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ABSTRACT

In our prior work toward Bartnik's static vacuum extension conjecture for near Euclidean boundary data, we establish a sufficient condition, called static regular, and confirm that large classes of boundary hypersurfaces are static regular. In this paper, we further improve some of those prior results. Specifically, we show that any hypersurface in an open and dense subfamily of a certain general smooth one-sided family of hypersurfaces (not necessarily a foliation) is static regular. The proof uses some of our new arguments motivated from studying the conjecture for boundary data near an arbitrary static vacuum metric.

I. INTRODUCTION

Let $n \geq 3$ and (M, g) be an n-dimensional Riemannian manifold. We say that (M, g) is *static vacuum* (or g is a *static vacuum metric* on M) if there is a scalar-valued function u on M satisfying

$$\begin{aligned} -u\mathrm{Ric}_g + \nabla_g^2 u &= 0, \\ \Delta_g u &= 0. \end{aligned} \tag{1.1}$$

Such u is called a *static potential*. The class of static vacuum metrics has played a fundamental role in general relativity because when $u > 0$, the triple (M, g, u) gives rise to a Ricci flat spacetime $(\mathbb{R} \times M, -u^2 dt^2 + g)$ that has a global Killing vector field ∂_t.

A very important example of asymptotically flat, static vacuum metrics is the family of (Riemannian) Schwarzschild metrics g_m,

$$g_m = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{S^{n-1}} \qquad \text{defined on } \mathbb{R}^n \backslash B_{(2m)^{\frac{1}{n-2}}},$$

with the static potential $u_m = \sqrt{1 - \frac{2m}{r^{n-2}}}$, where $g_{S^{n-1}}$ is the standard metric on the unit sphere S^{n-1}. Note that the Schwarzschild metrics are rotationally symmetric. When $m = 0$, the Schwarzschild metric becomes the Euclidean metric. When $m > 0$, the Schwarzschild manifold has a minimal hypersurface boundary, precisely at $u = 0$. In fact, the Schwarzschild metrics are the only asymptotically flat, static vacuum 3-manifolds with such a property by the celebrated uniqueness theorem of static black holes (see Refs. 1–3). Another family of static vacuum exact solutions was discovered by Weyl. The Weyl solutions are axially symmetric and have general asymptotics at infinity, but a subclass of them can have an asymptotically flat end. Those exact solutions can be characterized by certain conditions (e.g., having black hole boundary), and great efforts have been made toward the uniqueness and classification results of those static vacuum metrics. See, for example, Reiris and Peraza[4] and the references therein.

In contrast, Robert Bartnik conjectured the following "prescribing boundary value" problem for asymptotically flat, static vacuum manifolds [see Ref. 5 (Conjecture 7) and Ref. 6]. The conjecture was originated from his quasi-local mass program in 1989, for which we refer to the reader to the survey article of Anderson[7] for details. The conjecture itself is also of independent interest as a natural geometric partial differential equation (PDE) boundary value problem. Furthermore, progress toward this conjecture would give rise to new examples of asymptotically flat, static vacuum metrics and advance our understanding toward the structure of static vacuum metrics (Ref. 6).

Conjecture 1 (static extension conjecture). *Let (Ω, g_0) be a compact manifold with scalar curvature $R_{g_0} \geq 0$. Suppose the mean curvature H_{g_0} is positive somewhere on the boundary Σ. Then, there exists a unique asymptotically flat, static vacuum manifold (M, g) with the boundary $\partial M \cong \Sigma$ satisfying*

$$\begin{aligned} g_0^\top &= g^\top \\ H_{g_0} &= H_g \end{aligned} \qquad \text{on } \Sigma.$$

Here, $(\cdot)^\top$ denotes the restriction on the tangent bundle of Σ.

Convention: The mean curvature H_g of a hypersurface Σ in a Riemannian manifold (M, g) is defined as $H_g = \operatorname{div}_g \nu$, where ν is the unit normal vector of Σ. When (M, g) is asymptotically flat, we choose ν to point to infinity [and thus, the unit normal for Σ in (Ω, g_0) points outward].

We shall refer to the geometric boundary data $(g_0^\top, H_{g_0})$ as the *Bartnik boundary data*. Let us also remark on the assumption that H_{g_0} is positive somewhere. The conjecture would fail without this assumption because such an extension, if exist, would contain a minimal hypersurface homologous to the boundary (at least in dimensions $n \leq 7$), and the extension must be Schwarzschild by the uniqueness theorem, which put strong restriction on $g_0^\top$. See Ref. 8 for $n = 3$. For $n \leq 7$, by minimal surface theory, there is an outermost minimal hypersurface homologous to the boundary. From the result of Martin, Miao, and the second author of this paper [Ref. 9, Theorem 1], the static potential $u = 0$ on the outmost minimal hypersurface. From there, one applies the generalization of uniqueness of static black holes in higher dimensions by Gibbons, Ida, and Shiromizu.[10]

Even with the mean curvature assumption, it is highly speculated that Conjecture 1 does not hold, in general, as stated. Let Ω be a bounded open subset in $\mathbb{R}^n$. Observed by Refs. 11 and 12, if the boundary $\Sigma = \partial\overline{\Omega}$ is only *inner* embedded, i.e., Σ touches itself from the exterior region $\mathbb{R}^n \setminus \Omega$, the induced data $(\bar{g}^\top, H_{\bar{g}})$ is valid Bartnik boundary data, but $(\bar{g}, 1)$ in $\mathbb{R}^n \setminus \Omega$ is *not* a valid static vacuum extension as $\mathbb{R}^n \setminus \Omega$ is not a manifold with the boundary. One can further arrange so that the mean curvature $H_{\bar{g}}$ is positive everywhere. Those inner embedded hypersurfaces are conjectured to be counter-examples to Conjecture 1 by Ref. 11 (Conjecture 5.2) (see also Ref. 7), though it is not clear whether there could be another static vacuum extension far way from $(\mathbb{R}^n \setminus \Omega, \bar{g}, 1)$. Nevertheless, positive results to Conjecture 1, under suitable assumptions, will provide a structure theory for the space of static vacuum metrics (parameterized by their Bartnik boundary data). It also connects the fundamental problem on isometric embeddings of hypersurfaces into a static vacuum manifold with prescribed mean curvature. In particular, that question apparently has intriguing connections to the work of Chen, Wang, Wang, and Yau[13] where the notion of quasi-local energy defined via isometric embeddings into a reference static metric is proposed, extending the celebrated Wang–Yau quasi-local mass with respect to the Minkowski spacetime.[14]

There are some positive results toward Conjecture 1. The existence and local uniqueness is proven for $n = 3$ and for (g_0, H_{g_0}) sufficiently close to the induced Bartnik boundary data on a round sphere from the Euclidean metric, i.e., (g_0, H_{g_0}) sufficiently close to $(g_{S^2}, 2)$. See the work of Miao,[15] Anderson–Khuri,[12] and Anderson.[16] In recent work,[17] we give a general framework to tackle Conjecture 1 and confirm the existence and local uniqueness of Conjecture 1 for large classes of boundary data, including those close to the induced boundary data on either any star-shaped hypersurfaces or quite general perturbed hypersurfaces in the Euclidean space. In this paper, we improve Theorem 7 in Ref. 17 by employing new arguments in our recent work.[18] The new results are presented as Theorem 7, Corollary 8, and Theorem 9.

To describe the new results, we first recall the basic notations and definitions and review relevant results from Ref. 17.

Let Ω be a bounded open subset in $\mathbb{R}^n$ whose boundary $\Sigma = \partial\overline{\Omega}$ is a connected, embedded smooth hypersurface in $\mathbb{R}^n$. We denote by $\bar{g}$ the Euclidean metric in $\mathbb{R}^n$ with $\bar{g}_{ij} = \delta_{ij}$ (with respect to a fixed Cartesian coordinate chart). Our analytic framework is based on the weighted Hölder spaces $\mathcal{C}^{k,\alpha}_{-q}(\mathbb{R}^n \setminus \Omega)$ (see its definition in Sec. 2.1 of Ref. 17), and we always assume the Hölder exponent $\alpha \in (0, 1)$ and the fall-off rate $q \in (\frac{n-2}{2}, n-2)$ for asymptotical flatness. We denote by $D\mathrm{Ric}|_{\bar{g}}(h)$ the linearization of the Ricci curvature at $\bar{g}$; namely, let $g(t)$ be an arbitrary family of Riemannian metrics on $\mathbb{R}^n \setminus \Omega$ so that $g(0) = \bar{g}$ and $g'(0) = h$, then $D\mathrm{Ric}|_{\bar{g}}(h) := \frac{d}{dt}\big|_{t=0} \mathrm{Ric}_{g(t)}$. Similarly, we define the linearizations of the mean curvature and second fundamental form on Σ by $DH|_{\bar{g}}(h)$ and $DA|_{\bar{g}}(h)$, respectively. We will omit the subscript $\bar{g}$ in those linearizations when the context is clear.

Definition 2. The boundary Σ is said to be *static regular in* $\mathbb{R}^n \setminus \Omega$ if for any pair of a symmetric $(0, 2)$-tensor h and a scalar-valued function v satisfying $(h, v) \in \mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n \setminus \Omega)$ and

$$\begin{aligned} -D\mathrm{Ric}(h) + \nabla^2 v = 0, \quad \Delta v = 0 \quad &\text{in } \mathbb{R}^n \setminus \Omega, \\ h^\top = 0, \quad DH(h) = 0 \quad &\text{on } \Sigma, \end{aligned} \tag{1.2}$$

we must have $DA(h) = 0$ on Σ.

The following fundamental result obtained in Ref. 17 says that "static regular" is a sufficient condition for existence and local uniqueness.

Theorem 3 ([Ref. 17, Theorem 3]). *Suppose the boundary Σ is static regular in $\mathbb{R}^n\backslash\Omega$. Then, there exist positive constants ϵ_0, C such that for each $\epsilon \in (0,\epsilon_0)$, if (τ,ϕ) satisfies $\|(\tau,\phi)-(\bar{g}^\top, H_{\bar{g}})\|_{\mathcal{C}^{2,\alpha}(\Sigma)\times\mathcal{C}^{1,\alpha}(\Sigma)} < \epsilon$, then there exists an asymptotically flat pair (g,u) with $\|(g,u)-(\bar{g},1)\|_{\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\backslash\Omega)} < C\epsilon$ such that (g,u) is a static vacuum pair in $\mathbb{R}^n\backslash\Omega$ having the Bartnik boundary data $(g^\top, H_g) = (\tau,\phi)$ on Σ.*

Furthermore, the solution (g,u) is geometrically unique in a neighborhood $\mathcal{U}$ of $(\bar{g},1)$ in the $\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\backslash\Omega)$-norm.

We remark that the "local uniqueness" in the above theorem is precisely described under the static-harmonic gauge and the orthogonal gauge. Since we will not explicitly use them in the present paper, we refer to the reader to the discussion right after Theorem 3 in Ref. 17.

In Ref. 17, we furthermore show that large classes of hypersurfaces in $\mathbb{R}^n$ are static regular. In particular, we show that static regular hypersurfaces are "dense" in the following concrete sense. A family of embedded hypersurfaces $\{\Sigma_t\} \subset \mathbb{R}^n$ is said to form a *smooth generalized foliation* if the deformation vector X of $\{\Sigma_t\}$ is smooth and on each Σ_t, $\bar{g}(X,\nu) = \zeta$, where $\zeta > 0$ in a dense subset of Σ_t and ν is the unit normal of Σ_t. In other words, $\{\Sigma_t\}$ is slightly more general than a foliation in that the leaves can overlap on a nowhere dense subset.

Theorem 4 ([Ref. 17, Theorem 7]). *Let $\delta > 0$, $t \in [-\delta,\delta]$, and each $\Omega_t \subset \mathbb{R}^n$ be a bounded open subset with the hypersurface boundary $\Sigma_t = \partial\overline{\Omega_t}$ embedded in $\mathbb{R}^n$. Suppose the boundaries $\{\Sigma_t\}$ form a smooth generalized foliation. Then, there is an open dense subset $J \subset (-\delta,\delta)$ such that Σ_t is static regular in $\mathbb{R}^n\backslash\Omega_t$ for all $t \in J$.*

The above theorem has the following strong consequence because of the dilation property of the Euclidean static vacuum pair:

Corollary 5 ([Ref. 17, Corollary 8]). *Let Ω be a bounded open subset in $\mathbb{R}^n$ whose boundary $\Sigma = \partial\overline{\Omega}$ is a star-shaped hypersurface. Then, Σ is static regular in $\mathbb{R}^n\backslash\Omega$.*

The purpose of this paper is to extend Theorem 4.

Definition 6. A collection of embedded hypersurfaces $\{\Sigma_t\} \subset \mathbb{R}^n$ is a *smooth one-sided family of hypersurfaces foliating along $\hat{\Sigma}_t \subset \Sigma_t$ with relatively simply connected $\Sigma_t\backslash\hat{\Sigma}_t$* if the deformation vector X of $\{\Sigma_t\}$ is smooth, and on each Σ_t, $\bar{g}(X,\nu) = \zeta \geq 0$ with $\zeta > 0$ in a dense subset of each $\hat{\Sigma}_t \subset \Sigma_t$ satisfying $\pi_1(\Sigma_t,\hat{\Sigma}_t) = 0$.

For a subset U in M, the condition $\pi_1(M,U) = 0$ says that U is connected and the inclusion map $U \hookrightarrow M$ induces a surjection $\pi_1(U) \to \pi_1(M)$. In our setting, we clearly have $\pi_1(\mathbb{R}^n\backslash\Omega,\Sigma) = 0$. Thus, the additional condition for a subset $\hat{\Sigma} \subset \Sigma$ to satisfy $\pi_1(\Sigma,\hat{\Sigma}) = 0$ implies $\pi_1(\mathbb{R}^n\backslash\Omega,\hat{\Sigma}) = 0$. In loose terms, the later condition implies that for any point $x \in \mathbb{R}^n\backslash\Omega$, all paths from x to $\hat{\Sigma}$ are (homotopically) equivalent. See Fig. 1. This property is used in Theorem 2.5 to ensure certain global extensions of local vector fields.

Note that a smooth generalized foliation $\{\Sigma_t\}$ defined earlier is necessarily a smooth one-sided family of hypersurfaces foliating along $\hat{\Sigma}_t \subset \Sigma_t$ with relatively simply connected $\Sigma_t\backslash\hat{\Sigma}_t$ (by letting $\hat{\Sigma}_t = \Sigma_t$). However, a smooth one-sided family $\{\Sigma_t\}$ in the above sense may not form a foliation because the leaves can overlap on $\Sigma_t\backslash\hat{\Sigma}_t$. The following theorem generalizes Theorem 4.

Theorem 7. *Let $\delta > 0$, $t \in [-\delta,\delta]$, and each $\Omega_t \subset \mathbb{R}^n$ be a bounded open subset with the hypersurface boundary $\Sigma_t = \partial\overline{\Omega_t}$ embedded in $\mathbb{R}^n$. Suppose the boundaries $\{\Sigma_t\}$ form a smooth one-sided family of hypersurfaces foliating along $\hat{\Sigma}_t \subset \Sigma_t$ with relatively simply connected $\Sigma_t\backslash\hat{\Sigma}_t$. Then, there is an open dense subset $J \subset (-\delta,\delta)$ such that Σ_t is static regular in $\mathbb{R}^n\backslash\Omega_t$ for all $t \in J$.*

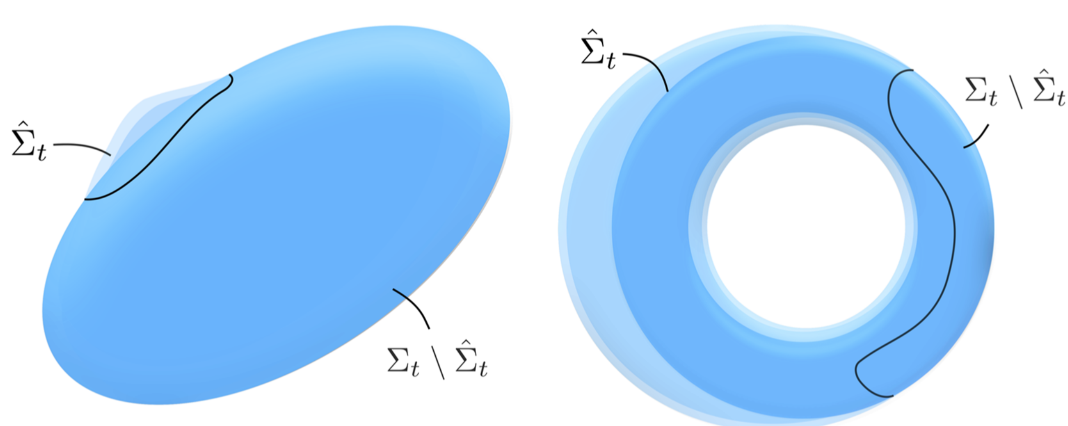

FIG. 1. Each figure illustrates Definition 6 that $\{\Sigma_t\}$ foliates along $\hat{\Sigma}_t$ with relatively simply connected $\Sigma_t\backslash\hat{\Sigma}_t$. In the left figure, a one-sided family of (topological) spheres $\{\Sigma_t\}$ is shown where $\hat{\Sigma}_t$ can be very small. In other words, Σ_t can largely overlap on $\Sigma_t\backslash\hat{\Sigma}_t$. The right figure illustrates a one-sided family of (topological) tori with $\pi_1(\Sigma_t,\hat{\Sigma}_t) = 0$.

In the special case that Σ is simply connected (e.g., Σ is a topological sphere), we trivially have $\pi_1(\Sigma,\hat\Sigma)=0$ for any nonempty connected subset $\hat\Sigma$. Slight perturbation on $\hat\Sigma$ produces a one-sided family $\{\Sigma_t\}$ with $\Sigma_0=\Sigma$ that foliates along small subsets $\hat\Sigma_t\subset\Sigma_t$ so that $\Sigma_t\setminus\hat\Sigma_t$ coincides with $\Sigma\setminus\hat\Sigma$ for all t. See the left figure in Fig. 1. Theorem 7 says that Σ_t is static regular for t in an open and dense set. Together with Theorem 3, we give the following corollary that one can solve for static vacuum extensions whose boundary data are arbitrarily close to the induced boundary data $(\bar g^\top, H_{\bar g})$ on Σ, except a small subset $\hat\Sigma\subset\Sigma$.

Corollary 8. Let $\Sigma=\partial\overline{\Omega}$ be a simply connected, closed, embedded hypersurface in $\mathbb{R}^n$. Given any nonempty open subset $\hat\Sigma\subset\Sigma$ and any $\delta>0$, there exists $(\tau_0,\phi_0)\in\mathcal{C}^{2,\alpha}(\Sigma)\times\mathcal{C}^{1,\alpha}(\Sigma)$ and constants $\epsilon_0, C>0$ satisfying

$$
\begin{aligned}
&(\tau_0,\phi_0)=\left(\bar g^\top, H_{\bar g}\right)\ \ on\ \ \Sigma\setminus\hat\Sigma\\
&\left\|(\tau_0,\phi_0)-\left(\bar g^\top, H_{\bar g}\right)\right\|_{\mathcal{C}^{2,\alpha}(\hat\Sigma)\times\mathcal{C}^{1,\alpha}(\hat\Sigma)}<\delta
\end{aligned}
$$

such that for each $\epsilon\in(0,\epsilon_0)$, if (τ,ϕ) satisfies $\|(\tau,\phi)-(\tau_0,\phi_0)\|_{\mathcal{C}^{2,\alpha}(\Sigma)\times\mathcal{C}^{1,\alpha}(\Sigma)}<\epsilon$, then there exists an asymptotically flat pair (g,u) with $\|(g,u)-(\bar g,1)\|_{\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)}<C\epsilon$ such that (g,u) is a static vacuum pair in $\mathbb{R}^n\setminus\Omega$ having the Bartnik boundary data $(g^\top, H_g)=(\tau,\phi)$ on Σ.

Furthermore, the solution (g,u) is geometrically unique in a neighborhood $\mathcal{U}$ of $(\bar g,1)$ in the $\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$-norm.

The proof to Theorem 7 involves several new arguments used in our recent work for general asymptotically flat, static vacuum background metrics.[18] One of the key arguments is the following theorem, which can be viewed as a uniqueness theorem for "localized" boundary data.

Theorem 9. *Let $\hat\Sigma$ be an open subset of Σ (can be the entire Σ) satisfying $\pi_1(\Sigma,\hat\Sigma)=0$. Let $(h,v)\in\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ solve*

$$
\begin{cases}
-D\mathrm{Ric}(h)+\nabla^2 v=0\\
\Delta v=0
\end{cases}\qquad in\ \ \mathbb{R}^n\setminus\Omega,
$$

$$
\begin{cases}
h^\top=0\\
DA(h)=0\\
D(\nabla_\nu A)(h)=0
\end{cases}\qquad on\ \ \hat\Sigma,
$$

where $D(\nabla_\nu A)(h)$ denotes the linearization of $\nabla_\nu A$. Then, there is a vector field $X\in\mathcal{C}^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ satisfying $X=0$ on $\hat\Sigma$ and $X-K\in\mathcal{C}^{3,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ for some Euclidean Killing vector field K (possibly zero) such that

$$
h=L_X\bar g\quad and\quad v=0\quad \text{in}\ \mathbb{R}^n\setminus\Omega.
$$

Furthermore, if $h^\top=0$ and $DH(h)=0$ everywhere on Σ, then $X=0$ everywhere on Σ and thus $DA(h)=0$ on Σ.

Theorem 9 says that the solutions must be "trivial" in the sense that (h,v) must arise from "infinitesimal" diffeomorphisms. More precisely, we let X be a vector field as in the above theorem and let ϕ_t be the family of diffeomorphisms on $\mathbb{R}^n\setminus\Omega$ generated from X (in particular, ϕ_t is the identity map on Σ for all t). Then, the family of static vacuum pairs $(g_t,u_t)=\phi_t^*(\bar g,1)$ as the pull-back pairs of $(\bar g,1)$ would also satisfy (1.1) and have the same boundary data $(\bar g^\top, A_{\bar g},\nabla_\nu A_{\bar g})$ on $\hat\Sigma$ (in fact, on the entire Σ).[12] The linearization of (g_t,u_t) becomes $(L_X\bar g, X(1))=(L_X\bar g,0)$, which satisfies the linearized system in Theorem 9. On the other hand, Theorem 9 says that those are the *only* solutions.

The rest of this paper is organized as follows: Theorem 9 is proved in Sec. II, and then Theorem 7 is proved in Sec. III.

II. LOCALIZED BOUNDARY DATA

The major motivation for the definition of static regular, Definition 2, is the following uniqueness theorem for Cauchy boundary data from Ref. 17:

Theorem 2.1. *Let $(h,v)\in\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ solve*

$$
\begin{cases}
-D\mathrm{Ric}(h)+\nabla^2 v=0\\
\Delta v=0
\end{cases}\qquad in\ \ \mathbb{R}^n\setminus\Omega,
$$

$$
\begin{cases}
h^\top=0\\
DA(h)=0
\end{cases}\qquad on\ \ \Sigma.
$$

Then, there is a vector field $X \in C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ satisfying $X = 0$ on Σ and $X - K \in C^{3,\alpha}_{1-q}(\mathbb{R}^n\setminus\Omega)$ for some Euclidean Killing vector field K (possibly zero) such that

$$h = L_X\bar{g} \quad \text{and} \quad v = 0 \ \text{ in } \mathbb{R}^n\setminus\Omega.$$

Proof. From the Proof of Lemma 4.8 in Ref. 17, we see that $v = 0$ in $\mathbb{R}^n\setminus\Omega$. Therefore, h is a Ricci flat deformation in the sense that $D\mathrm{Ric}(h) = 0$ in $\mathbb{R}^n\setminus\Omega$. Then, by Theorem 2.8 of Ref. 17, we get the desired conclusion. □

The goal of this section is to prove Theorem 9, whose main difference from the above theorem is that the boundary conditions for Theorem 9 are "localized" only on a subset $\hat{\Sigma} \subset \Sigma$ satisfying $\pi_1(\Sigma, \hat{\Sigma}) = 0$.

We will first establish some basic results, and then Theorem 9 follows immediately after proving Propositions 2.4 and 2.6.

We say that a symmetric $(0,2)$-tensor h is said to satisfy the *geodesic gauge (of order 2)* on Σ if

$$h(\nu,\cdot) = 0, \quad (\nabla_\nu h)(\nu,\cdot) = 0, \quad (\nabla^2_\nu h)(\nu,\cdot) = 0 \quad \text{on } \Sigma,$$

where ν is the unit normal vector of Σ parallelly extended into a collar neighborhood of Σ.

Following the same argument as in Ref. 17 (Lemma 2.5), we see that any tensor h can be "transformed" to satisfy the geodesic gauge.

Lemma 2.2 [cf. Ref. 17 (Lemma 2.5)]. *Let $h \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}\setminus\Omega)$ be a symmetric $(0,2)$-tensor. Then, there exists a vector field $V \in C^{3,\alpha}$ with $V = 0$ on Σ and V vanishing outside a collar neighborhood of Σ such that $k := h + L_V\bar{g}$ satisfies the geodesic gauge on Σ.*

The following lemma gives an analytic interpretation for the geometric boundary conditions of Theorem 9.

Lemma 2.3. Let $\hat{\Sigma}$ be an open subset of the boundary Σ (can be the entire Σ). Suppose $h \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$satisfies the geodesic gauge and

$$h^\top = 0, \quad DA(h) = 0, \quad D(\nabla_\nu A)(h) = 0 \quad \text{on } \hat{\Sigma}.$$

Then,

$$h = 0, \quad \nabla h = 0, \quad \nabla^2 h = 0 \quad \text{on } \hat{\Sigma}.$$

Proof. The first identity is an immediate consequence of $h^\top = 0$ and the geodesic gauge.

To show the second identity, it suffices to show $(\nabla_\nu h)^\top = 0$ because $(\nabla_{\mathrm{arbitrary}}h)(\nu,\cdot) = 0$ and $(\nabla_{\mathrm{tangential}}h)^\top = 0$ from $h = 0$ and geodesic gauge. Recall the formula [see Ref. 17, Eq. (2.3)],

$$DA(h) = \frac{1}{2}(\nabla_\nu h)^\top + A \circ h - \frac{1}{2}L_\omega g^\top - \frac{1}{2}h(\nu,\nu)A, \tag{2.1}$$

where the one-form ω is defined by $\omega(\cdot) = h(\nu,\cdot)$, A is the second fundamental form of $\Sigma \subset (\mathbb{R}^n, \bar{g})$, and $(A \circ h)_{ab} = \frac{1}{2}(A_{ac}h^c_b + A_{bc}h^c_a)$. Therefore, the assumption $DA(h) = 0$ implies that $(\nabla_\nu h)^\top = 0$ and, thus, $\nabla h = 0$.

To show $\nabla^2 h = 0$, we just need to show that $(\nabla^2_\nu h)^\top = 0$ because $\nabla_{\mathrm{tangential}}(\nabla h) = 0$ and $\nabla_\nu\nabla_{\mathrm{tangential}}h = \nabla_{\mathrm{tangential}}\nabla_\nu h$ plus terms involving h and ∇h, which are all zero on $\hat{\Sigma}$. Note that since h satisfies the geodesic gauge, we also have $\nabla_\nu(DA)(h) = D(\nabla_\nu A)(h)$ on $\hat{\Sigma}$. To see this, we compute, for tangential vectors e_a, e_b to Σ,

$$\begin{aligned}
(D(\nabla_\nu A)(h))(e_a,e_b) &:= \frac{d}{dt}\Big|_{t=0}\left(\nabla_{\nu_{g(t)}}A_{g(t)}\right)(e_a,e_b) \\
&= (\nabla_\nu(DA)(h))(e_a,e_b) + \left((\nabla_{\nu_{g(t)}})'A\right)(e_a,e_b) \\
&= (\nabla_\nu(DA)(h))(e_a,e_b),
\end{aligned} \tag{2.2}$$

where in the second line $(\nabla_{\nu_{g(t)}})' := \frac{d}{dt}\big|_{t=0}\nabla_{\nu_{g(t)}}$, and one can verify that $\left((\nabla_{\nu_{g(t)}})'A\right)(e_a,e_b)$ because $h = 0$ and $\nabla h = 0$ on $\hat{\Sigma}$.

To conclude, we get $\nabla_\nu(DA)(h) = 0$ on $\hat{\Sigma}$ using (2.2) and the assumption that $D(\nabla_\nu A)(h) = 0$ on $\hat{\Sigma}$. Covariant differentiating (2.1) in ν, we obtain $(\nabla^2_\nu h)^\top = 0$. □

Proposition 2.4. Let $\hat{\Sigma}$ be an open subset of Σ (can be the entire Σ). Let $(h,v) \in C^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ solve

$$\begin{cases} -D\mathrm{Ric}(h) + \nabla^2 v = 0 \\ \Delta v = 0 \end{cases} \quad \text{in } \mathbb{R}^n\setminus\Omega,$$

$$\begin{cases} h^\top = 0 \\ DA(h) = 0 \\ D(\nabla_\nu A)(h) = 0 \end{cases} \quad \text{on } \hat{\Sigma}.$$

Then, $v \equiv 0$ *and* $D\mathrm{Ric}(h) = 0$ *in* $\mathbb{R}^n \setminus \Omega$.

Proof. By Lemma 2.2, we may assume that h satisfies the geodesic gauge. The boundary conditions and Lemma 2.3 imply that $h = 0$, $\nabla h = 0, \nabla^2 h = 0$ on $\hat{\Sigma}$.

Note that $D\mathrm{Ric}(h)$ involves h and its derivatives up to the second order; precisely, in local coordinates {see, e.g., Ref. 17 [Eq. (2.1)]},

$$(D\mathrm{Ric}|_g(h))_{ij} = -\frac{1}{2}g^{k\ell}h_{ij;k\ell} + \frac{1}{2}g^{k\ell}(h_{ik;\ell j} + h_{jk;\ell i}) - \frac{1}{2}(\mathrm{tr}h)_{ij} + \frac{1}{2}(R_{i\ell}h_j^\ell + R_{j\ell}h_i^\ell) - R_{ik\ell j}h^{k\ell} \quad \text{for all } i,\ j = 0, 1, \ldots, n-1.$$

Restricting $-D\mathrm{Ric}(h) + \nabla^2 v = 0$ on the boundary, we see $\nabla^2 v = 0$ on $\hat{\Sigma}$.

Next, we define the function $f = \frac{\partial v}{\partial x_i}$ for some $i = 1, \ldots, n$, where $(x_1, \ldots, x_n)$ are the Cartesian coordinates. Since v is harmonic, f is also a harmonic function in $\mathbb{R}^n \setminus \Omega$. The conditions that $\nabla^2 v = 0$ on $\hat{\Sigma}$ imply

$$\nabla^\Sigma f = 0 \quad \text{and} \quad \nu(f) = 0 \qquad \text{on } \hat{\Sigma}.$$

The first identity implies $f \equiv c$ for some constant c in a connected open subset of $\hat{\Sigma}$. Together with the second identity and uniqueness of the Cauchy boundary value for the harmonic equation, we conclude that $f \equiv c$ everywhere in $\mathbb{R}^n \setminus \Omega$. Since $f \to 0$ by the fall-off rate of v, we see that $f = \frac{\partial v}{\partial x_i}$ is identically zero. Since i is arbitrary, we see that $\nabla v = 0$, and thus, v is constant. Since $v \to 0$ at infinity, we conclude that v is identically zero. □

In the above proposition, we have shown $D\mathrm{Ric}(h) = 0$ in $\mathbb{R}^n \setminus \Omega$. Proposition 2.6 generalizes Theorem 2.8 of Ref. 17, where $\hat{\Sigma}$ was assumed to be the entire boundary Σ. The key argument is the following extension theorem for h-Killing vector fields, which extends the classical result of Nomizu for the case that h is identically zero.[20]

Theorem 2.5 ([Ref. 18, Theorem 7], Cf. [Ref. 21, Lemma 2.6]). *Let* (M, g) *be a connected, analytic Riemannian manifold. Let* h *be an analytic, symmetric* $(0,2)$*-tensor on* M. *Let* $U \subset M$ *be a connected open subset satisfying* $\pi_1(M, U) = 0$. *Then, if* $h = L_X g$ *in* U, *there is a unique global vector field* Y *such that* $Y = X$ *in* U *and* $h = L_Y g$ *in the whole manifold* M.

Proposition 2.6. Let $\hat{\Sigma}$ *be an open subset of* Σ *satisfying* $\pi_1(\Sigma, \hat{\Sigma}) = 0$. *Let* $h \in \mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n \setminus \Omega)$ *satisfy*

$$D\mathrm{Ric}(h) = 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega$$

$$\begin{cases} h^\top = 0 \\ DA(h) = 0 \end{cases} \qquad \text{on } \hat{\Sigma}.$$

Then, there is a vector field $X \in \mathcal{C}^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$ *satisfying* $X = 0$ *on* $\hat{\Sigma}$ *and* $X - K \in \mathcal{C}^{3,\alpha}_{-q}(\mathbb{R}^n \setminus \Omega)$ *for some Euclidean Killing vector field* K *(possibly zero) such that*

$$h = L_X \bar{g} \ \text{in } \mathbb{R}^n \setminus \Omega.$$

Furthermore, if $h^\top = 0$ *and* $DH(h) = 0$ *everywhere on* Σ, *then* $X = 0$ *everywhere on* Σ *and, thus,* $DA(h) = 0$ *on* Σ.

Proof. We may without loss of generality assume that h satisfies the geodesic gauge on Σ. We extend h by 0 across $\hat{\Sigma}$ into some small open subset $U \subset \Omega$ so that the "extended" manifold $\hat{M} = (\mathbb{R}^n \setminus \Omega) \cup \overline{U}$ has a smooth embedded boundary $\partial \hat{M}$ and $\pi_1(\hat{M}, U) = 0$. Denote the extension of h by $k \in \mathcal{C}^1_{\mathrm{loc}}(\hat{M})$,

$$k = \begin{cases} h & \text{in } \mathbb{R}^n \setminus \Omega, \\ 0 & \text{in } U. \end{cases}$$

Let $Z \in \mathcal{C}^{1,\alpha}_{1-q}(\hat{M})$ be a vector field that weakly solves $\Delta Z = \beta k$ in $\hat{M}$ with $Z = 0$ on $\partial \hat{M}$, where the Bianchi operator $\beta k = -\mathrm{div}_{\bar{g}} k + \frac{1}{2} d(\mathrm{tr}_{\bar{g}} k)$, or, equivalently, $k + L_Z \bar{g}$ weakly solves $\beta(k + L_Z \bar{g}) = 0$ in $\hat{M}$. Together with the assumption that $D\mathrm{Ric}(h) = 0$ in $\mathbb{R}^n \setminus \Omega$ and the boundary condition $h = 0, \nabla h = 0$ on Σ, we have that $k + L_Z \bar{g}$ is a weak solution to $\Delta(k + L_Z \bar{g}) = 0$ in $\hat{M}$. So far, the argument has followed closely Ref. 17 (Theorem 2.8), to which we refer to the analytic details.

However, in the current setting, we cannot conclude that $k + L_Z \bar{g}$ is identically zero as in Ref. 17 (Theorem 2.8). (In Ref. 17, it was possible to extend the harmonic $k + L_X \bar{g}$ globally on the entire $\mathbb{R}^n$.) Here, we apply Weyl's lemma to see that $k + L_Z \bar{g}$ is analytic in $Int(\hat{M})$. Since $k + L_Z \bar{g} = L_Z \bar{g}$ in U (remember $k \equiv 0$ there), by Theorem 2.5, there is a unique vector field Y such that $Y = Z$ in U and

$$k + L_Z \bar{g} = L_Y \bar{g} \text{ in } \hat{M}.$$

To summarize, we obtain $X = Y - Z$ with $X = 0$ on $\hat{\Sigma}$ and

$$h = L_X \bar{g} \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

Note that $X \in \mathcal{C}^{3,\alpha}_{\text{loc}}(\mathbb{R}\setminus\Omega)$ because of the regularity h.

The rest of the conclusions follow from basic arguments as in Ref. 18, so we just give a sketch below. To show the desired asymptotic of X toward infinity, one first considers the ordinary differential equation (ODE) for X along any ray to infinity to show that $X = o(|x|^2)$. Then, writing the equation $D\text{Ric}(L_X\bar{g}) = 0$ in the harmonic gauge gives a harmonic expansion for X. Thus, X is asymptotic to a Euclidean Killing vector field K using the fall-off rate $L_X\bar{g} = h \in \mathcal{C}^{2,\alpha}_{-q}$.

Finally, to show that $X = 0$ on Σ under the added assumptions $h^\top = 0$ and $DH(h) = 0$ on Σ, we write $X = \eta\nu + X^\top$, where $X^\top$ is tangential to Σ. The assumptions $h^\top = 0$ and $DH(h) = 0$ on Σ imply that $\eta, X^\top$ satisfies a linear PDE system on Σ. Since $\eta, X^\top$ are identically zero on $\hat{\Sigma}$, by unique continuation, they are identically zero everywhere on Σ. □

Proof of Theorem 9. Let (h, v) be as in the statement of Theorem 9. By Proposition 2.4, $v \equiv 0$ in $\mathbb{R}^n\setminus\Omega$, and thus, h satisfies the assumptions in Proposition 2.6, which implies the desired conclusion. □

III. A SMOOTH ONE-SIDED FAMILY OF HYPERSURFACES

Let $\delta > 0$ and let $\Omega_t \subset \mathbb{R}^n$, $t \in [-\delta, \delta]$, be bounded open subsets such that their boundaries Σ_t are connected, embedded hypersurfaces and $\{\Sigma_t\}$ form a smooth one-sided family foliating along $\hat{\Sigma}_t \subset \Sigma_t$ with relatively simply connected $\Sigma_t \setminus \hat{\Sigma}_t$. In particular, their deformation vector X is smooth, and on each Σ_t, $\bar{g}(X, \nu) = \zeta \geq 0$ with $\zeta > 0$ in a dense subset of $\hat{\Sigma}_t \subset \Sigma_t$ satisfying $\pi_1(\Sigma_t, \hat{\Sigma}_t) = 0$. Let $\psi_t : \mathbb{R}^n\setminus\Omega_t \to \mathbb{R}^n$ be the flow of X. Let us denote $\Omega = \Omega_0, \Sigma = \Sigma_0$, and $\hat{\Sigma} = \hat{\Sigma}_0$. Then, $\Omega_t = \psi_t(\Omega), \Sigma_t = \psi_t(\Sigma)$. Denote by $g_t = \psi_t^*(\bar{g}|_{\mathbb{R}^n\setminus\Omega_t})$ the pull-back metric defined on $\mathbb{R}^n\setminus\Omega$. We also note $g_0 = \bar{g}$.

Let us define a family of linear operators, with respect to g_t, as

$$L_t : \mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega) \to \mathcal{C}^{0,\alpha}_{-q-2}(\mathbb{R}^n\setminus\Omega) \times \mathcal{B}(\Sigma),$$

$$L_t(h, v) = \begin{cases} \begin{cases} -D\text{Ric}|_{g_t}(h) + \nabla^2_{g_t} v \\ \Delta_{g_t} v \end{cases} & \text{in } \mathbb{R}^n\setminus\Omega, \\ \begin{cases} h^\top \\ DH|_{g_t}(h) \end{cases} & \text{on } \Sigma. \end{cases}$$

Here, $\mathcal{B}(\Sigma) = \mathcal{C}^{2,\alpha}(\Sigma) \times \mathcal{C}^{1,\alpha}(\Sigma)$ is the function space for the boundary operator. Note that each L_t is the pull-back of the operator corresponding to the boundary value problem (1.2) in $\mathbb{R}^n\setminus\Omega_t$.

In Ref. 17, we observed that the kernel spaces Ker L_t have the following properties:

Proposition 3.1 [cf. Ref. 17 (Proposition 6.6)]. *There is an open dense subset $J \subset (-\delta, \delta)$ such that for every $a \in J$ and every $(h, v) \in \text{Ker}\, L_a$, there is a sequence $\{t_j\}$ in J such that $t_j \searrow a$, $(h(t_j), v(t_j)) \in \text{Ker}\, L_{t_j}$, and $(p, z) \in \mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ such that, as $t_j \searrow a$,*

$$(h(t_j), v(t_j)) \to (h, v),$$
$$\frac{(h(t_j), v(t_j)) - (h, v)}{t_j - a} \to (p, z),$$

where both convergence are taken in the $\mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$-norm.

Remark 3.2. In Ref. 17 (Proposition 6.6), we actually proved the statement for the kernel of the corresponding "gauged" operators, which extends directly to the above statement.

Theorem 3.3. *Let $J \subset (-\delta, \delta)$ be the open dense subset as in Proposition 3.1. Then, for every $a \in J$ and every $(h, v) \in \text{Ker}\, L_a$, we have*

$$DA|_{g_a}(h) = 0 \quad \text{and} \quad D(\nabla_\nu A)|_{g_a}(h) = 0 \qquad \text{on} \quad \Sigma_a^+,$$

where $\Sigma_a^+ = \{x \in \Sigma : \psi_a^*(\zeta|_{\Sigma_a})(x) > 0\}$. [In other words, $\psi_a(\Sigma_a^+)$ is the subset of Σ_a on which $\zeta > 0$.]

Proof. By re-parameterizing, we may assume $a = 0$, and hence, $g_a = \bar{g}, \Sigma_a = \Sigma$, and we denote by $L_a = L$ and $\Sigma_a^+ = \Sigma^+$. We may also without loss of generality assume that h satisfies the geodesic gauge. As proven in Ref. 17 (Theorem 7 and Theorem 7′), $(p - L_X h, z - X(v))$ is a static vacuum deformation in $\mathbb{R}^n \setminus \Omega$ satisfying the boundary conditions on Σ,

$$\begin{aligned} (p - L_X h)^\top &= -2\zeta DA(h), \\ DH(p - L_X h) &= \zeta A \cdot DA(h). \end{aligned} \tag{3.1}$$

Recall a consequence of the Green-type identity from Ref. 17 (Corollary 3.5): If both $(h, v), (k, w) \in \mathcal{C}^{2,\alpha}_{-q}(\mathbb{R}^n \setminus \Omega)$ are static vacuum deformations at $(\bar{g}, 1)$ and h satisfies $h^\top = 0, DH(h) = 0$ on Σ, then

$$\int_\Sigma \left\langle \left(vA + DA(h) - \nu(v)\bar{g}^\top, 2v \right), \left(k^\top, DH(k) \right) \right\rangle d\sigma = 0.$$

We apply the previous identity by substituting $(k, w) := (p - L_X h, z - X(v))$ and using the boundary conditions (3.1) to obtain

$$\begin{aligned} 0 &= \int_\Sigma \left\langle \left(vA + DA(h) - \nu(v)\bar{g}^\top, 2v \right), \left(-2\zeta DA(h), \zeta A \cdot DA(h) \right) \right\rangle d\sigma \\ &= -\int_\Sigma 2\zeta |DA(h)|^2 \, d\sigma \end{aligned}$$

where we compute $\bar{g}^\top \cdot DA(h) = 0$ to get the last identity. Thus, we show that $DA(h) = 0$ on Σ^+.

To summarize our argument, we have shown that for any $a \in J$ and for any $(h, v) \in \operatorname{Ker} L_a$, we must have $DA|_{g_a}(h) = 0$ on Σ_a^+.

Applying $DA(h) = 0$ on Σ^+ to (3.1), the static vacuum deformation $(k, w) = (p - L_X h, z - X(v))$ defined earlier satisfies $k^\top = 0$ and $DH(k) = 0$ everywhere on Σ. In particular, $(k, w) \in \operatorname{Ker} L$, and thus, $DA(k) = 0$ on Σ^+. We show that $DA(k) = \nabla_\nu(DA(h))$ on Σ^+: Using $DA(k) = DA(p - L_X h)$ and

$$p - L_X h = \lim_{t_j \to 0} \frac{1}{t_j}\left(h(t_j) - h \right) - \lim_{t_j \to 0} \frac{1}{t_j}\left(\psi_{t_j}^* h - h \right) = \lim_{t_j \to 0} \frac{1}{t_j}\left(h(t_j) - \psi_{t_j}^* h \right),$$

we compute on Σ^+,

$$\begin{aligned} 0 = DA(p - L_X h) &= DA\left(\lim_{t_j \to 0} \frac{1}{t_j}\left(h(t_j) - \psi_{t_j}^* h \right) \right) \\ &= \lim_{t_j \to 0} \frac{1}{t_j} DA|_{g_{t_j}}\left(h(t_j) - \psi_{t_j}^* h \right) \\ &= -\lim_{t_j \to 0} \frac{1}{t_j} DA|_{g_{t_j}}\left(\psi_{t_j}^* h \right) \\ &= -\lim_{t_j \to 0} \frac{1}{t_j} \psi_t^* \left(DA(h)|_{\Sigma_{t_j}} \right) \\ &= -L_X(DA(h)) \\ &= -\zeta \nabla_\nu(DA(h)), \end{aligned}$$

where in the second equality, we use $h(t_j) - \psi_{t_j}^* h = 0$ when $t_j = 0$; in the third equality, we use $DA|_{g_{t_j}}(h(t_j)) = 0$ on $\Sigma_{t_j}^+$ because $(h(t_j), v(t_j)) \in \operatorname{Ker} L_{t_j}$ and $\Sigma_{t_j}^+ \to \Sigma^+$ as $t_j \to 0$; and in the last equality, we use $DA(h) = 0$ on Σ^+.

To conclude the proof, we computed as in (2.2) to get

$$D(\nabla_\nu A)(h) = \nabla_\nu(DA(h)) = 0 \quad \text{on } \Sigma^+.$$

Proof of Theorem 7. Let $\{\Sigma_t\}$, $t \in [-\delta, \delta]$, be given as in the theorem. Let J be the open dense subset of J from Proposition 3.1. We will show that for any $a \in J$, Σ_a is static regular in $\mathbb{R}^n \setminus \Omega_a$. Let $(h, v) \in \operatorname{Ker} L_a$. We apply Theorem 3.3 to see that $DA|_{g_a}(h) = 0$ and $D(\nabla_\nu A)|_{g_a}(h) = 0$ on Σ^+ and hence on $\hat{\Sigma}$. Then, we can apply Theorem 9 to conclude that $DA(h) = 0$ on the entire Σ. It completes the proof. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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