longrightarrow & K \left(\bigoplus_{i \in g^{-1}(\diamond)} O(h^{-1}(i)) \right) \\ & \searrow & \uparrow \\ C \left(O \left(\bigoplus_{i \in g^{-1}(\diamond)} h^{-1}(i) \right) \right) & \longrightarrow & D(f^{-1}(\diamond)) \end{array}$$

See Fig. 5 for a pictorial representation.

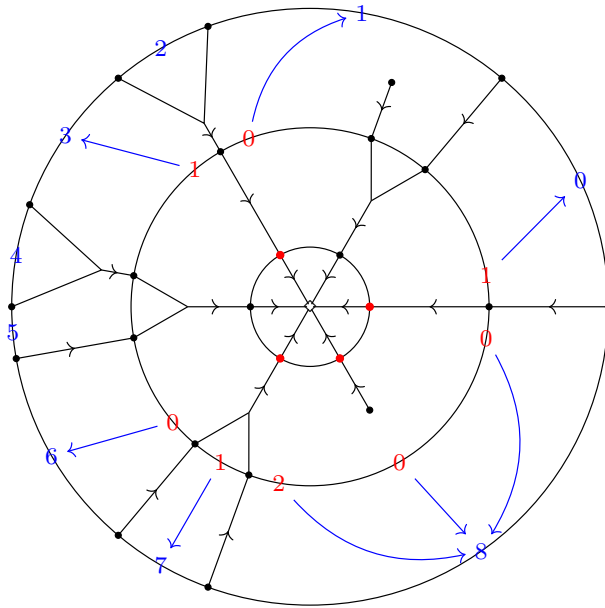


Fig. 5 A morphism in Ω represented as a composite of three morphisms in Ass_{CY} , $T \xrightarrow{\bar{g}} V \xrightarrow{h} U \xrightarrow{g} \diamond$. The chosen subset $P \subset U$ is marked by red points. The corresponding interstice sets $I(h^{-1}(i))$ are written in red numbers, and the set $D(f^{-1}(\diamond))$ in blue numbers. The induced morphism $\mathcal{L}(\mu) : \bigcup^P I(h^{-1}(i)) \rightarrow D(f^{-1}(\diamond))$ is drawn in blue. Note that the unmarked points in U are the reason that we do not necessarily get a morphism $C(\star_{i \in P} I(h^{-1}(i)) \rightarrow D(f^{-1}(\diamond)))$

3.2.1 Localizing

As with the constructions of Λ^* and $\mathcal{L}$, the proof of localization is more combinatorially involved. We will, however, still show that $\mathcal{L}$ is a localization at the morphisms of E using Lemma 2.15.

Definition 3.12 Let $M \in \Lambda^*$. We denote by Ω_M^E the subcategory of the weak fiber Ω_M whose morphisms are morphisms in E .

Proposition 3.13 For every M in Λ^* ,

1. Ω_M^E has an initial object.
2. The inclusion $\Omega_M^E \hookrightarrow \Omega_M$ is cofinal.

As before, we defer the combinatorial to Appendix B. Proposition B.1 shows that the weak fibers Ω_M^E have initial objects, and Propositions B.2 and B.3 show that the inclusions

$$\Omega_M^E \subset \Omega_M$$

are cofinal.

Corollary 3.14 The functor $\mathcal{L} : \Omega \rightarrow \Lambda^*$ is an ∞ -categorical localization at the morphisms of E .

3.2.2 Algebra Conditions

It remains only for us to identify the conditions on $F : \Lambda^* \rightarrow \mathcal{C}$ which corresponding to $F \circ \mathcal{L}$ satisfying conditions (3)–(5) in Corollary 3.6. These will uniquely characterize the image of $\text{Alg}_{\text{Sp}}^{\text{CY}}(\mathcal{C})$ in $\text{Fun}(\Lambda^*, \mathcal{C})$. It is immediate that condition (3) is equivalent to the condition that F display $F(\{[m_i]\}_{i \in S})$ canonically as products of the $F([m_i])$. Similarly, it is clear that condition (4) corresponds to the requirement that $F(\langle n \rangle \rightarrow \{[n]\})$ is an equivalence for all n .

Construction 3.15 Given $\langle n \rangle$ in Λ^* , we define a morphism

$$\sigma_n : \langle n \rangle \rightarrow \{[1]_{(i,i+1)}\}_{(i,i+1) \in D(\langle n \rangle)}$$

in Λ^* as follows. Take the canonical cyclic order on $D(\langle n \rangle)$, and define

$$\bigcup^{D(\langle n \rangle)} [1]_{(i,i+1)} \rightarrow \langle n \rangle$$

sending

$$\begin{aligned} 0 \in [1]_{(i,i+1)} &\mapsto i \\ 1 \in [1]_{(i,i+1)} &\mapsto i + 1. \end{aligned}$$

Note that given an object $X \in \Omega_{\langle n \rangle}$ in the fiber over $\langle n \rangle$, σ_n is simply the image of the source morphism in Ω .

Similarly, given an object $\{[m_i]\}_{i \in S}$ in Λ^* , define two morphisms

$$\begin{aligned} t_{\{m_i\}} : \{[m_i]\}_{i \in S} &\rightarrow \{[1]_i\}_{i \in S} \\ s_{\{m_i\}} : \{[m_i]\}_{i \in S} &\rightarrow \{[1]_{(j,j+1)}\}_{(j,j+1) \in \bigoplus_{i \in S} I(\{m_i\})} \end{aligned}$$

in Λ^* as follows. We define $t_{\{m_i\}} := (\text{id}_S, \{f_i\})$ where $f_i : [1]_i \rightarrow [m_i]$ is given by the formula

$$\begin{aligned} f_i(0) &= 0 \\ f_i(1) &= m_i \end{aligned}$$

We define $s_i := (\phi, \{g_i\})$, where

$$\phi : \bigoplus_{i \in S} I([m_i]) \rightarrow S$$

sends $I([m_i])$ to i , and the morphism

$$g_i : \bigoplus_{(j,j+1) \in \bigoplus_{i \in S} I([m_i])} [1]_{(j,j+1)} \rightarrow [m_i]$$

is given by

$$\begin{aligned} g_i(0 \in [1]_{(j,j+1)}) &= j \\ g_i(1 \in [1]_{(j,j+1)}) &= j+1 \end{aligned}$$

Note that, given an object $X \in \Omega_M$ in the fiber over $M := \{[m_i]\}_{i \in S}$, the morphisms s_M and t_M are simply the images under $\mathcal{L}$ of the source and target morphisms, respectively.

Lemma 3.16 *Given a functor $G : \Lambda^* \rightarrow \mathcal{C}$, $G \circ \mathcal{L}$ satisfies condition 5 of Corollary 3.6 if and only if the following two conditions on G are satisfied:*

1. *For any $\{[m_i]\}_{i \in S}$, and any $\{[n_{(j,j+1)}]\}_{(j,j+1) \in \bigoplus_{i \in S} I([m_i])}$ the diagram*

$$\begin{array}{ccc} & \{\star_{(j,j+1) \in I([m_i])} [n_{(j,j+1)}]\}_{i \in S} & \\ \swarrow & & \searrow \\ \{[n_{(j,j+1)}]\}_{(j,j+1) \in \bigoplus_{i \in S} I([m_i])} & & \{[m_i]\}_{i \in S} \\ \searrow \scriptstyle t_N & & \swarrow \scriptstyle s_M \\ & \{[1]_{(j,j+1)}\}_{(j,j+1) \in \bigoplus_{i \in S} I([m_i])} & \end{array}$$

is sent to a pullback under G .

2. *For $\langle n \rangle$ and any $\{[m_{(j,j+1)}]\}_{(j,j+1) \in D(\langle n \rangle)}$ the diagram*

$$\begin{array}{ccc} & C(\star_{(j,j+1) \in D(\langle n \rangle)} [m_{(j,j+1)}]) & \\ \swarrow & & \searrow \\ \{[m_{(j,j+1)}]\}_{(j,j+1) \in D(\langle n \rangle)} & & \langle n \rangle \\ \searrow \scriptstyle t_N & & \swarrow \scriptstyle \sigma_M \\ & \{[1]_j\}_{j \in D(\langle n \rangle)} & \end{array}$$

is sent to a pullback diagram under G .

Proof Using the same technique as in the proof of Proposition 2.19, we can reduce condition 5 to a statement about pullback squares along source and target maps. The diagrams of the lemma are then the images under $\mathcal{L}$ of the requisite pullback diagrams. $\square$

We can now explicitly characterize the subcategory of $\text{Fun}(\Lambda^*, \mathcal{C})$ corresponding to $\text{Alg}_{\text{Sp}}^{\text{CY}}(\mathcal{C})$:

Definition 3.17 We denote by $\text{Fun}^{\text{alg}}(\Lambda^*, \mathcal{C})$ the full sub-category on those functors which

1. Send $\{[m_i]\}_{i \in S}$ together with the projections to $[m_i]$ to product diagrams.
2. Send the morphisms $\langle n \rangle \rightarrow \{[n]\}$ to equivalences.

3. Send the diagrams from 3.16 to pullback diagrams.

Corollary 3.18 *There is an equivalence of ∞ -categories*

$$\mathrm{Alg}_{\mathrm{Sp}}^{\mathrm{CY}}(\mathbb{C}) \simeq \mathrm{Fun}^{\mathrm{alg}}(\Lambda^{\star}, \mathbb{C}).$$

3.3 Extension and restriction

We now come to final step of the proof: a pair of adjunctions

$$\mathrm{Fun}(\Lambda^{\star}, \mathbb{C}) \xrightleftharpoons{\quad} \mathrm{Fun}(\Lambda_{\Delta}^{\mathrm{op}}, \mathbb{C})$$

and

$$\mathrm{Fun}(\Lambda_{\Delta}^{\mathrm{op}}, \mathbb{C}) \xrightleftharpoons{\quad} \mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathbb{C})$$

which descend to an equivalence

$$\mathrm{Fun}^{\mathrm{alg}}(\Lambda^{\star}, \mathbb{C}) \simeq 2\text{-}\mathrm{Seg}_{\Lambda}(\mathbb{C}).$$

Definition 3.19 We define a category Λ_{Δ} to be the Grothendieck construction of the functor

$$\Delta^1 \xrightarrow{\{K\}} \mathrm{Cat}.$$

explicitly, $\mathrm{ob}(\Lambda_{\Delta}) = \mathrm{ob}(\Lambda) \amalg \mathrm{ob}(\Delta)$, with morphisms

- $f : [n] \rightarrow [m]$ morphism in Δ
- $f : \langle n \rangle \rightarrow \langle m \rangle$ morphism in Λ
- $f : [n] \rightarrow \langle m \rangle$ given by a morphism $f : K([n]) \rightarrow \langle m \rangle$ in Λ .

The category $(\Lambda_{\Delta})^{\mathrm{op}}$ can be identified with the full subcategory of $\Lambda^{\star}$ on the objects $\{[m]\}$ and $\langle n \rangle$.

Construction 3.20 By taking restriction and right Kan extension along the inclusion $\Lambda_{\Delta}^{\mathrm{op}} \subset \Lambda^{\star}$, we get an adjunction

$$\iota_{*} \mathrm{Fun}(\Lambda^{\star}, \mathbb{C}) \leftrightarrow \mathrm{Fun}(\Lambda_{\Delta}^{\mathrm{op}}, \mathbb{C}) : \iota_{!}$$

of ∞ -categories.

Definition 3.21 Denote by $\mathrm{Fun}^{\times}(\Lambda^{\star}, \mathbb{C})$ the full ∞ -subcategory of $\mathrm{Fun}(\Lambda^{\star}, \mathbb{C})$ on those functors which satisfy (1) from Definition 3.17.

Proposition 3.22 *The adjunction of Construction 3.20 restricts to an equivalence of ∞ -categories*

$$\mathrm{Fun}^{\times}(\Lambda^{\star}, \mathbb{C}) \simeq \mathrm{Fun}(\Lambda_{\Delta}, \mathbb{C})$$

Proof Since there are no morphisms $\{[m_i]\}_{i \in S} \rightarrow \langle n \rangle$ in Λ^* , this is, *mutatis mutandis*, the same as the proof of 2.21. $\square$

Construction 3.23 We have a full subcategory $F : \Lambda \subset \Lambda_\Delta$. We can similarly define a functor

$$H : \Lambda_\Delta \rightarrow \Lambda$$

by acting as K on Δ and as the identity on all other objects and morphisms. This defines an adjunction

$$F : \Lambda \leftrightarrow \Lambda_\Delta : H$$

It is easy to see that H is a reflective localization at the morphisms $[n] \rightarrow \langle n \rangle$ given by isomorphisms $K([n]) \cong \langle n \rangle$.

Proposition 3.24 *There is an equivalence of ∞ -categories*

$$\mathrm{Fun}^{\mathrm{alg}}(\Lambda^*, \mathbb{C}) \simeq 2\text{-}\mathrm{Seg}_\Lambda(\mathbb{C}).$$

Proof Proposition 3.22 and Construction 3.23 show us that $\mathrm{Fun}(\Lambda^{\mathrm{op}}, \mathbb{C})$ is equivalent, as an ∞ -category, to the full subcategory of $\mathrm{Fun}^\times(\Lambda^*, \mathbb{C})$ satisfying (1) and (2) from Definition 3.17. The relation between the 2-Segal condition and condition 3 from Definition 3.17 follows from a similar argument to the proof of Proposition 2.22. $\square$

We can then summarize our results in the following theorem:

Theorem 3.25 *There is an equivalence of ∞ -categories*

$$\mathrm{Alg}_{\mathrm{Sp}}^{\mathrm{CY}}(\mathbb{C}) \simeq 2\text{-}\mathrm{Seg}_\Lambda(\mathbb{C}).$$

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A The localization: associative case

This appendix is given over to the proof that the functor $\mathcal{L} : \Omega \rightarrow \Delta^*$ constructed in Sect. 2 satisfies the conditions of Lemma 2.15, and thus is an ∞ -categorical localization at the morphisms of E . By necessity, this involves fairly intricate combinatorial verifications.

A.1 Decomposing morphisms

Construction A.1 Given a morphism

$$f : [m] \rightarrow [n]$$

in Δ , we can uniquely decompose it as follows: Let $[1] =: [1_i] \subset [m]$ be the interval $\{i - 1 \leq i\}$, and let $[n_i] \subset [n]$ be the interval $\{f(i - 1) \leq f(i)\}$. Moreover, let $[n_{left}]$ and $[n_{right}]$ be the intervals $\{0 \leq f(0)\}$ and $\{f(m) \leq n\}$ in $[n]$ respectively. Then f is completely determined by the decomposition of $[n]$, since, given such a decomposition, we can reconstruct f by defining $f_i : [1_i] \rightarrow [n_i]$ to be the unique map preserving maximal and minimal elements, so that f is the composition

$$f = f_1 \star \cdots \star f_m : [1_1] \star \cdots \star [1_m] \rightarrow [n_1] \star \cdots \star [n_m] \hookrightarrow [n_{left}] \star [n_1] \star \cdots \star [n_m] \star [n_{right}].$$

We can clarify the indexing of the decomposition of $[n]$ by noting that the pairs $(i - 1, i)$ considered above are precisely the inner interstices of $[m]$. Hence, we have decomposed f as a morphism

$$\star_{(i-1,i) \in \mathbb{I}([m])} \{i - 1, i\} \rightarrow \star_{(i-1,i) \in \mathbb{I}([m])} [n_i].$$

Definition A.2 Given a morphism $\gamma : [n] \rightarrow [m]$ in Δ , we can uniquely factor γ as

$$[n] \xrightarrow{\gamma_1} [m_\gamma] \xrightarrow{\gamma_2} [m]$$

where $[m] = [k] \oplus [m_\gamma] \oplus [\ell]$. Applying O , we get

$$O([m]) \rightarrow O([m_\gamma]) \rightarrow O[n].$$

Where $O([m]) \rightarrow O([m_\gamma])$ acts as projection onto a sub-interval. We call $O([m_\gamma])$ the *minimal interval* of γ .

Lemma A.3 *Given an interval $\{i, j\} \subset [n]$ and a morphism $\eta : (\{i, j\} \subset [n]) \rightarrow (\{r, r + k\} \subset [m])$ in $\Delta^{\mathbb{I}}$, let $[p, \dots, q]$ be the minimal interval of $\gamma := \text{res}(\eta)$. Then $\eta|_{[p+1, \dots, q-1]} = O(\gamma)|_{[p+1, \dots, q-1]}$.*

Proof If $[p + 1, \dots, q - 1]$ is empty, the statement is vacuously true. Otherwise, note that for $s \in [p + 1, \dots, q - 1]$, the requirement that $\text{res}(\eta) = \gamma$ means that $\gamma(\eta(s)) \leq s < \gamma(\eta(s) + 1)$. Such an $\eta(s)$ always exists, and this inequality uniquely determines $\eta(s)$. (Note that, for p or q in $[p, \dots, q]$, we only have one-half of the inequality so that uniqueness need not hold.) $\square$

With these lemmata in hand, we can return to the proof of the localization result.

A.2 Constructing morphisms

We now prove the first criterion of Lemma 2.15.

Lemma A.4 *The object $\{0, k\} \subset [k] \xrightarrow{f_M} [m]$ is an initial object in Ω_M^E .*

Proof Given another object

$$\{i, j\} \subset [n] \xrightarrow{f} [m']$$

in Ω_M^E , and a morphism

$$\begin{array}{ccc} \{0, k\} & \subseteq & [k] \xrightarrow{f_M} [m] \\ & \phi \downarrow & \uparrow h \\ \{i, j\} & \subseteq & [n] \xrightarrow{f} [m'] \end{array}$$

ϕ must be the inclusion of $[i, \dots, j]$, since any such morphism in E will induce an isomorphism $[k] \rightarrow [i, \dots, j]$. Moreover, h is clearly uniquely determined by the condition that it maps $[f(i), f(i+1), \dots, f(j)]$ isomorphically to $[m]$. $\square$

To show the second criterion of Lemma 2.15, we must show that the inclusion $\Omega_M^E \hookrightarrow \Omega_M$ is cofinal. This amounts to showing, for any $g : \mathcal{L}(Z) \rightarrow M$ in Ω_M , the category $(\Omega_M^E)_{g/}$ is contractible. We will do this by showing that $(\Omega_M^E)_{g/}$ has an initial object.

To this end, we suppose we are given an object

$$Z := \left\{ \{i, j\} \subset [n] \xrightarrow{f} [\ell] \right\}$$

in Ω whose image under $\mathcal{L}$ is $([\ell_{i+1}], \dots, [\ell_j])$, and a morphism

$$g : ([\ell_{i+1}], \dots, [\ell_j]) \rightarrow ([m_0], \dots, [m_{k-1}])$$

in Δ^* . Write $\gamma : [k-1] \rightarrow [i+1, \dots, j] \in \Delta$ and $\bar{g} : [m_0] \star \dots \star [m_{k-1}] \rightarrow [\ell_{i+1}] \star \dots \star [\ell_j]$ for the morphisms defining g . Denote by $[n_c] := [p, \dots, q] \subset \{i, j\} \subset [n]$ the minimal interval of γ and by $\psi : [i, \dots, j] \rightarrow [n_c]$ the projection as above, and let $\{0, k\} \subset [k] \xrightarrow{f_M} [m] := [m_0] \star \dots \star [m_{k-1}]$ be the minimal object in Ω representing the target.

Note that, by definition, the morphism $\bar{g}$ has image contained in $[\ell_{p+1}] \star \dots \star [\ell_q] =: [\ell_c]$. We introduce some notation for specific decompositions:

$$\begin{aligned} [n] &= [n_\ell] \star [n_c] \star [n_r] \\ [\ell] &= [\ell_\ell] \star [\ell_c] \star [\ell_r] \end{aligned}$$

Lemma A.5 *There is a morphism in Ω*

$$\begin{array}{ccc} \{p, q\} & \subseteq & [p, \dots, q] \xrightarrow{f|_{[p, q]}} [\ell_c] \\ & \downarrow \nu & \uparrow \bar{g}' \\ \{0, k\} & \subseteq & [1] \star [k] \star [1] \xrightarrow{f'_M} [\ell^1] \star [m] \star [\ell^2] \end{array}$$

which extends to a morphism $\mu_{Z, M}$ in Ω covering g

$$\mu_{Z, M} := \left\{ \begin{array}{ccc} Z = \{i, j\} & \subseteq & [n_\ell] \star [n_c] \star [n_r] \longrightarrow [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & \downarrow & \uparrow \\ Z_M : \{0, k\} \subseteq [n_\ell] \star [1] \star [k] \star [1] \star [n_r] & \longrightarrow & [\ell_\ell] \star [\ell^1] \star [m] \star [\ell^2] \star [\ell_r] \end{array} \right\}$$

Moreover, given any other morphism $Z \rightarrow X$ covering g , there is a unique morphism $Z_M \rightarrow X$ in E such that the diagram

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ Z_M & \xrightarrow{\quad} & X \end{array}$$

commutes.

Proof In the first diagram, we define the map ν on $[p+1, \dots, q-1]$ to be the unique map from Lemma A.3 dual to γ under res , and send the endpoints to the endpoints of $[1] \star [k] \star [1]$. Then we write

$$[\ell_c] = [\ell^1] \star [\ell_c^m] \star [\ell^2],$$

where $[\ell_c^m]$ is the minimal interval containing the image of $\bar{g} : [m] \rightarrow [\ell]$. Note that $\bar{g} : [m] \rightarrow [\ell_c^m]$ hits both endpoints. We then define

$$\bar{g}' := \text{id}_{[\ell^1]} \star \bar{g} \star \text{id}_{[\ell^2]} : [\ell^1] \star [m] \star [\ell^2] \rightarrow [\ell_c]$$

(which then, by definition, hits both endpoints), and

$$f'_M : [1] \star [k] \star [1] \rightarrow [\ell^1] \star [m] \star [\ell^2]$$

to be f_M on $[k]$, and to send endpoints to endpoints. Then we can decompose the diagram as

$$\begin{array}{ccc} \{p, q\} & \subseteq & [1]_{p+1} \star \dots \star [1]_q \xrightarrow{f|_{[p, q]}} [\ell_{p+1}] \star \dots \star [\ell_q] \\ & \downarrow \nu & \uparrow \bar{g}' \\ \{0, k\} & \subseteq & [1] \star [k_{p+1}] \star \dots \star [k_q] \star [1] \xrightarrow{f'_M} [\ell^1] \star [m_{p+1}] \star \dots \star [m_q] \star [\ell^2] \end{array}$$

by decomposing the morphisms v , $f|_{\{p,q\}}$, and $f'_M \circ v$. The condition that the diagram commute is then equivalent to the conditions that, (1) for each $r \in \{p+2, \dots, q-1\}$, the endpoints of $[m_r]$ are sent to the endpoints of $[\ell_r]$ by $\bar{g}$, and (2) that $\bar{g}$ sends the endpoints of $[\ell^1] \star [m_{p+1}]$ and $[m_q] \star [\ell^2]$ to the endpoints of $[\ell_{p+1}]$ and $[\ell_q]$, respectively. Since

$$[m_r] = \star_{a \in \mathbb{I}([k_r])} [f'_M(a-1), f'_M(a-1)+1, \dots, f'_M(a)]$$

we see that case (1) is true by the definition of Δ^* . Case (2) is true by construction.

This diagram is defined so that the maps v , f'_M , $\bar{g}'$, and $f|_{\{p,q\}}$ preserve endpoints. Therefore, we can take the appropriate star products with the morphisms $\text{id}_{[n_\ell]}$, $\text{id}_{[n_r]}$, $\text{id}_{[\ell_\ell]}$, $\text{id}_{[\ell_r]}$, $f|_{[n_\ell]} : [n_\ell] \rightarrow [\ell_\ell]$, and $f|_{[n_r]} : [n_r] \rightarrow [\ell_r]$ to get a commutative diagram

$$\begin{array}{ccc} Z = \{i, j\} & \subseteq & [n_\ell] \star [n_c] \star [n_r] \longrightarrow [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & \downarrow & \uparrow \\ Z_M = \{0, k\} & \subseteq & [n_\ell] \star [1] \star [k] \star [1] \star [n_r] \longrightarrow [\ell_\ell] \star [\ell^1] \star [m] \star [\ell^2] \star [\ell_r] \end{array}$$

By construction, the morphism $\text{res}(v) : [k-1] \rightarrow \langle i+1, \dots, j \rangle$ is γ , and the morphism $\bar{g}'$ restricts to $\bar{g}$ on $[m]$, so this diagram determines a morphism in Ω covering g . Call this morphism $\mu_{Z,M} : Z \rightarrow Z_M$.

Now suppose we are given a morphism

$$\begin{array}{ccc} Z = \{i, j\} & \subseteq & [n] \xrightarrow{f} [\ell] \\ & \rho \downarrow & \uparrow w \\ X = \{0, k\} & \subseteq & [a] \xrightarrow{h} [b] \end{array}$$

covering g . We can decompose this into

$$\begin{array}{ccc} Z = \{i, j\} & \subseteq & [n_\ell] \star [n_c] \star [n_r] \xrightarrow{f} [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ & \rho \downarrow & \uparrow w \\ Z_M = \{0, k\} & \subseteq & [a_\ell] \star [a_c] \star [a_r] \xrightarrow{h} [b_\ell] \star [b_c] \star [b_r] \end{array}$$

where $\{0, k\} \subset [a_c]$. By Lemma A.3, we know that ρ is uniquely determined on all of $[n_c]$ except the endpoints. This allows us to further decompose the diagram

$$\begin{array}{ccc} [n_c] & \xrightarrow{f} & [\ell_c] \\ \rho \downarrow & & \uparrow w \\ [a_c] & \xrightarrow{h} & [b_c] \end{array}$$

as a diagram where the bottom map is a star product with f_M .

$$\begin{array}{ccc} [n_c] & \xrightarrow{f} & [\ell_c] \\ \rho \downarrow & & \uparrow w \\ [a_c^1] \star [k] \star [a_c^2] & \xrightarrow{h} & [b_c^1] \star [m] \star [b_c^2] \end{array}$$

If there is morphism $Z_M \rightarrow X$ in E commuting with the morphisms $Z \rightarrow X$ and $\mu_{Z,M}$, it must, in particular, restrict to a commutative diagram

$$\begin{array}{ccc} [n_c] & \xrightarrow{\quad} & [\ell_c] \\ \downarrow & \searrow & \uparrow \\ [1] \star [k] \star [1] & \xrightarrow{\quad} & [\ell^1] \star [m] \star [\ell^2] \\ & \searrow & \swarrow \\ & [a_c^1] \star [k] \star [a_c^2] & \xrightarrow{h} [b_c^1] \star [m] \star [b_c^2] \end{array}$$

Moreover, since the morphism is in E , the bottom square must restrict to the commutative diagram

$$\begin{array}{ccc} [k] & \xrightarrow{f_M} & [m] \\ \text{id} \downarrow & & \uparrow \text{id} \\ [k] & \xrightarrow{f_M} & [m] \end{array}$$

As a result, the component morphism $[1] \star [k] \star [1] \rightarrow [a_c^1] \star [k] \star [a_c^2]$ is uniquely determined by the commutativity of the left-hand triangle. Additionally, since $w : [b] \rightarrow [\ell]$ must restrict to $\bar{g}$ on $[m]$, we can decompose w as a star product

$$w = w^1 \star \bar{g} \star w^2 : [b_c^1] \star [m] \star [b_c^2] \rightarrow [\ell^1] \star [\ell_c^m] \star [\ell^2]$$

Therefore, the component morphism

$$[b_c^1] \star [m] \star [b_c^2] \rightarrow [\ell^1] \star [m] \star [\ell^2]$$

is uniquely determined, and must be $w^1 \star \text{id}_{[m]} \star w^2$.

We now extend back to the full diagram

$$\begin{array}{ccc} [n_\ell] \star [n_c] \star [n_r] & \xrightarrow{\quad} & [\ell_\ell] \star [\ell_c] \star [\ell_r] \\ \downarrow & \searrow & \uparrow \\ [n_\ell] \star [1] \star [k] \star [1] \star [n_r] & \xrightarrow{\quad} & [\ell_\ell] \star [\ell^1] \star [m] \star [\ell^2] \star [\ell_r] \\ & \searrow & \swarrow \\ & [a_\ell] \star [a_c^1] \star [k] \star [a_c^2] \star [a_r] & \rightarrow [b_\ell] \star [b_c^1] \star [m] \star [b_c^2] \star [b_r] \end{array}$$

and note that, since the vertical components of the back square restrict to identities on $[n_\ell]$, $[n_r]$, $[\ell_\ell]$, and $[\ell_r]$, the bottom square is uniquely determined by the morphisms $[n_\ell] \rightarrow [a_\ell]$, $[n_r] \rightarrow [a_r]$, $b_\ell] \rightarrow [\ell_\ell]$, and $[b_r] \rightarrow [\ell_r]$. So there is a unique morphism $Z_M \rightarrow X$ in Ω with the desired properties. $\square$

B The localization: Calabi–Yau case

We can immediately verify that each Ω_M^E has an initial object — showing the first criterion of Lemma 2.15.

Proposition B.1 *For every M in $\Lambda^\star$, there is an initial element in Ω_M^E .*

Proof We will complete the proof in two cases:

Suppose first that $M = \{[m_i]\}_{i \in P}$. Then the weak fiber only involves morphisms in $\mathcal{A}ss \subset \mathcal{A}ss_{CY}$. We define a set

$$T := \coprod_{i \in P} \mathbb{I}([m_i])$$

and a morphism $f_M : T \rightarrow P$ by setting $f_M(\mathbb{I}([m_i])) = i$. The canonical isomorphisms

$$\eta_i : O(\mathbb{I}([m_i])) \cong [m_i]$$

equip $P \subset P \xleftarrow{f_M} T$ with the structure of an object of Ω_M^E . Given an element

$$P \subset U \xleftarrow{f} V$$

and an isomorphism $\phi_i : O(f^{-1}(i)) \cong [m_i]$, we define a unique morphism μ in Ω_M^E given by

$$\begin{array}{ccccc} P & \subseteq & P & \xleftarrow{f_M} & T \\ & & g \uparrow & & \downarrow \bar{g} \\ P & \subseteq & U & \xleftarrow{f} & V \end{array}$$

as follows. Since this must be a morphism in E , we see that g must map P identically to P , and send $U \setminus P$ to the basepoint. On fibers, we consider the isomorphisms

$$\eta_i^{-1} \circ \phi_i : O(f^{-1}(i)) \rightarrow O(I[m_i])$$

Since O is fully faithful, this lifts to a unique isomorphism $I(\phi_i) : I([m_i]) \cong f^{-1}(i)$. We therefore see that $\bar{g}$ must be the coproduct of these morphisms if μ is to be a morphism in the weak fiber. It is immediate that this does, indeed, define a morphism in Ω_M^E .

Now suppose instead $M = \langle m \rangle$. We define $f_M : D(\langle m \rangle) \rightarrow \diamond$ to be the morphism with $f_M^{-1}(\diamond) = D(\langle m \rangle)$. Since D is an equivalence, we choose the isomorphism

$$\eta : D^2(\langle m \rangle) \cong \langle m \rangle$$

Suppose given another element

$$\{1\} \subset \diamond \xleftarrow{f} T$$

with $\phi : D(f^{-1}(\diamond)) \cong \langle n \rangle$ in the weak fiber. We define a unique morphism $\mu \in \Omega_M^E$ given by

$$\begin{array}{ccc} \{1\} & \subseteq & \diamond \xleftarrow{f_M} D(\langle m \rangle) \\ & \uparrow g & \downarrow \bar{g} \\ \{1\} & \subseteq & \diamond \xleftarrow{f} T \end{array}$$

as follows. The morphism g must be the identity, so we need only define $\bar{g}$. The condition that μ be in the weak fiber implies that $\eta_i \circ D(\bar{g}|_{D(\langle m \rangle)}) = \phi_i$, i.e. $D(\bar{g}|_{D(\langle m \rangle)}) = \eta_i^{-1} \circ \phi_i$. However, since D is fully faithful, this condition defines a unique isomorphism $D(\langle m \rangle) \cong f^{-1}(\diamond)$, determining $\bar{g}$, and thus μ , uniquely. $\square$

B.1 Cofinality

We now prove the second criterion of Lemma 2.15: that the inclusion $\Omega_M^M \rightarrow \Omega_{/M}$ is cofinal. As before, we do this by finding an initial object in each slice of the inclusion. Unlike the associative case, however, we must do this in two steps, depending on whether $M = \{[m_i]\}_{i \in P}$ or $M = \langle m \rangle$.

Proposition B.2 *Suppose given an object $M = \{[m_i]\}_{i \in P}$ in $\Lambda^\star$, an object*

$$Z := \left\{ Q \subset S \xleftarrow{f_Z} T \right\}$$

in Ω , and a morphism

$$(\phi, \{\gamma_i\}_{i \in Q}) : \mathcal{L}(Z) \rightarrow M$$

in $\Lambda^\star$. Then there is an element $X_{M,Z}$ in Ω_M^E and a morphism $\Phi : Z \rightarrow X_{M,Z}$ in Ω covering $(\phi, \{\gamma_i\}_{i \in Q})$ such that, for any other morphism $\Psi : Z \rightarrow X$ covering $(\phi, \{\gamma_i\}_{i \in Q})$, there is a unique morphism $\tau : X_{M,Z} \rightarrow X$ which makes the diagram

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & & \searrow \Psi \\ X_{M,Z} & \xrightarrow{\tau} & X \end{array}$$

commute.

Proof There are two cases to consider, corresponding to whether or not $S = \diamond$.

Case 1: First suppose $S \in \mathcal{A}ss$. In this case, we construct $X_{M,Z}$ as follows. Let

$$P \subset P \xleftarrow{f_M} U$$

be the object constructed in Proposition B.2. Then, in particular, $\phi : P \rightarrow Q \subset S$.

For each $i \in Q$, we have a morphism

$$\gamma_i : \bigoplus_{j \in \phi^{-1}(i)} [m_j] \rightarrow O(f_Z^{-1}(i))$$

For each $j \in \phi^{-1}(i)$ denote by $\gamma_i([m_j])$ the smallest subinterval of $O(f_Z^{-1}(i))$ containing the image of $[m_j]$ under γ_i . Then $\gamma_i|_{[m_j]} : \gamma_i([m_j]) \rightarrow \gamma_i([m_j])$ preserves boundary, and thus corresponds to a map $\bar{g}_j : I(\gamma_i([m_j])) \rightarrow I([m_j])$ of linearly ordered sets. Moreover, $\bar{g}_j$ fits into a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{f_Z} & I(\gamma_i([m_j])) \\ \phi \uparrow & & \downarrow \bar{g}_j \\ P & \xleftarrow{f_M} & I([m_j]) \end{array}$$

in $\mathcal{A}ss$. We here use the identification of $I(\gamma_i([m_j]))$ with a subset of T .

Since, by definition, $U = \coprod_{j \in P} I([m_j])$, we can then write down a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{f_Z} & \coprod_{i,j} I(\gamma_i([m_j])) \\ \phi \uparrow & & \downarrow \coprod_{i,j} \bar{g}_j \\ P & \xleftarrow{f_M} & \coprod_{j \in P} I([m_j]) \end{array} \quad (7)$$

in $\mathcal{A}ss$.

For each $i \in Q$, this restricts to a diagram of ordered sets

$$\begin{array}{ccc} \{i\} & \xleftarrow{f_Z} & \coprod_j I(\gamma_i([m_j])) \\ \phi \uparrow & & \downarrow \coprod_j \bar{g}_j \\ \phi^{-1}(i) & \xleftarrow{f_M} & \coprod_{j \in \phi^{-1}(i)} I([m_j]) \end{array}$$

We denote $L_i := f_Z^{-1}(i) \setminus \coprod_{j \in \phi^{-1}(i)} I(\gamma_i([m_j]))$, and proceed as follows.

- For $p, p+1$ in $\phi^{-1}(i)$, if there is at least one $k \in L_i$ such that

$$I(\gamma_i([m_p])) < k < I(\gamma_i([m_{p+1}]))$$

we define a new element r_p and append it to $\phi^{-1}(i)$ between p and $p + 1$.

- If there exists $k \in L_i$ such that

$$k < I(\gamma_i([m_p]))$$

for all $p \in \phi^{-1}(i)$, then we append a new minimal element r_{min} to $\phi^{-1}(i)$.

- If there exists $k \in L_i$ such that

$$I(\gamma_i([m_p])) < k$$

for all $p \in \phi^{-1}(i)$, then we append a new maximal element to $\phi^{-1}(i)$.

Call the resulting set $W_i \supset \phi^{-1}(i)$. We then set

$$R_i := U \sqcup L_i$$

and define $f_i : R_i \rightarrow W_i$ to act as f_M on U and on L_i to send

- $k \mapsto r_p$ if

$$I(\gamma_i([m_p])) < k < I(\gamma_i([m_{p+1}]))$$

- $k \mapsto r_{min}$ if

$$k < I(\gamma_i([m_p]))$$

for all $p \in \phi^{-1}(i)$

- $k \mapsto r_{max}$ if

$$I(\gamma_i([m_p])) < k$$

for all $p \in \phi^{-1}(i)$

We make f_i into a morphism in $\mathcal{A}ss$ by taking the linear order induced by L_i on the fibers over the r_p , r_{min} and r_{max} . We then define

$$\bar{g}^i : f_Z^{-1}(i) \rightarrow R_i$$

to act as $\coprod_{j \in \phi^{-1}(i)} \bar{g}_j$ on $\coprod_{j \in \phi^{-1}(i)} I(\gamma_i([m_j]))$ and as the identity on L_i . We further define $\phi_i : W_i \rightarrow \{i\}$ to send every element to i . We thus have a commutative diagram

$$\begin{array}{ccccc} \{i\} & \subseteq & \{i\} & \xleftarrow{f_Z} & \coprod_j I(\gamma_i([m_j])) \\ & & \uparrow \phi_i & & \downarrow \bar{g}^i \\ P \cap \phi^{-1}(i) & \subseteq & W_i & \xleftarrow{f_i} & R_i \end{array}$$

in $\mathcal{A}ss$, which covers the morphism $\gamma_i : \bigoplus_{j \in \phi^{-1}(i)} [m_j] \rightarrow O(f_Z^{-1}(i))$. Taking the coproduct over $i \in \text{Im}(\phi)$ gives us a morphism

$$\begin{array}{ccc} \text{Im}(\phi) & \subseteq & \text{Im}(\phi) \xleftarrow{f_Z} f_Z^{-1}(\text{Im}(\phi)) \\ & \uparrow \coprod_i \phi_i & \downarrow \coprod_i \bar{g}^i \\ P \cap \phi^{-1}(i) & \subseteq & \coprod_i W_i \xleftarrow{\coprod_i f_i} \coprod_i R_i \end{array}$$

Finally, we set

$$W = \left(\coprod_{i \in \text{Im}(\phi)} W_i \right) \sqcup (S \setminus \text{Im}(\phi))$$

and

$$R = \left(\coprod_{i \in \text{Im}(\phi)} R_i \right) \sqcup (T \setminus f_Z^{-1}(\text{Im}(\phi)))$$

We then define morphisms:

- $g : W \rightarrow S$ to act as $\coprod_{i \in \text{Im}(\phi)} \phi_i$ on $\coprod_{i \in \text{Im}(\phi)} W_i$ and as the identity otherwise.
- $f_{M,Z} : R \rightarrow W$ to act as f_i on R_i and as f_Z on $T \setminus f_Z^{-1}(\text{Im}(\phi))$.
- $\bar{g} : T \rightarrow R$ to act as $\coprod_{i \in \text{Im}(\phi)} \bar{g}^i$ on $f_Z^{-1}(\text{Im}(\phi))$ and the identity elsewhere.

By construction, this defines a commutative diagram

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f_Z} & T \\ & & g \uparrow & & \downarrow \coprod_i \bar{g} \\ P & \subseteq & W & \xleftarrow{f_{M,Z}} & R \end{array} \quad (8)$$

in $\mathcal{A}ss$, covering $(\phi, \{\gamma_i\})$, and the bottom row is in Ω_M . We therefore define $X_{M,Z}$ to be the bottom row, and Φ to be the morphism defined by the diagram (8).

To check the remaining universal property, we let

$$P \subset A \xleftarrow{f_X} B$$

and $\beta_i : O(f_X^{-1}(i)) \cong [m_i]$ be another element in Ω_M^E , and let ν be a morphism

$$\begin{array}{ccccc} Q & \subseteq & S & \xleftarrow{f_Z} & T \\ & & \rho \uparrow & & \downarrow \bar{\rho} \\ P & \subseteq & A & \xleftarrow{f_X} & B \end{array}$$

covering $(\phi, \{\gamma_i\}_{i \in Q})$.

For each $i \in \text{Im}(\phi)$, the identity on P and the condition nothing be sent to the basepoint uniquely determines a map of ordered sets

$$\zeta_i : \rho^{-1}(i) \rightarrow W_i.$$

Moreover, the ζ_i together with the restriction of ρ to $A \setminus \rho^{-1}(\text{Im}(\phi))$ uniquely determines a map

$$\zeta : A \rightarrow W$$

such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & S \\ & \searrow \zeta & \nearrow g \\ & W & \end{array}$$

commutes. Note that $\zeta|_P$ induces the identity $P \rightarrow P$.

Moreover, for each $i \in \text{Im}(\phi)$ the isomorphisms $I(\beta_i)$ on $I([m_j])$ and restriction $\bar{\rho}_i : L_i \rightarrow f_X^{-1}(\rho^{-1}(i))$ uniquely determine a map

$$\bar{\zeta}_i : R_i \rightarrow f_X^{-1}(\rho^{-1}(i)).$$

These, together with the restriction of $\bar{\rho}$ to $T \setminus f_Z^{-1}(\text{Im}(\phi))$ uniquely determine a morphism

$$\bar{\zeta} R \rightarrow B$$

such that the diagram

$$\begin{array}{ccc} B & \xleftarrow{\bar{\rho}} & T \\ & \nwarrow \bar{\zeta} & \nearrow \bar{g} \\ & R & \end{array}$$

commutes, and the restriction of $\bar{\zeta}$ to $f_X^{-1}(P)$ is the isomorphism $\coprod_i I(\beta_i)$.

We therefore have constructed a unique morphism

$$(\zeta, \bar{\zeta}) : X_{Z,M} \rightarrow X$$

in Ω_M^E such that the diagram

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & & \searrow \Psi \\ X_{M,Z} & \xrightarrow{(\zeta, \bar{\zeta})} & X \end{array}$$

commutes.

Case 2: Now suppose that $S = \diamond$. Then ϕ is completely determined by a cyclic order on P , and γ is a morphism

$$\gamma : \bigcup^S [m_i] \rightarrow D(f_Z^{-1}(\diamond)).$$

We note that, given any morphism

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{p} & V \\ & & g \uparrow & & \downarrow \bar{g} \\ A & \subseteq & B & \xleftarrow{\ell} & C \end{array}$$

a choice of linear order on $g^{-1}(\diamond)$ compatible with the cyclic order uniquely determines a factorization

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{p} & V \\ & & \uparrow & & \downarrow \text{id}_V \\ \{1\} & \subseteq & \{1\} & \xleftarrow{p} & V \\ & & g \uparrow & & \downarrow \bar{g} \\ A & \subseteq & B & \xleftarrow{\ell} & C \end{array}$$

Similarly, given a morphism $(\psi, \eta) : \langle n \rangle \rightarrow \{[n_i]\}_{i \in S}$, a choice of linear order on S compatible with the cyclic order uniquely determines a factorization

$$\langle n \rangle \rightarrow \{[n]\} \rightarrow \{[n_i]\}.$$

We can therefore choose a linear order on P and define Y to be the object

$$\{1\} \subset \{1\} \xleftarrow{f_Z} T.$$

Then take (ϕ_Y, γ_Y) to be the unique morphism yielding a factorization

$$\mathcal{L}(\phi, \gamma) : D(f_Z^{-1}(\diamond)) \rightarrow K(f_Z^{-1}(\diamond)) \xrightarrow{(\phi_Y, \gamma_Y)} \{[m_i]\}_{i \in P}$$

We can then construct $X_{M,Y}$ as in case 1. It is immediate that

$$\begin{array}{ccccc} \{1\} & \subseteq & \diamond & \xleftarrow{f_Z} & T \\ & & \uparrow & & \downarrow \text{id}_T \\ \{1\} & \subseteq & \{1\} & \xleftarrow{f_Z} & T \\ & & g \uparrow & & \downarrow \bar{g} \\ P & \subseteq & W_Y & \xleftarrow{f_{M,Y}} & R_Y \end{array}$$

defines a morphism Φ in Ω covering (ϕ, γ) .

Now suppose given any other morphism $\Psi = (\psi, \bar{\psi}) : Z \rightarrow X$ covering (ϕ, γ) . A choice of linear order on $\psi^{-1}(\diamond)$ compatible with the chosen linear order on P uniquely factors Ψ through Y . We therefore get a morphism $\tau : X_{M,Y} \rightarrow X$ such that the diagram

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & & \searrow \Psi \\ X_{M,Y} & \xrightarrow{\tau} & X \end{array}$$

commutes.

To see that this morphism is unique, suppose that $(\xi, \bar{\xi}), (\zeta, \bar{\zeta}) : X_{M,Y} \rightarrow X$ are two such morphisms. Then, choosing a linear order on $\psi^{-1}(\diamond)$ compatible with the chosen linear order on P uniquely factors the diagram as

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & \downarrow Y & \searrow \Psi \\ X_{M,Y} & \xrightarrow{(\xi, \bar{\xi}), (\zeta, \bar{\zeta})} & X \end{array}$$

But, by case 1, there is a unique morphism making the bottom triangle commute. Therefore, $(\xi, \bar{\xi}) = (\zeta, \bar{\zeta})$, proving the proposition. $\square$

In the second case, that of $M = \langle m \rangle$, the computation is somewhat simpler.

Proposition B.3 *Suppose given an object $M = \langle m \rangle$ in Λ^* , an object*

$$Z := \left\{ Q \subset S \xleftarrow{f_Z} T \right\}$$

in Ω , and a morphism

$$(\phi, \{\gamma_i\}_{i \in Q}) : \mathcal{L}(Z) \rightarrow M$$

in Λ^ . Then there is an element $X_{M,Z}$ in Ω_M^E and a morphism $\Phi : Z \rightarrow X_{M,Z}$ in Ω covering $(\phi, \{\gamma_i\}_{i \in Q})$ such that, for any other morphism $\Psi : Z \rightarrow X$ covering $(\phi, \{\gamma_i\}_{i \in Q})$, there is a unique morphism $\tau : X_{M,Z} \rightarrow X$ which makes the diagram*

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & & \searrow \Psi \\ X_{M,Z} & \xrightarrow{\tau} & X \end{array}$$

commute.

Proof We first note that $S = \diamond$, since otherwise no such morphism $(\phi, \{\gamma_i\}_{i \in Q})$ can exist. Consequently, $\phi = \text{id}_\diamond$, and γ is a morphism of cyclically ordered sets $\langle m \rangle \rightarrow D(f_Z^{-1}(\diamond))$. We can therefore take $X_{Z,M}$ to be the object

$$\{1\} \subseteq \diamond \xleftarrow{f_M} D(\langle m \rangle)$$

constructed in the proof of Proposition B.1. We then get a commutative diagram

$$\begin{array}{ccc} \{1\} & \subseteq & \diamond \xleftarrow{f_Z} T \\ & \text{id} \uparrow & \downarrow \bar{g} \\ \{1\} & \subseteq & \diamond \xleftarrow{f_M} D(\langle m \rangle) \sqcup (T \setminus f_Z^{-1}(\diamond)) \end{array}$$

where $\bar{g}$ acts as $D(\gamma)$ on $f_Z^{-1}(\diamond)$ and the identity on $T \setminus f_Z^{-1}(\diamond)$. This morphism in Ω clearly covers (id, γ) .

Given $X \in \Omega_M^E$ and $\Psi : Z \rightarrow X$, represented by a diagram

$$\begin{array}{ccc} \{1\} & \subseteq & \diamond \xleftarrow{f_Z} T \\ & \text{id} \uparrow & \downarrow \bar{\ell} \\ \{1\} & \subseteq & \diamond \xleftarrow{f_X} A \end{array}$$

by B.1 that there is a unique morphism

$$\begin{array}{ccc} \{1\} & \subseteq & \diamond \xleftarrow{f_M} D(\langle m \rangle) \\ & \text{id} \uparrow & \downarrow \bar{h} \\ \{1\} & \subseteq & \diamond \xleftarrow{f_X} A \end{array}$$

in Ω_M^E . Via the restriction of $\bar{\ell}$ to $T \setminus f_Z^{-1}(\diamond)$, this extends to a morphism

$$\begin{array}{ccc} \{1\} & \subseteq & \diamond \xleftarrow{f_M} D(\langle m \rangle) \sqcup (T \setminus f_Z^{-1}(\diamond)) \\ & \text{id} \uparrow & \downarrow \bar{\xi} \\ \{1\} & \subseteq & \diamond \xleftarrow{f_X} A \end{array}$$

in Ω_M^E .

Since all of the left-hand vertical morphisms are required to be identities, we only need to check that $\bar{\xi} \circ \bar{g} = \bar{\ell}$, which is true by construction. The requirement that $\bar{\xi}$ define a morphism in Ω_M^E uniquely determines $\bar{\xi}$ on $D(\langle m \rangle)$ and the requirement that $\bar{\xi} \circ \bar{g} = \bar{\ell}$ uniquely determines $\bar{\xi}$ on $T \setminus f_Z^{-1}(\diamond)$. $\square$

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