J. DIFFERENTIAL GEOMETRY
119 (2021) 161-182

SMOOTHLY BOUNDED DOMAINS COVERING FINITE VOLUME MANIFOLDS

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Abstract

In this paper we prove: if a bounded domain with C^2 boundary covers a manifold which has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then the domain is biholomorphic to the unit ball. This answers a question attributed to Yau. Further, when the domain is convex we can assume that the boundary only has $C^{1,\epsilon}$ regularity.

1. Introduction

Given a domain $\Omega \subset \mathbb{C}^d$ let $\operatorname{Aut}(\Omega)$ denote the biholomorphism group of Ω. When Ω is bounded, H. Cartan proved that $\operatorname{Aut}(\Omega)$ is a Lie group (with possibly infinitely many connected components) and acts properly on Ω.

An old theorem of Wong and Rosay [**28**, **27**] states that if $\Omega \subset \mathbb{C}^d$ is a bounded domain with C^2 boundary and $\operatorname{Aut}(\Omega)$ acts co-compactly on Ω, then Ω is biholomorphic to the unit ball. The following conjecture has been attributed to Yau (see [**28**, p. 257] or [**21**, Conjecture 1.15]).

Conjecture 1.1 (Yau). Let $\Omega \subset \mathbb{C}^d$ $(d \geq 2)$ be a bounded pseudoconvex domain whose boundary is C^2. Assume that Ω has a (open) quotient of finite-volume (in the sense of Kähler-Einstein volume). Then Ω is biholomorphic to the unit ball in $\mathbb{C}^d$.

Considering bounded domains that cover finite volume non-compact manifolds seems more natural than studying those that cover compact manifolds. For instance, it is well known that $\mathcal{T}_g$, the Teichmüller space of hyperbolic surfaces with genus g, is biholomorphic to a bounded domain and has a finite volume quotient but not a compact quotient. Further, Griffiths constructed the following examples.

Key words and phrases. Rigidity, finite volume complex manifolds, Bergman metric, Kähler-Einstein metric, Kobayashi metric, biholomorphism group.

Received March 12, 2018.

Theorem 1.2 ([**12**, Theorem I, Proposition 8.12])**.** *Suppose V is an irreducible, smooth, quasi-projective algebraic variety over the complex numbers. For any $x \in V$ there exists a Zariski neighborhood U of x such that $\widetilde{U}$, the universal cover of U, is biholomorphic to a bounded pseudoconvex domain in $\mathbb{C}^d$. Moreover, the Kobayashi-Eisenman volume of U is finite.*

In this paper we answer Yau's question:

Theorem 1.3. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^2 boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω. If $\Gamma\backslash\Omega$ has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then Ω is biholomorphic to the unit ball.*

Remark 1.4. Recently, Liu and Wu [**21**], see Theorem 1.12 below, established the above theorem with the additional assumptions that

1) $d = 2$ and Ω is convex, or
2) $d > 2$, Ω is convex, and Γ is irreducible.

It is well known that Teichmüller spaces admit finite volume quotients and so Theorem 1.3 provides a new proof of the following result.

Corollary 1.5 (Yau [**29**, p. 328])**.** *Let $\mathcal{T}_g$ denote the Teichmüller space of hyperbolic surfaces with genus g. If $g \geq 2$, then $\mathcal{T}_g$ is not biholomorphic to a bounded pseudoconvex domain with C^2 boundary.*

Remark 1.6.

1) A theorem of Bers [**2**] says that $\mathcal{T}_g$ is biholomorphic to a bounded domain.
2) Recently, Gupta and Seshadri [**13**, Theorem 1.2] provided a new proof of Corollary 1.5 which relies on the ergodicity of the Teichmüller geodesic flow.

If $\Omega \subset \mathbb{C}^d$ is a bounded domain, $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω, and $\Gamma\backslash\Omega$ is a quasi-projective variety, then a result of Griffiths implies that $\Gamma\backslash\Omega$ has finite volume with respect to the Kobayashi-Eisenman volume (see Proposition 8.12 and the discussion following Question 8.13 in [**12**]). So we have the following corollary of Theorem 1.3.

Corollary 1.7. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^2 boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω. If $\Gamma\backslash\Omega$ is a quasi-projective variety, then Ω is biholomorphic to the unit ball.*

The proof of Theorem 1.3 uses the Levi form of the boundary and hence does not easily generalize to domains whose boundaries have less than C^2 regularity. However, by assuming our domain is convex we can lower the required regularity to $C^{1,\epsilon}$ for any $\epsilon > 0$.

Theorem 1.8. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{1,\epsilon}$ boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω. If $\Gamma \backslash \Omega$ has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then Ω is biholomorphic to the unit ball.*

REMARK 1.9.

1) The proof will use a recent result of Liu and Wu [**21**], see Theorem 1.12 below, and a result in [**31**], see Theorem 1.14 below.
2) It is conjectured that a bounded convex domain with a finite volume quotient (with no assumptions on the regularity of $\partial\Omega$) must be a bounded symmetric domain, see for instance [**21**, Conjecture 1.12].
3) Using Theorem 1.8, the hypothesis of Corollary 1.7 can be modified to assume that Ω is a bounded convex domain with $C^{1,\epsilon}$ boundary. Theorem 1.8 also can be used to show that $\mathcal{T}_g$ ($g \geq 2$) is not biholomorphic to a convex domain with $C^{1,\epsilon}$ boundary, however, a recent result of Markovic [**22**] implies that $\mathcal{T}_g$ is not biholomorphic any convex domain when $g \geq 2$ (with no regularity assumptions on the boundary of the convex domain).

1.1. Outline of the proofs. We will use a theorem of Wong and Rosay to prove Theorem 1.3.

Theorem 1.10 (Wong-Rosay Ball Theorem [**28**, **27**])**.** *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain. Assume that $\partial\Omega$ is C^2 and strongly pseudoconvex in a neighborhood of $\xi \in \partial\Omega$. If there exists some $z_0 \in \Omega$ and a sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(z_0) \to \xi$, then Ω is biholomorphic to the unit ball.*

When Ω is a bounded domain with C^2 boundary, then there exists some $\xi \in \partial\Omega$ which is strongly pseudoconvex (see Observation 4.1 below). If $\mathrm{Aut}(\Omega)$ acts co-compactly on Ω then it is easy to show that there exists some $z_0 \in \Omega$ and a sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(z_0) \to \xi$. So one has the following Corollary to Theorem 1.10:

Corollary 1.11. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with C^2 boundary. If $\mathrm{Aut}(\Omega)$ acts co-compactly on Ω, then Ω is biholomorphic to the unit ball.*

In the case when Ω only admits a finite volume quotient, finding $z_0 \in \Omega$ and a sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(z_0)$ converges to a certain boundary point $\xi \in \partial\Omega$ is much harder. We accomplish this task by considering the behavior of the Bergman distance and in particular the shape of horospheres near a strongly pseudoconvex point. The squeezing function also plays an important role in understanding the complex geometry of Ω.

For convex domains, there are precise estimates for the Kobayashi distance and so in the proof of Theorem 1.8 we consider horospheres with respect to the Kobayashi distance (instead of the Bergman distance). Since the hypothesis of Theorem 1.8 only assumes $\partial\Omega$ has $C^{1,\epsilon}$ boundary there is no hope of using the Wong-Rosay Ball Theorem. Instead we use two recent results about the automorphism group of convex domains. Before stating these results we need a few definitions:

1) Given a bounded domain $\Omega \subset \mathbb{C}^d$ let $\mathrm{Aut}_0(\Omega)$ denote the connected component of the identity in $\mathrm{Aut}(\Omega)$.
2) When $\Omega \subset \mathbb{C}^d$ is a bounded domain, the *limit set of* Ω, denoted $\mathcal{L}(\Omega)$, is the set of points $x \in \partial\Omega$ where there exists some $z \in \Omega$ and a sequence $\varphi_n \in \mathrm{Aut}(\Omega)$ such that $\varphi_n(z) \to x$.
3) Given a convex domain $\Omega \subset \mathbb{C}^d$ with C^1 boundary and $x \in \partial\Omega$, let $T_x^{\mathbb{C}}\partial\Omega \subset \mathbb{C}^d$ be the complex affine hyperplane tangent to $\partial\Omega$ at x. Then the *closed complex face of* x *in* $\partial\Omega$ is the set $T_x^{\mathbb{C}}\partial\Omega \cap \partial\Omega$.

Liu and Wu recently proved the following rigidity result.

Theorem 1.12 (Liu-Wu [**21**]). *Suppose* $\Omega \subset \mathbb{C}^d$ *is a bounded convex domain,* $\Gamma \leq \mathrm{Aut}(\Omega)$ *is a discrete group acting freely on* Ω*, and* $\Gamma\backslash\Omega$ *has finite volume with respect to either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume. If either:*

1) $\Gamma \leq \mathrm{Aut}_0(\Omega)$,
2) $\mathrm{Aut}_0(\Omega) \neq 1$ *and* Γ *is irreducible,*
3) Ω *has* C^1 *boundary and* Γ *is irreducible,*
4) $d = 2$ *and* $\mathrm{Aut}_0(\Omega) \neq 1$*, or*
5) $d = 2$ *and* Ω *has* C^1 *boundary,*

then Ω *is biholomorphic to a bounded symmetric domain.*

Remark 1.13. By Frankel's rescaling method [**8**, **17**], part (3) is a consequence of part (2). The rescaling method also implies that part (5) is a consequence of part (4).

We recently proved the following result.

Theorem 1.14 ([**31**]). *Suppose* $\Omega \subset \mathbb{C}^d$ *is a bounded convex domain with* $C^{1,\epsilon}$ *boundary. If* $\mathcal{L}(\Omega)$ *intersects at least two different closed complex faces of* $\partial\Omega$*, then*

1) $\mathrm{Aut}(\Omega)$ *has finitely many components,*
2) *there exists a compact normal subgroup* $N \leq \mathrm{Aut}_0(\Omega)$ *such that* $\mathrm{Aut}_0(\Omega)/N$ *is a non-compact simple Lie group with real rank one.*

Hence, to prove Theorem 1.8 it is enough to show that $\mathcal{L}(\Omega)$ intersects at least two different closed complex faces of $\partial\Omega$. In that case, Theorem 1.14 implies that $\Gamma_0 := \Gamma \cap \mathrm{Aut}_0(\Omega)$ has finite index in Γ and hence $\Gamma_0\backslash\Omega$ has finite volume. Then Theorem 1.12 implies that Ω is biholomorphic to a bounded symmetric domain. Finally, Mok and

Tsai [23] proved that: if D is a bounded symmetric domain which is convex and has C^1 boundary, then D is biholomorphic to the unit ball. So Ω is biholomorphic to the ball.

Acknowledgments. I would like to thank a referee for helpful corrections and comments which improved this paper. This material is based upon work supported by the National Science Foundation under grant DMS-1760233.

2. Preliminaries

2.1. Notations. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain, then:

1) Let $k_\Omega : \Omega \times \mathbb{C}^d \to \mathbb{R}_{\geq 0}$ denote the infinitesimal Kobayashi metric, $K_\Omega : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ denote the Kobayashi distance on Ω, and Vol_K denote the Kobayashi-Eisenman volume form.
2) Let g_B denote the Bergman metric on Ω, B_Ω denote the Bergman distance on Ω, and Vol_B denote the Riemannian volume form associated to g_B. We will also let $b_\Omega : \Omega \times \mathbb{C}^d \to \mathbb{R}$ denote the norm associated to g_B, that is
$$b_\Omega(x; v) = \sqrt{g_B(v, v)},$$
when $v \in T_x\Omega$.
3) Let g_{KE} denote the Kähler-Einstein metric on Ω with Ricci curvature -1 constructed by Cheng-Yau [4] when Ω has C^2 boundary and Mok-Yau [24] in general. And let Vol_{KE} denote the Riemannian volume form associated to g_{KE}.

Throughout the paper $\|\cdot\|$ will denote the standard Euclidean norm on $\mathbb{C}^d$. Given $z_0 \in \mathbb{C}^d$ and $r > 0$ define
$$\mathbb{B}_d(z_0; r) = \{z \in \mathbb{C}^d : \|z - z_0\| < r\}.$$
Finally, given a domain $\Omega \subset \mathbb{C}^d$ and $z \in \Omega$ define
$$\delta_\Omega(z) = \inf\{\|w - z\| : w \in \partial\Omega\}.$$

2.2. The squeezing function. Given a domain $\Omega \subset \mathbb{C}^d$ let $s_\Omega : \Omega \to (0, 1]$ be the *squeezing function on* Ω, that is
$$s_\Omega(z) = \sup\{r : \text{there exists an one-to-one holomorphic map } f : \Omega \to \mathbb{B}_d(0;1) \text{ with } f(z) = 0 \text{ and } \mathbb{B}_d(0;r) \subset f(\Omega)\}.$$

In this section we recall a result of S.K. Yeung.

Theorem 2.1 ([30, Theorem 2])**.** *Suppose $s > 0$ and $d > 0$. Then there exists $C > 1$ and $\iota_0, \epsilon, \kappa > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a pseudoconvex* *domian,* $z_0 \in \Omega$*,* $s_\Omega(z_0) > s$*, and*

$$B_\epsilon := \{z \in \Omega : B_\Omega(z_0, z) \leq \epsilon\},$$

then

1) B_ϵ *is a compact subset of* Ω*,*
2) g_B*,* g_{KE}*, and* k_Ω *are all* C*-bi-Lipschitz on* B_ϵ*,*
3) *the sectional curvature of* g_B *is bounded in absolute value by* κ *on* B_ϵ*,*
4) *the injectivity radius of* g_B *is bounded below by* ι_0 *on* B_ϵ*, and*
5) *if* Vol *denotes either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume, then*

$$\frac{1}{C} r^{2d} \leq \mathrm{Vol}\Big(\{z \in \Omega : B_\Omega(z, z_0) \leq r\}\Big) \leq C r^{2d}$$

for all $r \in [0, \epsilon]$*.*

Parts (1)–(4) follow from [**30**, Theorem 2]. In [**30**, Theorem 2] it is assumed that $s_\Omega(z) > s$ for all $z \in \Omega$, however, all the arguments are local in nature and can be easily modified to prove parts (1)–(4) in the above Theorem. Part (2) also follows from the proof of [**20**, Theorem 7.2].

Part (5) is a consequence of the definition and part (2): since $s_\Omega(z_0) > s$ we can assume that $z_0 = 0$ and

$$\mathbb{B}_d(0; s) \subset \Omega \subset \mathbb{B}_d(0; 1).$$

Then

$$k_{\mathbb{B}_d(0;1)} \leq k_\Omega \leq k_{\mathbb{B}_d(0;s)}$$

on $\mathbb{B}_d(0; s)$. Then, from the well known explicit description of the Kobayashi metric on the ball and part (2), we see that there exists $C_1 > 0$ such that g_B, g_{KE}, and k_Ω are all C_1-bi-Lipschitz to the Euclidean metric on $\mathbb{B}_d(0; s/2)$. So we can find $C, \epsilon > 0$ such that: if Vol denotes either the Bergman volume or the Kähler-Einstein volume, then

$$\frac{1}{C} r^{2d} \leq \mathrm{Vol}\Big(\{z \in \Omega : B_\Omega(z, z_0) \leq r\}\Big) \leq C r^{2d}$$

for all $r \in [0, \epsilon]$. Next let Vol_K denote the Kobayashi-Eisenman volume on Ω and for $\tau > 0$ let $\mathrm{Vol}_{\mathbb{B}_d(0;\tau)}$ denote the Kobayashi-Eisenman volume on $\mathbb{B}_d(0; \tau)$. Then by definition

$$\mathrm{Vol}_{\mathbb{B}_d(0;1)}(A) \leq \mathrm{Vol}_K(A) \leq \mathrm{Vol}_{\mathbb{B}_d(0;s)}(A)$$

for all subsets $A \subset \mathbb{B}_d(0; s)$. So from the well known explicit description of the Kobayashi-Eisenman volume for the ball, part (2), and by possibly modifying C, ϵ we can also assume that

$$\frac{1}{C} r^{2d} \leq \mathrm{Vol}_K\Big(\{z \in \Omega : B_\Omega(z, z_0) \leq r\}\Big) \leq C r^{2d}$$

for all $r \in [0, \epsilon]$.

2.3. Invariant metrics near a strongly pseudoconvex point. We will use the following well known facts about invariant metrics near a strongly pseudoconvex point.

Theorem 2.2. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain. Assume that $\partial\Omega$ is C^2 and strongly pseudoconvex in a neighborhood of $\xi \in \partial\Omega$. Then there exists an neighborhood U of ξ in $\overline{\Omega}$ and $C > 1$ such that:*

1) *k_Ω and g_B are C-bi-Lipshitz to each other on $U \cap \Omega$,*
2) *$k_\Omega(x; v) \geq C^{-1} \|v\| \, \delta_\Omega(x)^{-1/2}$ for all $x \in U \cap \Omega$ and $v \in \mathbb{C}^d$, and*
3) *g_B has negative sectional curvature on $U \cap \Omega$.*

Proof. Fix open neighborhoods $V_2 \Subset V_1$ of ξ such that there exist a holomorphic embedding $\varphi : V_1 \to \mathbb{C}^d$ with $\varphi(V_2 \cap \Omega)$ a convex domain which is strongly convex near $\varphi(\xi)$.

By [**7**, Theorem 2.1] there exists a neighborhood V_3 of ξ such that $V_3 \Subset V_2$ and

$$k_\Omega(x; v) \leq k_{V_2 \cap \Omega}(x; v) \leq 2 k_\Omega(x; v)$$

for all $x \in V_3$ and $v \in \mathbb{C}^d$ (notice that the first inequality is by definition). Further, by [**5**, Theorem 1] there exists $C_0 > 1$ such that

$$\frac{1}{C_0} b_{\Omega \cap V_2}(x; v) \leq b_\Omega(x; v) \leq C_0 b_{\Omega \cap V_2}(x; v)$$

for all $x \in V_3$ and $v \in \mathbb{C}^d$.

Now since $V_2 \cap \Omega$ is biholomorphic to a convex domain, a result of Frankel [**9**] implies that $b_{\Omega \cap V_2}$ and $k_{\Omega \cap V_2}$ are C_1-bi-Lipschitz to each other for some $C_1 > 1$. So we see that k_Ω and g_B are C-bi-Lipshitz to each other on $V_3 \cap \Omega$ for some $C > 1$.

Given a domain $\mathcal{O} \subset \mathbb{C}^d$, $x \in \mathcal{O}$, and nonzero $v \in \mathbb{C}^d$ define

$$\delta_{\mathcal{O}}(x; v) = \inf\{\|y - x\| : y \in \partial\Omega \cap (x + \mathbb{C}\, v)\}.$$

Since $\mathcal{C} = \varphi(V_2 \cap \Omega)$ is convex, a result of Graham [**10**, **11**] says that

$$\frac{\|v\|}{2\delta_{\mathcal{C}}(x; v)} \leq k_{\mathcal{C}}(x; v) \leq \frac{\|v\|}{\delta_{\mathcal{C}}(x; v)}$$

for all $x \in \mathcal{C}$ and $v \in \mathbb{C}^d$. Then, since $\mathcal{C}$ is strongly convex at $\varphi(\xi)$, there exists a neighborhood W of $\varphi(\xi)$ and some $C_2 > 0$ such that

$$C_2 \frac{\|v\|}{\delta_{\mathcal{C}}(x)^{1/2}} \leq k_{\mathcal{C}}(x; v)$$

for all $x \in W$ and $v \in \mathbb{C}^d$. Since $V_2 \Subset V_1$, the map $\varphi : V_1 \to \mathbb{C}^d$ is bi-Lipschitz on V_2, so by possibly shrinking V_3 and increasing C we can assume that

$$\frac{1}{C}\frac{\|v\|}{\delta_\Omega(x)^{1/2}} \leq k_\Omega(x; v)$$

for all $x \in V_3$ and $v \in \mathbb{C}^d$.

Finally, part (3) follows from [**18**, Theorem 1]. q.e.d.

2.4. Completeness of the Bergman metric. We will use the following fact about the Bergman metric:

Theorem 2.3 (Ohsawa [**26**]). *If $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^1 boundary, then the Bergman metric is a complete Riemannian metric on Ω.*

Remark 2.4. It is also known that the Bergman metric is complete on the more general class of hyperconvex domains, see [**15**] and [**3**].

2.5. A local version of E. Cartan's fixed point theorem. E. Cartan proved that a compact group G acting by isometries on (X, g) a complete simply connected Riemannian manifold with non-positive sectional curvature always has a fixed point. One proof, see for instance [**6**, p. 21], uses the following lemma: if $K \subset X$ is compact, then the function

$$f(x) = \sup\{d(x, k) : k \in K\}$$

has a unique minimum in X. In this section we observe a local version of this lemma which will allow us to show that a certain compact subgroup has a fixed point in the proof of Theorem 1.3.

Given a complete Riemannian manifold (X, g), $x_0 \in X$, and $R > 0$ let $B_{(X,g)}(x_0, R)$ denote the open metric ball of radius R centered at x_0.

Proposition 2.5. *Suppose (X, g) is a complete Riemannian manifold, $x_0 \in X$, $R > 0$, the metric g has non-positive sectional curvature on $B_{(X,g)}(x_0, 8R)$, and g has injectivity radius at least $16R$ at each point in $B_{(X,g)}(x_0, 8R)$. If $K \subset B_{(X,g)}(x_0, R)$ is compact, then the function*

$$f(x) = \sup\{d(x, k) : k \in K\}$$

has a unique minimum in X.

The following proof is nearly identical to the proof of the Lemma on p. 21 in [**6**], but we provide the details for the reader's convenience.

Proof. When $\alpha \in [0, 4]$, every two points in $B_{(X,g)}(x_0, \alpha R)$ are joined by a unique geodesic and this geodesic is contained in $B_{(X,g)}(x_0, 2\alpha R)$.

Since g is non-positively curved on $B_{(X,g)}(x_0, 8R)$ and has injectivity radius at least $16R$ at each point in $B_{(X,g)}(x_0, 8R)$ the Rauch comparison theorem implies (see [**14**, p. 73]): if $\mathcal{T}$ is a geodesic triangle contained in $B_{(X,g)}(x_0, 4R)$ with side lengths a, b, c then

$$a^2 + b^2 - 2ab\cos\theta \leq c^2, \tag{1}$$

where θ is the angle at the vertex opposite to the side of length c.

Since f is a proper continuous function there exists at least one minimum. Since $f(x) > R$ when $x \in X \setminus B_{(X,g)}(x_0, 2R)$ and $f(x_0) \leq R$ any minimum of f is in $B_{(X,g)}(x_0, 2R)$.

Suppose for a contradiction that there exists two distinct minimum points x, y of f. Let $\sigma : [0, T] \to X$ denote the unique geodesic with $\sigma(0) = x$ and $\sigma(T) = y$. Let $m = \sigma(T/2)$. Then consider some $k \in K$ and let $\gamma : [0, S] \to X$ denote the unique geodesic in X with $\gamma(0) = m$ and $\gamma(S) = k$. Since

$$\angle_m(-\sigma'(T/2), \gamma'(0)) + \angle_m(\sigma'(T/2), \gamma'(0)) = \pi,$$

by relabelling x, y we can assume that $\theta := \angle_m(\sigma'(T/2), \gamma'(0)) \geq \pi/2$. Then Equation (1) implies that

$$d(y, k)^2 \geq d(y, m)^2 + d(m, k)^2 - 2d(y, m)d(m, k)\cos\theta > d(m, k)^2.$$

So $d(m, k) < d(y, k) \leq f(y)$. Since $k \in K$ was arbitrary and K is compact, we then have $f(m) < f(y)$ which is a contradiction. q.e.d.

3. An estimate for the Bergman distance

Theorem 3.1. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain. Assume that $\partial\Omega$ is C^2 and strongly pseudoconvex in a neighborhood of $\xi \in \partial\Omega$. If $z_0 \in \Omega$ and $\epsilon_0 > 0$, then there exists $\epsilon \in (0, \epsilon_0)$ and $R > 0$ such that*

$$B_\Omega(z, w) \geq B_\Omega(z, z_0) + B_\Omega(z_0, w) - R$$

for all $z, w \in \Omega$ with $\|z - \xi\| < \epsilon$ and $\|w - \xi\| > 2\epsilon$.

Remark 3.2. This says that if z is near ξ and w is far away from ξ, then z and w can be joined by a path that passes through z_0 and is length minimizing up to an error of R.

The following argument is based on the proof of [**16**, Lemma 36] which establishes a similar estimate for the Kobayashi distance.

Proof. By Theorem 2.2 there exists an neighborhood U of ξ and some $C > 1$ such that

$$\frac{1}{C} k_\Omega(x; v) \leq b_\Omega(x; v) \leq C k_\Omega(x; v)$$

and

$$\frac{1}{C} \frac{\|v\|}{\delta_\Omega(x)^{1/2}} \leq b_\Omega(x; v) \tag{2}$$

for all $x \in \Omega \cap U$ and $v \in \mathbb{C}^d$.

By definition

$$k_\Omega(x;v) \leq \frac{\|v\|}{\delta_\Omega(x)}$$

and so

$$b_\Omega(x;v) \leq C\frac{\|v\|}{\delta_\Omega(x)}$$

for $x \in U \cap \Omega$ and $v \in \mathbb{C}^d$. Then since $\partial\Omega$ is C^2 near ξ one can consider parametrizations of inward pointing normal lines to show that there exists $\alpha, \beta > 0$ and a neighborhood $V \subset U$ of ξ such that

$$B_\Omega(z_0, z) \leq \alpha + \beta \log \frac{1}{\delta_\Omega(z)} \tag{3}$$

for all $z \in V \cap \Omega$.

Now fix $\epsilon \in (0, \epsilon_0)$ such that

$$\mathbb{B}_d(\xi; 2\epsilon) \subset V.$$

Consider points $z, w \in \Omega$ with $\|z - \xi\| < \epsilon$ and $\|w - \xi\| > 2\epsilon$. Let $\sigma : [0, T] \to \Omega$ be a geodesic (with respect to the Bergman distance) joining z and w. Define

$$T_0 = \max\left\{t \in [0,T] : \sigma([0,t]) \subset \overline{\mathbb{B}_d(z;\epsilon)}\right\}.$$

Since $\|w - z\| > \|w - \xi\| - \|z - \xi\| > \epsilon$, we must have

$$\|z - \sigma(T_0)\| = \epsilon. \tag{4}$$

Further,

$$\sigma([0,T_0]) \subset \overline{\mathbb{B}_d(z;\epsilon)} \subset \mathbb{B}_d(\xi; 2\epsilon) \subset V.$$

Then let $\tau \in [0, T_0]$ be such that

$$\delta_\Omega(\sigma(\tau)) = \max\{\delta_\Omega(\sigma(t)) : t \in [0, T_0]\}. \tag{5}$$

Now for $t \in [0, T_0]$ we have

$$\begin{aligned} |t - \tau| = B_\Omega(\sigma(t), \sigma(\tau)) &\leq B_\Omega(\sigma(t), z_0) + B_\Omega(z_0, \sigma(\tau)) \\ &\leq 2\alpha + \beta \log \frac{1}{\delta_\Omega(\sigma(t))\delta_\Omega(\sigma(\tau))}. \end{aligned}$$

So

$$\delta_\Omega(\sigma(t)) \leq \sqrt{\delta_\Omega(\sigma(t))\delta_\Omega(\sigma(\tau))} \leq \exp\left(\frac{-|t-\tau| + 2\alpha}{2\beta}\right).$$

Now fix $M > 0$ such that

$$\int_M^\infty \exp\left(\frac{-r + 2\alpha}{4\beta}\right) dr < \epsilon/(4C). \tag{6}$$

Since σ is a geodesic $b_\Omega(\sigma(t);\sigma'(t)) \equiv 1$. Then by Equations (4) and (2)

$$\begin{aligned}\epsilon = \|z-\sigma(T_0)\| = \|\sigma(0)-\sigma(T_0)\| &\leq \int_0^{T_0} \left\|\sigma'(t)\right\| dt \\ \leq C\int_0^{T_0} b_\Omega(\sigma(t);\sigma'(t))\delta_\Omega(\sigma(t))^{1/2}dt &= C\int_0^{T_0} \delta_\Omega(\sigma(t))^{1/2}dt.\end{aligned}$$

Then by Equations (5) and (6)

$$\begin{aligned}\epsilon &\leq C\int_{[0,T_0]\cap(\tau-M,\tau+M)} \delta_\Omega(\sigma(t))^{1/2}dt + C\int_{[0,T_0]\cap(\tau-M,\tau+M)^c} \delta_\Omega(\sigma(t))^{1/2}dt \\ &\leq C\int_{[0,T_0]\cap(\tau-M,\tau+M)} \delta_\Omega(\sigma(\tau))^{1/2}dt + 2C\int_M^\infty \exp\left(\frac{-r+2\alpha}{4\beta}\right)dr \\ &\leq 2CM\delta_\Omega(\sigma(\tau))^{1/2} + \epsilon/2.\end{aligned}$$

So

$$\delta_\Omega(\sigma(\tau))^{1/2} \geq \epsilon/(4CM).$$

Then by Equation (3)

$$\begin{aligned}B_\Omega(z,w) &= B_\Omega(z,\sigma(\tau)) + B_\Omega(\sigma(\tau),w) \\ &\geq B_\Omega(z,z_0) + B_\Omega(z_0,w) - 2B_\Omega(z_0,\sigma(\tau)) \\ &\geq B_\Omega(z,z_0) + B_\Omega(z_0,w) - R,\end{aligned}$$

where

$$R = 2\alpha + 4\beta \log\frac{4CM}{\epsilon}.$$

Notice that R does not depend on z or w, so the proof is complete.

q.e.d.

4. Proof of Theorem 1.3

For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with C^2 boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω. Further, assume that $\mathrm{Vol}(\Gamma\backslash\Omega) < +\infty$ where Vol is either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume.

Given a Lebesgue measurable set $A \subset \Omega$, we will also let $\widetilde{\mathrm{Vol}}(A)$ denote the volume relative to the associated measure on Ω. Notice, that if $\pi : \Omega \to \Gamma\backslash\Omega$ is the natural covering map and $\pi|_A$ is injective, then

$$\widetilde{\mathrm{Vol}}(A) = \mathrm{Vol}(\pi(A)). \tag{7}$$

By translating and scaling Ω, we may assume that $0 \in \Omega$, $\Omega \subset \mathbb{B}_d(0;1)$, and $\partial\Omega \cap \partial\,\mathbb{B}_d(0;1) \neq \emptyset$. Then by rotating Ω we may assume that $(1,0,\dots,0) \in \partial\Omega$. Then, since $\partial\Omega$ is C^2, there exists some

$r \in (0,1)$ such that

$$\mathbb{B}_d((r,0,\dots,0);1-r) \subset \Omega \subset \mathbb{B}_d(0;1).$$

Observation 4.1. $\xi = (1,0,\dots,0)$ is a strongly pseudoconvex point of $\partial\Omega$.

Proof. This is a consequence of the fact that $(1,0,\dots,0) \in \partial\Omega$ and $\Omega \subset \mathbb{B}_d(0;1)$, see for instance [**13**, Lemma 4.1]. q.e.d.

Observation 4.2. Let $w_t = (t,0,\dots,0) \in \mathbb{C}^d$. Then there exists some $s_0 > 0$ such that $s_\Omega(w_t) \geq s_0$ for $t \in [r,1)$.

Proof. For $t \in [r,1)$ consider the transformation

$$\varphi(z_1,\dots,z_d) = \left(\frac{z_1 - t}{tz_1 - 1}, \frac{(1-t^2)^{1/2}}{tz_1 - 1} z_2, \dots, \frac{(1-t^2)^{1/2}}{tz_1 - 1} z_d \right).$$

Then $\varphi \in \mathrm{Aut}(\mathbb{B}_d(0;1))$ and $\varphi(0) = w_t$. We claim that

$$\varphi(\mathbb{B}_d(0;s_0)) \subset \Omega,$$

where

$$s_0 = \frac{1-r}{12\sqrt{d}}.$$

Suppose $z \in \mathbb{B}_d(0;s_0)$. Then $\|z\| < 1/2$ and so

$$|tz_1 - 1| > 1/2.$$

Then

$$\begin{aligned}
\left| \frac{z_1 - t}{tz_1 - 1} - r \right|^2 &= \left| (t-r) + \frac{(1-t^2)z_1}{tz_1 - 1} \right|^2 \\
&\leq (t-r)^2 + \frac{2(t-r)(1-t^2)}{|tz_1 - 1|} |z_1| + \frac{(1-t^2)^2}{|tz_1-1|^2} |z_1|^2 \\
&\leq (t-r)^2 + 8(1-t)|z_1| + 8(1-t)|z_1|^2 \\
&\leq (t-r)^2 + 8(1-t)|z_1| + 4(1-t)|z_1| \\
&\leq (t-r)^2 + 12(1-t)|z_1|.
\end{aligned}$$

We also have

$$(t-r)^2 - (1-r)^2 = (2r-1-t)(1-t) \leq (r-1)(1-t)$$

and so

$$\left| \frac{z_1 - t}{tz_1 - 1} - r \right|^2 \leq (1-r)^2 + (r-1)(1-t) + 12(1-t)|z_1|.$$

Further,

$$\left| \frac{(1-t^2)^{1/2}}{tz_1 - 1} z_i \right|^2 \leq 8(1-t)|z_i|^2 \leq 4(1-t)|z_i|.$$

So

$$\begin{aligned} \|\varphi(z) - w_r\|^2 &\leq (1-r)^2 + (r-1)(1-t) + 12(1-t)(|z_1| + \cdots + |z_d|) \\ &\leq (1-r)^2 + (r-1)(1-t) + 12\sqrt{d}(1-t)\,\|z\| \\ &< (1-r)^2 \end{aligned}$$

by our choice of s_0. So $\varphi(z) \in \mathbb{B}_d(w_r; 1-r) \subset \Omega$. Since $z \in \mathbb{B}_d(0; s_0)$ was arbitrary, we then have

$$\varphi(\mathbb{B}_d(0; s_0)) \subset \Omega.$$

Then $\varphi^{-1}(w_t) = 0$ and

$$\mathbb{B}_d(0; s_0) \subset \varphi^{-1}(\Omega) \subset \mathbb{B}_d(0; 1),$$

so $s_\Omega(w_t) \geq s_0$. q.e.d.

Fix a sequence $r_n \nearrow 1$ and consider the points $y_n = (r_n, 0, \dots, 0) \in \Omega$. Then by Observation 4.2 and Theorem 2.1, there exists some $C_0, \epsilon_0 > 0$ such that

$$\widetilde{\mathrm{Vol}}\left(\{z \in \Omega : B_\Omega(z, y_n) < \epsilon\}\right) \geq C_0 \epsilon^{2d} \tag{8}$$

for any $n \geq 0$ and $\epsilon \in (0, \epsilon_0]$.

For each $n \geq 0$, define

$$\delta_n = \min_{\gamma \in \Gamma \setminus \{1\}} B_\Omega(y_n, \gamma y_n).$$

Then the quotient map $\pi : \Omega \to \Gamma\backslash\Omega$ restricts to an embedding on

$$B_n = \{z \in \Omega : B_\Omega(z, y_n) < \delta_n/2\}.$$

So by Equations (7) and (8)

$$\mathrm{Vol}(\pi(B_n)) = \widetilde{\mathrm{Vol}}(B_n) \geq C_0 \min\left\{\epsilon_0^{2d}, (\delta_n/2)^{2d}\right\}.$$

After passing to a subsequence we can assume that

$$\lim_{n\to\infty} \delta_n = \delta \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We will consider two cases depending on whether δ is zero or not. Informally, $\delta > 0$ means that the sequence $\pi(y_n)$ is trapped in the "thick part" of the quotient space $\Gamma\backslash\Omega$ while $\delta = 0$ means that $\pi(y_n)$ escapes to infinity along a cusp in $\Gamma\backslash\Omega$. The second case is the more difficult one and is itself divided into two cases. Very informally: In Case 2 (a), one obtains a sequence of "parabolic automorphisms" whose fixed points at infinity converge to ξ while in Case 2 (b) one finds a "parabolic automorphism" whose fixed point is exactly ξ.

Case 1: $\delta \neq 0$. Let $r = \min\{\epsilon_0, \delta/4\}$ and define

$$B_n' := \{z \in \Omega : B_\Omega(z, y_n) < r\}.$$

Then there exists $N \geq 0$ such that $B_n' \subset B_n$ for all $n \geq N$. Further, by Equations (7) and (8)

$$\mathrm{Vol}(\pi(B_n')) = \widetilde{\mathrm{Vol}}(B_n') \geq C_0 r^{2d},$$

when $n \geq N$.

Let $M := \Gamma\backslash\Omega$ and let d be the distance on M making the covering map $(\Omega, B_\Omega) \to (M, d)$ a local isometry. We claim that the set $\{\pi(y_n) : n \in \mathbb{N}\}$ is relatively compact in M. Suppose not. Then we construct an increasing sequence

$$n_1 < n_2 < n_3 < \dots$$

such that

$$\min_{1 \leq k < j} d\Big(\pi(y_{n_k}), \pi(y_{n_j})\Big) > 2r$$

for every $j \geq 2$. Since $(\Omega, B_\Omega) \to (M, d)$ is 1-Lipschitz, we have

$$\pi(B_{n_j}') \subset \{z \in M : d(z, \pi(y_{n_j})) < r\}.$$

Thus the sets

$$\pi(B_{n_1}'), \pi(B_{n_2}'), \pi(B_{n_3}'), \dots$$

are pairwise disjoint. So

$$\mathrm{Vol}(\Gamma\backslash\Omega) \geq \sum_{j=1}^{\infty} \mathrm{Vol}(\pi(B_n')) \geq \sum_{j=1}^{\infty} C r^{2d} = \infty,$$

which is a contradiction. Hence, the set $\{\pi(y_n) : n \in \mathbb{N}\}$ must be relatively compact in $\Gamma\backslash\Omega$.

Then for each n, there exist some $\gamma_n \in \Gamma$ such that the set $\{\gamma_n y_n : n \in \mathbb{N}\}$ is relatively compact in Ω. Then we can pass to a subsequence such that $\gamma_n y_n \to y \in \Omega$. Using Montel's theorem and passing to a subsequence we can assume that γ_n^{-1} converges locally uniformly to a holomorphic map $f : \Omega \to \overline{\Omega}$. Then

$$\xi = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \gamma_n^{-1}(\gamma_n y_n) = f(y) = \lim_{n\to\infty} \gamma_n^{-1} y.$$

So Ω is biholomorphic to the ball by Theorem 1.10.

Case 2: $\delta = 0$. For each n select $\gamma_n \in \Gamma$ such that

$$B_\Omega(\gamma_n y_n, y_n) = \delta_n.$$

Case 2(a): The set $\{\gamma_1, \gamma_2, \dots\}$ is infinite. Since Γ is discrete, by passing to a subsequence we can suppose that $\gamma_n \to \infty$ in $\mathrm{Aut}(\Omega)$. Fix some $z_0 \in \Omega$. Using the fact that $\mathrm{Aut}(\Omega)$ acts properly on Ω and passing to another subsequence we can assume that $\gamma_n^{-1} z_0 \to \eta \in \partial\Omega$. Since (Ω, B_Ω) is a proper metric space we must have

$$\lim_{n\to\infty} B_\Omega(\gamma_n^{-1} z_0, z_0) = \infty.$$

We claim that $\eta = \xi$. Suppose not, then by Theorem 3.1 there exists $R > 0$ such that

$$B_\Omega(\gamma_n^{-1} z_0, z_0) + B_\Omega(z_0, y_n) - B_\Omega(\gamma_n^{-1} z_0, y_n) \leq R$$

for every $n \geq 0$. However,

$$\begin{aligned} B_\Omega(\gamma_n^{-1} z_0, z_0) &+ B_\Omega(z_0, y_n) - B_\Omega(\gamma_n^{-1} z_0, y_n) \\ &= B_\Omega(\gamma_n^{-1} z_0, z_0) + B_\Omega(z_0, y_n) - B_\Omega(z_0, \gamma_n y_n) \\ &\geq B_\Omega(\gamma_n^{-1} z_0, z_0) - B_\Omega(\gamma_n y_n, y_n) \\ &= B_\Omega(\gamma_n^{-1} z_0, z_0) - \delta_n. \end{aligned}$$

So

$$\begin{aligned} R &\geq \limsup_{n\to\infty} B_\Omega(\gamma_n^{-1} z_0, z_0) + B_\Omega(z_0, y_n) - B_\Omega(\gamma_n^{-1} z_0, y_n) \\ &= \limsup_{n\to\infty} B_\Omega(\gamma_n^{-1} z_0, z_0) = \infty. \end{aligned}$$

So we have a contradiction and hence $\xi = \eta$. So Ω is biholomorphic to the unit ball by Theorem 1.10.

Case 2(b): The set $\{\gamma_1, \gamma_2, \dots\}$ is finite. By passing to a subsequence we can suppose that $\gamma_n = \gamma$ in for all $n \in \mathbb{N}$. Fix some $z_0 \in \Omega$ and consider the functions

$$b_n(z) = B_\Omega(z, y_n) - B_\Omega(y_n, z_0).$$

Since $b_n(z_0) = 0$ and each b_n is 1-Lipschitz (with respect to the Bergman distance) we can pass to a subsequence such that $b_n \to b$ locally uniformly. Then

$$\begin{aligned} b(\gamma^{-1} z) &= \lim_{n\to\infty} \left[B_\Omega(\gamma^{-1} z, y_n) - B_\Omega(y_n, z_0)\right] \\ &= \lim_{n\to\infty} \left[B_\Omega(z, \gamma y_n) - B_\Omega(y_n, z_0)\right] = b(z) \end{aligned}$$

since

$$\limsup_{n\to\infty} |B_\Omega(z, \gamma y_n) - B_\Omega(z, y_n)| \leq \limsup_{n\to\infty} B_\Omega(y_n, \gamma y_n) = \lim_{n\to\infty} \delta_n = 0.$$

So

$$b(\gamma^{-n} z_0) = b(z_0) = 0$$

for all $n \in \mathbb{N}$.

Observation 4.3. For any $t_0 \in \mathbb{R}$

$$\overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\mathrm{Euc}} \cap \partial\Omega = \{\xi\}.$$

Proof. We first observe that

$$\overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\mathrm{Euc}} \cap \partial\Omega \neq \emptyset.$$

Let $\sigma_n : [0, T_n] \to \Omega$ be a unit speed geodesic (with respect to the Bergman distance) such that $\sigma_n(0) = z_0$ and $\sigma_n(T_n) = y_n$. Since $y_n \to \xi \in \partial\Omega$ and B_Ω is a proper distance on Ω, we have $T_n \to \infty$. Since $\sigma_n(0) = z_0$, by the Arzelà-Ascoli theorem there exists $n_j \to \infty$ such that σ_{n_j} converges to a geodesic $\sigma : [0, \infty) \to \Omega$. We claim that $b(\sigma(t)) = -t$. Notice that if $t \leq T_n$, then

$$\begin{aligned} b_n(\sigma_n(t)) &= B_\Omega(\sigma_n(t), y_n) - B_\Omega(y_n, z_0) \\ &= B_\Omega(\sigma_n(t), \sigma_n(T_n)) - B_\Omega(\sigma_n(T_n), \sigma_n(0)) \\ &= (T_n - t) - (T_n - 0) = -t. \end{aligned}$$

Further,

$$b(\sigma(t)) = \lim_{n\to\infty} b_n(\sigma(t)) = \lim_{j\to\infty} b_{n_j}(\sigma_{n_j}(t)) = -t$$

since b_n converges locally uniformly to b. Then, since $\overline{\Omega}$ is compact, there exists $t_j \to \infty$ such that $\sigma(t_j)$ converges to some $\eta_1 \in \overline{\Omega}$. Since $B_\Omega(\sigma(t_j), z_0) = t_j$, we must have $\eta_1 \in \partial\Omega$. Then

$$\eta_1 \in \overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\text{Euc}} \cap \partial\Omega.$$

Next we show that

$$\overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\text{Euc}} \cap \partial\Omega = \{\xi\}.$$

Suppose $w_m \in b^{-1}\Big((-\infty, t_0]\Big)$ and $w_m \to \eta_2 \in \partial\Omega$. If $\eta_2 \neq \xi$, then Theorem 3.1 implies that there exists $R > 0$ such that

$$B_\Omega(w_m, z_0) + B_\Omega(z_0, y_n) - B_\Omega(w_m, y_n) \leq R$$

for every $n \geq 0$. Then

$$t_0 \geq b(w_m) = \lim_{n\to\infty} [B_\Omega(w_m, y_n) - B_\Omega(z_0, y_n)] \geq B_\Omega(w_m, z_0) - R.$$

However, $B_\Omega(w_m, z_0) \to \infty$ since B_Ω is a proper distance on Ω. So we have a contradiction. Hence, $\eta_2 = \xi$ and

$$\overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\text{Euc}} \cap \partial\Omega = \{\xi\}.$$

q.e.d.

Using the previous observation, if $\gamma^{-n} z_0$ is unbounded in Ω, then there exists $n_k \to \infty$ such that $\gamma^{-n_k} z_0 \to \xi$. Hence, in this case, Ω is biholomorphic to the unit ball by Theorem 1.10.

It remains to consider the case where the sequence $\gamma^{-n} z_0$ is bounded in Ω. Since Γ is discrete and acts properly on Ω, in this case

$$M := \text{order}(\gamma) < \infty.$$

We claim that γ has a fixed point in Ω. First, notice that

$$B_\Omega(\gamma^m y_n, y_n) \leq (M-1)\delta_n$$

for all $m \in \mathbb{Z}$. By Theorem 2.1, there exists some $\tau > 0$ such that the injectivity radius of g_Ω is bounded below by τ on each $U_n = \{z \in \Omega : B_\Omega(z_0, y_n) \leq \tau\}$. By Theorem 2.2, g_B is negatively curved on U_n when n is large. Then since $\delta_n \to 0$, Proposition 2.5 implies that when n is large the function

$$f_n(x) = \sup\{B_\Omega(\gamma^m y_n, x) : m = 0, 1, \dots, M-1\}$$

has a unique minimum c_n in Ω. Since

$$\gamma \left\{y_n, \gamma y_n, \gamma^2 y_n, \dots, \gamma^{M-1} y_n\right\} = \left\{y_n, \gamma y_n, \gamma^2 y_n, \dots, \gamma^{M-1} y_n\right\},$$

we then have $\gamma c_n = c_n$. So γ has a fixed point in Ω. Since Γ acts freely on Ω, we have a contradiction.

5. The convex case

Before starting the proof of Theorem 1.8 we will recall some results about convex domains.

As in Section 2.2, let $s_\Omega : \Omega \to (0, 1]$ denote the squeezing function on a domain $\Omega \subset \mathbb{C}^d$. For convex domains, the squeezing function is bounded from below by a positive constant which only depends on dimension.

Theorem 5.1 ([**9**, **19**, **25**])**.** *For any $d > 0$ there exists some $s = s(d) > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a bounded convex domain, then $s_\Omega(z) \geq s$ for all $z \in \Omega$.*

We will also use the following fixed point theorem.

Theorem 5.2 ([**8**, Theorem 12.2])**.** *Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded convex domain and $K \leq \mathrm{Aut}(\Omega)$ is a compact group. Then there exists a point $z \in \Omega$ such that $k(z) = z$ for all $k \in K$.*

Finally, we need the following facts about the Kobayashi distance.

Proposition 5.3. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain. Then the metric space (Ω, K_Ω) is proper and Cauchy complete.*

For a proof of Proposition 5.3 see for instance [**1**, Proposition 2.3.45].

Theorem 5.4 ([**32**, Theorem 4.1])**.** *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{1,\epsilon}$ boundary. If $\xi, \eta \in \partial\Omega$ and $T^{\mathbb{C}}_\xi \partial\Omega \neq T^{\mathbb{C}}_\eta \partial\Omega$, then*

$$\limsup_{x \to \xi, y \to \eta} \left(K_\Omega(x, z_0) + K_\Omega(z_0, y) - K_\Omega(x, y)\right) < \infty$$

for some (hence, any) $z_0 \in \Omega$.

Remark 5.5. This says that a point x near ξ and point y near η can be joined by a path that passes through z_0 and is length minimizing up to a bounded error.

5.1. Proof of Theorem 1.8. For the rest of the section suppose that $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $C^{1,\epsilon}$ boundary and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a discrete group acting freely on Ω. Further, assume that $\mathrm{Vol}(\Gamma\backslash\Omega) < +\infty$ where Vol is either the Bergman volume, the Kähler-Einstein volume, or the Kobayashi-Eisenman volume.

Using Theorem 1.12 and Theorem 1.14 it is enough to show that $\mathcal{L}(\Omega)$ intersects at least two different closed complex faces of $\partial\Omega$. We will prove the stronger result that the limit set intersects every closed complex face of $\partial\Omega$.

Lemma 5.6. *If $\xi \in \partial\Omega$, then $\mathcal{L}(\Omega) \cap T_\xi^{\mathbb{C}}\partial\Omega \neq \emptyset$.*

The proof of the Lemma is nearly identical to the proof of Theorem 1.3, but we provide a detailed argument for the reader's convenience.

Proof. By replacing Ω with an affine translate, we may assume that $\xi = (1, 0, \dots, 0)$ and $0 \in \Omega$. Then fix a sequence $r_n \nearrow 1$ and consider the points $y_n = (r_n, 0, \dots, 0) \in \Omega$. For each $n \in \mathbb{N}$ define

$$\delta_n = \min_{\gamma \in \Gamma\backslash\{1\}} K_\Omega(y_n, \gamma y_n).$$

Now for each $n \in \mathbb{N}$ the quotient map $\pi : \Omega \to \Gamma\backslash\Omega$ restricts to an embedding on

$$B_n = \{z \in \Omega : K_\Omega(z, y_n) < \delta_n/2\}.$$

Further, by Theorem 5.1 and Theorem 2.1 there exists some $C, \epsilon_0 > 0$ such that

$$\mathrm{Vol}(\pi(B_n)) \geq C \min\{\epsilon_0^{2d}, \delta_n^{2d}\}.$$

After passing to a subsequence we can assume that

$$\lim_{n\to\infty} \delta_n = \delta \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Case 1: $\delta \neq 0$. Repeating the argument in Case 1 of the proof of Theorem 1.3 shows that the set $\{\pi(y_n) : n \in \mathbb{N}\}$ is relatively compact in $\Gamma\backslash\Omega$. So for each n, there exist some $\gamma_n \in \Gamma$ such that the set $\{\gamma_n y_n : n \in \mathbb{N}\}$ is relatively compact in Ω. Then we can pass to a subsequence such that $\gamma_n y_n \to y \in \Omega$. Then $\gamma_n^{-1} y \to \xi$. So $\xi \in \mathcal{L}(\Omega)$.

Case 2: $\delta = 0$. Then select $\gamma_n \in \Gamma$ such that

$$K_\Omega(\gamma_n y_n, y_n) = \delta_n.$$

Case 2(a): The set $\{\gamma_1, \gamma_2, \dots\}$ is infinite. Since Γ is discrete, by passing to a subsequence we can suppose that $\gamma_n \to \infty$ in $\mathrm{Aut}(\Omega)$. Fix some $z_0 \in \Omega$. By passing to another subsequence we can assume that $\gamma_n^{-1} z_0 \to \eta \in \partial\Omega$. Since (Ω, K_Ω) is a complete proper metric space we must have

$$\lim_{n\to\infty} K_\Omega(z_0, \gamma_n^{-1} z_0) = \infty.$$

We claim that $\eta \in T_\xi^{\mathbb{C}} \partial\Omega$. Suppose not, then by Theorem 5.4 there exists $R > 0$ such that

$$K_\Omega(\gamma_n^{-1} z_0, z_0) + K_\Omega(z_0, y_n) - K_\Omega(\gamma_n^{-1} z_0, y_n) \leq R$$

for all $n \geq 0$. However,

$$\begin{aligned} K_\Omega(\gamma_n^{-1} z_0, z_0) &+ K_\Omega(z_0, y_n) - K_\Omega(\gamma_n^{-1} z_0, y_n) \\ &= K_\Omega(\gamma_n^{-1} z_0, z_0) + K_\Omega(z_0, y_n) - K_\Omega(z_0, \gamma_n y_n) \\ &\geq K_\Omega(\gamma_n^{-1} z_0, z_0) - K_\Omega(\gamma_n y_n, y_n) \\ &= K_\Omega(\gamma_n^{-1} z_0, z_0) - \delta_n. \end{aligned}$$

And so

$$\begin{aligned} R &\geq \limsup_{n\to\infty} K_\Omega(\gamma_n^{-1} z_0, z_0) + K_\Omega(z_0, y_n) - K_\Omega(\gamma_n^{-1} z_0, y_n) \\ &\geq \lim_{n\to\infty} K_\Omega(\gamma_n^{-1} z_0, z_0) - \delta_n = \infty. \end{aligned}$$

Thus we have a contradiction and hence $\eta \in T_\xi^{\mathbb{C}} \partial\Omega$.

Case 2(b): The set $\{\gamma_1, \gamma_2, \dots\}$ is finite. By passing to a subsequence we can suppose that $\gamma_n = \gamma$ in for all $n \in \mathbb{N}$.

Fix some $z_0 \in \Omega$. If the set $\{\gamma^n(z_0) : n \in \mathbb{N}\}$ is relatively compact in Ω, then γ has a fixed point in Ω by Theorem 5.2. But Γ acts freely on Ω, so the set $\{\gamma^n(z_0) : n \in \mathbb{N}\}$ must be unbounded in Ω.

Next consider the functions

$$b_n(z) = K_\Omega(z, y_n) - K_\Omega(y_n, z_0).$$

Since $b_n(z_0) = 0$ and each b_n is 1-Lipschitz (with respect to the Kobayashi distance) we can pass to a subsequence such that $b_n \to b$ locally uniformly. Then

$$\begin{aligned} b(\gamma^{-1} z) &= \lim_{n\to\infty} \left[K_\Omega(\gamma^{-1} z, y_n) - K_\Omega(y_n, z_0) \right] \\ &= \lim_{n\to\infty} \left[K_\Omega(z, \gamma y_n) - K_\Omega(y_n, z_0) \right] = b(z) \end{aligned}$$

since

$$\begin{aligned} \limsup_{n\to\infty} \left| K_\Omega(z, \gamma y_n) - K_\Omega(z, y_n) \right| &\leq \limsup_{n\to\infty} K_\Omega(y_n, \gamma y_n) \\ &= \limsup_{n\to\infty} \delta_n = 0. \end{aligned}$$

So

$$b(\gamma^{-n} z_0) = b(z_0) = 0$$

for all $n \in \mathbb{N}$.

Observation 5.7. For any $t_0 \in \mathbb{R}$

$$\overline{b^{-1}\Big((-\infty, t_0]\Big)}^{\text{Euc}} \cap \partial\Omega \subset T_\xi^{\mathbb{C}}\partial\Omega.$$

Proof. Suppose $w_m \in b^{-1}\Big((-\infty, t_0]\Big)$ and $w_m \to \eta \in \partial\Omega$. If $\eta \notin T_\xi^{\mathbb{C}}\partial\Omega$, then Theorem 5.4 implies that there exists $R > 0$ such that

$$K_\Omega(w_m, z_0) + K_\Omega(z_0, y_n) - K_\Omega(w_m, y_n) \leq R.$$

Then

$$b(w_m) = \lim_{n\to\infty} [K_\Omega(w_m, y_n) - K_\Omega(z_0, y_n)] \geq K_\Omega(w_m, z_0) - R.$$

However, $K_\Omega(w_m, z_0) \to \infty$ since K_Ω is a proper distance on Ω. So we have a contradiction. q.e.d.

Using the previous observation, there exists $n_k \to \infty$ such that

$$\lim_{k\to\infty} d_{\text{Euc}}\left(\gamma^{-n_k} z_0, T_\xi^{\mathbb{C}}\partial\Omega\right) = 0.$$

So $\mathcal{L}(\Omega) \cap T_\xi^{\mathbb{C}}\partial\Omega \neq \emptyset$.

q.e.d.

Lemma 5.8. *Ω is biholomorphic to the unit ball.*

Proof. Since Ω is bounded, there exists $x, y \in \partial\Omega$ such that $T_x^{\mathbb{C}}\partial\Omega \neq T_y^{\mathbb{C}}\partial\Omega$. By Lemma 5.6, the limit set $\mathcal{L}(\Omega)$ intersects both $T_x^{\mathbb{C}}\partial\Omega$ and $T_y^{\mathbb{C}}\partial\Omega$. So by Theorem 1.14, the group $\text{Aut}(\Omega)$ has finitely many components. So $\Gamma^0 := \Gamma \cap \text{Aut}_0(\Omega)$ has finite index in Γ and hence the quotient $\Gamma^0\backslash\Omega$ also has finite volume. So by Theorem 1.12 part (1), Ω is a bounded symmetric domain. Since Ω is convex and has C^1 boundary a result of Mok and Tsai [**23**] implies that Ω is biholomorphic to the unit ball. q.e.d.

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