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Virtual Critical Regularity of Mapping Class Group Actions on the Circle

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Abstract
We show that if G_1 and G_2 are non-solvable groups, then no $C^{1,\tau}$ action of $(G_1 \times G_2) * \mathbb{Z}$ on S^1 is faithful for $\tau > 0$. As a corollary, if S is an orientable surface of complexity at least three then the critical regularity of an arbitrary finite index subgroup of the mapping class group $\mathrm{Mod}(S)$ with respect to the circle is at most one, thus strengthening a result of the first two authors with Baik.

Keywords Free group · Non-solvable group · Smoothing · Critical regularity · Mapping class group · Invariant measure

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1 Introduction

Let G be a group, and let M be a smooth manifold. For $k \in \mathbb{N}$ and $\tau \in [0, 1]$, we denote by $\mathrm{Diff}^{k,\tau}(M)_0$ the group of C^k diffeomorphisms of M whose k^{th} derivatives are τ–Hölder continuous and are isotopic to the identity.

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The *critical regularity of G with respect to M* is defined to be

$$\mathrm{CritReg}_M(G) = \sup\{k + \tau \mid k \in \mathbb{N}, \tau \in [0, 1] \text{ and } G \text{ injects into } \mathrm{Diff}^{k,\tau}(M)_0\}.$$

By convention, $\mathrm{Homeo}(M)_0 = \mathrm{Diff}^0(M)_0$, and if G admits no injective homomorphism into $\mathrm{Homeo}(M)_0$ then $\mathrm{CritReg}_M(G) = -\infty$.

1.1 Main Results

In this article, we concentrate on computing the critical regularity of certain groups in the case $M = S^1$, and we will suppress M from the notation; therefore, we write

$$\mathrm{CritReg}(G) := \mathrm{CritReg}_{S^1}(G).$$

Note that $\mathrm{Homeo}(S^1)_0 = \mathrm{Homeo}_+(S^1)$, where the right hand side denotes the group of orientation preserving homeomorphisms of S^1. Our main result is as follows.

Theorem 1.1 *If G_1 and G_2 are non-solvable groups, then*

$$\mathrm{CritReg}((G_1 \times G_2) * \mathbb{Z}) \leq 1.$$

Every countable subgroup G of $\mathrm{Homeo}^+(S^1)$ is topologically conjugate to a group of bi–Lipschitz homeomorphisms of S^1 by [6]. Moreover, the group $G * \mathbb{Z}$ admits an embedding into $\mathrm{Homeo}^+(S^1)$ by [3].

It follows that

$$\mathrm{CritReg}(G) = \mathrm{CritReg}(G * \mathbb{Z}) \geq 1.$$

The following is now an immediate corollary of the main theorem.

Corollary 1.2 *We have*

$$\mathrm{CritReg}((F_2 \times F_2) * \mathbb{Z}) = 1.$$

We note that the group $(F_2 \times F_2) * \mathbb{Z}$ admits a faithful C^1–action on S^1 and on $I := [0, 1]$, as does every finitely generated residually torsion–free nilpotent group [8, 11, 24], and so Corollary 1.2 is optimal.

Corollary 1.2 allows us to compute the critical regularity of many mapping class groups of surfaces. Recall that if S is an orientable surface of genus g and with n punctures, boundary components, and marked points, we write $\mathrm{Mod}(S)$ for the group of isotopy classes of homeomorphisms of S that preserve the punctures, boundary components, and marked points (pointwise). We use $\xi(S)$ for the *complexity* of S, which is defined by

$$\xi(S) = 3g - 3 + n.$$

If $g \geq 2$ and $n = 1$ then $\mathrm{Mod}(S)$ acts faithfully on S^1, and if S has a boundary component then $\mathrm{Mod}(S)$ acts faithfully on I [4, 10, 23]. It was shown in [9] that the critical regularity of $\mathrm{Mod}(S)$ is at most two, provided that $g \geq 3$. This was strengthened in [18, 25], where it was shown that the critical regularity of $\mathrm{Mod}(S)$ is at most one. These latter results in fact showed that any C^1 action of the full mapping class group on S^1 factors through a finite group.

For finite index subgroups $H < \mathrm{Mod}(S)$, the critical regularity question is more complicated because finite index subgroups of mapping class groups are poorly understood. The first two authors and Baik [2] proved that if $\xi(S) \geq 2$, then every finite index subgroup H of $\mathrm{Mod}(S)$ satisfies $\mathrm{CritReg}(H) \leq 2$, answering a question of Farb in [7]. In [25], it is shown that *if* every finite index subgroup of the mapping class group has finite abelianization when $g \geq 3$ (i.e. *if* the Ivanov Conjecture holds), then $\mathrm{CritReg}(H) \leq 1$ for $H < \mathrm{Mod}(S)$ of finite index and S of genus at least 6 (and in fact no faithful C^1 action exists).

Whereas Corollary 1.2 does not rule out the existence of a faithful C^1 action of a finite index subgroup of the mapping class group, it does show that the critical regularity of a finite index subgroup of $\mathrm{Mod}(S)$ is bounded above by one.

Corollary 1.3 *Let S be a surface with $\xi(S) \geq 3$, and let $\tau > 0$. If H is a finite index subgroup of* $\mathrm{Mod}(S)$ *then it admits no faithful $C^{1,\tau}$–action on the circle; in particular* $\mathrm{CritReg}(H) \leq 1$.

To see how Corollary 1.2 implies Corollary 1.3, note that under the hypotheses on S, there are two subsurfaces S_1 and S_2 of S which are homeomorphic to tori with a single boundary component. The mapping class groups of both S_1 and S_2 contain copies of the group F_2, and the corresponding copies of F_2 commute with each other. Adjoining a pseudo-Anosov mapping class ψ of S gives a 5–generated group, which after passing to powers of the generators if necessary, furnishes a copy of $(F_2 \times F_2) * \mathbb{Z}$ in $\mathrm{Mod}(S)$ (by the main result of [17], for instance). Corollary 1.3 then follows immediately.

Note that Corollary 1.2 also implies an analogous version of Corollary 1.3 for the groups $Aut(F_n)$ and $Out(F_n)$, since whenever $n \geq 4$, these groups contain copies of mapping class groups that fall under the purview of Corollary 1.3. We will not comment on these points any further, since unlike mapping class groups of surfaces, automorphism groups of free groups are not known to have any natural actions on the circle.

We emphasize that Corollary 1.3 does not rule out the possibility that such a subgroup H may admit a faithful action on S^1 by C^1 diffeomorphisms. Whether or not such an action exists appears to be beyond the reach of current technology. The reader may consult section 8.3.5 of [16], for instance. For certain restricted classes of finite index subgroups, it is known that H cannot act faithfully by diffeomorphisms; see [26].

It is currently an open question for which surface S a finite index subgroup of $\mathrm{Mod}(S)$ admits a faithful C^0–action on S^1. In the case when H is such a finite index subgroup we have from the above corollary that $\mathrm{CritReg}(H) = 1$. We note that it is usually quite difficult to compute the critical regularity of a particular group whose critical regularity is known to be finite. For a survey of results, the reader is directed to [5, 12, 14, 19, 20].

Corollary 1.3 follows immediately from Corollary 1.2 after observing that under the assumption that $\xi(S) \geq 3$, the group $\mathrm{Mod}(S)$ and all of its finite index subgroups contain copies of $(F_2 \times F_2) * \mathbb{Z}$ (cf. [2, 15, 17]).

In the case where $\xi(S) \leq 1$, the mapping class group of S is virtually free, so that $\mathrm{CritReg}(H) = \infty$ for a suitable finite index subgroup H of $\mathrm{Mod}(S)$. The only case that is left out from Corollary 1.3 is exactly when $\xi(S) = 2$:

Question 1.4 *Let S be a twice–punctured torus or a five–times punctured sphere. Does some finite index subgroup of* $\mathrm{Mod}(S)$ *admit a faithful $C^{1,\tau}$ action on S^1 with $\tau > 0$?*

1.2 A Dynamical Perspective on the Main Result

For the remainder of this section, we frame the discussion of this article in a more precise manner, and while doing so introduce some relevant concepts. Let G be a group acting on a space X, and define the *(open) support* of $g \in G$ by

$$\operatorname{supp} g := X \setminus Fix g.$$

The *support of G* is the set

$$\operatorname{supp} G := \bigcup_{g \in G} \operatorname{supp} g.$$

We call each point in

$$Fix G := \bigcap_{g \in G} Fix g$$

a *global fixed point* of G.

Let us say G admits a *disjointly supported pair* (or, G is *non–overlapping*) if there exist nontrivial elements $g, h \in G$ satisfying

$$\operatorname{supp} g \cap \operatorname{supp} h = \varnothing.$$

In [15], the authors proved the following result, which was partially based on the methods in [20]:

Theorem 1.5 ([15], Theorem 1.1) *If G_1 and G_2 are non-solvable groups, and if $\tau > 0$, then there is no faithful $C^{1,\tau}$ action of $(G_1 \times G_2) * \mathbb{Z}$ on a compact interval.*

In that paper, the main technical result was the following.

Theorem 1.6 ([15, Section 4.1]) *Let $\tau > 0$ be a real number, and let $k \geq 3$ be an integer such that $\tau(1+\tau)^{k-2} \geq 1$. If G_1 and G_2 are groups that are not solvable of degree at most k, and if*

$$G := G_1 \times G_2 \longrightarrow \mathrm{Diff}_+^{1,\tau}([0,1])$$

is an embedding, then G contains a disjointly supported pair.

Theorem 1.5 follows from Theorem 1.6 by an application of the *abt*–Lemma from [13]:

Proposition 1.7 (The *abt*–Lemma) *Let* $M \in \{I, S^1\}$ *and let* $a, b \in \mathrm{Diff}^1_+(M)$ *be such that*

$$\operatorname{supp} a \cap \operatorname{supp} b = \varnothing.$$

Then if $t \in \mathrm{Diff}^1_+(M)$ *is arbitrary, the group* $\langle a, b, t\rangle$ *is not isomorphic to* $\mathbb{Z}^2 * \mathbb{Z}$.

Proposition 1.7 implies that if a group G *always* has elements with disjoint supports whenever acting on I or S^1 by diffeomorphisms of some regularity, then $G * \mathbb{Z}$ *never* acts faithfully by diffeomorphisms on I or S^1 of that regularity. Thus, to prove Theorem 1.1, it will suffice for us to establish the following:

Proposition 1.8 *Let* $\tau > 0$*, and let* G_1 *and* G_2 *be non-solvable groups. If* ϕ *is a* $C^{1,\tau}$*–action of* $G_1 \times G_2$ *on* S^1*, then* G *admits a disjointly supported pair.*

We will argue Proposition 1.8 by showing directly that the commutator subgroup of $G_1 \times G_2$ admits a global fixed point, thus reducing to Theorem 1.5.

2 Preliminaries

For a direct product of groups

$$G = G_1 \times G_2,$$

we identify G_1 with $G_1 \times \{1\}$ and G_2 with $\{1\} \times G_2$, so that G_i is a normal subgroup of G for $i \in \{1, 2\}$; moreover, we have $G = \langle G_1, G_2\rangle$.

Now, suppose a subgroup $G \leq \mathrm{Homeo}^+(S^1)$ is given. A Borel probability measure μ on S^1 is said to be G*–invariant* if for all $g \in G$ and for all measurable $A \subset S^1$, we have $\mu(A) = \mu(g^{-1}A)$. The *support* of μ, denoted as $\operatorname{supp}\mu$, means the largest closed subset $X \subset S^1$ such that every open subset of X has positive measure.

Recall that the *rotation number*

$$\mathrm{rot}\colon \mathrm{Homeo}_+(S^1) \longrightarrow \mathbb{R}/\mathbb{Z}$$

is defined as follows. Let $f \in \mathrm{Homeo}_+(S^1)$, and lift f to $F \in \mathrm{Homeo}_+(\mathbb{R})$. Note that such a lift is always periodic, and that any two such lifts differ by an integer translation. One chooses an arbitrary $x \in \mathbb{R}$ and writes

$$\mathrm{rot}(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \pmod{\mathbb{Z}}.$$

It is not difficult to check that the definition is independent of all the choices made.

A standard fact is that an orientation preserving homeomorphism of S^1 has nonzero rotation number if and only if it has no fixed points [1, 21]. We will appeal to the following basic fact relating rotation numbers and invariant measures; note that the second part of the proposition is an immediate consequence of the first.

Proposition 2.1 (see [21], Theorem 2.2.10) *If* $G \leq \mathrm{Homeo}_+(S^1)$ *admits an invariant measure* μ*, then the restriction*

$$\mathrm{rot}\restriction_G\colon G \to \mathbb{R}/\mathbb{Z}$$

is a group homomorphism satisfying

$$\mathrm{rot}(g) = \mu[x, g(x))$$

for all $g \in G$ *and* $x \in S^1$. *Moreover, the kernel of this homomorphism fixes every point in* $\operatorname{supp}\mu$.

We now recall some ideas from [15] that will be crucial in the proof of our main result. Following [22], we say that two elements $f, g \in \mathrm{Homeo}_+(\mathbb{R})$ are *crossed* if there exist point $u < w < v$ in $\mathbb{R}$ such that:

(1) $g^n(u) < w < f^n(v)$ for all $n \in \mathbb{Z}$;
(2) There is an $N \in \mathbb{Z}$ such that $g^N(v) < w < f^N(u)$.

A group action of G on $\mathbb{R}$ by orientation preserving homeomorphisms is called *Conradian* if it admits no crossed elements.

Lemma 2.2 ([15], Lemma 3.10 (Centralizer–Conradian Lemma)) *Let* $\tau > 0$, *and let* $G \leq \mathrm{Diff}_+^{1,\tau}([0,1])$. *If* c *is a central element of* G, *then the restriction of* G *to* $\operatorname{supp} c$ *is Conradian.*

The relationship between $C^{1,\tau}$ actions and Conradian actions is elucidated by the following technical fact.

Lemma 2.3 ([15], Lemmas 3.4) *If* $\tau, u > 0$ *are real numbers, and if* $k \geq 2$ *is an integer satisfying* $\tau(1+\tau)^{k-2} \geq u$, *then* $\mathrm{Diff}_+^{1,\tau}([0,1])$ *does not contain a* (k, u)*–nesting.*

Briefly speaking, a (k, u)*–nesting* is a finite set $S \subseteq \mathrm{Homeo}^+([0,1])$ such that for some infinite sequence $(s_1, s_2, \ldots)$ of elements from S, for some nested open intervals

$$J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_k,$$

and for some choices

$$\{t_{n,i} \mid 2 \leq i \leq k, n \geq 0\} \subseteq S,$$

one has

$$\sum_{n \geq 0} |s_n \cdots s_2 s_1 J_1|^u < \infty,$$

together with

$$t_{n,i} w_n J_i \cap w_n J_i = \varnothing, \quad t_{n,i} w_n J_{i-1} = w_n J_{i-1},$$

where here $w_n = s_n \cdots s_1$. A (k, u)–nesting is a feature of an action that is weaker than the classical notion of a "k–level structure" [20].

Lemma 2.4 ([15], Lemma 3.13) *Let* $G \leq \mathrm{Homeo}_+([0,1])$ *be a Conradian group such that* $G^{(k)} \neq 1$ *for some* $k \geq 2$. *If* c *is a central element of* G *fixing no points in* $(0,1)$, *then* G *contains a* $(k,1)$*–nesting.*

One may take the (k, u)–nesting as a black box for the purpose of this paper, and only note the following immediate consequence of the three preceding lemmas.

Lemma 2.5 *Let $\tau > 0$ be a real number and $k \geq 3$ be an integer such that $\tau(1+\tau)^{k-2} \geq 1$. If c is a central element of $G \leq \mathrm{Diff}_+^{1,\tau}([0,1])$ fixing no points in $(0,1)$, then $G^{(k)} = 1$.*

A fixed point a of $g \in \mathrm{Diff}_+^1(S^1)$ is called a *hyperbolic fixed point* if $g'(a) \neq 1$. The following deep theorem of Deroin–Kleptsyn–Navas (which is a generalization of a result due to Sacksteder) will be an important ingredient for us.

Theorem 2.6 ([6]) *If a subgroup G of $\mathrm{Diff}_+^1(S^1)$ preserves no probability measure on S^1, then G contains an element g such that $Fix g$ is nonempty, finite, and consists entirely of hyperbolic fixed points.*

Remark 2.7 For a group $G \leq \mathrm{Homeo}^+(S^1)$ that does not admit a finite orbit, there uniquely exists a smallest, nonempty, closed G–invariant set Λ_G, called the *limit set* of G; see Theorem 2.1.1 in [21], for instance. The limit set is either S^1 or a Cantor set, the latter of which is called the *exceptional minimal set* of G. In Theorem 2.6, we can find a point $x \in \Lambda_G \setminus \mathrm{Fix} g$, since the limit set Λ_G is necessarily infinite. Consider now the component J of $\mathrm{supp} g$ containing x. The G–invariance of Λ_G implies that

$$\partial J = g^{\pm\infty}(x) \subseteq \mathrm{Fix} g \cap \Lambda_G.$$

In other words, we can always find a hyperbolic fixed point of g in Λ_G.

3 Establishing the Main Result

A group G is said to be *solvable of degree at most k* if the subgroup $G^{(k)}$, the k–th term in the derived series, is trivial. As noted in the introduction Theorem 1.1 will follow from Proposition 1.8, which in turn is an immediate consequence of the stronger result given below.

Theorem 3.1 *Let $k \geq 3$ be an integer, and let $\tau > 0$ be a real number satisfying $\tau(1+\tau)^{k-2} \geq 1$. If G_1 and G_2 are groups that are not solvable of degree at most $(k+1)$, then every faithful $C^{1,\tau}$–action of $G_1 \times G_2$ on S^1 admits a disjointly supported pair. In particular, we have that*

$$\mathrm{CritReg}\left((G_1 \times G_2) * \mathbb{Z}\right) \leq 1 + \tau.$$

Note that the second part of the theorem follows from the first along with the *abt*–Lemma (Proposition 1.7). The lemma below is a key step in the proof of the first part.

Lemma 3.2 *Let k and τ be as in Theorem 3.1. If a group $H \leq \mathrm{Diff}_+^{1,\tau}(S^1)$ can be written as a direct product $H = H_1 \times H_2$, and if H_1 does not preserve a probability measure on S^1, then H_2 is solvable of degree at most $(k+1)$.*

Proof From Theorem 2.6 and Remark 2.7, we can find some $c \in H_1$ and $a \in \Lambda_{H_1} \cap \mathrm{Fix} c$ such that c fixes finitely many points and such that $c'(a) \neq 1$. For all $h \in H_2$, the point $h(a)$ is also a hyperbolic fixed point of c with the derivative $c'(a)$ since

$$c' \circ h(a) = (c \circ h)'(a)/h'(a) = h'(c(a)) \cdot c'(a)/h'(a) = c'(a).$$

It follows that $H_2(a)$ does not have an accumulation point, and in particular is finite. As H_2 admits an invariant probability measure (with atoms at points of $H_2(a)$), we see from Proposition 2.1 that $K := [H_2, H_2]$ fixes the point a.

Let U_1 and U_2 be the two components of $\mathrm{supp}\, c$ containing a on their boundaries.

The group K preserves each U_i, since K permutes the components of $\mathrm{supp}\, c$ and fixes the point a. Applying Lemma 2.5 to the restriction

$$(\langle c \rangle \times K) \restriction_{\overline{U_i}},$$

we see that $K^{(k)}$ acts trivially on U_i for $i = 1, 2$.

Suppose V is a component of the support of K, not intersecting $U_1 \cup U_2$. Since a lies in the limit set of H_1, we can find some $h_1 \in H_1$ such that

$$h_1(V) \subseteq U_1 \cup U_2.$$

Let $g \in K^{(k)}$ and $v \in V$ be arbitrary. Since g acts trivially on $h_1(v)$, we have that

$$g(v) = h_1^{-1} \circ g \circ h_1(v) = h_1^{-1} \circ h_1(v) = v.$$

Combined with the preceding paragraph, this proves that

$$H_2^{(k+1)} = K^{(k)} = 1.$$

□

We note the following general observation regarding topological actions. The first part of the lemma is well-known. Presumably, the second part is also known to experts, but the authors could not find a written reference for it.

Lemma 3.3 *Let $H = H_1 \times H_2$ be a subgroup of* $\mathrm{Homeo}^+(S^1)$.

(1) *If each H_i admits a global fixed point, then so does H.*
(2) *If each H_i preserves a Borel probability measure on S^1, then so does H.*

Proof (3.3) Suppose not. Since $Fix H_1 \cap Fix H_2 = \varnothing$, we can find some $b \in Fix H_1 \cap \mathrm{supp}\, H_2$. Let J be the component of $\mathrm{supp}\, H_2$ containing b. There exists a sequence $\{h_n\}$ in H_2 such that

$$b' := \lim_{n \to \infty} h_n(b) \in \partial J.$$

Then $b' \in Fix H_1 \cap Fix H_2$, which is a contradiction.

(3.3) Let μ_i be a probability measure preserved by H_i. By Proposition 2.1, the restriction of rot to each H_i is a homomorphism.

Suppose first that $\mathrm{rot}(H_1) \cup \mathrm{rot}(H_2)$ is a discrete subset of $\mathbb{R}/\mathbb{Z}$. This means that K_i, the kernel of the map $\mathrm{rot} : H_i \longrightarrow \mathbb{Q}/\mathbb{Z}$, has finite index in H_i ($i = 1, 2$). Since each K_i admits a global fixed point, so does $K_1 \times K_2$. This latter group has finite index in H, and so H has a finite orbit and preserves a probability measure.

We now assume that $\mathrm{rot}(H_1) \cup \mathrm{rot}(H_2)$ is indiscrete in $\mathbb{R}/\mathbb{Z}$. Without loss of generality, $\mathrm{rot}(H_1)$ is a dense subgroup of $\mathbb{R}/\mathbb{Z}$. By a result of Plante (see Proposition 2.2 of [27]), it follows that H_1 preserves a *unique* Borel probability measure μ_1. Finally, if $h_2 \in H_2$ and $h_1 \in H_1$, then

$$h_1^* h_2^* \mu_1 = h_2^* h_1^* \mu_1 = h_2^* \mu_1.$$

The uniqueness of μ_1 implies that $h_2^* \mu_1 = \mu_1$. In other words we have shown that μ_1 is also H_2–invariant, and so also H–invariant. □

Proof of Theorem 3.1 We may assume that the given group $G := G_1 \times G_2$ is a subgroup of $\mathrm{Diff}_+^{1,\tau}(S^1)$. □

If some G_i does not admit an invariant probability measure, we apply Lemma 3.2 to obtain a contradiction. So, we will assume that each G_i preserves a probability measure. Lemma 3.3 implies that G also preserves a probability measure μ.

By Proposition 2.1 the rotation number is trivial on the group

$$H := [G, G] = [G_1, G_1] \times [G_2, G_2].$$

Moreover, the support of μ is contained in the global fixed point set of H, which is therefore nonempty. So, the inclusion $H \hookrightarrow \mathrm{Diff}_+^{1,\tau}(S^1)$ factors through an injection $H \hookrightarrow \mathrm{Diff}_+^{1,\tau}([0,1])$. By Theorem 1.6, it follows that H admits a disjointly supported pair. Along with the *abt*–Lemma (Proposition 1.7), this completes the proof.

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Declarations

Conflict of Interest The authors declare no competing interests.

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