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A coarse embedding theorem for homological filling functions

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Abstract

We demonstrate under appropriate finiteness conditions that a coarse embedding induces an inequality of homological Dehn functions. Applications of the main results include a characterization of what finitely presentable groups may admit a coarse embedding into a hyperbolic group of geometric dimension 2, characterizations of finitely presentable subgroups of groups with quadratic Dehn function with geometric dimension 2, and to coarse embeddings of nilpotent groups into other nilpotent groups of the same growth and into hyperbolic groups.

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1 | INTRODUCTION

A coarse embedding of one metric space into another generalizes the notion of a quasi-isometry. It allows for the metric to be distorted by any functions that tend to infinity, rather than just linear functions allowed in quasi-isometries.

Definition 1.1. We say that a map $f : X \to Y$ is a *coarse embedding* if there are functions $\rho_-, \rho_+ : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} \rho_-(x) = \lim_{x\to\infty} \rho_+(x) = \infty$ and where the following inequality holds:

$$\rho_-(d(x,y)) \leqslant d(f(x), f(y)) \leqslant \rho_+(d(x,y)).$$

In geometric group theory, a coarse embedding is a geometric version of subgroup containment. Indeed, if H is a finitely generated subgroup of G, then the inclusion map is a coarse embed-

ding. Thus, there are questions about which results on subgroups remain true when passing to coarse embeddings.

In general, coarse embedding are wilder than subgroups. For instance, every finitely generated, infinite group admits a coarse embedding of $\mathbb{Z}$, however, there are infinite finitely generated torsion groups. Moreover, in the class of hyperbolic groups, there are coarse embeddings of $\mathbb{Z}^2$ into hyperbolic groups while it certainly cannot be a subgroup.

For the statement of our theorems, we let $\mathrm{Fill}^k_G(\ell)$ be the k-dimensional homological Dehn function for G, that is, the difficulty of filling $(k-1)$-cycles with k-chains. Additionally, if $f, g : \mathbb{N} \to \mathbb{N}$ are functions, we say that $f \prec g$ if there is a constant $C > 0$ such that for all n we have that $f(n) \leqslant Cg(Cn + C) + Cn + C$. We say that $f \sim g$ when $f \prec g$ and $g \prec f$. For a group G, we denote $K(G, 1)$ as a choice of an Eilenberg–Maclane space for G. We say a group G has *geometric dimension 2* if the minimal dimension of a $K(G, 1)$ is 2. Finally, we say that a group G is of type F_n if it admits a $K(G, 1)$ with a compact n-skeleton.

In this paper, we study the preservation of homological filling functions under coarse embeddings. It is known that there are groups $H \subset G$ for which the homological filling function vary greatly. For instance, consider the inclusion $\mathrm{GL}_3(\mathbb{Z}) \hookrightarrow \mathrm{GL}_5(\mathbb{Z})$, the former has exponential Dehn function [4, Chapter 11] whereas the latter has quadratic Dehn function [17].

This type of example does not exist if G is assumed to have cohomological dimension 2. See the remark after Theorem 4.6 of [5].

Theorem 1.2. *Let G be a group a cohomological dimension 2, and let H be a finitely presented subgroup. Then* $\mathrm{Fill}^2_H(\ell) \prec \mathrm{Fill}^2_G(\ell)$.

In this paper, we take a more geometric approach and prove a similar theorem for coarse embeddings. As we cannot use the algebra of cohomology, we must replace the cohomological dimension assumption with the corresponding geometric dimension assumption. We prove the following.

Theorem 4.2. *Let G be a finitely presented group of geometric dimension 2, and suppose that H is a finitely presented group which admits a coarse embedding into G. Then* $\mathrm{Fill}^2_H(\ell) \prec \mathrm{Fill}^2_G(\ell)$.

Due to [5, Theorem 5.2], we know that a finitely presented group G is word hyperbolic if and only if $\mathrm{Fill}^2_G(\ell) \sim \ell$. Thus, the first consequence of the above theorem is a geometric generalization of Gersten's characterization of finitely presented subgroups of hyperbolic groups of geometric dimension 2. [5, Theorem 5.4].

Theorem 4.3. *Let G be a hyperbolic group of geometric dimension 2, and suppose H is a finitely presented group that admits a coarse embedding into G. Then H is a hyperbolic group.*

For the next corollary, we collect a few observations. First note that if $\delta_G(\ell)$ is the Dehn function of G, then $\mathrm{Fill}^2_G(\ell) \prec \delta_G(\ell)$. Moreover, [12, Proposition 8] implies that if $\mathrm{Fill}^2_G(\ell)$ grows strictly slower than ℓ^2, then G is hyperbolic. Thus, we may use Theorem 4.3 to obtain a characterization of subgroups of groups with quadratic Dehn function of geometric dimension 2.

Corollary 1.3. *Let G be be a finitely presented group of geometric dimension 2 with quadratic Dehn function. If H is a finitely presented group that admits a coarse embedding into G, then either H is a hyperbolic group or H has quadratic Dehn function. In particular, all finitely presented subgroups of G are either hyperbolic or have quadratic Dehn function.*

The next theorem generalizes [7, Theorem 1.1] to the context of coarse embeddings. It says that under appropriate finiteness conditions the higher dimensional homological Dehn filling functions are non-decreasing under coarse embeddings.

Theorem 4.4. *Let $k \geqslant 1$. Suppose that G admits a finite $(k+1)$-dimensional $K(G,1)$ and that H is a finitely presented group of type F_{k+1}. If H admits a coarse embedding into G, then* $\mathrm{Fill}_H^{k+1}(\ell) \prec \mathrm{Fill}_G^{k+1}(\ell)$.

Since the inclusion of a horosphere in hyperbolic $(k+1)$-space gives a coarse embedding of $\mathbb{Z}^k$ into $\pi_1(M)$ when M is a closed $(k+1)$-dimensional real hyperbolic manifold, the above theorem cannot be improved. In contrast, the following application of Theorem 4.4 demonstrates that there exists no coarse embedding of a torsion free, finitely generated nilpotent group whose integral cohomological dimension is $k+1$ into $\pi_1(M)$. Thus, whenever a torsion free, finitely generated nilpotent admits a coarse embedding into $\pi_1(M)$, its integral cohomological dimension must be strictly less than that of the integral cohomological dimension of M. This follows from the fact that [10, Theorem 4] implies all homological filling functions for hyperbolic groups are linear and that [15, IV 5.8 Theorem] and [6] together imply the top-dimensional homological filling functions for torsion free, finitely generated nilpotent group are superlinear. Similarly, [9, Theorem 3.3] implies that torsion free, finitely generated nilpotent groups of integral cohomological dimension $k+1$ cannot coarsely embed into a nonamenable integral Poincaré duality group of dimension $k+1$.

Corollary 1.4. *Let G be either a hyperbolic group that admits a finite $(k+1)$-dimensional $K(G,1)$ or a nonamenable integral Poincaré duality group of dimension $k+1$. Then a torsion-free, finitely generated nilpotent group of integral cohomological dimension $k+1$ cannot admit a coarse embedding into G.*

Our next application of Theorem 4.4 is to coarse embeddings between torsion free, finitely generated nilpotent groups G and H with integral cohomological dimensions equal to n. Since asymptotic dimension, integral cohomological dimension, and Hirsch length are all equal for torsion free, finitely generated nilpotent groups, one must appeal to other invariants such as word growth or the asymptotic cone to obstruct the existence of coarse embeddings between G and H. For instance, when G and H equal polynomial degrees of word growth, [3, Theorem 1.2] implies that there exist no coarse embeddings from either G to H or vice versa when G and H have distinct asymptotic cones. When G and H have word growth of polynomial degrees d_1 and d_2, respectively, where $d_1 < d_2$, monotonicity of word growth under coarse embeddings implies that there are no coarse embeddings of H into G. Since [15, IV 5.8 Theorem] and [6] together imply that $\mathrm{Fill}_G^n(\ell) \sim \ell^{\frac{d_1}{d_1-1}}$ and $\mathrm{Fill}_H^n(\ell) \sim \ell^{\frac{d_2}{d_2-1}}$, Theorem 4.4 implies that G does not admit a coarse embedding into H. These facts together imply a nearly complete classification of what torsion free, finitely generated nilpotent groups of the same integral cohomological dimension admit coarse embeddings into each other.

Corollary 1.5. *Let G and H be torsion free, finitely generated nilpotent groups of the same integral cohomological dimension. If the asymptotic cones of G and H are not isomorphic as Lie groups, then G and H cannot admit coarse embeddings into one another.*

The only case that remains is when two torsion free, finitely generated nilpotent groups G and H who have the same asymptotic cone. These are groups will share many properties such

as growth rate and integral cohomological dimension. Moreover, by [15, IV 5.8 Theorem], we have that $\mathrm{Fill}^n_G(\ell) \sim \mathrm{Fill}^n_H(\ell)$ where n is the topological dimension of the asymptotic cone. In certain cases, such as when G and H are cocompact lattices in the associated asymptotic cone, we have that G and H are bi-Lipschitz. Hence, they naturally admit coarse embeddings into each other. However, when either G or H is not a lattice in the asymptotic cone, then whether a coarse embedding or even a Lipschitz embedding exists is open.

Finally, we apply the techniques of Theorem 4.2, to obtain characterizations of finitely presented subgroups of infinitely presented small cancelation groups.

Theorem 5.5. *Let G be finitely generated group that admits an infinite $C'(1/6)$-small cancelation presentation where no relator is a proper power. If H is finitely presented group that admits a coarse embedding into G, then H is a hyperbolic group. In particular, all finitely presented subgroups of G are hyperbolic.*

Thus, small cancelation groups contain no finitely presented subgroup obstructions to hyperbolicity. However, there are many examples of small cancelation groups that cannot coarsely embed into hyperbolic groups [8].

1.1 | Structure of the article

In Section 2, we introduce basic definitions and embedding results between CW complexes associated to Lipschitz embeddings between finitely generated groups. Using these embedding results, Section 3 introduces homological filling functions and relates homological fillings of codimension 0 subcomplexes with fillings in the ambient complex. Sections 4 and 5 give the proofs of the main results. Section 6 finishes with some questions.

2 | EMBEDDING LEMMAS

In this section, we will obtain various embedding results between classifying spaces.

Throughout we will be interested in CW complexes with G-actions. To make the article easier, we will focus on a particular subset of CW complexes namely, combinatorial complexes.

Definition 2.1. Combinatorial complexes and combinatorial maps between them are defined recursively on dimension. Zero-dimensional CW complexes are defined to be combinatorial as are arbitrary maps between them. In general, a continuous map between CW complexes is said to be *combinatorial* if it the image of each open cell of the domain is an open cell of the target. A *combinatorial $(k+1)$-complex* is a CW complex whose k-skeleton is a combinatorial k complex and whose attaching maps ϕ_e of $(k+1)$-cells e are combinatorial maps for suitable combinatorial structures on $\partial D^{k+1}_e = S^k$.

Every simplicial complex is a combinatorial complex. Since every CW complex is homotopy equivalent to a simplicial complex, we see that, up to homotopy, the class of combinatorial complexes is equivalent to the class of CW complexes.

When given a combinatorial complex X, we denote its k-skeleta as $X^{(k)}$. We denote S^k as the k-sphere and D^k as the k-disk.

Definition 2.2. We say that a map $f : H \to G$ is a *coarse embedding* if there are nondecreasing functions $\rho_-, \rho_+ : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} \rho_-(x) = \lim_{x\to\infty} \rho_+(x) = \infty$ and where the following inequality holds:

$$\rho_-(d(x,y)) \leqslant d(f(x), f(y)) \leqslant \rho_+(d(x,y)).$$

Throughout $f : H \to G$ will be a coarse embedding of groups. The following is useful in that we can replace the upper bound by a linear function.

Proposition 2.3. *Let $f : H \to G$ be a coarse embedding. Then f is $\rho_+(1)$-Lipschitz.*

Proof. Let x, y be arbitrary elements of H. There is a sequence of elements $x = z_0, z_1, \dots, z_\ell = y$ such that z_i and z_{i+1} are at distance 1 apart and $\ell = d(x,y)$. Since z_i and z_{i+1} are at distance 1, we have that $d(f(z_i), f(z_{i+1})) \leqslant \rho_+(1)$, and so, $d(f(x), f(y)) \leqslant \sum_i d(f(z_i), f(z_{i+1})) \leqslant \ell\rho_+(1) = \rho_+(1)d(x,y)$. □

Proposition 2.4. *Let $f : H \to G$ be a coarse embedding. Then the preimage of each $f^{-1}(g)$ is uniformly bounded*

Proof. Since ρ_- is nondecreasing and $\lim_{x\to\infty} \rho_-(x) = \infty$, there is an l such that $\rho_-(x) > 0$ for all $x \geqslant l$. Thus if $f(h_1) = f(h_2)$, then $d(h_1, h_2) \leqslant l$. Thus the preimage of g is bounded by the size of the ball of radius l in H. □

Definition 2.5. A *Cayley n-complex* for a group G is an n-dimensional cell complex X equipped with a free, cellular G-action such that $\pi_i(X) = 1$ for all $i < n$. We say that X is *cocompact* if there are finitely many orbits of k-cells for each k.

A Cayley n-complex for G is exactly the n-skeleton of the universal cover of a $K(G,1)$. Indeed, by adding cells of dimension greater than n, we can obtain a contractible complex with a free, cellular G action. Thus, the quotient is a $K(G,1)$.

Remark 2.6. If a Cayley n-complex has one orbit of vertices, then the 1-skeleton is a Cayley graph for the group. By collapsing a G invariant fores,t we can always arrange this situation.

The following lemma follows immediately from the definition of type F_n.

Lemma 2.7. *A group G is of type F_n if and only if it admits a cocompact Cayley n-complex.*

This next proposition demonstrates that under appropriate finiteness conditions we may extend the coarse embedding $f : H \to G$ to an inclusion of cocompact Cayley n-complexes.

Proposition 2.8. *Let H be a group of type F_n for $n > 1$, and let X be a cocompact Cayley n-complex for H with a single orbit of vertices.*

Let G be a group of type F_{n-1}, and let Z be Cayley n-complex for G with a cocompact $(n-1)$-skeleton and a single orbit of vertices.

Let $f : H \to G$ be a coarse embedding. Then there exists a Cayley n-complex Y for G such that the following hold.

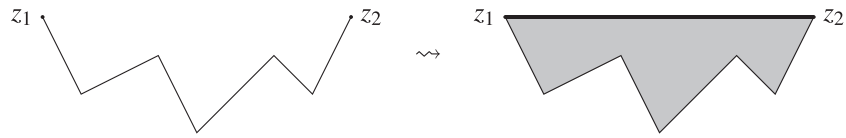

FIGURE 1 An extra edge is added between every pair of points z_1, z_2 at distance $\leqslant C$. The resulting copy of S^1 is then filled with a disk

- *Y has cocompact $(n-1)$-skeleton.*
- *Z is a deformation retract of Y.*
- *There is a map $\phi : X \to Y$ extending f, which is injective on $n-1$-skeleta.*
- *There exists $N \geqslant 0$ such that each n-cell of X is mapped to a collection of $\leqslant N$ n-cells in Y.*

Proof. There is a one-to-one correspondence between the vertex set of X (respectively, Z) and the group H (respectively, G). Thus, we can view f as a map of 0-skeleta.

Let P be the constant from Proposition 2.4. Let Δ be a maximal tree in the simplex with P vertices. Let x be a base vertex of Δ. Let Z_0 be the space obtained from Z by taking a copy of Δ denoted Δ_w for each vertex w of Z and identifying $x \in \Delta_w$ with w. Since we have wedged a contractible space at each vertex of Z, we can see that Z is a deformation retract of Z_0. Also we can extend the G action by insisting $g \cdot \Delta_w = \Delta_{g \cdot w}$. Thus, Z_0 is still a Cayley n-complex. Also Z_0 still has cocompact $(n-1)$-skeleton as we have only added finitely many orbits of vertices and finitely many orbits of edges.

For each $w \in G \cdot v$, there is an injection $f^{-1}(w)$ to the vertices of Δ_w. Thus, we obtain an injection f' from the 0-skeleton of X to the 0-skeleton of Z_0. Moreover, if $d(f(h_1), f(h_2)) \leqslant D$, then $d(f'(h_2), f'(h_2)) \leqslant D + 2$.

Since f is $\rho_+(1)$-Lipschitz, the end points of each edge of X are mapped by f' to points at distance at most $C = \rho_+(1) + 2$ apart. Let Z_1' be the space obtained from Z_0 by adding an edge between every pair of points z_1, z_2 such that $d(z_1, z_2) \leqslant C$. Since Z_0 has a cocompact 1-skeleton, we only add finitely many orbits of edges in this process. For each orbit of edges $[e]$ added to Z_0, let γ_e be a minimal length path in Z_0 between the endpoints of e. Now attach a 2-cell to Z_1' with boundary $\gamma_e \bar{e}$. Let Z_1 be the complex obtained by adding these disks equivariantly. Since we only added finitely many orbits of edges to Z_0 to obtain Z_1', we only add finitely many orbits of 2-cells to Z_1' to obtain Z_1. We can now extend f' to an injective cellular map $X^{(1)} \to Z_1^{(1)}$. See figure 1 for a depiction of this procedure.

We now proceed by induction. Suppose that we have built a complex Z_k which has a cellular embedding that extends f' to the k-skeleton of X for some $k < n-1$ and where Z is a deformation retract of Z_k. Moreover, assume G acts cocompactly on the $(n-1)$-skeleton of Z_k. Since X is a cocompact Cayley n-complex, there are finitely many orbits of $(k+1)$-cells. Let $D_1, \dots, D_l$ be representatives of these orbits. Let L be the maximum number of cells in the boundary of D_i.

Let S be the collection of cellular maps $S^k \to Z_k$ such that the image contains $\leqslant L$ k-cells. Let Z_{k+1}' be the complex obtained by attaching a $k+1$-cell to each such map. Since H acts cocompactly on the $(k+1)$-skeleton of Z_k, we only attach finitely many orbits of cells to obtain Z_{k+1}'. Thus, H still acts cocompactly on the $(k+1)$-skeleton of Z_{k+1}', and moreover, we have a cellular embedding $X^{(k+1)} \to Z_{k+1}'$.

Now for each $(k+1)$-cell c added to Z_k, its boundary is null-homotopic in Z_k. Therefore, it bounds a $(k+1)$-cell d in Z_k. We can now attach a $(k+2)$-cell to Z_{k+1}' with boundary which

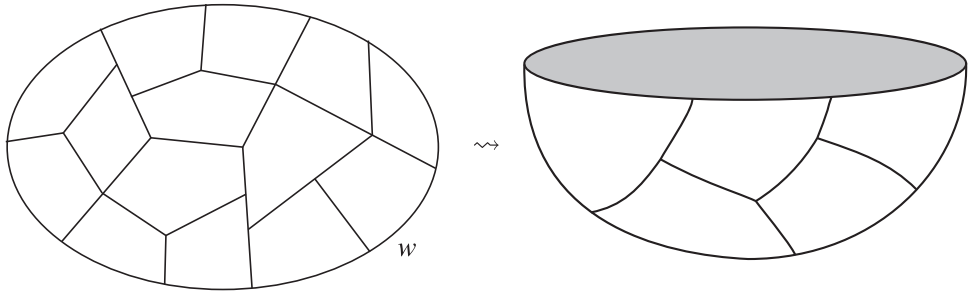

FIGURE 2 The 2-dimensional case from the proof of Proposition 2.8. We add an extra 2-cell for each disk diagram with boundary w, where $|w| \leqslant L$. We then fill the resulting sphere with a disk

consists of c on the upper hemisphere and d on the lower hemisphere. Let the complex obtained this way be Z_{k+1}.

See figure 2, for a depiction of the 2-dimensional case.

We will now show that Z_k is a deformation retract of Z_{k+1}. Let $T = Z_{k+1}/Z_k$ be the space obtained by collapsing Z_k to a point. We can understand the space T by building it as follows. Let $S = Z'_{k+1}/Z_k$. Then S has the homotopy type of a wedge of k-spheres as we obtained Z_{k+1} from Z by attaching k-cells. For each sphere in S, we attached a $(k+1)$-cell to Z'_{k+1}. After collapsing Z_k, the attaching map is homotopic to the identity $S^k \to S^k$. Thus, we obtain T by attaching $(k+1)$-cells to S one for each sphere in S. Hence, T has the homotopy type of a wedge of copies of D^{k+1} which is contractible. Since Z_k is a cellular subcomplex of Z_{k+1} with contractible quotient, we see that Z_k is a deformation retract of Z_{k+1}. Thus, by induction Z is a deformation retract of Z_{k+1}.

Now assume that we have a cellular embedding at the level of $(n-1)$-skeleta. By assumption, there are finitely many orbits of n-cells in X. Let $\mu_{i,h} : S^{n-1} \to X$ be the attaching map of the ith n-cell based at h. The number of cells in the image of $\mu_{i,h}$ is uniformly bounded as S^{n-1} is compact and there are only finitely many orbits of n-cells. Let L be the bound on the number of $(n-1)$-cells in $\mu_{i,h}(S^{n-1})$. We have a cellular embedding at the level of $(n-1)$-skeleta; thus, $\phi \circ \mu_{i,h}(S^{n-1})$ contains at most L cells. Since Z is a Cayley n-complex, we have that $\phi \circ \mu_{i,h}$ bounds a copy of D^n. Thus, we can extend the map ϕ to the cell $D_{i,h}$ attached via $\mu_{i,h}$ by sending the attached cell to the copy of D^n bounded by $\phi \circ \mu_{i,h}$. The number of n-cells in the image of $D_{i,h}$ is bounded by $\delta(L)$ where δ is the $(n-1)$-st filling function of Z. □

Remark 2.9. By subdividing the n-cells of X, we may obtain a combinatorial map $X' \to Y$.

In the presence of a coarse embedding, we may quantify how badly injectivity fails at the level of n-skeleta.

Lemma 2.10. *Let X be a cocompact Cayley n-complex for H. Let Z be a Cayley n-complex for G with cocompact $n-1$ skeleton. Let Y and $\phi : X \to Y$ be the complex and map as in Proposition 2.8. Let X' be the subdivision from Remark 2.9. Suppose further that f is a coarse embedding.*

Then there is a constant L such that if $x_1, x_2 \in X'$ are such that $f(x_1) = f(x_2)$, then $d(x_1, x_2) < L$.

Proof. Let x_1, x_2 be points of X' that are identified in Y. Since ϕ is an injection on $X^{(n-1)} \subset X'$, we see that x_1, x_2 must belong to the interior of an n-cell. Let N be the constant from Proposition 2.8.

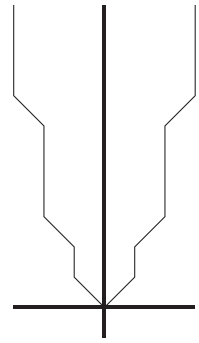

FIGURE 3 The map $n \mapsto (\log_2(|n|+1), n)$ is a coarse embedding which is not a quasi-isometry. However, it is easy to see that the image of the induced map on Cayley graphs with the induced path metric is quasi-isometric to $\mathbb{R}$

We can see that there are vertices v_1, v_2 of X such that $d(f(v_i), f(x_i)) \leqslant N$. Thus, $d(f(v_1), f(v_2)) \leqslant 2N$. Let $L' = \sup\{t \in \mathbb{R} \mid \rho_-(t) \leqslant 2N+1\}$. We can now see that $d(v_1, v_2) \leqslant L'$ and $d(x_1, x_2) \leqslant L' + 2N = L$ □

In fact, the conclusions of this lemma do not require coarse embedding. See figure 4 for an example where the conclusions of this lemma hold but the map is very far form a coarse embedding.

Theorem 2.11. *Let X, Y, ϕ be as in Proposition 2.8. Let $M = \phi(X) \subset Y$. Give $M^{(1)}$ the graph metric and extend this to M. Then X is quasi-isometric to M.*

Proof. Let X' be the complex obtained from Remark 2.9. Since $\phi : X' \to Y$ is a cellular map, we see that it is Lipschitz when restricted to 1-skeleta which gives us the upper bound in the quasi-isometry.

To obtain the lower bound, let x_1, x_2 be two vertices in X'. Let $y_i = \phi(x_i)$. Let $d_1, \dots, d_\lambda$ be a minimal path from y_1 to y_2 in $M^{(1)}$. Hence, we have $d(y_1, y_2) = \lambda$. Since $\phi : X' \to M$ is surjective and cellular, we can find an edge e_i in X' such that $\phi(e_i) = d_i$.

By Lemma 2.10, we must have that $d(\tau(e_i), \iota(e_{i+1})) \leqslant L$. Also, $d(x_1, \iota(e_1)), d(\tau(e_\lambda), x_2) \leqslant L$. Thus, we can find a path in $X^{(1)}$ from x_1 to x_2 with at most $L\lambda + \lambda + 2L$ edges. That gives an upper bound for $d(x_1, x_2)$, and thus, we obtain the following inequality:

$$\frac{d(x_1, x_2)}{L+1} - \frac{2L}{L+1} \leqslant d(y_1, y_2).$$

□

The reader should have the following example in mind. Let $Y = \mathbb{H}^3$ and $X = \mathbb{R}^2$. Let $\phi : X \to Y$ be the inclusion of X as a horosphere in Y. Then $\phi(X)$ and M are isometric. However, ϕ is not a quasi-isometric embedding. Another example of a coarse embedding $\mathbb{Z} \to \mathbb{Z}^2$ is in figure 3.

3 | HOMOLOGICAL FILLING FUNCTIONS

Throughout this section, we denote $C_d(X, \mathbb{Z})$ as the cellular d-dimensional chain group X. Define a norm on $C_d(X, \mathbb{Z})$ by $|\sum_\sigma n_\sigma \sigma| = \sum_\sigma |n_\sigma|$, where the sum runs over all d-cells of X. Note that

this is well defined as these are finite sums. Finally, let $Z_d(X,\mathbb{Z})$ be the $\mathbb{Z}$-module of integral d-cycles of X, and let $\partial : C_{d+1}(X,\mathbb{Z}) \to Z_d(X,\mathbb{Z})$ be the boundary map. Finally, we let $H_d(X,\mathbb{Z}) = \ker(\partial_d)/\mathrm{Im}(\partial_{d+1})$ be the dth integral homology group of X.

Definition 3.1. Let S be a collection of $(n-1)$-cycles in a space X. Suppose that each element of S is the boundary of an n-chain. Define the *filling volume* of $s \in S$ to be

$$\mathrm{FVol}_X(s) = \min\left\{ \sum_i |\alpha_i| \,\middle|\, \sum_i \alpha_i\sigma_i \in C_n(X,\mathbb{Z}) \text{ where } \partial\left(\sum_i \alpha_i\sigma_i\right) = s \right\}.$$

We define the *S-restricted n*th filling function of S to be

$$\mathrm{Fill}^n_{X,S}(\ell) = \max\{\mathrm{FVol}_X(s) \mid |s| \leqslant \ell, s \in S\}.$$

In the case that S is the collection of all $(n-1)$-cycles in X, we obtain the usual homological filling function for X which we denote as $\mathrm{Fill}^n_X(\ell)$. Also, for any collection S of $(n-1)$-cycles, we have that $\mathrm{Fill}^n_{X,S}(\ell) \prec \mathrm{Fill}^n_X(\ell)$. In the case that X has a group action that is cocompact on $X^{(n-1)}$, then $\mathrm{Fill}^n_X(\ell)$ is well defined.

The following lemma follows from [1, 16] where the action of a group was not used to show invariance under quasi-isometry. We produce it here for convenience.

Lemma 3.2. *Let X, ϕ be as in Proposition 2.8 and $M = \phi(X)$ as in Theorem 2.11. If S be the set of all $(n-1)$-cycles in X, then*

$$\mathrm{Fill}^n_X(\ell) = \mathrm{Fill}^n_{X,S}(\ell) \sim \mathrm{Fill}^n_{M,S}(\ell).$$

Proof. Let X' be the complex from Remark 2.9. Each n-cell of X is subdivided into at most N n-cells of X' where N is the constant from Proposition 2.8. Thus, we obtain $\mathrm{Fill}^n_{X,S}(\ell) \sim \mathrm{Fill}^n_{X',S}(\ell)$.

Since we have a cellular map $X' \to M$, a filling in X' maps to a filling in M, and thus, $\mathrm{FVol}_{X'}(s) \geqslant \mathrm{FVol}_M(s)$ for every $s \in S$. As such, we obtain the upper bound $\mathrm{Fill}^n_{M,S}(\ell) \prec \mathrm{Fill}^n_{X',S}(\ell)$.

To prove the other direction, we construct a quasi-inverse to ϕ, that is, a (K,C)-quasi-isometry ψ such that $\psi(\phi(x)) \leqslant C$ for all $x \in X$. Let g be the inverse of $\phi|_{X^{(n-1)}}$. By construction, ϕ is injective on $X^{(n-1)}$. Therefore, g is well defined. Now for each vertex of $M \backslash \phi(X^{(n-1)})$ define $\psi(v)$ to be any of the closest vertices of $\phi(X^{(n-1)})$. Since X is $(n-1)$-connected, we can extend this to higher cells by sending a cell to a minimal filling of its boundary.

Let δ^m be the mth isoperimetric function for X. Since at each stage of the process of Proposition 2.8, we were only taking finitely many cells for our fillings, we see that there exists a k such that the boundary of each ℓ-cell of M contains $\leqslant k\,(\ell-1)$-cells of M. Thus, the boundary of each n-cell of M is sent to at most $k(\delta^{n-2}(\dots k(\delta^0(K+C)\dots)$ $(n-1)$-cells of X. Hence, each n-cell is sent to at most $\delta^{n-1}(k(\delta^{n-2}(\dots k(\delta^0(K+C)\dots)$ n-cells, and so, we see that fillings are changed by at most a constant multiple. Therefore, the functions are equivalent. □

The above lemma allows us to study fillings in M and obtain information about fillings in X. However, we need relate this to fillings in Y. In general, there will be more efficient fillings in Y that are not contained in M. For instance, in the previous example where M is a horosphere in

$\mathbb{H}^3$, the fillings in M are quadratic in the boundary length but there are linear fillings in $\mathbb{H}^3$. The next lemma shows that this issue vanishes for top-dimensional filling functions.

Lemma 3.3. *Suppose that Y is an n-dimensional complex with $H_n(Y,\mathbb{Z}) = 0$. Let M be a subspace and c be an $(n-1)$-cycle in M. Suppose that c is the boundary of an n-cycle $d = \sum_i \alpha_i\sigma_i$ in M. Then $\mathrm{FVol}_M(c) = \mathrm{FVol}_Y(c)$.*

Proof. We show the stronger statement that homological fillings of $(n-1)$-cycles in Y are unique.

Since Y is n-dimensional and $H_n(Y,\mathbb{Z}) = 0$, we see that $\ker(\partial_n) = 0$. Now suppose we had another filling d' for c. We then see that $d - d'$ is an element of $\ker(\partial_n)$ and hence is trivial. Thus, $d = d'$, and the result follows. □

We can now use this to bound the homological filling function of M in terms of that of Y.

Corollary 3.4. *Let Y, M be as in Lemma 3.3. Then $\mathrm{Fill}_M^{n-1}(\ell) \prec \mathrm{Fill}_Y^{n-1}(\ell)$.*

4 | PROOF OF MAIN THEOREMS

In this section, we will prove Theorems 4.2, 4.3, and 4.4. We start with the definition of the nth homological filling function for a group of type F_k.

Definition 4.1. Let G be a group acting properly, cocompactly, and by cellular automorphisms on a k-connected cell complex X. The kth homological Dehn filling function of G is the function $\mathrm{Fill}_G^k(\ell)\colon \mathbb{N}\to\mathbb{N}$ given by $\mathrm{Fill}_G^k(\ell) = \mathrm{Fill}_X^k(\ell)$.

We remark that Young [16] proved that $\mathrm{Fill}_G^k(\ell)$ is a well-defined invariant of a group meaning that if G acts properly, cocompactly, and by cellular automorphisms on two k-connected cell complexes X and Y, then $\mathrm{Fill}_X^k(\ell) \sim \mathrm{Fill}_Y^k(\ell)$.

We start with Theorem 4.2 whose statement we recall for the convenience of the reader.

Theorem 4.2. *Let G be a finitely presented group of geometric dimension 2, and suppose that H is a finitely presented group which admits a coarse embedding into G. Then $\mathrm{Fill}_H^2(\ell) \prec \mathrm{Fill}_G^2(\ell)$.*

Proof. We split into two cases, first assuming that G admits a compact $K(G,1)$ of dimension 2. Let K be the universal cover of this $K(G,1)$, that is, K is a contractible 2-dimensional cell-complex with a free, proper, and cocompact action of G. Let $f\colon H\to G$ be the given coarse embedding, and let X be the universal cover of the presentation complex of H associated to some finite presentation of H. By Proposition 2.3, we have that f is C-Lipschitz for some constant C. By Proposition 2.8, there exists a Cayley 2-complex Y for G that contains K as a subcomplex which has a cocompact 2-skeleton and a map $\phi\colon X\to Y$ which extends f and is an injection on 1-skeleta. Moreover, we have that each 2-cell is mapped to a collection at most N 2-cells in Y, and by subdividing if necessary, we may assume that ϕ is a cellular map. Define $M = \phi(X)$. Letting S be the set of 1-cycles of X in Y, Lemma 3.2 implies that $\mathrm{Fill}_{M,S}^2(\ell) \sim \mathrm{Fill}_{X,S}^2(\ell)$. Since Corollary 3.4 implies that $\mathrm{Fill}_M^2(\ell) \prec \mathrm{Fill}_Y^2(\ell)$, we have that $\mathrm{Fill}_X^2(\ell) \prec \mathrm{Fill}_Y^2(\ell)$. Therefore, we have that $\mathrm{Fill}_H^2(\ell) \prec \mathrm{Fill}_G^2(\ell)$ as desired.

Now suppose that G does not admit a cocompact $K(G,1)$ of dimension 2. By contracting a maximal tree in $K(G,1)^{(1)}$, we may assume that there is a single vertex in $K(G,1)$. Thus, this classifying space is the presentation 2-complex of an infinite aspherical presentation $\langle S \mid R\rangle$. Let K be the universal cover of this presentation 2-complex.

Since G is finitely generated, there exists a finite subset of S that generates G. Let $\{x_1,\dots,x_\ell\} = A$ be such a subset. We have an inclusion ψ of the Cayley graph Γ of G with respect to A to the complex K. We can find a finite presentation of the form $\langle A \mid w_1,\dots,w_m\rangle$. Let Z be the universal cover of the presentation complex associated to this presentation.

We can extend ψ to a map $\phi : Z \to K$ as follows. Each relation w_i is trivial in H and thus bounds a disk in H. This disk is spanned by the relators in R. We can now map the disk with boundary w_i to the image of a disk diagram for w_i in K. We now extend this equivariantly to all of Z.

Let $M = \phi(Z)$. Since ϕ is G-equivariant, we see that the action of G on Z gives an action of G on M. Since every point of M is at bounded distance from a point in the image of Γ, we see that this action is proper and cocompact. Thus, Z is quasi-isometric to M. By Lemma 3.2, we have that $\mathrm{Fill}^2_Z(\ell) \sim \mathrm{Fill}^2_M(\ell)$.

Using Proposition 2.8, we can obtain a map $X \to Z$ where X is the universal cover of a presentation complex for H. We can compose with the map $Z \to M$. Let L be the image of X in M. Lemma 2.10 shows that L and X are quasi-isometric and so $\mathrm{Fill}^2_X(\ell) \sim \mathrm{Fill}^2_L(\ell)$.

Since K is contractible, we have that $H_2(K,\mathbb{Z}) = 0$. Since there are no 3-cells in K, we see that $\partial : C_2(K,\mathbb{Z}) \to C_1(K,\mathbb{Z})$ is injective. Thus, if we restrict to $C_2(L,\mathbb{Z})$, we also obtain an injection. Therefore, $H_2(L,\mathbb{Z}) = 0$.

Now, by Lemma 3.3, we see that $\mathrm{Fill}^2_L(\ell) \prec \mathrm{Fill}^2_M(\ell)$. Thus, $\mathrm{Fill}^2_H(\ell) \prec \mathrm{Fill}^2_G(\ell)$. □

Theorem 4.3 is a direct consequence of Theorem 4.2. However, we felt that Theorem 4.3 is interesting in its own right and demonstrates that hyperbolicity of groups of geometric dimension 2 passes to finitely presented groups which admit a coarse embedding to the given hyperbolic group. This is contrast to higher geometric dimensions where there exist hyperbolic groups which admit coarse embeddings of $\mathbb{Z}^2$.

Theorem 4.3. *Let G be a hyperbolic group of geometric dimension 2, and suppose H is a finitely presented group that admits a coarse embedding into H. Then H is a hyperbolic group.*

Proof. Due to [5, Theorem 5.2], we know that a finitely presented group G is word hyperbolic if and only if $\mathrm{Fill}^2_H(\ell) \sim \ell$. Theorem 4.2 implies that $\mathrm{Fill}^2_H(\ell) \prec \mathrm{Fill}^2_G(\ell) \sim \ell$. Therefore, $\mathrm{Fill}^2_H(\ell) \sim \ell$ which implies that H is hyperbolic as desired. □

The proof of Theorem 4.4 is very similar to the proof of Theorem 4.2 in the case where G has a compact $K(G,1)$ of dimension 2. However, we include its proof for completeness.

Theorem 4.4. *Let $k \geqslant 1$ Suppose that G admits a finite $(k+1)$-dimensional $K(H,1)$ and that H is a finitely presented group of type F_{k+1}. If H admits a coarse embedding into G, then $\mathrm{Fill}^{k+1}_H(\ell) \prec \mathrm{Fill}^{k+1}_G(\ell)$.*

Proof. Let $f : H \to G$ be the given coarse embedding, and let X be the universal cover of a $K(H,1)$ which a compact $(k+1)$-skeleta. By definition, we have that f is an injective C-Lipschitz map for some constant $C > 0$. By Lemma 2.8, there exists a Cayley k-complex Y for G with

cocompact k-skeleton and a map $\phi : X \to Y$ which extends f and is an injection on k-skeleta. Moreover, we have that each $(k+1)$-cell is mapped to a collection at most N $(k+1)$-cells in Y, and by subdividing if necessary, we may assume that ϕ is a cellular map. Define $M = \phi(X)$. Letting S be the set of k-cycles of X in Y, Lemma 3.2 implies that $\mathrm{Fill}^{k+1}_{M,S}(\ell) \sim \mathrm{Fill}^{k+1}_{X,S}(\ell)$. Since Corollary 3.4 implies that $\mathrm{Fill}^{k+1}_{M}(\ell) \prec \mathrm{Fill}^{k+1}_{Y}(\ell)$, we have that $\mathrm{Fill}^{k+1}_{X}(\ell) \prec \mathrm{Fill}^{k+1}_{Y}(\ell)$. Therefore, we have $\mathrm{Fill}^{k+1}_{H}(\ell) \prec \mathrm{Fill}^{k+1}_{G}(\ell)$ as desired. □

5 | SMALL CANCELATION GROUPS

In this section, we prove Theorem 5.5. For the following, we let $F(A)$ be the free group on the set A.

Definition 5.1. Let A be a finite set. Let R be a subset of $F(A)$. We say that $p \in F(A)$ is a *piece* in R if there exists cyclic permutations $s_i \neq s_j$ of relators $r_i, r_j \in R$ such that $s_i = pu$ and $s_j = pv$.

We say that a presentation $\langle A \mid R\rangle$ satisfies the *small cancelation condition* $C'(\lambda)$ if given a piece p in a relator r, we have that $|p| < \lambda |r|$.

We refer the reader to [11, Section V] for a full treatment of small cancelation theory. The following will be key to our theorem.

Theorem 5.2. *If $G = \langle A \mid R\rangle$ satisfies $C'(\frac{1}{6})$ and no relator is a proper power, then:*

- *the presentation 2-complex of G is aspherical;*
- *G has geometric dimension 2;*
- *If R is finite, then G is hyperbolic.*

The first two points follow from [2, 13]. The latter follows from the following theorem which shows that G has linear Dehn function.

Theorem 5.3 [11, Theorem V.4.5]. *Let $G = \langle A \mid R\rangle$ be a small cancelation group. Let w be a word in A which represents the trivial element in G. Then w contains a sub-word which is more than half a relator.*

This allows us to understand the filling function of the universal cover of the presentation complex.

Corollary 5.4. *Let $\langle A \mid R\rangle$ be a $C'(\frac{1}{6})$ small cancelation group. Let Z be the universal cover of the presentation complex. Then $\mathrm{Fill}^2_Z(\ell) \sim \ell$.*

We are now ready to prove our theorem.

Theorem 5.5. *Let $G = \langle A \mid R\rangle$ be a $C'(\frac{1}{6})$ presentation where no relator is a proper power. If H is a finitely presented group that admits a coarse embedding into G, then H is hyperbolic. In particular, all finitely presented subgroups of G are hyperbolic.*

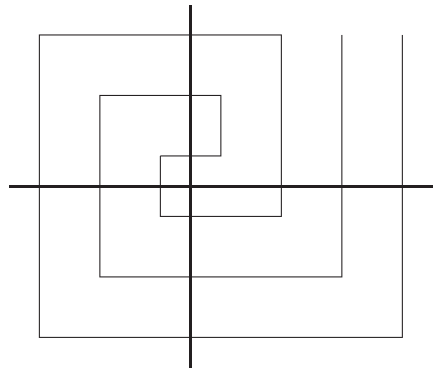

FIGURE 4 An example of a Lipschitz injection $\mathbb{Z} \to \mathbb{Z}^2$. In this example for any D there are points in $\mathbb{Z}$ of distance $\geqslant D$ such that there images have distance 1. However, the induced path metric on the subcomplex is quasi-isometric to $\mathbb{Z}$

Proof. Let H be a finitely presented group with a coarse embedding $f : H \to G$. Let Z be the universal cover of the presentation complex for G. Let X be the universal cover of a presentation complex for H. By Proposition 2.8, we have a cellular map $\phi : X \to Y$ where Y is a Cayley 2-complex for H containing Z as a deformation retract. Thus, Y is contractible.

Let $M = \phi(X)$. Then Theorem 2.11 shows that X and M are quasi-isometric, and by Lemma 3.2, they have equivalent filling functions.

Since Y is aspherical, we see by Lemma 3.3 that $\mathrm{Fill}^2_M(\ell) \prec \mathrm{Fill}^2_Y(\ell)$. The latter is equivalent to a linear function by Corollary 5.4. Thus, X satisfies a linear filling function, and hence, H is a hyperbolic group by [5, Theorem 3.1]. □

6 | FUTURE QUESTIONS

We finish with some discussion of questions of interest to the authors. Throughout this article, we considered coarse embeddings. However, one could weaken this condition by considering Lipschitz embeddings which are not coarse embeddings. In many cases, our theorems still hold. For example, in figure 4, we show an example where f is a Lipschitz embedding which is not a coarse embedding but the subcomplex is quasi-isometric to the group. This gives evidence to a possible strengthening of Theorem 4.2 and 4.4 where Lipschitz embeddings take the place of coarse embeddings. We do not know of a counterexample at this time. Therefore, we have the following questions.

Question 6.1. Suppose that H is a group of type F_{k+1}. Suppose that G admits a finite $(k+1)$-dimensional $K(G,1)$. If H admits a Lipschitz embedding into G, does there exist an inequality of $(k+1)$-th dimensional homological Dehn functions?

If not, can one find an example of groups G and H as above where there exist a Lipschitz embedding of H into G where H has strictly higher $(k+1)$-th dimensional homological Dehn function than G?

Through Theorem 4.3, we generalize [5, Theorem 5.4] to coarse embeddings. However, one may ask if the same holds for a general Lipschitz embedding.

Question 6.2. Let G be a hyperbolic group of geometric dimension 2. If H admits a Lipschitz embedding into G, is H necessarily hyperbolic?

Another finiteness question that we are interested in involves cohomological and homological dimension over a ring R. As originally proved by Sauer [14], if we are given a coarse embedding of H into G where both G and H have finite cohomological (homological) dimension over R, then the cohomological (homological) dimension of H is bounded by that of G. We may ask a similar question for Lipschitz embeddings.

Question 6.3. Suppose that H admits a Lipschitz embedding into G and that G and H both have finite cohomological (homological) dimension over a commutative ring R. Is the R-cohomological (R-homological) dimension of H bounded above by that of G?

If not, can one find an example of a Lipschitz embedding of groups G and H where H has strictly greater R-cohomological dimension than that of G?

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JOURNAL INFORMATION

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REFERENCES

1. J. M. Alonso, X. Wang, and S. J. Pride, *Higher-dimensional isoperimetric (or Dehn) functions of groups*, J. Group Theory **2** (1999), no. 1, 81–112.
2. I. M. Chiswell, D. J. Collins, and J. Huebschmann, *Aspherical group presentations*, Math. Z. **178** (1981), no. 1, 1–36.
3. D. B. Cohen and M. Pengitore, *Translation-like actions of nilpotent groups*, J. Topol. Anal. **11** (2019), no. 2, 357–370.
4. D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.
5. S. M. Gersten, *Subgroups of word hyperbolic groups in dimension 2*, J. Lond. Math. Soc. (2) **54** (1996), no. 2, 261–283.
6. M. Gruber, *Filling invariants of stratified nilpotent Lie groups*, Math. Z. **293** (2019), no. 1–2, 39–79.
7. R. G. Hanlon and E. Martínez Pedroza, *A subgroup theorem for homological filling functions*, Groups Geom. Dyn. **10** (2016), no. 3, 867–883.
8. D. H. and A. Sisto, *Groups with no coarse embeddings into hyperbolic groups*, New York J. Math. **23** (2017), 1657–1670.
9. D. Kielak and P. Kropholler, *Isoperimetric inequalities for poincaré duality groups*, arXiv:2008.07812, 2020.
10. U. Lang, *Higher-dimensional linear isoperimetric inequalities in hyperbolic groups*, Int. Math. Res. Notices (2000), no. 13, 709–717.
11. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer, Berlin, 2001. (Reprint of the 1977 edition.)
12. I. Mineyev, *Bounded cohomology characterizes hyperbolic groups*, Q. J. Math. **53** (2002), no. 1, 59–73.

13. A. Yu. Ol'shanskiĭ, *Geometry of defining relations in groups*, volume 70 of Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991. (Translated from the 1989 Russian original by Yu. A. Bakhturin.)
14. R. Sauer, *Homological invariants and quasi-isometry*, Geom. Funct. Anal. **16** (2006), no. 2, 476–515.
15. N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, volume 100 of Cambridge Tracts in Mathematics, Cambridge Univ. Press, Cambridge, 1992.
16. R. Young, *Homological and homotopical higher-order filling functions*, Groups Geom. Dyn. **5** (2011), no. 3, 683–690.
17. R. Young, *The Dehn function of* SL(n; $\mathbb{Z}$), Ann. of Math. (2) **177** (2013), no. 3, 969–1027.