A NOTE ON UTILITY MAXIMIZATION WITH PROPORTIONAL TRANSACTION COSTS AND STABILITY OF OPTIMAL PORTFOLIOS

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Abstract. The aim of this short note is to establish a limit theorem for the optimal trading strategies in the setup of the utility maximization problem with proportional transaction costs. This limit theorem resolves the open question from [4]. The main idea of our proof is to establish a uniqueness result for the optimal strategy. The proof of the uniqueness is heavily based on the dual approach which was developed recently in [6, 7, 8].

Key words. Utility Maximization, Proportional Transaction Costs, Shadow Price Process

1. Preliminaries and the Limit Theorem.

1.1. Utility Maximization with Proportional Transaction Costs. We consider a model with one risky asset which we denote by $S = (S_t)_{0\leq t\leq T}$, where $T < \infty$ is a fixed finite time horizon. We assume that the investor has a bank account that, for simplicity, bears no interest. The process S is assumed to be an adapted, strictly positive and continuous process (not necessarily a semi-martingale) defined on a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where the filtration $\mathcal{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual assumptions (right continuity and completeness).

Let $\kappa \in (0, 1)$ be a constant. Consider a model in which every purchase or sale of the risky asset at time $t \in [0, T]$ is subject to a proportional transaction cost of rate κ. A trading strategy is an adapted process $\gamma = (\gamma_t)_{0\leq t\leq T}$ of bounded variation with right-continuous paths; note that it automatically has left limits and hence is RCLL (right-continuous with left limits). The random variable γ_t denotes the number of shares held at time t. We use the convention $\gamma_{0-} = 0$. Moreover, we require that $\gamma_T = 0$ which means that we liquidate the portfolio at the maturity date.

Let $\gamma_t := \gamma^+_t - \gamma^-_t$, $t \in [0, T]$ be the Jordan decomposition into two non-decreasing processes $(\gamma^+_t)_{0\leq t\leq T}$ and $(\gamma^-_t)_{0\leq t\leq T}$ describing the positive variation and negative variation, respectively. Because the bid price process is $(1 - \kappa)S$ and the ask price process is $(1 + \kappa)S$, the liquidation value of a trading strategy γ at time t is given by

$$V^\gamma_t := (1-\kappa)\int_0^t S_u d\gamma^-_u - (1+\kappa)\int_0^t S_u d\gamma^+_u + (1-\kappa)S_t(\gamma_t)^+ - (1+\kappa)S_t(\gamma_t)^-$$

where $(\gamma_t)^+ := \max(0, \gamma_t)$ and $(\gamma_t)^- := \max(0, -\gamma_t)$ (beware that these are not the same variables as γ^+_t, γ^-_t above). Note that the integrals take into account the possible transaction at $t = 0$. Namely, we define

$$\int_0^t S_u d\gamma^-_u := S_0\gamma^-_0 + \int_{(0,t]} S_u d\gamma^-_u \quad \text{and} \quad \int_0^t S_u d\gamma^+_u := S_0\gamma^+_0 + \int_{(0,t]} S_u d\gamma^+_u .$$

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By rearranging the terms, we get

$$V_t^\gamma = \gamma_t S_t - \int_0^t S_u d\gamma_u - \kappa|\gamma_t|S_t - \kappa\int_0^t S_u|d\gamma_u|, \quad t\in[0,T]. \tag{1.1}$$

Observe that the wealth process $(V_t^\gamma)_{0\le t\le T}$ is RCLL like γ and $\gamma_T=0$ implies $V_{T-}^\gamma = V_T^\gamma$. For any initial capital $x>0$, we denote by $\mathcal{A}(x)$ the set of all trading strategies γ which satisfy the admissibility condition $x+V_t^\gamma\ge 0$, for all $t\in[0,T]$.

We will assume that the process S is sticky (Definition 2.2 in [11]) and satisfies a slight strengthening of the condition of "two-way crossing" (TWC) (Definition 3.1 in [3]). For completeness, we formulate the assumptions explicitly.

Assumption 1.1. *The process S is sticky with respect to the filtration $\mathcal{F}$. That is, for any $\delta>0$ and a stopping time $\tau\le T$ (with respect to $\mathcal{F}$) such that $\mathbb{P}(\tau<T)>0$, we have*

$$\mathbb{P}\left(\sup_{\tau\le u\le T}|S_u-S_\tau|<\delta, \tau<T\right)>0.$$

Assumption 1.2. *The process S satisfies the (TWC) property with respect to the filtration $\mathcal{F}$, if, for any stopping time $\sigma\le T$, we have*

$$\inf\{t>\sigma : S_t>S_\sigma\} = \inf\{t>\sigma : S_t<S_\sigma\} = \sigma \quad a.s.$$

Remark 1.3. *Let us remark that Assumptions 1.1–1.2 hold true for reasonable semi–martingale models and important non semi–martingale models such as the exponential fraction Brownian motion (see [11] for Assumption 1.1 and [3, 16] for Assumption 1.2). Moreover, in [11] the author proved that Assumption 1.1 implies the absence of arbitrage with the presence of proportional transaction costs, and so this is a quite natural assumption. Assumption 1.2 is more technical and its financial interpretation is linked to arbitrage opportunities with simple strategies in a frictionless setup (for details see [3]).*

Next, we introduce our utility maximization problem. Let $U:(0,\infty)\to\mathbb{R}$ be an increasing, strictly concave, continuously differentiable utility function, satisfying the Inada conditions $U'(0)=\infty$ and $U'(\infty)=0$, as well as the condition of "reasonable asymptotic elasticity" introduced in [14]

$$AE(U) := \limsup_{x\to\infty}\frac{xU'(x)}{U(x)}<1.$$

For a given initial capital $x>0$, we consider the optimization problem

$$u(x) := \sup_{\gamma\in\mathcal{A}(x)}\mathbb{E}_{\mathbb{P}}[U(x+V_T^\gamma)]. \tag{1.2}$$

1.2. Approximating Sequence of Models. For any n, let $S^n=(S_t^n)_{0\le t\le T}$ be a strictly positive, continuous process defined on some filtered probability space $(\Omega^n,\mathbb{F}^n,(\mathcal{F}_t^n)_{0\le t\le T},\mathbb{P}^n)$, where the filtration $\mathcal{F}^n := (\mathcal{F}_t^n)_{0\le t\le T}$ satisfies the usual assumptions . For the n–th model, a trading strategy is a right continuous adapted processes $\gamma^n=(\gamma_t^n)_{0\le t\le T}$ of bounded variation satisfying $\gamma_T^n=0$. As before, we use the convention that $\gamma_{0-}^n=0$. Similarly to (1.1) the corresponding liquidation value is given by

$$V_t^{\gamma^n} := \gamma_t^n S_t^n - \int_0^t S_u^n d\gamma_u^n - \kappa|\gamma_t^n|S_t^n - \kappa\int_0^t S_u^n|d\gamma_u^n|, \quad t\in[0,T].$$

For any $x > 0$ we denote by $\mathcal{A}^n(x)$ the set of all trading strategies γ^n which satisfy $x + V_t^{\gamma^n} \geq 0$, for all $t \in [0,T]$. Set

$$u^n(x) := \sup_{\gamma^n \in \mathcal{A}^n(x)} \mathbb{E}_{\mathbb{P}^n} \left[U\left(x + V_T^{\gamma^n} \right) \right].$$

As in [4] we assume the following natural assumption.

Assumption 1.4. *There exist $\varepsilon \in (0,\kappa)$ and probability measures $\mathbb{Q} \sim \mathbb{P}$, $\mathbb{Q}^n \sim \mathbb{P}^n$, $n \in \mathbb{N}$ with the following properties:*

1) There exists a local $\mathbb{Q}$–martingale $(M_t)_{0 \leq t \leq T}$ and for any $n \in \mathbb{N}$ there exists a local $\mathbb{Q}_n$–martingale $(M_t^n)_{0 \leq t \leq T}$ such that

$$|M_t - S_t| \leq (\kappa - \varepsilon) S_t \quad \mathbb{P} \ a.s., \quad \forall t \in [0,T]$$

and for any n

$$|M_t^n - S_t^n| \leq (\kappa - \varepsilon) S_t^n, \quad \mathbb{P}^n \ a.s., \quad \forall t \in [0,T].$$

2) The sequence of probability measures $\mathbb{P}^n$, $n \in \mathbb{N}$, is contiguous to the sequence $\mathbb{Q}^n$, $n \in \mathbb{N}$. Namely, for any sequence of events $A^n \in \mathcal{F}^n$, $n \in \mathbb{N}$ if $\lim_{n\to\infty} \mathbb{Q}^n(A^n) = 0$ then $\lim_{n\to\infty} \mathbb{P}^n(A^n) = 0$.

Remark 1.5. *Let us notice that condition 1) in Assumption 1.4 is a priori a robust no-arbitrage condition (for details see [12]) and condition 2) in Assumption 1.4 can be viewed as an asymptotic no-arbitrage condition for large markets (for details see [13]).*

Next, we formulate an assumption which guarantees uniform integrability.

Assumption 1.6. *One (or more) of the following conditions hold:*

(i) U is bounded from above.

(ii) There exist a constant $q > \frac{1}{1-AE(U)}$ and a sequence of pairs $(\hat{\mathbb{Q}}^n, \hat{M}^n)$, $n \in \mathbb{N}$ such that for any n, $\hat{\mathbb{Q}}^n \sim \mathbb{P}^n$, $(\hat{M}_t^n)_{0 \leq t \leq T}$ is a $\hat{\mathbb{Q}}^n$–local martingale, for all $t \in [0,T]$ we have $|\hat{M}_t^n - S_t^n| \leq \kappa S_t^n$ $\mathbb{P}^n$-a.s. and

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\hat{\mathbb{Q}}^n} \left[\left(\frac{d\mathbb{P}^n}{d\hat{\mathbb{Q}}^n} \right)^q \right] < \infty.$$

The verification of the second condition in the above assumption requires an explicit representation of consistent price systems. For the case where the market models are semi–martingales defined on the Brownian probability space and satisfy some regularity assumptions this condition holds true (for details see Example 2.8 in [4]).

Lemma 1.7. *Assume that Assumption 1.6 holds true. Then, for any $x > 0$, the set $\left\{ U^+ \left(x + V_T^{\gamma^n} \right) \right\}_{n \in \mathbb{N}, \gamma^n \in \mathcal{A}^n(x)}$ is uniformly integrable, where $U^+ := \max(U, 0)$.*

Proof. The statement is obvious if U is bounded. Thus, assume that the second condition in Assumption 1.6 holds true. From Lemma 6.3 in [14] it follows that there exists a constant L such that $U(v) \leq L(1 + v^q)$ for all v. Hence, the result follows from Proposition 2.7 in [4]. ∎

1.3. Meyer–Zheng Topology and Extended Weak Convergence. Any RCLL function $f \in \mathbb{D}[0,T] := \mathbb{D}([0,T];\mathbb{R})$ can be extended to a function $f : \mathbb{R}_+ \to \mathbb{R}$ by $f(t) := f(T)$ for all $t \geq T$. The Meyer–Zheng topology, introduced in [15], is a relative topology, on the image measures on graphs $(t, f(t))$ of trajectories $t \to f(t)$, $t \in \mathbb{R}_+$ under the measure $\lambda(dt) := e^{-t}dt$ (called pseudo-paths), induced by the weak topology of probability laws on the compactified space $[0,\infty] \times \overline{\mathbb{R}}$. From Lemma 1 in [15], it follows that the Meyer–Zheng topology on the space $\mathbb{D}[0,T]$ is given by the metric

$$d_{MZ}(f,g) := \int_0^T \min(1, |f(t) - g(t)|)dt + |f(T) - g(T)|, \quad f, g \in \mathbb{D}[0,T].$$

We denote the corresponding space by $\mathbb{D}_{MZ}[0,T]$.

Next, we formulate our convergence assumptions.

Assumption 1.8. *For any $k \in \mathbb{N}$, let $\mathbb{D}([0,T];\mathbb{R}^k)$ be the space of all RCLL functions $f : [0,T] \to \mathbb{R}^k$ equipped with the Skorokhod topology (for details see [2]). We assume that there exists $m \in \mathbb{N}$ and a stochastic processes $X^n : \Omega^n \to \mathbb{D}([0,T];\mathbb{R}^m)$, $n \in \mathbb{N}$, $X : \Omega \to C([0,T];\mathbb{R}^m)$ (i.e. X is continuous) which satisfy the following:*

(i) The filtrations $(\mathcal{F}^n_t)_{0 \leq t \leq T}$, $n \in \mathbb{N}$ and $(\mathcal{F}_t)_{0 \leq t \leq T}$, are the usual filtrations (right continuous and completed by the corresponding probability measure) generated by X^n, $n \in \mathbb{N}$ and X, respectively.

(ii) We have the weak convergence

$$((S^n, X^n), \mathbb{P}^n) \Rightarrow ((S, X), \mathbb{P}) \quad \text{on} \quad \mathbb{D}([0,T];\mathbb{R}^{m+1}).$$

The above relation means that the joint distribution of (S^n, X^n) under $\mathbb{P}^n$ converge to the joint distribution of (S, X) under $\mathbb{P}$.

(iii) We have the extended weak convergence $(X^n, \mathbb{P}^n) \Rrightarrow (X, \mathbb{P})$. This means (see [1]) that, for any k and a continuous bounded function $\psi : \mathbb{D}([0,T];\mathbb{R}^m) \to \mathbb{R}^k$, we have

$$((X^n, Y^n), \mathbb{P}^n) \Rightarrow ((X, Y), \mathbb{P}) \quad \text{on} \quad \mathbb{D}([0,T];\mathbb{R}^{m+k}),$$

where

$$Y^n_t := \mathbb{E}_{\mathbb{P}^n}\left[\psi(X^n)|\mathcal{F}^n_t\right] \quad \text{and} \quad Y_t := \mathbb{E}_{\mathbb{P}}\left[\psi(X)|\mathcal{F}_t\right], \quad t \in [0,T].$$

1.4. The Main Result. We are ready to state our limit theorem.

Theorem 1.9. *Let $x > 0$. Then we have,*

$$u(x) = \lim_{n\to\infty} u^n(x). \tag{1.3}$$

Moreover, let $\hat{\gamma}^n \in \mathcal{A}^n(x)$, $n \in \mathbb{N}$ be a sequence of asymptotically optimal portfolios, namely

$$\lim_{n\to\infty} \left(u^n(x) - \mathbb{E}_{\mathbb{P}^n}\left[U\left(x + V^{\hat{\gamma}^n}_T\right)\right]\right) = 0. \tag{1.4}$$

Then,

$$((S^n, \hat{\gamma}^n), \mathbb{P}^n) \Rightarrow \left((S, \gamma^{opt}); \mathbb{P}\right) \quad \text{on the space } \mathbb{D}([0,T]) \times \mathbb{D}_{MZ}[0,T], \tag{1.5}$$

where γ^{opt} is the unique optimal portfolio for the optimization problem (1.2).

We finish this section with the following remark.

REMARK 1.10. *Assumptions 1.4, 1.6, 1.8 are analogues (for the current setup) of similar assumptions in [4] and are needed for the proof of (1.3). This proof follows exactly the lines of the proof from [4]. In order to prove the "new" result (1.5) we establish a uniqueness result, that is Proposition 2.1. For the proof of Proposition 2.1 we need to assume Assumptions 1.1-1.2.*

2. The Uniqueness Result. In this section we prove that for a given initial capital, the problem of utility maximization from terminal wealth has a unique optimal trading strategy. Although, for strictly concave utility the uniqueness of the optimal terminal wealth is straightforward, the uniqueness of the optimal trading strategy is far from obvious and was an open question for the general setup we consider in the present note (see Remark 6.9 in [18]). It is important to mention the paper [9] where the authors proved a uniqueness result for consumption-investment problems in the presence of proportional transaction costs where the price of the assets is given by a geometric Lévy process.

PROPOSITION 2.1. *Let $x > 0$, be the initial capital. Then, there exists a unique optimal portfolio $\gamma^{opt} = (\gamma^{opt}_t)_{0\le t\le T}$ to the optimization problem (1.2).*

Proof. From Theorem 2.3 in [8], there exists a semi–martingale $\hat{S} \in [(1-\kappa)S, (1+\kappa)S]$ and $\gamma^1 \in \mathcal{A}(x)$ such that γ^1 is a solution to (1.2) and $(\gamma^1_{t-})_{0\le t\le T}$ is a solution to the frictionless problem

$$\mathbb{E}_{\mathbb{P}}\left[U\left(x+\int_0^T \gamma^1_{t-}d\hat{S}_t\right)\right] = \sup_{\theta}\mathbb{E}_{\mathbb{P}}\left[U\left(x+\int_0^T \theta_t d\hat{S}_t\right)\right] \tag{2.1}$$

where the supremum is taken over all $\hat{S}$–integrable predictable processes $\theta = (\theta_t)_{0\le t\le T}$ which satisfy the admissibility condition $x + \int_0^u \theta_t d\hat{S}_t \ge 0$ for all $u \in [0,T]$. Moreover, we have

$$\int_0^T \gamma^1_{t-}d\hat{S}_t = V_T^{\gamma^1}. \tag{2.2}$$

Step I: Assume by contradiction that there exists an optimal solution $\gamma^2 \neq \gamma^1$ which solves the utility maximization problem (1.2). First, let us notice that $V_T^{\gamma^1} = V_T^{\gamma^2}$. Indeed, if by contraction there is no equality then from the fact that U is increasing and strictly concave we obtain that for the strategy $\gamma := (\gamma^1+\gamma^2)/2$

$$\mathbb{E}_{\mathbb{P}}\left[U\left(x+V_T^{\gamma}\right)\right] > \frac{\mathbb{E}_{\mathbb{P}}\left[U\left(x+V_T^{\gamma^1}\right)\right] + \mathbb{E}_{\mathbb{P}}\left[U\left(x+V_T^{\gamma^2}\right)\right]}{2}$$

which contradicts the optimality of γ^1, γ^2. Thus, from (2.2) and the fact that $\hat{S} \in [(1-\kappa)S, (1+\kappa)S]$ it follows that

$$\int_0^T \gamma^1_{t-}d\hat{S}_t = V_T^{\gamma^1} = V_T^{\gamma^2} \le \int_0^T \gamma^2_{t-}d\hat{S}_t.$$

We conclude that

$$\int_0^T \gamma^1_{t-}d\hat{S}_t = \int_0^T \gamma^2_{t-}d\hat{S}_t. \tag{2.3}$$

Let us prove that

$$\int_0^u \gamma^1_{t-} d\hat{S}_t = \int_0^u \gamma^2_{t-} d\hat{S}_t, \quad \forall u \in [0,T]. \tag{2.4}$$

Assume by contradiction that (2.4) does not hold. Then, without loss of generality we can assume that there exist $\epsilon > 0$, a stopping time $\Theta \le T$ and an event of positive probability $A \in \mathcal{F}_\Theta$ such that

$$\int_0^\Theta \gamma^1_{t-} d\hat{S}_t - \int_0^\Theta \gamma^2_{t-} d\hat{S}_t > \epsilon \quad \text{ on the event } A. \tag{2.5}$$

Define a strategy $(\gamma^3_t)_{0\le t\le T}$ by

$$\gamma^3_t := \gamma^1_t \ \text{ for } \ t < \Theta$$

and

$$\gamma^3_t := (1-\mathbb{I}_A)\gamma^1_t + \mathbb{I}_A \gamma^2_t \ \text{ for } \ t \ge \Theta.$$

From (2.5) and the relation $\hat{S} \in [(1-\kappa)S, (1+\kappa)S]$ it follows that for any $u \in [0,T]$

$$\begin{aligned}
&\int_0^u \gamma^3_{t-} d\hat{S}_t \\
&= (1-\mathbb{I}_A\mathbb{I}_{u>\Theta}) \int_0^u \gamma^1_{t-} d\hat{S}_t + \mathbb{I}_A\mathbb{I}_{u>\Theta} \left(\int_0^\Theta \gamma^1_{t-} d\hat{S}_t + \int_\Theta^u \gamma^2_{t-} d\hat{S}_t \right) \\
&\ge (1-\mathbb{I}_A\mathbb{I}_{u>\Theta}) \int_0^u \gamma^1_{t-} d\hat{S}_t + \mathbb{I}_A\mathbb{I}_{u>\Theta} \int_0^u \gamma^2_{t-} d\hat{S}_t \\
&\ge (1-\mathbb{I}_A\mathbb{I}_{u>\Theta}) V^{\gamma^1}_u + \mathbb{I}_A\mathbb{I}_{u>\Theta} V^{\gamma^2}_u .
\end{aligned}$$

Thus, γ^3 satisfies the admissibility condition $x + \int_0^u \gamma^3_{t-} d\hat{S}_t \ge 0$ for all $u \in [0,T]$. Moreover, for $u = T$, by applying (2.3) we obtain

$$\begin{aligned}
&\int_0^T \gamma^3_{t-} d\hat{S}_t \\
&= (1-\mathbb{I}_A) \int_0^T \gamma^1_{t-} d\hat{S}_t + \mathbb{I}_A \left(\int_0^\Theta \gamma^1_{t-} d\hat{S}_t + \int_\Theta^T \gamma^2_{t-} d\hat{S}_t \right) \\
&\ge \int_0^T \gamma^1_{t-} d\hat{S}_t + \epsilon \mathbb{I}_A .
\end{aligned}$$

This contradicts the fact that γ^1 is the optimal solution for (2.1) and so, (2.4) follows.

Step II: Since by contradiction $\gamma^1 \ne \gamma^2$, there exists $\epsilon > 0$ such that the stopping time

$$\sigma = \sigma(\epsilon) := T \wedge \inf\{t : |\gamma^1_t - \gamma^2_t| > \epsilon\}$$

satisfies

$$\mathbb{P}(\sigma < T) > 0. \tag{2.6}$$

Next, define the stopping time

$$\tau := \inf\{t > \sigma : |\gamma^1_t - \gamma^2_t| < \epsilon/2\}.$$

Observe that $\gamma^1_T = \gamma^2_T = 0$ implies $\tau \le T$ a.s. on the event $\sigma < T$. Clearly, on the interval $(\sigma, \tau]$ we have $|\gamma^1_{t-} - \gamma^2_{t-}| \ge \frac{\epsilon}{2}$ and so from the associativity of the stochastic integral (see Section 2 in [17]) and (2.4) we conclude that for any $t \in (\sigma, \tau]$

$$\hat{S}_\tau - \hat{S}_t = \int_t^\tau \frac{1}{\gamma^1_{u-} - \gamma^2_{u-}} d\left(\int_0^u \left(\gamma^1_{v-} - \gamma^2_{v-}\right) d\hat{S}_v\right) = 0.$$

Thus, ($\hat{S}$ is right continuous) $\hat{S}$ is constant on $[\sigma, \tau]$. Since $\hat{S} \in [(1-\kappa)S, (1+\kappa)S]$ then from Assumption 1.2 we get $\hat{S}_\sigma \in ((1-\kappa)S_\sigma, (1+\kappa)S_\sigma))$ (i.e. the shadow price is strictly between the bid price and the ask price). From Assumption 1.1 and (2.6), it follows that for the event

$$B := \left\{(1-\kappa)S_t < \hat{S}_\sigma < (1+\kappa)S_t, \quad \forall t \in [\sigma, \tau]\right\}$$

we have $\mathbb{P}\left(B \cap \{\sigma < T\}\right) > 0$. Finally, since $\hat{S}$ is constant on the interval $[\sigma, \tau]$, we observe that on the event $B \cap \{\sigma < T\}$ the interval $[\sigma, \tau]$ is a no-trading region for any solution of (1.2) (see Theorem 3.5 in [6] and Remark 2.13 in [7]). Hence on the event $B \cap \{\sigma < T\}$, $\gamma^1_{[\sigma,\tau]}$ and $\gamma^2_{[\sigma,\tau]}$ are (random) constants. In particular

$$\gamma^1_\sigma - \gamma^2_\sigma = \gamma^1_\tau - \gamma^2_\tau \quad \text{on} \quad B \cap \{\sigma < T\}. \tag{2.7}$$

On the other hand, from the definition of σ, τ and the right continuity of γ^1, γ^2 it follows that

$$|\gamma^1_\sigma - \gamma^2_\sigma| \ge \epsilon \quad \text{and} \quad |\gamma^1_\tau - \gamma^2_\tau| \le \epsilon/2 \quad \text{on} \quad \{\sigma < T\}$$

which is a contradiction to (2.7). ∎

3. Proof of Theorem 1.9 . We start with the following lower semi–continuity result.

LEMMA 3.1. *For any $x > 0$ we have*

$$u(x) \le \lim\inf_{n\to\infty} u^n(x).$$

Proof. The proof is done by using the same approximating arguments as in Lemma 4.2 in [4]. Observe that since our utility function is not state dependent, then Assumption 2.5(i) in [4] is trivially satisfied. Moreover the continuity of u which is essential for the proof (and was established in Lemma 4.1 in [4]) is a well known fact for the current setup (see Theorem 3.2 in [6]). ∎

Next, we have the following result.

LEMMA 3.2. *Let $x > 0$ and $\gamma^{(n)} \in \mathcal{A}^n(x)$, $n \in \mathbb{N}$ be a sequence of admissible trading strategies. The sequence $((X^n, S^n, \gamma^n), \mathbb{P}^n)$ is tight on the space $\mathbb{D}([0,T]; \mathbb{R}^{m+1}) \times \mathbb{D}_{MZ}[0,T]$ and so from Prohorov's theorem (see [2]) it is relatively compact. Moreover, any cluster point is of the form $((X, S, \hat{\gamma}), \mathbb{P})$ and satisfies the following conditional independence property:*

Let $(\mathcal{F}^{X,\hat{\gamma}}_t)_{0\le t\le T}$ be the usual filtration (right continuous and $\mathbb{P}$-completed) generated by X and $\hat{\gamma}$. Then, for any $t < T$, $\mathcal{F}^{X,\hat{\gamma}}_t$ and $\mathcal{F}_T$ are conditionally independent given $\mathcal{F}_t$. As before $\mathcal{F} := (\mathcal{F}_t)_{0\le t\le T}$ is the usual filtration generated by X.

Proof. The proof is the same as the proof of Lemma 4.3 in [4] and it is based on Assumption 1.4 (Assumption 2.3 in [4]) and the extended weak convergence Assumption 1.8 (Assumption 2.9 in [4]). ∎

Now, we are ready to prove Theorem 1.9.

Proof. Let $x > 0$ and let $\hat{\gamma}^n \in \mathcal{A}^n(x)$, $n \in \mathbb{N}$ be a sequence of portfolios which satisfy (1.4). By passing to a subsequence, we assume without loss of generality that $\lim_{n\to\infty} u^n(x)$ exists.

Step I: From Proposition 2.1, there exists a unique solution to (1.2), denote it by γ^{opt}. From the tightness of the sequence $((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)$, $n \in \mathbb{N}$ (Lemma 3.2), it follows that in order to prove (1.5) it is sufficient to show that the only cluster point of this sequence is (S, X, γ^{opt}).

From Lemma 3.2, any cluster point is of the form $((X, S, \hat{\gamma}), \mathbb{P})$ where $\hat{\gamma}$ satisfies the conditional independence property which is formulated in this lemma. Let $\hat{\mathcal{A}}(x)$ be the set of all $(\mathcal{F}_t^{X,\hat{\gamma}})_{0\leq t\leq T}$–adapted processes $\gamma = (\gamma_t)_{0\leq t\leq T}$ of bounded variation with right continuous paths which satisfy $\gamma_T = 0$ and $x + V_t^{\gamma} \geq 0$, for all t. The term V^{γ} is defined as in (1.1). Introduce the optimization problem

$$\hat{u}(x) := \sup_{\gamma\in\hat{\mathcal{A}}(x)} \mathbb{E}_{\mathbb{P}}[U(x + V_T^{\gamma})]. \tag{3.1}$$

By exploiting the uniform integrability result given by Lemma 1.7 (this is Assumption 2.5(ii) in [4]), and applying the same arguments as in Section 4.2 in [4], we obtain that $\hat{\gamma} \in \hat{\mathcal{A}}(x)$ and satisfies

$$\mathbb{E}_{\mathbb{P}}[U(x + V_T^{\hat{\gamma}})] \geq \lim_{n\to\infty} u^n(x). \tag{3.2}$$

Moreover, applying the Jensen inequality and the conditional independence property given by Lemma 3.2 (Lemma 4.3 in [4]) in the same way as in Section 4.2 in [4], we obtain, for any $\gamma \in \hat{\mathcal{A}}(x)$, that

$$V_t^{\gamma^{\mathcal{F}}} \geq \mathbb{E}_{\mathbb{P}}[V_t^{\gamma}|\mathcal{F}_T], \quad \forall t \in [0, T], \tag{3.3}$$

where $\gamma^{\mathcal{F}}$ denotes the optional projection of the process γ with respect to $\mathcal{F}$ and it is well defined. In particular, (3.3) implies that $\gamma^{\mathcal{F}} \in \mathcal{A}(x)$.

Thus, from the Jensen inequality (for the concave function U), (3.3) and the trivial relation $\mathcal{A}(x) \subseteq \hat{\mathcal{A}}(x)$, we get

$$u(x) \geq \sup_{\gamma\in\hat{\mathcal{A}}(x)} \mathbb{E}_{\mathbb{P}}\left[U\left(x + V_T^{\gamma^{\mathcal{F}}}\right)\right] \geq \sup_{\gamma\in\hat{\mathcal{A}}(x)} E_{\mathbb{P}}\left[U\left(x + V_T^{\gamma}\right)\right] = \hat{u}(x) \geq u(x). \tag{3.4}$$

By applying the Jensen inequality, Lemma 3.1 and (3.2)–(3.3) it follows that

$$u(x) \geq \mathbb{E}_{\mathbb{P}}\left[U\left(x + V_T^{\hat{\gamma}^{\mathcal{F}}}\right)\right] \geq \mathbb{E}_{\mathbb{P}}\left[U\left(x + V_T^{\hat{\gamma}}\right)\right] \geq \lim_{n\to\infty} u^n(x) \geq u(x). \tag{3.5}$$

From (3.4)–(3.5) we get (1.3) and we conclude that $\hat{\gamma}, \gamma^{opt} \in \hat{\mathcal{A}}(x)$ are optimal portfolios for the optimization problem (3.1). Thus in order complete the proof it remains to argue that the uniqueness result Proposition 2.1 holds true where the filtration $\mathcal{F}$ is replaced with the filtration $\mathcal{F}^{X,\hat{\gamma}}$. For that end it remains to prove that Assumptions 1.1–1.2 hold true with respect to the filtration $\mathcal{F}^{X,\hat{\gamma}}$. This brings us to the second step.

Step II: We start with Assumption 1.1. From Lemma 3.1 in [5] it follows that we can restrict τ in Assumption 1.1 to be deterministic and the Assumption will remains the same. From [10] (see Chapter 2, Theorem 45) and the conditional independence property given by Lemma 3.2 it follows that for any t,

$$\mathbb{P}(S|\mathcal{F}_t) = \mathbb{P}(S|\mathcal{F}_t^{X,\hat{\gamma}})$$

and so Assumption 1.1 holds true with respect to $\mathcal{F}^{X,\hat{\gamma}}$.

Next, we treat Assumption 1.2. Assume by contradiction that the Assumption does not hold. Then, there exists a stopping time with respect to $\mathcal{F}^{X,\hat{\gamma}}$, $\sigma \leq T$ and $\epsilon > 0$ such that (without loss of generality we choose the positive direction)

$$\mathbb{P}\left(S_t - S_\sigma \geq 0 \quad \forall t \in [\sigma, \sigma + \epsilon]\right) > 0. \tag{3.6}$$

By enlarging the underlying probability space we assume (without loss of generality) that there exists a random variable U which is uniformly distributed on the interval $[0,1]$ and is independent of $\mathcal{F}$. Consider the process

$$Z_t := \mathbb{P}(\sigma \leq t | \mathcal{F}_T), \quad t \in [0,T].$$

Clearly, Z is a right continuous increasing process which satisfies $Z_T = 1$. Introduce the random time

$$\tau := \inf\{t : Z_t \geq U\}.$$

Observe that $Z_T = 1$ implies that $\tau \leq T$. Moreover, for any $t \in [0,T]$

$$\mathbb{P}(\tau \leq t | \mathcal{F}_T) = \mathbb{P}(Z_t \geq U | \mathcal{F}_T) = Z_t = \mathbb{P}(\sigma \leq t | \mathcal{F}_T).$$

We conclude,

$$((S,\sigma);\mathbb{P}) = ((S,\tau);\mathbb{P}). \tag{3.7}$$

Next, for any $u \in [0,1]$ define the random time $\tau_u := \inf\{t : Z_t \geq u\}$. From the conditional independence property given by Lemma 3.2, it follows that

$$Z_t = \mathbb{P}(\sigma \leq t | \mathcal{F}_t), \quad \forall t \in [0,T].$$

Hence, for any $u \in [0,1]$, τ_u is a stopping time with respect to the filtration $\mathcal{F}$. From (3.7) and the fact that U is independent of S it follows that

$$\begin{aligned}
&\mathbb{P}\left(S_t - S_\sigma \geq 0 \quad \forall t \in [\sigma, \sigma + \epsilon]\right) \\
&\mathbb{P}\left(S_t - S_\tau \geq 0 \quad \forall t \in [\tau, \tau + \epsilon]\right) \\
= &\int_0^1 \mathbb{P}\left(S_t - S_{\tau_u} \geq 0 \quad \forall t \in [\tau_u, \tau_u + \epsilon]\right) du = 0
\end{aligned}$$

where the last equality follows from Assumption 1.2 (for the filtration $\mathcal{F}$). We obtain a contradiction to (3.6), which completes the proof. ∎

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