



Minimal Realizations and Determinantal Representations in the Indefinite Setting

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Abstract. For a signature matrix J , we show that a rational matrix function $M(z)$ that is strictly J -contractive on the unit circle $\mathbb{T}$, has a strict $\tilde{J} \oplus J$ -contractive realization $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ for an appropriate signature matrix $\tilde{J}$; that is, $M(z) = D + zC(I - zA)^{-1}B$. As an application, we use this result to show that a two variable polynomial $p(z_1, z_2)$ of degree (n_1, n_2) , $n_2 = 1$, without roots on $\{(0, 0)\} \cup (\mathbb{T} \times \{0\}) \cup \mathbb{T}^2$ allows a determinantal representation

$$p(z_1, z_2) = p(0, 0) \det(I_{n_1+1} - KZ), \quad Z = z_1 I_{n_1} \oplus z_2 I_{n_2}, \quad (1)$$

where K is a strict $\tilde{J} \oplus J$ -contraction. This provides first evidence of a new conjecture that a two variable polynomial $p(z_1, z_2)$ of degree (n_1, n_2) has a determinantal representation (1) with K a strict $\tilde{J} \oplus J$ -contraction if and only if $p(z_1, z_2)$ has no roots in $\{(0, 0)\} \cup \mathbb{T}^2$.

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1. Introduction

For rational matrix functions in the one variable Schur class finite dimensional contractive realizations exist, a result due to Arov [7]. That is, if $M(z)$ is a rational matrix function that is analytic and contractive on the unit disk, there exists a contractive block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ so that

$$M(z) = D + zC(I - zA)^{-1}B, \quad |z| < 1.$$

The theory of realizations is of importance in control and systems theory and in interpolation problems, and it provides a useful tool in operator theory in

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general. The above realization result was used in obtaining a determinantal representation result for stable polynomials of two variables; see [17] (see also [12]). Namely, a two variable polynomial $p(z_1, z_2)$ of degree (n_1, n_2) without roots on the open bidisk $\mathbb{D}^2$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, can be shown to allow the representation

$$p(z_1, z_2) = p(0, 0) \det(I_{n_1+n_2} - KZ), \quad Z = z_1 I_{n_1} \oplus z_2 I_{n_2},$$

where K is a contraction. We will focus on the case when $p(z_1, z_2)$ does not have roots on the closed bidisk, in which case the matrix K can be chosen to be a strict contraction [12, 17].

In this paper we study generalizations of these results where matrices are no longer contractive in the traditional sense, but now with respect to an indefinite inner product. Equivalently, the condition $I - KK^*$ being positive definite (notation: $I - KK^* > 0$) is replaced by the condition that $J - KJK^* > 0$, where $J = J^* = J^{-1}$ is a signature matrix representing the indefinite inner product. The main result presented in Sect. 2 involves a minimal realization results for rational matrix functions that are strictly J -contractive. Next, in Sect. 3 we use these results to show that a two variable polynomial $p(z_1, z_2)$ of degree $(n_1, 1)$ without roots in $\{(0, 0)\} \cup (\mathbb{T} \times \{0\}) \cup \mathbb{T}^2$ allows a determinantal representation

$$p(z_1, z_2) = p(0, 0) \det(I_{n_1+n_2} - KZ), \quad Z = z_1 I_{n_1} \oplus z_2 I_{n_2},$$

where K is a J -contraction for an appropriate diagonal signature matrix J . The negative signature of J is determined by the number of roots of $p(z, z)$ inside the open unit disk.

2. A Realization Theorem for Rational Matrix Functions

A *signature matrix* J is an operator that is both self-adjoint and unitary, i.e. $J = J^* = J^{-1}$. A signature matrix diagonalizes as

$$J = \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix}.$$

If $J - KJK^* > 0$, then we call K a *strict J -contraction*.

Analogous to Arov's result in the definite case, we will find a strict J -contractive minimal realizations for rational matrix functions of the same type. Realizations of J -contractive operator valued functions were studied in depth in [1–6, 10, 19] in the context of reproducing kernel Pontryagin spaces, with the papers [5, 10, 19] studying general minimal passive realizations of operator valued generalized Schur functions where the state space as well as the incoming and outgoing spaces are Pontryagin spaces, the case we study in this section. The following result is a finite dimensional result addressing the strict J -contractive case and our main tool is the Hahn–Banach hyperplane separation theorem.

Theorem 2.1. *Suppose $M(z)$ is a square rational matrix function with no poles on $\mathbb{T}$ and J is a signature matrix. Then the following are equivalent:*

1. $J - M(z)JM(z)^* > 0$ for $|z| = 1$,
2. There is a strict $\begin{bmatrix} \tilde{J} \\ J \end{bmatrix}$ -contraction $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ for some signature matrix $\tilde{J}$ giving a minimal realization for $M(z)$, i.e.

$$M(z) = D + Cz(I - zA)^{-1}B. \quad (2)$$

Recall that the realization (2) is *minimal* if the matrix A is of minimal possible size. The size of this minimal A is the so-called *McMillan degree* of $M(z)$ [8].

The proof of (1) $\rightarrow$ (2) will rely on using the hyperplane separation theorem. Indeed, we will be concerned with finding a positive semidefinite element of the affine space

$$L_0 + \mathcal{W} = \left\{ \begin{bmatrix} T & 0 \\ 0 & J \end{bmatrix} - L \begin{bmatrix} T & 0 \\ 0 & J \end{bmatrix} L^* : T = T^* \right\}, \quad (3)$$

where $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is fixed and J is some diagonal signature matrix. Note that L_0 corresponds to $T = 0$, i.e.

$$L_0 = \begin{bmatrix} -BJB^* & -BJD^* \\ -DJB^* & J - DJD^* \end{bmatrix},$$

and then

$$\mathcal{W} = \left\{ \begin{bmatrix} T - ATA^* & -ATC^* \\ -CTA^* & -CTC^* \end{bmatrix} : T = T^* \right\}.$$

Let PSD denote the cone of positive semidefinite matrices, and PD denote the cone of positive definite matrices. We will use the trace inner product, i.e. $\langle X, Y \rangle = \text{tr}(XY^*)$. With this inner product, the Hermitian matrices of a fixed size form an inner product space over the reals. We will make use of the Hahn–Banach Separation Theorem on this inner product space to prove the following lemma.

Lemma 2.2. *Let $L_0 + \mathcal{W}$ be the affine space (3). If $\langle L_0, K \rangle > 0$ for all nonzero $K \in PSD \cap \mathcal{W}^\perp$, then $(L_0 + \mathcal{W}) \cap PD \neq \emptyset$.*

Proof. We will prove the contrapositive. Suppose $(L_0 + \mathcal{W}) \cap PD = \emptyset$. By the Hahn–Banach Separation Theorem, there exists a hyperplane that separates $L_0 + \mathcal{W}$ and PD . That is, there exists a matrix $K \neq 0$ and real number c such that $\langle X, K \rangle \leq c$ for all $X \in L_0 + \mathcal{W}$, and $\langle X, K \rangle \geq c$ for all $X \in PD$.

We first claim that we may choose c to be 0. Suppose for the sake of contradiction that $c > 0$, so we have $\langle X, K \rangle \geq c > 0$ for all $X \in PD$. If for some fixed $X_0 \in PD$ we have $\langle X_0, K \rangle = b$, we can choose $\epsilon < c/b$, so that $0 < \langle \epsilon X_0, K \rangle < c$. As $\epsilon X_0 \in PD$, this contradicts that $\langle X, K \rangle \geq c$ for all $X \in PD$. We must therefore have that $c \leq 0$. We next claim that we may choose $c = 0$. If for some $X_1 \in PD$ we have $\langle X_1, K \rangle < 0$, then $\lim_{n \rightarrow \infty} \langle nX_1, K \rangle = -\infty$, which would violate any inequality $\langle X, K \rangle \geq c$. And thus, $\langle X, K \rangle \geq 0$ for all $X \in PD$, justifying the choice $c = 0$.

Note also that $\langle X, K \rangle \geq 0$ for all $X \in PD$ implies that $K \in PSD$. Next we claim that $K \in \mathcal{W}^\perp$. Suppose, once more for the sake of contradiction,

that $K \notin \mathcal{W}^\perp$. Choose a $W \in \mathcal{W}$ such that $\langle K, W \rangle \neq 0$. Then $\langle L_0 + tW, K \rangle = \langle L_0, K \rangle + t\langle W, K \rangle$ takes on any real value when we let t vary. This violates that $\langle L_0 + tW, K \rangle \leq 0$ for all t . Hence, $K \in \mathcal{W}^\perp$.

Let $X \in L_0 + \mathcal{W}$. Then $X = L_0 + W$ for some $W \in \mathcal{W}$. Then $\langle L_0, K \rangle = \langle X, K \rangle \leq 0$, where we use that $K \in \mathcal{W}^\perp$. $\square$

Now let us suppose that $K \in PSD \cap \mathcal{W}^\perp$. Write $K = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}$, with $P = P^*, R = R^*$. We then have

$$\begin{aligned} \langle K, W \rangle &= 0 \quad \forall W \in \mathcal{W} \\ \iff \left\langle \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}, \begin{bmatrix} T - ATA^* & -ATC^* \\ -CTA^* & -CTC^* \end{bmatrix} \right\rangle &= 0, \quad \forall T = T^* \\ \iff \text{tr}PT - \text{tr}PATA^* - \text{tr}QCTA^* - \text{tr}Q^*ATC^* - \text{tr}RCTC^* &= 0, \forall T = T^* \\ \iff \langle P - A^*PA - A^*QC - C^*Q^*A - C^*RC, T \rangle &= 0, \quad \forall T = T^* \\ \iff P = A^*PA + A^*QC + C^*Q^*A + C^*RC \\ \iff P = [A^* \ C^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}. \end{aligned}$$

Thus the condition that $K \in \mathcal{W}^\perp$ is equivalent to $K = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}$, with $P = P^*, R = R^*$, and $P = [A^* \ C^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}$. We may similarly rewrite the condition that $\langle L_0, K \rangle > 0$. Observe the following.

$$\begin{aligned} \langle K, L_0 \rangle &= \left\langle \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}, \begin{bmatrix} -BJB^* & -BJD^* \\ -DJB^* & J - DJD^* \end{bmatrix} \right\rangle \\ &= -\text{tr}PBJB^* - \text{tr}QDJB^* - \text{tr}Q^*BJD^* + \text{tr}RJ - \text{tr}RDJD^* \\ &= \text{tr}RJ - \text{tr}B^*PBJ - \text{tr}B^*QDJ - \text{tr}D^*Q^*BJ - \text{tr}D^*RDJ \\ &= \left\langle R - [B^* \ D^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}, J \right\rangle. \end{aligned}$$

We now arrive at a lemma that is equivalent to Lemma 2.2.

Lemma 2.3. *If, for every nonzero $K = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geq 0$ with $P = P^*, R = R^*$, and*

$$P = [A^* \ C^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}, \quad (4)$$

we have

$$\left\langle R - [B^* \ D^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}, J \right\rangle > 0, \quad (5)$$

then there exists a positive definite element of the affine space

$$\left\{ \begin{bmatrix} T & 0 \\ 0 & J \end{bmatrix} - L \begin{bmatrix} T & 0 \\ 0 & J \end{bmatrix} L^* : T = T^* \right\}.$$

We will also make use of the following lemma.

Lemma 2.4. Suppose $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geq 0$ satisfies (4). Then $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}$ is a sum of positive semidefinite rank 1 matrices $\begin{bmatrix} P_\ell & Q_\ell \\ Q_\ell^* & R_\ell \end{bmatrix}$ with $P_\ell = \begin{bmatrix} A^* & C^* \end{bmatrix} \begin{bmatrix} P_\ell & Q_\ell \\ Q_\ell^* & R_\ell \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}$.

Proof. Since $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geq 0$, we can write $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} = \begin{bmatrix} G \\ H \end{bmatrix} \begin{bmatrix} G^* & H^* \end{bmatrix}$. Together with (4), this implies

$$GG^* = (A^*G + C^*H)(A^*G + C^*H)^*.$$

By Douglas' Lemma [9], there exists a unitary U such that $GU = (A^*G + C^*H)$. We diagonalize U as $U = V\Delta V^*$, with V, Δ unitary, Δ diagonal. Then

$$GV\Delta = (A^*G + C^*H)V.$$

Write $\Delta = \text{diag}(e^{i\theta_\ell})$, and let $\text{col}_\ell(X)$ denote the ℓ th column of a matrix X . Then

$$\text{col}_\ell(GV) \cdot e^{i\theta_\ell} = \text{col}_\ell((A^*G + C^*H)V). \quad (6)$$

We can then write

$$\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} = \sum_\ell \left(\text{col}_\ell \left(\begin{bmatrix} G \\ H \end{bmatrix} V \right) \right) \left(\text{col}_\ell \left(\begin{bmatrix} G \\ H \end{bmatrix} V \right) \right)^* =: \sum_\ell \begin{bmatrix} P_\ell & Q_\ell \\ Q_\ell^* & R_\ell \end{bmatrix},$$

where each $\begin{bmatrix} P_\ell & Q_\ell \\ Q_\ell^* & R_\ell \end{bmatrix}$ is rank 1, and (6) implies that

$$P_\ell = \begin{bmatrix} A^* & C^* \end{bmatrix} \begin{bmatrix} P_\ell & Q_\ell \\ Q_\ell^* & R_\ell \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}$$

for each ℓ . This completes the proof of the lemma. $\square$

Proof of Theorem 2.1. (2) $\rightarrow$ (1). Suppose there is a strict $\begin{bmatrix} \tilde{J} \\ J \end{bmatrix}$ -contraction $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ giving a minimal realization for $M(z)$. As $M(z)$ does not have poles on $\mathbb{T}$ and the realization is minimal, we have that $I - zA$ is invertible for $z \in \mathbb{T}$ (see, e.g., [8, Theorem 3.3]). By definition of a J -contraction, we may write

$$\begin{bmatrix} \tilde{J} \\ J \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{J} \\ J \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} > 0$$

where $P = P^*$ and $R = R^*$.

This gives the following identities:

$$\begin{aligned} P &= \tilde{J} - A\tilde{J}A^* - BJB^*, \\ Q &= -A\tilde{J}C^* - BJD^*, \\ R &= J - C\tilde{J}C^* - DJD^*. \end{aligned}$$

We then have (when z is not a pole of $M(z)$),

$$\begin{aligned}
& J - M(z)JM(z)^* \\
&= J - DJD^* - zC(I - zA)^{-1}BJD^* \\
&\quad - \bar{z}DJB^*(I - \bar{z}A^*)^{-1}C^* \\
&\quad - z\bar{z}C(I - zA)^{-1}BJB^*(I - \bar{z}A^*)^{-1}C^*, \\
&= R + C\tilde{J}C^* + zC(I - zA)^{-1}(Q + A\tilde{J}C^*) \\
&\quad + \bar{z}(Q^* + C\tilde{J}A^*)(I - \bar{z}A^*)^{-1}C^* \\
&\quad + z\bar{z}C(I - zA)^{-1}(P - \tilde{J} + A\tilde{J}A^*)(I - \bar{z}A^*)^{-1}C^*, \\
&= R + zC(I - zA)^{-1}Q + \bar{z}Q^*(I - zA)^{-1}C^* \\
&\quad + z\bar{z}C(I - zA)^{-1}P(I - \bar{z}A^*)^{-1}C^* \\
&\quad + C\tilde{J}C^* + zC(I - zA)^{-1}A\tilde{J}C^* + \bar{z}C\tilde{J}A^*(I - \bar{z}A^*)^{-1}C^* \\
&\quad + z\bar{z}C(I - zA)^{-1}(A\tilde{J}A^* - \tilde{J})(I - \bar{z}A^*)^{-1}C^*.
\end{aligned}$$

We now note that

$$\begin{aligned}
& R + zC(I - zA)^{-1}Q + \bar{z}Q^*(I - zA)^{-1}C^* + z\bar{z}C(I - zA)^{-1}P(I - \bar{z}A^*)^{-1}C^* \\
&= [zC(I - zA)^{-1} \ I] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} \bar{z}(I - \bar{z}A^*)^{-1}C^* \\ I \end{bmatrix}, \tag{7}
\end{aligned}$$

and

$$\begin{aligned}
& C\tilde{J}C^* + zC(I - zA)^{-1}A\tilde{J}C^* + \bar{z}C\tilde{J}A^*(I - \bar{z}A^*)^{-1}C^* \\
&\quad + z\bar{z}C(I - zA)^{-1}(A\tilde{J}A^* - \tilde{J})(I - \bar{z}A^*)^{-1}C^* \\
&= C(I - zA)^{-1} \left[(I - zA)\tilde{J}(I - \bar{z}A^*) + \bar{z}(I - zA)\tilde{J}A^* \right. \\
&\quad \left. + zA\tilde{J}(I - \bar{z}A^*) - z\bar{z}\tilde{J} + z\bar{z}A\tilde{J}A^* \right] (I - \bar{z}A^*)^{-1}C^*, \\
&= C(I - zA)^{-1} \left[(1 - z\bar{z})\tilde{J} \right] (I - \bar{z}A^*)^{-1}C^*. \tag{9}
\end{aligned}$$

Expression (7) is positive definite for any choice of z with $I - zA$ invertible.

Expression (9) is equal to zero when $|z| = 1$. We therefore have that

$$J - M(z)JM(z)^* > 0 \quad \text{for } |z| = 1.$$

(1) $\rightarrow$ (2). Suppose $J - M(z)JM(z)^* > 0$ for $|z| = 1$, and let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a minimal realization for $M(z)$. As $M(z)$ does not have poles on $\mathbb{T}$, the minimality assures that $I - zA$ is invertible when $z \in \mathbb{T}$. Any other minimal realization for $M(z)$ can be written as

$$\begin{bmatrix} S^{-1}AS & S^{-1}B \\ CS & D \end{bmatrix} = \begin{bmatrix} S^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S & \\ & I \end{bmatrix}$$

for some invertible S (see Theorem 3.1 in [8]). We would like to find an S that will transform our minimal realization into a strict $\begin{bmatrix} \tilde{J} \\ J \end{bmatrix}$ -contraction,

i.e.

$$\begin{bmatrix} \tilde{J} \\ J \end{bmatrix} - \begin{bmatrix} S^{-1}AS & S^{-1}B \\ CS & D \end{bmatrix} \begin{bmatrix} \tilde{J} \\ J \end{bmatrix} \begin{bmatrix} S^{-1}AS & S^{-1}B \\ CS & D \end{bmatrix}^* > 0,$$

or equivalently,

$$\begin{aligned} & \begin{bmatrix} \tilde{J} \\ J \end{bmatrix} - \begin{bmatrix} S^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S\tilde{J}S^* & \\ & J \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} S^{-*} & \\ & I \end{bmatrix} > 0 \\ & \iff \begin{bmatrix} S\tilde{J}S^* & \\ & J \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S\tilde{J}S^* & \\ & J \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* > 0. \end{aligned}$$

Now we observe that every invertible Hermitian matrix T can be factored as $T = V\hat{J}V^*$ where V is invertible and $\hat{J}$ is a diagonal signature matrix. Indeed, since T is invertible and Hermitian, it is unitarily diagonalizable with nonzero diagonal entries, $T = UDU^*$. We can factor $D = |D|^{1/2}\hat{J}|D|^{1/2}$, where the negative terms in $\hat{J}$ correspond to the negative entries of D . Then $T = U|D|^{1/2}\hat{J}|D|^{1/2}U^* = V\hat{J}V^*$.

So, if we set $T = S\tilde{J}S^*$, then we need only to show that there exists invertible $T = T^*$ such that

$$\begin{bmatrix} T & \\ & J \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T & \\ & J \end{bmatrix} \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} > 0. \quad (10)$$

It is worth noting that the above is a positive definite Stein equation, so if a solution T exists, it is necessarily invertible (see [18], Section (13.2)).

By Lemma 2.3, it is sufficient to show that for nonzero $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \geq 0$, if

$$P = [A^* \ C^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix},$$

then

$$\left\langle R - [B^* \ D^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}, J \right\rangle > 0.$$

By Lemma 2.4, without loss of generality we may assume that $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}$ is of rank 1. Let

$$\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x^* & y^* \end{bmatrix}.$$

Then

$$xx^* = P = [A^* \ C^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = (A^*x + C^*y)(x^*A + y^*C),$$

giving that $x = e^{i\theta}(A^*x + C^*y)$ for some θ . We can rewrite this as $(I - e^{i\theta}A^*)x = e^{i\theta}C^*y$, and thus

$$x = e^{i\theta}(I - e^{i\theta}A^*)^{-1}C^*y. \quad (11)$$

Note that $y \neq 0$, as otherwise $\begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} = 0$.

Now

$$\begin{aligned}
& \left\langle R - [B^* \ D^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}, J \right\rangle \\
&= \langle R, J \rangle - \left\langle [B^* \ D^*] \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}, J \right\rangle \\
&= y^* J y - [x^* \ y^*] \begin{bmatrix} B \\ D \end{bmatrix} J [B^* \ D^*] \begin{bmatrix} x \\ y \end{bmatrix} \\
&= y^* J y - x^* B J B^* x - x^* B J D^* y - y^* D J B^* x - y^* D J D^* y \\
&= y^* \left(J - D J D^* - e^{-i\theta} C (I - e^{-i\theta} A)^{-1} B J D^* - D J B^* (I - e^{i\theta} A^*)^{-1} C^* e^{i\theta} \right. \\
&\quad \left. - e^{-i\theta} C (I - e^{-i\theta} A)^{-1} B J B^* (I - e^{i\theta} A^*)^{-1} C^* e^{i\theta} \right) y \\
&= y^* \left(J - M(e^{-i\theta}) J M(e^{-i\theta})^* \right) y.
\end{aligned}$$

Since we assumed $J - M(z) J M(z)^* > 0$ for $|z| = 1$, the above expression is positive and equation (5) holds.

We therefore have that there exists a matrix T for which equation (10) holds, and we are done. $\square$

Note that the proof of (2) $\rightarrow$ (1) does not use the finite dimensionality (as long as one assumes that $\mathbb{T}$ is in the resolvent set of A); compare also with [19, Proposition 2.4].

3. Determinantal Representations

The problem of writing polynomials as determinants of linear matrix polynomials has attracted a lot of attention in the past few decades; see, e.g., [11–13, 15–17, 20]. In this section, we will focus on determinantal representations with an indefinite norm constraint. It was shown in [17] that a nonzero bivariate polynomial $p(z_1, z_2)$ of degree (n_1, n_2) is without roots in $\mathbb{D}^2$ if and only if it allows the determinantal representation

$$p(z_1, z_2) = p(0, 0) \det(I_{n_1+n_2} - KZ), \quad Z = z_1 I_{n_1} \oplus z_2 I_{n_2},$$

with the $(n_1 + n_2) \times (n_1 + n_2)$ matrix K being a contraction. In addition, $p(z_1, z_2)$ is without roots in $\overline{\mathbb{D}}^2$ if and only if it allows the above determinantal representation with K a strict contraction. The proof was streamlined in [12].

In [11] a different determinantal representation appears, where it was shown that if a bivariate polynomial $p(z, w)$ with bidegree (m, n) has its roots in

$$(\mathbb{D} \times \mathbb{C}) \cup \mathbb{T}^2 \cup ((\mathbb{C} \setminus \overline{\mathbb{D}}) \times \mathbb{C}), \text{ or } (\mathbb{C} \times \mathbb{D}) \cup \mathbb{T}^2 \cup (\mathbb{C} \times (\mathbb{C} \setminus \overline{\mathbb{D}})),$$

then we can write

$$p(z, w) = p(0, 0) \det \left(U \begin{bmatrix} w I_m & & \\ & z I_{n_1} & \\ & & I_{n_2} \end{bmatrix} - \begin{bmatrix} I_m & & \\ & I_{n_1} & \\ & & z I_{n_2} \end{bmatrix} \right),$$

where U is unitary, n_2 is the number of zeros of $p(z, 0)$ in $\mathbb{D}$, and $n_1 = n - n_2$. For more on this type of determinantal representation and related results see also [15, 16].

In this section we provide evidence for the following conjecture.

Conjecture 3.1. *Let $p(z_1, z_2)$ be a nonzero polynomial of degree (n_1, n_2) . Then $p(z_1, z_2)$ is without roots in $\{(0, 0)\} \cup \mathbb{T}^2$ if and only if there is a diagonal signature matrix J and strict J -contraction K so that*

$$p(z_1, z_2) = p(0, 0) \det(I - K(z_1 I_{n_1} \oplus z_2 I_{n_2})). \quad (12)$$

The if direction is easily checked. Clearly, we cannot have that $p(0, 0) = 0$, since in that case (12) implies that $p(z_1, z_2) \equiv 0$. If now (z_1, z_2) is in the bitorus $\mathbb{T}^2$, we get that $Z = z_1 I_{n_1} \oplus z_2 I_{n_2}$ is unitary, and since Z commutes with J , we get that

$$J - KZJZ^*K = J - KJK^* > 0.$$

Due to a standard Stein equation result (see, e.g., [18]) this gives that KZ has as many eigenvalues in the open unit disk as J has negative eigenvalues, and as many eigenvalues outside the closed unit disk as J has positive eigenvalues. In particular, KZ does not have the number 1 as an eigenvalue, and thus we find that $p(z_1, z_2) \neq 0$ when $(z_1, z_2) \in \mathbb{T}^2$.

Using Theorem 2.1, we will prove the following.

Theorem 3.2. *Suppose $p(z_1, z_2)$ is a bivariate polynomial with bidegree $(n_1, 1)$ and $p(z_1, z_2) \neq 0$ on $\{(0, 0)\} \cup (\mathbb{T} \times \{0\}) \cup \mathbb{T}^2$. Then p admits a determinantal representation*

$$p(z_1, z_2) = p(0, 0) \det(I_{n_1+1} - KZ),$$

where $Z = z_1 I_{n_1} \oplus z_2$, and K is a strict $\begin{bmatrix} \tilde{J} & 0 \\ 0 & J \end{bmatrix}$ -contraction with $\tilde{J} = \begin{bmatrix} I_{n_1-k} & 0 \\ 0 & -I_k \end{bmatrix}$ where k is the (constant) number of roots of $p(z_1, z_2)$ inside $\mathbb{D}$ for fixed $z_2 \in \mathbb{T}$, and

$$J = \begin{cases} -1 & \text{if the one root of } p(z_1, z_2) \text{ for fixed } z_1 \in \mathbb{T} \text{ is inside } \mathbb{D}, \\ 1 & \text{if the root is outside } \mathbb{D}. \end{cases}$$

We first need the following observation.

Lemma 3.3. *Suppose $p(z_1, z_2)$ is a bivariate polynomial with bidegree (n_1, n_2) and $p(z_1, z_2) \neq 0$ on $\mathbb{T}^2$. Then, if we let z_1 vary over $\mathbb{T}$, the number of roots of the polynomial $p_{z_1}(z_2) := p(z_1, z_2)$ has k inside $\mathbb{D}$ and $n_2 - k$ roots outside $\mathbb{D}$, where k is independent of z_1 . A similar statement holds when interchanging the roles of z_1 and z_2 .*

Proof. We need to show that the number k is independent of $z_1 \in \mathbb{T}$. If that were not the case, then since the roots of a polynomial vary continuously with its coefficients, there would exist $z_1 \in \mathbb{T}$ where the number of roots inside $\mathbb{D}$ changes value, but that can only happen when a root crosses the unit circle. Thus, there would be $z_2 \in \mathbb{T}$ such that $p(z_1, z_2) = 0$, which is a contradiction. $\square$

Proof of Theorem 3.2. Without loss of generalization, we assume that $p(0, 0) = 1$. Let us write $p(z_1, z_2) = p_0(z_1) + z_2 p_1(z_1)$. First, we assume that $p_0(z_1)$ and $p_1(z_1)$ do not have a common factor, and we put $M(z_1) = -\frac{p_1(z_1)}{p_0(z_1)}$. As $p_0(z_1)$ and $p_1(z_1)$ do not have a common factor, the poles of $M(z_1)$ are exactly the roots of $p_0(z_1)$. The condition that $p(z_1, z_2) \neq 0$ on $\mathbb{T}^2$, gives that for $|z_1| = 1$, we have that the root of $p_{z_1}(z_2) := p(z_1, z_2)$ is either always inside $\mathbb{D}$ or outside $\mathbb{D}$. With the choice of J as stated in the theorem, we now obtain that $J - M(z_1)JM(z_1)^* > 0$ for $|z_1| = 1$.

Since $p(z_1, z_2)$ has no roots in $\mathbb{T} \times \{0\}$, we have that $M(z_1)$ has no poles on $\mathbb{T}$. Apply now Theorem 2.1, giving that $M(z_1)$ has a realization $M(z_1) = D + z_1 C(I - z_1 A)^{-1} B$, with

$$K := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

a strict $\tilde{J} \oplus J$ -contraction. We now claim that

$$p(z_1, z_2) = \det(I - K(z_1 I_{n_1} \oplus z_2)).$$

Indeed, with $n_2 = 1$,

$$\begin{aligned} p(z_1, z_2) &= p_0(z_1) \det(I_{n_2} - z_2 M(z_1)) \\ &= p_0(z_1) \det(I_{n_2} - z_2 D - z_2 C z_1 (I_{n_1} - A z_1)^{-1} B) \\ &= \frac{p_0(z_1)}{\det(I_{n_1} - A z_1)} \det \begin{bmatrix} I_{n_1} - A z_1 & -B z_2 \\ -C z_1 & I_{n_2} - z_2 D \end{bmatrix} \\ &= \frac{p_0(z_1)}{\det(I_{n_1} - A z_1)} \det \left(I_{n_1+n_2} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} z_1 I_{n_1} & \\ & z_2 I_{n_2} \end{bmatrix} \right). \end{aligned} \quad (13)$$

The term $\frac{p_0(z_1)}{\det(I_{n_1} - A z_1)}$ is equal to 1 as the poles of $M(z_1)$ coincide exactly with the roots of $p_0(z_1)$ and the realization is minimal, and we have have $p(z_1, z_2) = \det(I_{n_1+n_2} - KZ)$ where K and Z are as desired.

If p has a factor containing z_1 alone, we may factor p as $p(z_1, z_2) = h(z_1)q(z_1, z_2)$, with $h(0) = 1$, such that q has no factors containing z_1 alone. In this case we apply the previous paragraph to obtain a determinantal representation $q(z_1, z_2) = \det(I - K(z_1 I \oplus z_2))$ for q with K a strict $\tilde{J} \oplus J$ -contraction. Next, if $\{\lambda_i\}_{i=1}^n$ are the roots of $h(z_1)$, then

$$h(z_1) = \prod_{i=1}^n \left(1 - \frac{z_1}{\lambda_i}\right) = \det \left(I_n - \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix} z_1 \right) =: \det(I - \Lambda z_1).$$

Choosing $\hat{J}$ to be the diagonal signature matrix with +1's in the positions corresponding to the λ_i that lie outside $\mathbb{D}$ and -1's corresponding to the λ_i that lie inside $\mathbb{D}$, we obtain that Λ is a $\hat{J}$ -contraction. Now $\Lambda \oplus K$ is a strict $\hat{J} \oplus \tilde{J} \oplus J$ -contraction and $p(z_1, z_2) = \det(I - (\Lambda \oplus K)(z_1 I_{n_1} \oplus z_2))$, providing the desired determinantal representation of p . $\square$

The condition that $p(z_1, z_2)$ has no roots in $\mathbb{T} \times \{0\}$ is needed in the proof, as we use Theorem 2.1 and consequently we need $M(z_1)$ not to have poles on $\mathbb{T}$. However, polynomials $p(z_1, z_2)$ with roots in $\mathbb{T} \times \{0\}$ can still have the desired determinantal representations. Indeed, for $p(z_1, z_2) = 1 - z_1 + 3z_2$, we find for instance that

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, K = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix},$$

gives a desired determinantal representation. Also, for $p(z_1, z_2) = (1 - z_1)^2 + 5z_2$, the choice

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, K = \begin{pmatrix} 1.4783 & 2.1294 & -2.8379 \\ -0.1074 & 0.5217 & -0.3497 \\ -3.1181 & -3.2919 & 5.0000 \end{pmatrix},$$

works. It could be that an argument where we approximate $p_0(z_1)$ with polynomials without roots in $\mathbb{T}$ might work, as long as we would have a way of ensuring that the corresponding limit of strict $\tilde{J} \oplus J$ -contractions would still be a strict $\tilde{J} \oplus J$ -contractions (for instance, by proving that $\tilde{J} \oplus J - K(\tilde{J} \oplus J)K^* \geq \epsilon I > 0$ for some specified ϵ in Theorem 2.1).

The case when $n_2 > 1$ may be approached as follows. Write $p(z_1, z_2) = \sum_{k=0}^{n_2} p_k(z_1)z_2^k$, and write $\frac{p(z_1, z_2)}{p_0(z_1)}$ as $\det(I - z_2 M(z_1))$, where $M(z_1)$ corresponds to some appropriate linearization of $\sum_{k=0}^{n_2} z_2^k \frac{p_k(z_1)}{p_0(z_1)}$. The trick will now be to find an appropriate similarity $U(z_1)$ so that $U(z_1)M(z_1)U(z_1)^{-1}$ is strictly J -contractive. In [12] this approach was successfully executed in the definite case where the linearization chosen was a companion matrix, and the similarity $U(z_1)$ was found by performing a Fejér-Riesz factorization of a related Bezoutian. In [14] this approach was further pursued in the indefinite case, and resulted in a determinantal representation with a non-strict J -contraction in some cases (see [14, Theorem 5.1] for details). However, this approach does not lead to a strict J -contraction. It is possible that a different choice of linearization and/or similarity $U(z_1)$ could be more effective in this regard. As an aside, we mention that [14, Chapter 6] also contains a description and Matlab codes how to compute the strict J -contractive realizations and determinantal representations numerically.

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