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Intrinsic Flat Convergence of Points and Applications to Stability of the Positive Mass Theorem

Lan-Hsuan Huang, Dan A. Lee and Raquel Perales

Abstract. We prove results on intrinsic flat convergence of points—a concept first explored by Sormani (Commun Anal Geom 26(6):1317–1373, 2018). In particular, we discuss compatibility with Gromov–Hausdorff convergence of points—a concept first described by Gromov (Inst Hautes Études Sci Publ Math 53:53–73, 1981). We apply these results to the problem of stability of the positive mass theorem in mathematical relativity. Specifically, we revisit the article (Huang et al. in J Reine Angew Math 727:269–299, 2017) on intrinsic flat stability for the case of graphical hypersurfaces of Euclidean space: We are able to fill in some details in the proofs of Theorems 1.4 and Lemma 5.1 of Huang et al. (2017) and strengthen some statements. Moreover, in light of an acknowledged error in the proof of Theorem 1.3 of Huang et al. (2017), we provide an alternative proof that extends recent work of Allen and Perales (Intrinsic flat stability of manifolds with boundary where volume converges and distance is bounded below, 2020. arXiv:2006.13030).

1. Introduction

Questions concerning convergence of Riemannian manifolds with lower bounds on scalar curvature have attracted increasing attention in the past decade. From mathematical relativity, the question of *stability of the positive mass theorem* asks: If a sequence of complete asymptotically flat n-dimensional manifolds (M_j, g_j) with nonnegative scalar curvature has ADM masses converging to zero, in what sense must the sequence (M_j, g_j) converge to Euclidean space? In [16], Sormani and the second author observed that convergence fails in the Gromov–Hausdorff topology in general, but they conjectured that convergence

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holds (outside the apparent horizon) in the intrinsic flat topology of Sormani and Wenger [20] and established the conjecture in the spherically symmetric case.

There has been much recent progress on applying Sormani–Wenger's intrinsic flat convergence to scalar curvature convergence problems in certain special cases. In particular, the case of Riemannian manifolds that can be embedded as graphical hypersurfaces in Euclidean space has been studied in [7,8,11]. The advantage in the graphical setting is due to an observation of G. Lam [14] that the scalar curvature of a graphical hypersurface, which can be expressed as a divergence quantity, induces a "quasi-local mass" quantity on level sets of the graphical hypersurface. Further investigation of Lam's quasi-local mass leads to several intriguing properties of the hypersurfaces. For example, an alternative proof of rigidity of the positive mass theorem in this setting was given in [13]. From there, the first two authors obtained the stability of the positive mass theorem in Federer–Fleming's flat topology [10]. In [11], Sormani and the first two authors developed new tools to understand how the results of [10] relate to the intrinsic flat topology, and some of these tools have been applied by other works on intrinsic flat topology [1,2,5,6,8].

Some of the arguments in [11], especially regarding intrinsic flat convergence of points, were less mature at the time that [11] was written, but the ideas have influenced the study of pointed intrinsic flat convergence. In this note, we clarify those arguments and establish new results on intrinsic flat convergence of points and its compatibility with Gromov–Hausdorff convergence of points. These general results, proved in Sect. 2, are not specific to the graphical hypersurface setting, and we expect them to find further applications. In Sects. 3.2 and 3.3, we use these results to flesh out some missing details in the proofs of Theorems 1.4 and Lemma 5.1 of [11]. Separately, we also address an acknowledged error in the proof of Theorem 1.3 of [11] by providing an alternative proof using recent work of B. Allen and the third author [3]. In doing so, we verify that all of the results of [11] are true.

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2. Point Convergence in Gromov–Hausdorff or Intrinsic Flat Sense

In this section, we introduce new vocabulary and notation that will replace some of the less precise language regarding point convergence that was used in [11,19]. This will be convenient for our desired applications. Otherwise, we will use the same notation and definitions as in [11], with one main exception:

Notation. Throughout this paper, we use the notation $B(p, r)$ to denote the **closed** ball of radius r around p, and if there is no point specified, then $B(r)$ is just the closed ball of radius r around the origin in Euclidean space. One reason why we choose this convention is that if we regard a *closed* ball $B(p,r)$ in a complete Riemannian manifold as an integral current space $S(p,r)$, then the canonical set of $S(p,r)$, denoted $\mathrm{set}(S(p,r))$, can be identified with $B(p,r)$, whereas this does not work for open balls. The compactness of closed balls in complete Riemannian manifolds is also convenient.

Recall that if metric spaces (X_j,d_j) converge to a metric space (X_∞,d_∞) in the Gromov–Hausdorff sense, we write $(X_j,d_j)\xrightarrow{\mathrm{GH}}(X_\infty,d_\infty)$, or perhaps $X_j\xrightarrow{\mathrm{GH}}X_\infty$ when there is no chance for confusion. Similarly, if integral current spaces $M_j=(X_j,d_j,T_j)$ converge to an integral current space $M_\infty=(X_\infty,d_\infty,T_\infty)$ in the intrinsic flat sense, we write $M_j\xrightarrow{\mathrm{F}}M_\infty$. It is often convenient to see these convergences as occurring within a fixed metric space, so we introduce the following notation:

Definition 2.1. Consider metric spaces (X_j,d_j) and a choice of a separable complete metric space (Z,d) and metric–isometric embedding maps $\varphi_j : X_j\to Z$, for $j\in\mathbb{N}\cup\{\infty\}$.

We say that

$$X_j\xrightarrow[Z]{\mathrm{GH}}X_\infty$$

if and only if $\varphi_j(X_j)\to\varphi_\infty(X_\infty)$ in the Hausdorff sense in Z.

If $M_j=(X_j,d_j,T_j)$ are n-dimensional integral current spaces for $j\in\mathbb{N}\cup\{\infty\}$, we say that

$$M_j\xrightarrow[Z]{\mathrm{F}}M_\infty$$

if and only if ${\varphi_j}_\#(T_j)\to{\varphi_\infty}_\#(T_\infty)$ in the flat sense in Z.

The Z in this notation is intended to indicate that the maps φ_j have also been chosen despite not being written down explicitly.

2.1. Background

Here we rephrase various results from [19,20] using the language of Definition 2.1. We first state [19, Theorem 2.3], which is sometimes called Gromov's embedding theorem.

Theorem 2.2 (Gromov[1]). *If (X_j,d_j) are compact metric spaces for $j\in\mathbb{N}\cup\{\infty\}$, then $X_j\xrightarrow{\mathrm{GH}}X_\infty$ if and only if there exists a compact metric space Z (and embedding maps φ_j) such that $X_j\xrightarrow[Z]{\mathrm{GH}}X_\infty$.*

We also have the analogous statement for intrinsic flat convergence:

Theorem 2.3 [20, Theorem 4.2]. *If $M_j=(X_j,d_j,T_j)$ are n-dimensional integral current spaces for $j\in\mathbb{N}\cup\{\infty\}$, then $M_j\xrightarrow{\mathrm{F}}M_\infty$ if and only if there exists a separable complete metric space Z (and embedding maps φ_j) such that $M_j\xrightarrow[Z]{\mathrm{F}}M_\infty$.*

[1]The explanation for why this theorem follows from results of [9] can be found within the proof of [20, Theorem 4.2].

We introduce notation to deal with the concept of convergence of points in the Gromov–Hausdorff or intrinsic flat sense:

Definition 2.4. Consider metric spaces (X_j, d_j) and a choice of a separable complete metric space (Z, d) and metric–isometric embedding maps $\varphi_j : X_j \to Z$, for $j \in \mathbb{N} \cup \{\infty\}$.

For points $x_j \in X_j$ for $j \in \mathbb{N} \cup \{\infty\}$, we say that

$$(X_j, x_j) \xrightarrow[Z]{\text{GH}} (X_\infty, x_\infty)$$

if and only if $X_j \xrightarrow[Z]{\text{GH}} X_\infty$ and also $\varphi_j(x_j) \to \varphi_\infty(x_\infty)$ as points in Z.

If $M_j = (X_j, d_j, T_j)$ are n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$, then for points $x_j \in X_j$ for $j \in \mathbb{N}$, and $x_\infty \in \overline{X}_\infty$, we say that

$$(M_j, x_j) \xrightarrow[Z]{\text{F}} (M_\infty, x_\infty)$$

if and only if $M_j \xrightarrow[Z]{\text{F}} M_\infty$ and also $\varphi_j(x_j) \to \varphi_\infty(x_\infty)$ as points in Z.

Note that for the second part of the definition, x_∞ need not lie in X_∞, but the definition makes sense since φ_∞ extends to the completion $\overline{X}_\infty$.

The concept of point convergence in the intrinsic flat sense was first formulated in [19, Definition 3.1], which referred to "$x_j \in X_j$ converging to $x_\infty \in \overline{X}_\infty$." In our language, this means that there exist Z and φ_j such that $(M_j, x_j) \xrightarrow[Z]{\text{F}} (M_\infty, x_\infty)$. (See also [11, Definitions 2.4 and 2.11].)

If a sequence converges in both the Gromov–Hausdorff sense and the intrinsic flat sense, then there exists a complete separable metric space Z and isometric embeddings where both Hausdorff and flat convergence are realized [20, Theorem 3.20]. Furthermore, if the sequences converge to the same space we have the following.

Proposition 2.5 ([20, Theorem 3.20], c.f. [11, Remark 2.12]). *Let $M_j = (X_j, d_j, T_j)$ be compact n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$. Then, we have both $X_j \xrightarrow{\text{GH}} X_\infty$ and $M_j \xrightarrow{\text{F}} M_\infty$ if and only if there exists a separable complete metric space Z (and embedding maps φ_j) such that we have both $X_j \xrightarrow[Z]{\text{GH}} X_\infty$ and $M_j \xrightarrow[Z]{\text{F}} M_\infty$.*

We now state some useful facts about point convergence in the intrinsic flat sense that were proved by Sormani.

Lemma 2.6 [19, Lemma 3.4]. *Let $M_j = (X_j, d_j, T_j)$ be n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$. If $M_j \xrightarrow[Z]{\text{F}} M_\infty$, then for any $x_\infty \in \overline{X}_\infty$, there exist points $x_j \in X_j$ such that $(M_j, x_j) \xrightarrow[Z]{\text{F}} (M_\infty, x_\infty)$.*

For an integral current space (X, d, T), a point $x \in X$, and $r > 0$, we define:

$$S(x, r) := (\text{set}(T \llcorner B(x, r)), d, T \llcorner B(x, r)).$$

Lemma 2.7 [19, Lemma 4.1]. *Let $M_j = (X_j, d_j, T_j)$ be n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$. If $(M_j, x_j) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x_\infty)$, then there is a subsequence $x_{j_k} \in X_{j_k}$ such that for almost every $r > 0$, $S(x_{j_k}, r)$ and $S(x_\infty, r)$ are integral currents spaces, and*

$$(S(x_{j_k}, r), x_{j_k}) \xrightarrow[Z]{\mathrm{F}} (S(x_\infty, r), x_\infty)\,,$$

with embedding maps given by restriction.

Note that the conclusion only holds for a subsequence rather than the full sequence.

Theorem 2.8 ([19, Theorem 7.1]). *Let $M_j = (X_j, d_j, T_j)$ be n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$. Assume that $M_j \xrightarrow[Z]{\mathrm{F}} M_\infty$ and that there exist $\delta > 0$, a function $h : (0, \delta) \to (0, \infty)$, and a sequence $x_j \in X_j$ such that for almost every $r \in (0, \delta)$,*

$$\liminf_{j\to\infty} d_{\mathcal{F}}(S(x_j, r), \mathbf{0}) \geq h(r) > 0. \tag{1}$$

Then, there exist a subsequence x_{j_k} and a point $x_\infty \in \overline{X}_\infty$ such that

$$(M_{j_k}, x_{j_k}) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x_\infty).$$

2.2. Compatibility of Point Convergence and Intrinsic Flat Volume Convergence

Using the results described in Sect. 2.1, we prove some facts concerning point convergence which are needed for our applications, and which might be of independent interest. The first theorem concerns compatibility of point convergence $(M_j, x_j) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x_\infty)$ with respect to different choices of Z, and also compatibility with point convergence in converging subsets. Given an integral current space $M = (X, d, T)$ and a subset $V \subset X$, we define

$$M \llcorner V := (\mathrm{set}(T \llcorner V), d, T \llcorner V).$$

So for example, for $x \in X$ and $r > 0$, $S(x, r) := M \llcorner B(x, r)$.

Theorem 2.9. *Let $M_j = (X_j, d_j, T_j)$ be n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$, and for $j \in \mathbb{N}$, let $x_j \in V_j \subset X_j$ such that $M_j \llcorner V_j$ is a n-dimensional integral current space. Assume the following:*

1. *$(M_j \llcorner V_j, x_j) \xrightarrow[W]{\mathrm{F}} (N_\infty, x_\infty)$ for some integral current space N_∞, some point $x_\infty \in \overline{\mathrm{set}(N_\infty)}$, and some choice of W (and embedding maps).*
2. *There exists $\delta > 0$ such that the metric ball $B(x_j, \delta) \subset X_j$ is entirely contained in V_j for all large j.*

Then, for any choice of Z (and embedding maps) such that $M_j \xrightarrow[Z]{\mathrm{F}} M_\infty$, there exist a subsequence x_{j_k} and a point $x'_\infty \in \overline{X}_\infty$ such that

$$(M_{j_k}, x_{j_k}) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x'_\infty).$$

Remark 2.10. Note that the conclusion is nontrivial even when $V_j = X_j$, in which case the assumption (2) trivially holds. Note that even in this case, it need not be true that $x'_\infty = x_\infty$.

Proof. By assumption (2), for all $r \in (0, \delta)$ and all large j, we have

$$(M_j \llcorner V_j) \llcorner B^{V_j}(x_j, r) = M_j \llcorner B^{X_j}(x_j, r),$$

so we can unambiguously refer to both spaces as $S(x_j, r)$. So by Lemma 2.7 and assumption (1), there exists a subsequence x_{j_k} such that for almost every $r \in (0, \delta)$, $S(x_{j_k}, r) \xrightarrow[W]{\mathrm{F}} S(x_\infty, r)$, where $S(x_\infty, r) = N_\infty \llcorner B(x_\infty, r)$. So for large k, we obviously have

$$d_{\mathcal{F}}(S(x_{j_k}, r), \mathbf{0}) > \tfrac{1}{2} d_{\mathcal{F}}(S(x_\infty, r), \mathbf{0}) > 0.$$

Taking $h(r) := \frac{1}{2} d_{\mathcal{F}}(S(x_\infty, r), \mathbf{0})$, we see that $S(x_{j_k}, r)$ satisfies the hypotheses of Theorem 2.8. The result then follows from applying Theorem 2.8 to the convergence $M_{j_k} \xrightarrow[Z]{\mathrm{F}} M_\infty$ with points $x_{j_k} \in X_{j_k}$. $\square$

Proposition 2.5 tells us that if we have both Gromov–Hausdorff and intrinsic flat convergence, it is possible to find a common embedding space Z in which both types of convergence is "realized." Theorem 2.11(ii) below shows that if we have intrinsic flat convergence of spaces whose *boundaries* converge in the Gromov–Hausdorff sense, then again, we can see that both types of convergence are "realized" in the same embedding space. (Recall that n-dimensional intrinsic flat convergence always implies $(n-1)$-dimensional intrinsic flat convergence of the boundaries.) Roughly speaking, the proofs of [11, Theorem 1.4 and Lemma 5.1] were written in such a way that they assumed that this theorem is true.

Theorem 2.11. *Let $M_j = (X_j, d_j, T_j)$ be n-dimensional integral current spaces for $j \in \mathbb{N} \cup \{\infty\}$. Assume $M_j \xrightarrow[Z]{\mathrm{F}} M_\infty$, $\partial M_\infty \neq \mathbf{0}$ and, that we can decompose $\partial M_j = \partial_1 M_j + \partial_2 M_j$ such that $\partial_2 M_j \xrightarrow{\mathrm{F}} \mathbf{0}$. (In other words, some parts of the boundary are negligible in the intrinsic flat limit, though we also allow $\partial_2 M_j = \mathbf{0}$). Define $\Sigma_j := \mathrm{set}(\partial_1 M_j)$ and $\Sigma_\infty := \mathrm{set}(\partial M_\infty)$.*

The following statements hold:

(i) If $(M_j, x_j) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x_\infty)$, then $d_\infty(x_\infty, \Sigma_\infty) \geq \limsup_{j\to\infty} d_j(x_j, \Sigma_j)$.

(ii) If Σ_j, Σ_∞ are compact and $(\Sigma_j, d_j) \xrightarrow{\mathrm{GH}} (\Sigma_\infty, d_\infty)$, then $(\Sigma_j, d_j) \xrightarrow[Z]{\mathrm{GH}} (\Sigma_\infty, d_\infty)$.

Proof. We prove (i): Assume $(M_j, x_j) \xrightarrow[Z]{\mathrm{F}} (M_\infty, x_\infty)$ with embedding maps φ_j. For any $y \in \Sigma_\infty$, we will estimate $d_\infty(x_\infty, y)$ from below. Our hypotheses imply that $\partial_1 M_j \xrightarrow[Z]{\mathrm{F}} \partial M_\infty$, so we can apply Lemma 2.6 to see that there exists $y_j \in \Sigma_j$ such that $\varphi_j(y_j) \to \varphi_\infty(y)$ in Z. So

$$d_\infty(x_\infty, y) = d_Z(\varphi_\infty(x_\infty), \varphi_\infty(y)) = \lim_{j\to\infty} d_Z(\varphi_j(x_j), \varphi_j(y_j))$$

$$= \limsup_{j\to\infty} d_j(x_j, y_j) \geq \limsup_{j\to\infty} d_j(x_j, \Sigma_j).$$

The result follows by taking the infimum over $y \in \Sigma_\infty$.

We prove (ii): We assume that $M_j \xrightarrow[Z]{\mathrm{F}} M_\infty$ with embedding maps φ_j, and also that $\Sigma_j \xrightarrow{\mathrm{GH}} \Sigma_\infty$. Suppose, to get a contradiction, that $\varphi_j(\Sigma_j)$ does not converge to $\varphi_\infty(\Sigma_\infty)$ in the Hausdorff sense in Z. Then, there exists $\epsilon > 0$ such that one of the following two cases must occur:

- There exist a subsequence of Σ_j, still indexed by j, and points $z_j \in \Sigma_\infty$ such that
$$d_Z(\varphi_j(\Sigma_j), \varphi_\infty(z_j)) > \epsilon, \tag{2}$$
- There exist a subsequence of Σ_j, still indexed by j, and points $y_j \in \Sigma_j$ such that
$$d_Z(\varphi_j(y_j), \varphi_\infty(\Sigma_\infty)) > \epsilon. \tag{3}$$

We discuss the first case. By compactness of Σ_∞, there is a subsequential limit $z_\infty \in \Sigma_\infty$. By Lemma 2.6, there exist points $y_j \in \Sigma_j$ such that $\varphi_j(y_j) \to \varphi_\infty(z_\infty)$ in Z, but this contradicts equation (2).

We discuss the second case. Since Σ_j, Σ_∞ are compact and we have both $\Sigma_j \xrightarrow{\mathrm{GH}} \Sigma_\infty$ and $\partial_1 M_j \xrightarrow{\mathrm{F}} \partial M_\infty$, Proposition 2.5 tells us that there exist a separable complete metric space W and maps ψ_j such that $\Sigma_j \xrightarrow[W]{\mathrm{GH}} \Sigma_\infty$ and $\partial_1 M_j \xrightarrow[W]{\mathrm{F}} \partial M_\infty$. In particular, $d_W(\psi_j(y_j), \psi_\infty(\Sigma_\infty)) \to 0$. Since $\psi_\infty(\Sigma_\infty)$ is compact, it follows that there is a subsequence of $\psi_j(y_j)$, which we still index by j, that converges to something in $\psi_\infty(\Sigma_\infty)$. So there exists $y_\infty \in \Sigma_\infty$ such that

$$(\partial_1 M_j, y_j) \xrightarrow[W]{\mathrm{F}} (\partial M_\infty, y_\infty).$$

So by Theorem 2.9 applied to the convergence $\partial_1 M_j \xrightarrow[Z]{\mathrm{F}} \partial M_\infty$ (and V_j equal to the full space Σ_j), it follows that there exist a subsequence of y_j, still indexed by j and $y'_\infty \in \overline{\Sigma}_\infty = \Sigma_\infty$ such that

$$(\partial_1 M_j, y_j) \xrightarrow[Z]{\mathrm{F}} (\partial M_\infty, y'_\infty).$$

In particular, $\varphi_j(y_j) \to \varphi_\infty(y'_\infty) \in \varphi_\infty(\Sigma_\infty)$, which contradicts (3). □

Recall that in Lemma 2.7, the conclusion only holds for a subsequence and not necessarily for the original sequence. An elementary theorem of analysis says that if every subsequence has a subsequence that converges to the same thing, then the original sequence itself must also converge to the same thing. The reason why this principle does not apply to Lemma 2.7 is the "almost every" part of the conclusion: For any *fixed* radius r, we do not know that every subsequence has a converging subsequence. The following proposition explains how we can get around this problem when the integral current spaces are Riemannian manifolds and *intrinsic flat volume convergence* holds. We remark that a result such as this is needed to prove that convergence holds for the original sequence rather than just for a subsequence (even if one only wants the conclusion for almost every R). For example, see Theorem 3.3 below.

Theorem 2.12. *Let (M_j, g_j) be Riemannian manifolds with $x_j \in M_j$, for $j \in \mathbb{N} \cup \{\infty\}$. Assume that every subsequence of x_{j_k} of x_j has a subsequence $x_{j_{k_\ell}}$ such that for almost every $R > 0$,*

$$S(x_{j_{k_\ell}}, R) \xrightarrow{\mathrm{VF}} S(x_\infty, R).$$

Then, for all $R > 0$,

$$S(x_j, R) \xrightarrow{\mathrm{VF}} S(x_\infty, R).$$

Remark 2.13. In the following proof, we note where the Riemannian assumption is used, so that the reader can see when the result applies to more general spaces.

Proof. First we will prove that $S(x_j, R) \xrightarrow{\mathrm{F}} S(x_\infty, R)$ for all $R > 0$. To the contrary, suppose there exists a specific $R > 0$ such that $S(x_j, R)$ fails to converge to $S(x_\infty, R)$. So there exist $\epsilon > 0$ and a subsequence x_{j_k} such that for all k,

$$d_{\mathcal{F}}(S(x_{j_k}, R), S(x_\infty, R)) > \epsilon. \tag{4}$$

By our assumption, there exists a subsequence $x_{j_{k_\ell}}$ such that for almost every $r > 0$, $S(x_{j_{k_\ell}}, r) \xrightarrow{\mathrm{VF}} S(x_\infty, r)$. We select $R' > R$ close enough to R so that

$$\begin{aligned} d_{\mathcal{F}}(S(x_\infty, R), S(x_\infty, R')) &\leq \mathbf{M}\left(\overline{B(x_\infty, R') \setminus B(x_\infty, R)}\right) \\ &= \mathrm{Vol}(B(x_\infty, R')) - \mathrm{Vol}(B(x_\infty, R)) < \epsilon/4. \end{aligned} \tag{5}$$

(This is clearly possible since the limit space is Riemannian.) Of course, we can also select R' so that $S(x_{j_{k_\ell}}, R') \xrightarrow{\mathrm{VF}} S(x_\infty, R')$. In particular, for sufficiently large ℓ, we have

$$d_{\mathcal{F}}(S(x_{j_{k_\ell}}, R'), S(x_\infty, R')) < \epsilon/4. \tag{6}$$

By (4), (5), (6), and the triangle inequality, we see that for sufficiently large ℓ,

$$d_{\mathcal{F}}(S(x_{j_{k_\ell}}, R), S(x_{j_{k_\ell}}, R')) > \epsilon/2, \tag{7}$$

and this is the inequality that we will contradict.

For almost every $r > 0$, $\partial S(x_{j_{k_\ell}}, r) \xrightarrow{\mathrm{F}} \partial S(x_\infty, r)$, so we also have convergence of slices $\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle \xrightarrow{\mathrm{F}} \langle M_\infty, \rho_\infty, r\rangle$, where ρ_j denotes the distance function to the point x_j in M_j, for $j \in \mathbb{N} \cup \{\infty\}$. By lower semicontinuity of mass under intrinsic flat convergence,

$$\mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle) \leq \liminf_{\ell \to \infty} \mathbf{M}(\langle M_\infty, \rho_\infty, r\rangle).$$

Applying the Ambrosio–Kirchheim slicing theorem for the case of a distance function on a Riemannian manifold (in which case it is simply the co-area formula), and also Fatou's Lemma,

$$\begin{aligned}
\mathrm{Vol}(B(x_\infty, R')) &= \int_0^{R'} \mathbf{M}(\langle M_\infty, \rho_\infty, r\rangle)\, dr \\
&\le \int_0^{R'} \liminf_{\ell\to\infty} \mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle)\, dr \\
&\le \liminf_{\ell\to\infty} \int_0^{R'} \mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle)\, dr \\
&= \liminf_{\ell\to\infty} \mathrm{Vol}(B(x_{j_{k_\ell}}, R')) \\
&= \mathrm{Vol}(B(x_\infty, R')),
\end{aligned}$$

where we use the assumption of volume convergence in the last line. This equality implies that we must actually have the equality

$$\mathbf{M}(\langle M_\infty, \rho_\infty, r\rangle) = \liminf_{\ell\to\infty} \mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle),$$

for almost every $r < R'$.

Recall that we do not have good convergence properties for R, but we can use the co-area formula and Fatou again, combined with the above equality to obtain:

$$\begin{aligned}
&\limsup_{\ell\to\infty} \left[\mathrm{Vol}(B(x_{j_{k_\ell}}, R')) - \mathrm{Vol}(B(x_{j_{k_\ell}}, R))\right] \\
&\quad\le \limsup_{\ell\to\infty} \mathrm{Vol}(B(x_{j_{k_\ell}}, R')) - \liminf_{\ell\to\infty} \int_0^{R} \mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle)\, dr \\
&\quad\le \mathrm{Vol}(B(x_\infty, R')) - \int_0^{R} \liminf_{\ell\to\infty} \mathbf{M}(\langle M_{j_{k_\ell}}, \rho_{j_{k_\ell}}, r\rangle)\, dr \\
&\quad= \mathrm{Vol}(B(x_\infty, R')) - \int_0^{R} \mathbf{M}(\langle M_\infty, \rho_\infty, r\rangle)\, dr \\
&\quad= \mathrm{Vol}(B(x_\infty, R')) - \mathrm{Vol}(B(x_\infty, R)) \\
&\quad< \epsilon/4,
\end{aligned}$$

by assumption (5). Since

$$d_{\mathcal{F}}\left(S(x_{j_{k_\ell}}, R'), S(x_{j_{k_\ell}}, R)\right) \le \mathrm{Vol}(B(x_{j_{k_\ell}}, R')) - \mathrm{Vol}(B(x_{j_{k_\ell}}, R)),$$

this contradicts (7).

Finally, we deal with the possibility that the convergence $S(x_j, R) \xrightarrow{\mathrm{F}} S(x_\infty, R)$ holds, but volume convergence does not. If volume convergence fails, there exist $\epsilon > 0$ and a subsequence x_{j_k} such that

$$|\,\mathrm{Vol}(B(x_{j_k}, R)) - \mathrm{Vol}(B(x_\infty, R))| > \epsilon.$$

From here, we can use the same argument as above to get a contradiction in exactly the same way. $\square$

3. Application to [11]

We will briefly recall the main definitions of [11].

Definition 3.1. For $n \geq 3$, $r_0, \gamma, D > 0$, and $\alpha < 0$, define $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ to be the space of all smooth complete Riemannian manifolds (M^n, g) with nonnegative scalar curvature, possibly with boundary, that admit a smooth Riemannian isometric embedding $\Psi : M \longrightarrow \mathbb{E}^{n+1}$ such that for some open $U \subset B(r_0/2) \subset \mathbb{E}^n$, the image $\Psi(M)$ is the graph of a function $f \in C^\infty(\mathbb{E}^n \smallsetminus \overline{U}) \cap C^0(\mathbb{E}^n \smallsetminus U)$:

$$\Psi(M) = \{(x, f(x)) : \ x \in \mathbb{E}^n \smallsetminus U\}$$

with empty or minimal boundary:

$$\text{either } \partial M = \emptyset \text{ and } U = \emptyset,$$
$$\text{or } f \text{ is constant on each component of } \partial U \text{ and } \lim_{x \to \partial U} |Df(x)| = \infty,$$

and for almost every h, the level set

$$f^{-1}(h) \subset \mathbb{E}^n \text{ is strictly mean-convex and outward-minimizing,}$$

where strictly mean-convex means that the mean curvature is strictly positive, and outward-minimizing means that any region of $\mathbb{E}^n$ that contains the region enclosed by $f^{-1}(h)$ must have perimeter at least as large as $\mathcal{H}^{n-1}(f^{-1}(h))$.

In addition, we require uniform asymptotic flatness conditions:

$$|Df| \leq \gamma \text{ for } |x| \geq r_0/2 \text{ and } \lim_{x \to \infty} |Df| = 0.$$

If $n \geq 5$, we require that $f(x)$ approaches a constant as $x \to \infty$. If $n = 3$ or 4, we require that the graph is asymptotically Schwarzschild:[2]

$$\exists \Lambda, m \in \mathbb{R} \text{ such that } |f(x) - (\Lambda + S_m(|x|))| \leq \gamma |x|^\alpha \text{ for } |x| \geq r_0.$$

For $r \geq r_0$, we define

$$\Omega(r) := \Psi^{-1}(B(r) \times \mathbb{R}) \quad \text{and} \quad \Sigma(r) := \partial\Omega(r) \smallsetminus \partial M,$$

so that $\Omega(r)$ represents the part of M whose Ψ-image lies in the cylinder $B(r) \times \mathbb{R}$, and $\Sigma(r)$ represents the "outer" component of $\partial\Omega(r)$, which is the part of M whose Ψ-image lies in the cylindrical shell $\partial B(r) \times \mathbb{R}$.

Finally, we require a "bounded depth" assumption:

$$\sup \{d_M(p, \Sigma(r_0)) : p \in \Omega(r_0)\} \leq D.$$

For n-dimensional integral current spaces, we say that M_j converges to M_∞ in the *intrinsic flat volume* sense, or $M_j \xrightarrow{\text{VF}} M_\infty$, if we have intrinsic flat convergence, $M_j \xrightarrow{\text{F}} M_\infty$, as well as $\mathbf{M}(M_j) \to \mathbf{M}(M_\infty)$, where $\mathbf{M}$ denotes the mass of an integral current space (not to be confused with the unrelated concept of ADM mass). Recall that $\mathbf{M}$ is the same thing as Vol for Riemannian spaces.

An equivalent statement of [11, Theorem 1.3] is the following:

[2] See [10] for the definition of the function S_m.

Theorem 3.2. *Let $n \geq 3$, $r_0, \gamma, D > 0$, $\alpha < 0$, and $r \geq r_0$. Let $M_j \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ and adopt the notation in Definition 3.1 with a j-subscript. If the ADM masses of M_j converge to zero, then $\Omega_j(r)$ converges to the Euclidean ball $B(r)$ in the intrinsic flat volume sense. That is,*

$$\Omega_j(r) \xrightarrow{\text{VF}} B(r).$$

A. Cabrera Pacheco, C. Ketterer, and the third author discovered an error in the proof of this theorem in [11] while researching stability of tori with nonnegative scalar curvature [8]. More specifically, the error of [11] is the claim in Section 6 that "equality in the second inequality (6.1) for a 1-Lipschitz function implies that $\Psi_\infty : \Omega_\infty(r) \to B(r)$ must be an isometry." See [12] for more details. B. Allen and the third author were able to provide an alternative proof of Theorem 3.2 under the added assumption that M has no boundary [3, Section 7]. The alternative proof is an application of [3, Theorem 4.2] in conjunction with estimates from [11]. In Sect. 3.1, we will extend that argument to obtain a proof of Theorem 3.2 in full generality.

The following theorem is a consequence of Theorem 3.2 and Sect. 2.

Theorem 3.3. *Let $n \geq 3$, $r_0, \gamma, D > 0$, and $\alpha < 0$. Let $M_j \in \mathcal{G}_n(r_0, \gamma, D, \alpha)$ and adopt the notation in Definition 3.1 with a j-subscript. If the ADM masses of M_j converge to zero, then for any sequence of points $p_j \in \Sigma_j(r_0)$ and any $R > 0$, the geodesic ball $B(p_j, R) \subset M_j$ converges to the Euclidean ball $B(R)$ in the intrinsic flat volume sense. That is,*

$$B(p_j, R) \xrightarrow{\text{VF}} B(R).$$

A slightly weaker version of this theorem appears in [11] as Theorem 1.4. In the course of researching how to use [7] to prove an asymptotically hyperbolic version of Theorem 3.3, Cabrera Pacheco and the third author identified some parts of the proofs of [11, Theorem 1.4 and Lemma 5.1] that require further justification. In Sect. 3.2, we will explain in detail how to apply the results from Sect. 2 to prove Theorem 3.3, and to be thorough, we also discuss how to apply them to the proof of [11, Lemma 5.1] in Sect. 3.3, thereby legitimizing all of the results of [11].

We note that Theorem 3.3 can be also rephrased as "(M_j, p_j) converges to $\mathbb{E}^n$ in the *pointed* intrinsic flat volume sense." The general meaning of pointed intrinsic flat convergence of *locally* integral current spaces is explained in [18,21], but our statement does not require this formalism.

3.1. Proof of Theorem 3.2

Throughout the rest of the paper, we will often abusively refer to regions of Riemannian manifolds as sets, metric spaces, and integral current spaces, depending on what is convenient, as long as there is minimal chance of confusion.

Our task in this section is to adapt the proof from [3, Section 7] to the case of nontrivial boundary. We will find it convenient to use the following corollary of [3, Theorem 4.2], which easily follows from a simple scaling argument and the application of a diffeomorphism:

Theorem 3.4 (Allen-Perales). *Let $(\Omega_\infty, g_\infty)$ be a smooth compact Riemannian manifold, possibly with boundary, and let Ω_j be diffeomorphic to Ω_∞ via C^1 diffeomorphisms*

$$\Phi_j : \Omega_\infty \to \Omega_j, \tag{8}$$

such that Ω_j is equipped with a continuous metric g_j. Assume that this sequence has the following properties:

$$g_\infty(u, u) < \left(1 + \tfrac{1}{j}\right) g_j(d\Phi_j(u), d\Phi_j(u)), \tag{9}$$

for all tangent vectors u,

$$\operatorname{diam}(\Omega_j) \le L, \tag{10}$$

$$\operatorname{Vol}(\Omega_j) \to \operatorname{Vol}(\Omega_\infty), \tag{11}$$

$$\operatorname{Vol}(\partial\Omega_j) \le A, \tag{12}$$

for some constants L and A. Further assume that the interior of $(\Omega_\infty, g_\infty)$ is convex. Then, (Ω_j, g_j) converges to $(\Omega_\infty, g_\infty)$ in the intrinsic flat sense.

The first result of this type was proved by Lakzian-Sormani [15, Theorem 5.2]. (See also [17].) Improvements to the hypotheses were made by [4] for the case of no boundary.

Proof of Theorem 3.2. Let M_j be a sequence in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ whose ADM masses are approaching zero, and let $r \ge r_0$. Adopting the notation in Definition 3.1 with a j-subscript, M_j is Riemannian isometric (via the isometry Ψ_j) to the graph of a function $f_j : \mathbb{E}^n \setminus U_j \to \mathbb{R}$, and f_j is constant on (each component of) ∂U_j and $|Df_j| \to \infty$ at ∂U_j. For simplicity of presentation, we will assume without loss of generality that f_j is zero on ∂U_j.

Recall that $\Omega_j(r) = \Psi_j^{-1}(B(r) \times \mathbb{R})$ is the subset of M_j corresponding to the part of the graph of f_j lying within the cylinder of radius r. Furthermore, from the proof of [11, Theorem 3.1] we know that $\operatorname{diam}(\Omega_j(r)) \le L$ and $\operatorname{Vol}(\Omega_j(r)) \le A$ for some constants L and A, and by [11, Corollary 4.4] we know that $\operatorname{Vol}(\Omega_j(r)) \to \operatorname{Vol}(B(r))$.

In order to prove the result, we have to show that $\Omega_j(r)$ converges to the Euclidean ball $(B(r), g_{\mathbb{E}})$ in the intrinsic flat sense. We first paraphrase the argument from [3] in the no boundary case: When M_j has no boundary, we can define the diffeomorphism $\Phi_j : B(r) \to \Omega_j(r)$ to be the "graphing map"

$$\Phi_j(x) = \Psi_j^{-1}(x, f_j(x)). \tag{13}$$

Note that Φ_j is also the inverse of the map $\pi \circ \Psi_j$, where π is the projection map to $\mathbb{E}^n$. Now it is easy to see that the hypotheses of Theorem 3.4 are satisfied. Inequality (9) holds because a graphing map is distance nondecreasing. We already mentioned that (10), (12) (11) hold. Finally, the interior of a Euclidean ball is obviously convex. Hence we can apply Theorem 3.4, with $(\Omega_\infty, g_\infty) = (B(r), g_{\mathbb{E}})$, to conclude that $\Omega_j(r)$ converges to $B(r)$ in the intrinsic flat sense.

We will now generalize the previous argument to the case of nontrivial boundary. In this case, $\Omega_j(r)$ is not diffeomorphic to $B(r)$, so we cannot apply

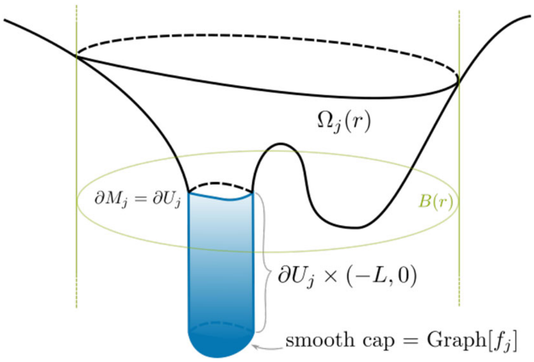

FIGURE 1. Space $\tilde{\Omega}_j$ is obtained from $\Omega_j(r)$ by appending a Riemannian cylinder $\partial M_j \times (-L, 0) \cong \partial U_j \times (-L, 0)$ and a smooth graphical cap

Theorem 3.4 directly to the sequence $\Omega_j(r)$. Instead, we will replace $\Omega_j(r)$ by a new sequence $\tilde{\Omega}_j$ obtained by "filling in" the boundary.

Let L be the uniform diameter bound for $\Omega_j(r)$ for all j mentioned above. The space $\tilde{\Omega}_j$ will be obtained from $\Omega_j(r)$ by first appending a cylinder $\partial M_j \times (-L, 0) \cong \partial U_j \times (-L, 0)$ to $\partial M_j \subset \Omega_j(r)$, and then, we will smoothly "cap" the other end of the cylinder. The cap may be regarded as being isometric to the graph of a function on U_j, which we will also refer to as f_j for simplicity, where f_j is constant on ∂U_j and $|Df_j| \to \infty$ as we approach ∂U_j from the inside. (Further details of the capping turn out to be inessential.) See Fig. 1.

Since the cylinder we add has length L, it is clear that $\Omega_j(r)$ embeds into $\tilde{\Omega}_j$ metric isometrically. Therefore,

$$d_{\mathcal{F}}(\Omega_j(r), \tilde{\Omega}_j) \leq d_F(\Omega_j(r), \tilde{\Omega}_j) \leq \mathrm{Vol}(\tilde{\Omega}_j \setminus \Omega_j(r)).$$

Now, since L is fixed and the ADM mass of M_j approaches zero, we can certainly ensure that $\tilde{\Omega}_j \setminus \Omega_j(r)$ has volume approaching zero as $j \to \infty$. Thus,

$$d_{\mathcal{F}}(\Omega_j(r), \tilde{\Omega}_j) \to 0.$$

In fact, since the ADM mass of M_j approaches zero, the Penrose inequality [14] implies that $\mathrm{Vol}(\partial U_j) = \mathrm{Vol}(\partial M_j) \to 0$, and so $\mathrm{Vol}(\partial U_j \times (-L, 0)) \to 0$. By appropriately choosing the capping function $f_j : U_j \to \mathbb{R}$, we will also have $\mathrm{Vol}(\mathrm{graph}[f_j|_{U_j}]) \to 0$. Note that if f_j were constant on U_j (though it cannot be chosen that way), this would follow from the isoperimetric inequality. To obtain the actual f_j, we can just smooth out the corner. So what remains to do is to ensure that we can apply Theorem 3.4 to the sequence $\tilde{\Omega}_j$ where the limit space has to be the Euclidean ball $B(r)$.

The new spaces $\tilde{\Omega}_j$ are diffeomorphic to $B(r)$, but we must choose the diffeomorphism carefully. The graphing map $\Phi_j(x) = \Psi_j^{-1}(x, f_j(x))$ defines a nice diffeomorphism away from ∂U_j, but it needs to be altered in a neighborhood of ∂U_j to obtain a new diffeomorphism $\tilde{\Phi}_j : B(r) \to \tilde{\Omega}_j$. The main task is to show that the property (9) holds for $\tilde{\Phi}_j$. Intuitively, this is not hard to do: We just need to "stretch" in the directions orthogonal to ∂U_j. The stretching can only help $\tilde{\Phi}_j$ to be distance increasing, but the slight change to the tangential directions will introduce a small error term. We describe the details in the following.

The following construction will depend on a parameter $\epsilon > 0$, which will depend on j, and we will see how small ϵ should be in order for the construction to work. For ϵ small enough, we consider a tubular neighborhood T_ϵ of ∂U_j in $\mathbb{E}^n$ diffeomorphic to $\partial U_j \times (-\epsilon, \epsilon)$ via the exponential map $(\theta, \rho) \mapsto (\theta + \rho\nu_j)$, where $\theta \in \partial U_j$, $\rho \in (-\epsilon, \epsilon)$, and ν_j is the outward unit normal to ∂U_j at θ. We will define $\tilde{\Phi}_j$ one piece at a time. For $x \in B(r) \smallsetminus T_\epsilon$, define $\tilde{\Phi}_j(x) = \Phi_j(x)$.

Let T_ϵ^+ denote the outer side of T_ϵ corresponding to $\partial U_j \times (0, \epsilon)$. We will define $\tilde{\Phi}_j$ on T_ϵ^+. Using the facts that $\lim_{x\to\partial U_j} |Df_j| = \infty$, $f_j|_{\partial U_j} = 0$, and graph$[f_j]$ is a submanifold, one can see that there exists $\delta \in (0, \epsilon)$ such that $\frac{\partial f_j}{\partial \rho} > 1$ for $\rho \in (0, \delta)$. In particular, note that f_j is increasing in ρ for small ρ. By further shrinking δ if necessary, it follows that for all $(\theta, t) \in \partial U_j \times (0, \delta)$, there exists a unique $\alpha(\theta, t) \in (0, \delta)$ such that

$$f_j(\theta + \alpha(\theta, t)\nu_j) = \frac{2L}{\epsilon} t,$$

and $\alpha(\theta, t)$ is increasing in t. Note that this implies that

$$\lim_{t\to 0^+} \alpha(\theta, t) = 0.$$

By the implicit function theorem, α is a smooth function of θ and t. More precisely,

$$\frac{\partial \alpha}{\partial t}(\theta, t) = \frac{2L}{\epsilon} \left[\frac{\partial f_j}{\partial \rho}(\theta + \alpha(\theta, t)\nu_j)\right]^{-1}. \tag{14}$$

$$\frac{\partial \alpha}{\partial s}(\theta, t) = -\frac{\partial f_j}{\partial s}(\theta + \alpha(\theta, t)\nu_j) \left[\frac{\partial f_j}{\partial \rho}(\theta + \alpha(\theta, t)\nu_j)\right]^{-1}, \tag{15}$$

where s denotes a coordinate on the surface ∂U_j.

Define $\tilde{\alpha}$ on T_ϵ^+ to smoothly interpolate between α and t such that $\tilde{\alpha}(\theta, t) = \alpha(\theta, t)$ for $t < \delta/2$ and $\tilde{\alpha}(\theta, t) = t$ for $t > \delta$. For δ small enough, $\frac{\partial \tilde{\alpha}}{\partial t} > 0$, and thus

$$\phi_j : \theta + t\nu_j \mapsto \theta + \tilde{\alpha}(\theta, t)\nu_j$$

defines a diffeomorphism from T_ϵ^+ to itself. Finally, we define

$$\tilde{\Phi}_j = \Phi_j \circ \phi_j$$

on T_ϵ^+. The interpolation guarantees that $\tilde{\Phi}_j = \Phi_j$ near the outer boundary of T_ϵ^+. Meanwhile, one can check that Eqs. (14) and (15) imply that if we extend $\tilde{\Phi}_j$ to be the identity map on the inner boundary of T_ϵ^+ (which is just ∂U_j), then the extended map is C^1 up to the boundary. This is the essential advantage that $\tilde{\Phi}_j$ has over Φ_j, whose derivatives blow up as we approach ∂U_j. (The factor of $\frac{2L}{\epsilon}$ is convenient for later.)

We now check that property (9) holds for $\tilde{\Phi}_j$ if we choose ϵ appropriately. For small enough ϵ, we can make sure that

$$\frac{\partial \tilde{\alpha}}{\partial t} > 2\left[\frac{\partial f_j}{\partial \rho}(\theta + \tilde{\alpha}(\theta, t)\nu_j)\right]^{-1}$$

on all of $\partial U_j \times (0, \epsilon)$. Although ϕ_j does some distance contracting in the ∂_t direction, this inequality guarantees that the contracting is counteracted by the stretching by Φ_j, or in other words, we have:

$$g_{\mathbb{E}}(\partial_\rho, \partial_\rho) \leq \tfrac{1}{2} g_j(d\tilde{\Phi}_j(\partial_\rho), d\tilde{\Phi}_j(\partial_\rho)). \tag{16}$$

Now consider a vector u tangent to the level set $\rho = t$ at a point $p = \theta + t\nu_j$. Select local coordinates $\theta_1, \ldots, \theta_{n-1}$ on ∂U_j so that each ∂_{θ_i} may be regarded as a vector field that is tangent to the level sets of ρ, and so that u is equal to ∂_{θ_1} at p. From the definition of ϕ_j, we can see that the component of $d\phi_j(u)$ that is tangent to the $\rho = \tilde{\alpha}(\theta, t)$ level set of $\phi_j(p)$ is precisely ∂_{θ_1} at $\phi_j(p)$. By taking ϵ small, the length distortion between ∂_{θ_1} on these two parallel level sets can be taken to be arbitrarily close to 1. Denoting that tangential component by $[d\phi_j(u)]^T$, we see that for ϵ small enough, we can force

$$\begin{aligned} g_{\mathbb{E}}(u,u) &< \left(1 + \tfrac{1}{j}\right) g_{\mathbb{E}}\left([d\phi_j(u)]^T, [d\phi_j(u)]^T\right) \\ &\leq \left(1 + \tfrac{1}{j}\right) g_{\mathbb{E}}(d\phi_j(u), d\phi_j(u)) \\ &\leq \left(1 + \tfrac{1}{j}\right) g_j(d\tilde{\Phi}_j(u), d\tilde{\Phi}_j(u)), \end{aligned}$$

where the last inequality follows because we already know that Φ_j is distance nondecreasing.

It remains to verify (9) for a general vector that has both a radial part ∂_ρ and a tangential part u. While the cross-term $g_j(d\tilde{\Phi}_j(\partial_\rho), d\tilde{\Phi}_j(u))$ could be potentially large, it is dominated by the radial inner product with a small error of tangential inner product by the Cauchy–Schwarz inequality.

Turning our attention to the inner side of T_ϵ, for $t \in (-\frac{\epsilon}{2}, 0)$, we define

$$\tilde{\Phi}_j(\theta + t\nu_j) = \left(\theta, \frac{2L}{\epsilon} t\right),$$

so that this part of $\tilde{\Phi}_j$ is a diffeomorphism from an inner tubular neighborhood to the cylinder $\partial U_j \times (-L, 0)$, and this map clearly satisfies (9) for small ϵ. We now see that the factor of $\frac{2L}{\epsilon}$ in (14) ensures that these two definitions of $\tilde{\Phi}_j$ match up in such a way that $\tilde{\Phi}_j$ is C^1 across the common boundary ∂U_j.

Finally, for $t \in (-\epsilon, -\frac{\epsilon}{2})$, we do something similar to what we did for $t \in (0, \epsilon)$, except now the diffeomorphism ϕ_j should be chosen to map the $t \in (-\epsilon, -\frac{\epsilon}{2})$ part of the tubular neighborhood to the entire $t \in (-\epsilon, 0)$ inner side of the tubular neighborhood. It all works out the same way since the graphing function f_j defining the "cap" also satisfies $\frac{\partial f_j}{\partial \rho} \to \infty$ as we approach ∂U_j from the inside.

Putting it all together, we have diffeomorphisms $\tilde{\Phi}_j : B(r) \to \tilde{\Omega}_j$ satisfying all hypotheses of Theorem 3.4 (noting that we still have a uniform diameter bound for $\tilde{\Omega}_j$ and volume convergence to $\mathrm{Vol}(B(r))$ as was explained in the first part of the proof), and the result follows. □

3.2. Discussion of Theorem 3.3

In this section, we will explain how Theorem 3.2 implies Theorem 3.3, using the results of Sect. 2. First, let us briefly summarize the original argument in [11]: Assume M_j as in the hypotheses above, and choose a large $\bar{R} > 0$. Theorem 3.2 tells us that $\Omega_j(r_0 + \bar{R}) \xrightarrow{\mathrm{F}} B(r_0 + \bar{R})$. Starting with a sequence $x_j \in \Sigma_j(r_0)$, we want to extract a subsequential limit in the sense that $(\Omega_j(r_0 + \bar{R}), x_j) \xrightarrow[Z]{\mathrm{F}} (B(r_0 + \bar{R}), x_\infty)$ for some x_∞ and Z. Then, we can invoke Lemma 2.6 to obtain the desired result. In [11, Lemma 5.1], it was shown that $\Sigma_j(r_0) \xrightarrow{\mathrm{GH}} \partial B(r_0)$, and this implies one can extract a subsequential limit in the sense that $(\Sigma_j(r_0), x_j) \xrightarrow[Z]{\mathrm{GH}} (\partial B(r_0), x_\infty)$ for some x_∞ and Z. Because of some imprecision of language in [11], it was implicitly assumed that this is good enough. In this section, we will fill in the details:

According to Theorem 2.11(ii), we can find x_∞ and Z such that both

$$(\Sigma_j(r_0), x_j) \xrightarrow[Z]{\mathrm{GH}} (\partial B(r_0), x_\infty) \quad \text{and} \quad (\Omega_j(r_0), x_j) \xrightarrow[Z]{\mathrm{F}} (B(r_0), x_\infty).$$

This is *almost* what we want, but in order to make the argument completely rigorous, we will use Gromov–Hausdorff convergence of an entire neighborhood of $\Sigma_j(r_0)$ rather than just $\Sigma_j(r_0)$. So we will need the following lemma on convergence of "coordinate annular regions" in the exterior part of M_j.

Lemma 3.5. *Assume M_j is a sequence in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ with ADM masses approaching zero. For $r > s > r_0/2$, define $\Omega_j(s, r) := \overline{\Omega_j(r) \setminus \Omega_j(s)} \subset M_j$ and $A(s, r) := \overline{B(r) \setminus B(s)} \subset \mathbb{E}^n$. Then, $\Omega_j(s, r)$ converges to $A(s, r)$ in both the Gromov–Hausdorff and intrinsic flat senses.*

Proof. We first claim that a subsequence of $\Omega_j(s, r)$ converges to some integral current space $\Omega_\infty(s, r)$ in both the Gromov–Hausdorff and intrinsic flat senses. Technically, this claim is all that is needed in order to prove Theorem 3.3. We provide a stronger conclusion in the statement of Lemma 3.5 simply because we can.

The claim is proved using the same argument used to prove (the first part of) Lemma 5.1 of [11]: The definition of $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ implies that the diffeomorphism $\pi \circ \Psi_j : \Omega_j(s, r) \to A(s, r) \times \{0\}$ has a bilipschitz constant which is uniform in s, r, and j, where π is the projection map from $\mathbb{E}^{n+1}$ onto $\mathbb{E}^n \times \{0\}$. Then we can apply Theorem A.1 of [11] to obtain the claim. The only complication is that Theorem A.1 of [11] is stated only for integral current spaces *without* boundary, but one can see from its proof that the conclusion will hold as long as the boundary mass is uniformly bounded, as explained in [2, Remark 2.22]. Specifically, this can be seen by considering the effect of an extra boundary term on page 294 of [11], which turns out to be negligible. Note that for our desired application, the boundary mass is just $\mathrm{Vol}(\partial\Omega_j(s,r))$, which we know is uniformly bounded because of the uniform Lipschitz bound on f_j.

To obtain the final conclusion, we can use (a much easier version of) the same argument that was used to prove Theorem 1.3 of [11] to see that $\Omega_j(s,r) \xrightarrow{\mathrm{F}} A(s,r)$, and hence $\Omega_\infty(s,r)$ must be isometric to $A(s,r)$. Since every subsequence of $\Omega_j(s,r)$ has a subsequence converging in both the Gromov–Hausdorff and intrinsic flat senses to $A(s,r)$, which is independent of choice of subsequence, the original sequence must converge to $A(s,r)$. □

Proof of Theorem 3.3. Assume M_j is a sequence in $\mathcal{G}_n(r_0,\gamma,D,\alpha)$ with ADM masses approaching zero, and let $p_j \in \Sigma_j(r_0)$. Choose some large $\bar{R} > 1$. Theorem 3.2 tells us that $\Omega_j(r_0+\bar{R}) \xrightarrow[Z]{\mathrm{F}} B(r_0+\bar{R})$ for some choice of Z (and maps), and this is the main ingredient of our proof. Our first task is to prove the following:

<u>Claim:</u> There is a subsequence of p_j (still indexed by j) such that for almost every $R \in (0, \bar{R}-1)$, we have $B(p_j,R) \xrightarrow{\mathrm{F}} B(R)$.

Without loss of generality, assume $r_0 > 2$. Applying Lemma 3.5, we know that $\Omega_j(r_0-1,r_0+1)$ converges in both the Gromov–Hausdorff and intrinsic flat senses to $A(r_0-1,r_0+1)$. By Proposition 2.5, there exist a separable complete metric space W and embeddings ψ_j such that

$$\Omega_j(r_0-1,r_0+1) \xrightarrow[W]{\mathrm{GH}} A(r_0-1,r_0+1) \text{ and}$$
$$\Omega_j(r_0-1,r_0+1) \xrightarrow[W]{\mathrm{F}} A(r_0-1,r_0+1).$$

Since $\psi_\infty(A(r_0-1,r_0+1))$ is compact, the Hausdorff convergence implies that there is subsequence of $\psi_j(p_j)$, still indexed by j, that converges to $\psi_\infty(p_\infty)$ for some $p_\infty \in A(r_0-1,r_0+1)$. Therefore

$$(\Omega_j(r_0-1,r_0+1),p_j) \xrightarrow[W]{\mathrm{F}} (A(r_0-1,r_0+1),p_\infty).$$

We apply Theorem 2.9 with $X_j = \Omega_j(r_0+\bar{R})$ and $V_j = \Omega_j(r_0-1,r_0+1)$ to obtain a subsequence, still indexed by j, and a point p'_∞ such that

$$\left(\Omega_j(r_0+\bar{R}),p_j\right) \xrightarrow[Z]{\mathrm{F}} \left(B(r_0+\bar{R}),p'_\infty\right). \tag{17}$$

By Theorem 2.11 (i), we also have

$$d_{\mathbb{E}}(p'_\infty, \partial B(r_0+\bar{R})) \geq \limsup_{j\to\infty} d_j\left(p_j, \Sigma_j(r_0+\bar{R})\right) \geq \bar{R}-1.$$

(Note that we apply Theorem 2.11 (i) with $\Sigma_j(r_0+\bar{R})$ as our "$\partial_1 M_j$" and ∂M_j as our "$\partial_2 M_j$," the latter of which we know vanishes in the intrinsic flat limit.)

For $R < \bar{R}-1$, we have $B(p_j, R) \subset \Omega_j(r_0+\bar{R})$ and $B(p'_\infty, R) \subset B(r_0+\bar{R})$, so we can apply Lemma 2.7 to (17) to obtain the Claim.

The proof of Lemma 2.7 in [19, Lemma 4.1] also shows, by looking at complements of balls rather than the balls themselves, that a further subsequence (still indexed by j) satisfies $\Omega_j(r_0+\bar{R}) \setminus B(p_j, R) \xrightarrow{\mathrm{F}} B(r_0+\bar{R}) \setminus B(p'_\infty, R)$. Using this, the volume convergence argument in [11, Theorem 1.4] tells us that $\mathrm{Vol}(B(p_j, R)) \to \mathrm{Vol}(B(R))$, and hence $B(p_j, R) \xrightarrow{\mathrm{VF}} B(R)$ for almost every $R \in (0, \bar{R}-1)$.

Finally, since $\bar{R} > 1$ was arbitrary, a diagonalization argument shows that there exists a subsequence such that for almost all $R > 0$, $B(p_j, R) \xrightarrow{\mathrm{VF}} B(R)$. Then, we invoke Theorem 2.12 to see that for *all* $R > 0$, the *original* sequence satisfies $B(p_j, R) \xrightarrow{\mathrm{VF}} B(R)$. □

3.3. Discussion of Lemma 5.1 of [11]

In this section, we explain how Theorem 2.11(ii) is used in the proof of [11, Lemma 5.1]. It is only relevant to the second part of [11, Lemma 5.1], which says the following:

Lemma 3.6. *Assume M_j is a sequence in $\mathcal{G}_n(r_0, \gamma, D, \alpha)$ with ADM masses approaching zero. Then, the map $\Psi_\infty : \mathrm{set}(\Omega_\infty(r)) \to B(r) \times \{0\}$ restricted to $\Sigma_\infty(r) := \mathrm{set}(\partial\Omega_\infty(r))$ is a bilipschitz map onto $\partial B(r) \times \{0\}$.*

We briefly recall the construction of $\Omega_\infty(r)$ and Ψ_∞ in [11, Theorem 3.1]: There exist a subsequence, still indexed by j, an integral current space $\Omega_\infty(r)$, and a choice of Z, φ_j such that $\Omega_j(r) \xrightarrow[Z]{\mathrm{F}} \Omega_\infty(r)$. Then, Ψ_∞ was defined so that, after taking an appropriate subsequence, for any $x \in \mathrm{set}(\Omega_\infty(r))$ and any sequence $x_j \in \Omega_j(r)$,

$$\text{if} \quad (\Omega_j(r), x_j) \xrightarrow[Z]{\mathrm{F}} (\Omega_\infty(r), x), \quad \text{then} \quad \Psi_\infty(x) = \lim_{j\to\infty} \Psi_j(x_j).$$

In particular, the definition of Ψ_∞ depends on the choice of Z (and choice of subsequence). We know $\mathrm{Lip}(\Psi_\infty) \le 1$ since each $\mathrm{Lip}(\Psi_j) \le 1$, and we know the image of Ψ_∞ lies in $B(r) \times \{0\}$ by [11, Lemma 4.5]. In other words, $\pi \circ \Psi_\infty = \Psi_\infty$, where π is the projection map to $\mathbb{E}^n \times \{0\}$. Finally, since $\Psi_j(\Sigma_j(r)) \subset \partial B(r) \times \mathbb{R}$, we also have $\Psi_\infty(\Sigma_\infty(r)) \subset \partial B(r) \times \{0\}$.

Proof. We will prove that $\Psi_\infty|_{\Sigma_\infty(r)} : \Sigma_\infty(r) \to \partial B(r) \times \{0\}$ is bilipschitz by constructing a Lipschitz inverse. We define $\Phi_j : \partial B(r) \times \{0\} \to \Sigma_j(r)$ to be the inverse of $\pi \circ \Psi_j$, where π is the projection map. The first part of [11, Lemma 5.1] says that $\Sigma_j(r) \xrightarrow{\mathrm{GH}} \Sigma_\infty(r)$. Since $\partial\Omega_j(r) = \Sigma_j(r) \cup \partial M_j$, and ∂M_j vanishes in the intrinsic flat limit, we can apply Theorem 2.11(ii) to see that $\Sigma_j(r) \xrightarrow[Z]{\mathrm{GH}} \Sigma_\infty(r)$, where Z and φ_j are the same metric space and maps that were used to construct Ψ_∞. (In the original proof of [11, Lemma 5.1], it was implicitly assumed that one could use the same Z and φ_j as in the construction of Ψ_∞.)

Since there is a uniform Lipschitz bound for Φ_j, we can extract a subsequence, still indexed by j, such that Φ_j converges to a Lipschitz map

$$\Phi_\infty : \partial B(r) \times \{0\} \to \Sigma_\infty(r),$$

where Φ_∞ is defined so that for all $y \in \partial B(r) \times \{0\}$, $(\Sigma_j(r), \Phi_j(y)) \xrightarrow[Z]{\text{GH}} (\Sigma_\infty(r), \Phi_\infty(y))$, or in other words, $(\varphi_\infty \circ \Phi_\infty)(y) = \lim_{j\to\infty}(\varphi_j \circ \Phi_j)(y)$. The proof is completed by showing that Φ_∞ is the inverse map of $\Psi_\infty|_{\Sigma_\infty(r)}$.

The rest of the argument proceeds as in [11, Lemma 5.1]. We know that

$$\Phi_j \circ \pi \circ \Psi_j = id : \Sigma_j(r) \to \Sigma_j(r)$$
$$\pi \circ \Psi_j \circ \Phi_j = id : \partial B(r) \times \{0\} \to \partial B(r) \times \{0\},$$

and then the desired result follows from taking limits. We explain this in detail below:

For any $x \in \Sigma_\infty(r)$, Lemma 2.6 implies there exists $x_j \in \Sigma_j(r)$ such that the following holds in Z:

$$\begin{aligned}\varphi_\infty(x) &= \lim_{j\to\infty} \varphi_j(x_j) = \lim_{j\to\infty} \varphi_j((\Phi_j \circ \pi \circ \Psi_j)(x_j)) \\ &= (\varphi_\infty \circ \Phi_\infty)\left(\lim_{j\to\infty}(\pi \circ \Psi_j)(x_j)\right) = \varphi_\infty((\Phi_\infty \circ \pi \circ \Psi_\infty)(x)),\end{aligned}$$

where we used our definitions of Φ_∞ and Ψ_∞. So $\Phi_\infty \circ \Psi_\infty = id$ on $\Sigma_\infty(r)$.

Meanwhile, for any $y \in \partial B(r) \times \{0\}$, by definition of Φ_∞, $\varphi_\infty(\Phi_\infty(y)) = \lim_{j\to\infty} \varphi_j(\Phi_j(y))$ in Z, and then by definition of Ψ_∞, $\Psi_\infty(\Phi_\infty(y)) = \lim_{j\to\infty} \Psi_j(\Phi_j(y))$ in $\partial B(r)\times\mathbb{R}$. Hence $(\pi\circ\Psi_\infty\circ\Phi_\infty)(y) = \lim_{j\to\infty}(\pi\circ\Psi_j\circ\Phi_j)(y) = y$. So $\Psi_\infty \circ \Phi_\infty = id$ on $\partial B(r) \times \{0\}$. □

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Lan-Hsuan Huang
University of Connecticut
Storrs
USA
e-mail: lan-hsuan.huang@uconn.edu

Dan A. Lee
CUNY Graduate Center and Queens College
New York
USA
e-mail: dan.lee@qc.cuny.edu

Raquel Perales
CONACyT Research Fellow at the National Autonomous University of Mexico
Oaxaca
Mexico
e-mail: raquel.perales@im.unam.mx

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