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Covering 2-colored complete digraphs by monochromatic d-dominating digraphs

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Abstract

A digraph is d-dominating if every set of at most d vertices has a common out-neighbor. For all integers $d \geq 2$, let $f(d)$ be the smallest integer such that the vertices of every 2-edge-colored (finite or infinite) complete digraph (including loops) can be covered by the vertices of at most $f(d)$ monochromatic d-dominating subgraphs. Note that the existence of $f(d)$ is not obvious – indeed, the question which motivated this paper was simply to determine whether $f(d)$ is bounded, even for $d = 2$. We answer this question affirmatively for all $d \geq 2$, proving $4 \leq f(2) \leq 8$ and $2d \leq f(d) \leq 2d\left(\frac{d^d - 1}{d - 1}\right)$ for all $d \geq 3$. We also give an example to show that there is no analogous bound for more than two colors. Our result provides a positive answer to a question regarding an infinite analogue of the Burr-Erdős conjecture on the Ramsey numbers of d-degenerate graphs. Moreover, a special case of our result is related to properties of d-paradoxical tournaments.

KEYWORDS

infinite digraphs, monochromatic cover, paradoxical tournaments, Ramsey

1 | INTRODUCTION

Throughout this note, a directed graph (or digraph for short) is a pair (V, E), where V can be finite or infinite and $E \subseteq V \times V$ (so in particular, loops are allowed). A digraph is complete if $E = V \times V$. For $v \in V$, we write $N^+(v) = \{u : (v, u) \in E\}$ and $N^-(v) = \{u : (u, v) \in E\}$. For a positive integer k, we define $[k] := \{1, \ldots, k\}$.

Let $G = (V, E)$ be a digraph. For $X, Y \subseteq V$ we say that X dominates Y if $(x, y) \in E$ for all $x \in X, y \in Y$. We say that G is d-dominating if for all $S \subseteq V$ with $1 \le |S| \le d$, S dominates some $w \in V$. Note that it is possible for $w \in S$, in which case we must have $(w, w) \in E$. Reversing all edges of a d-dominating digraph gives a d-dominated digraph. These notions are well studied for tournaments (see Section 4).

Note that regardless of whether $G = (V, E)$ is a graph or a digraph, if $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, we will write $H \subseteq G$ and we will always refer to H as a subgraph of G rather than making a distinction between "subgraph" and "subdigraph."

A cover of a digraph $G = (V, E)$ is a set of subgraphs $\{H_1, \ldots, H_t\}$ such that $V(G) = \bigcup_{i \in [t]} V(H_i)$. By a 2-coloring of $G = (V, E)$, we will always mean a 2-coloring of the edges of G; that is, a function $c: E \to [2]$. Given a 2-coloring of G, we let E_i be the set of edges receiving color i (i.e., $E_i = c^{-1}(\{i\})$) and $G_i = (V, E_i)$ for $i \in [2]$. A cover of G by monochromatic subgraphs is a cover $\{H_1, \ldots, H_t\}$ of G such that for all $i \in [t]$ there exists $j \in [2]$ such that $H_i \subseteq G_j$.

The following problem was raised in [4, problem 6.6] (see Section 3 for the context in which this problem was raised).

Problem 1.1. Given a 2-colored complete digraph K, is it possible to cover K with at most four monochromatic 2-dominating subgraphs? (If not four, some other fixed number?)

Our main result is a positive answer for the qualitative part of Problem 1.1 in a more general form.

Theorem 1.2. Let d be an integer with $d \ge 2$. In every 2-colored complete digraph K, there exists a cover of K with at most $2 \times \sum_{i=1}^{d} d^i = 2d\left(\frac{d^d - 1}{d - 1}\right)$ monochromatic d-dominating subgraphs. In case of $d = 2$ there exists a cover of K with at most eight monochromatic 2-dominating subgraphs.

For all integers $d \ge 1$, let $f(d)$ be the minimum number of monochromatic d-dominating subgraphs needed to cover an arbitrarily two-colored complete digraph. Note that obviously $f(1) = 2$ since the two sets of monochromatic loops provide an optimal cover. For $d \ge 2$, Theorem 1.2 shows that $f(d)$ is well-defined. Example 1.3 below (adapted from [4, proposition 6.3]) combined with Theorem 1.2 gives

$$4 \le f(2) \le 8 \text{ and } 2d \le f(d) \le 2d\left(\frac{d^d - 1}{d - 1}\right) \text{ for all integers } d \ge 3. \tag{1}$$

Example 1.3. Let K be a complete digraph on at least $2d$ vertices and partition $V(K)$ into nonempty sets $R_1, \ldots, R_d$ and $B_1, \ldots, B_d$, color all edges inside R_i red, all edges inside B_i blue, all edges from R_i to B_j red, all edges from B_i to R_j blue, all edges between R_i and R_j with $i \ne j$ blue, and all edges between B_i and B_j with $i \ne j$ red. One can check that every monochromatic d-dominating subgraph of K is entirely contained inside one of the sets $R_1, \ldots, R_d, B_1, \ldots, B_d$.

Finally, the following example shows that for $d \geq 2$ there is no analogue of Theorem 1.2 for more than two colors (c.f. [4, example 2.3]).

Example 1.4. Let V be a totally ordered set and let K be the complete digraph on V where for all $i \in V$, (i, i) is green and for all $i, j \in V$ with $i < j$ (i, j) is red and (j, i) is blue. Note that for $d \geq 2$ the only monochromatic d-dominating subgraphs are the green loops and thus no bound can be put on the number of monochromatic d-dominating subgraphs needed to cover V.

While we have completely solved the qualitative problem, we would be very interested to see an improvement in the quantitative bounds (1) given above.

2 | COVERING DIGRAPHS, PROOF OF THEOREM 1.2

For a graph G, we denote the order of a largest clique (pairwise adjacent vertices) in G by $\omega(G)$. Given a 2-colored complete digraph K and a set $U \subseteq V(K)$, define $G[U]_{\text{blue}}$ to be the graph on U where $\{u, v\} \in G[U]_{\text{blue}}$ if and only if (u, v) and (v, u) are blue in K; define $G[U]_{\text{red}}$ analogously.

For all positive integers ω and d, let $f(0, d) = 0$ and let $f(\omega, d)$ be the smallest positive integer D such that if K is a 2-colored complete digraph on vertex set V where every loop has the same color, say red, and $\omega(G[V]_{\text{blue}}) = \omega$, then V can be covered by at most D monochromatic d-dominating subgraphs.

Lemma 2.1.

(1) $f(1, 2) = 1$.
(2) $f(\omega, d) \leq d(f(\omega - 1, d) + 1)$ for all $1 \leq \omega \leq d$ (in particular, $f(1, d) \leq d$). In fact, all d-dominating subgraphs in the covering have the same color as the loops.

Note that the upper bound $\omega \leq d$ is not strictly necessary, but we include it here for clarity since in the next lemma, we will prove a stronger result when $\omega \geq d + 1$.

Proof. Let K be a 2-colored complete digraph on vertex set V where all loops have the same color, say red.

(1) is trivial since for all distinct $u, v \in V$ both (u, u) and (v, v) are red and $\omega(G[V]_{\text{blue}}) = 1$ implies that either (u, v) or (v, u) is red.

To see (2), note first that we may assume that K itself is not spanned by a red d-dominating subgraph, otherwise we are done. This is witnessed by a set $U = \{u_1, \ldots, u_d\} \subseteq V$, such that there is no $w \in V$ with (u_i, w) red for all $i \in [d]$.

For all $i \in [d]$ we define

$$W_i = \{v \in V : (v, u_i) \text{ is red}\}.$$

Note that $u_i \in W_i$ and $K[W_i]$ is spanned by a red d-dominating subgraph for all $i \in [d]$. Set $V' = V \setminus (\cup_{i \in [d]} W_i)$ and define

$$T_i = \{v \in V' : (u_i, v) \text{ is blue}\}.$$

Note, that by the definition of V', (v, u_i) is also blue for all $v \in T_i$ and $i \in [d]$. Moreover, from the selection of U, every vertex in V' receives a blue edge from some vertex in U and therefore $V' = \cup_{i=1}^{d} T_i$.

Note that if $\omega = 1$, then $T_i = \emptyset$ for all $i \in [d]$ and thus $\cup_{i \in [d]} W_i$ is a cover of K with d red d-dominating subgraphs; that is, $f(1, d) \leq d = d(f(0, d) + 1)$.

Otherwise, we have that $\omega(K[T_i]_{\text{blue}}) \leq \omega - 1$ and thus K is covered by at most

$$d + d \cdot f(\omega - 1, d) = d(f(\omega - 1, d) + 1)$$

red d-dominating subgraphs. □

Lemma 2.2. Let K be a 2-colored complete digraph K where R is the set of red loops and B is the set of blue loops. If $\omega(G[R]_{\text{blue}}) \geq d + 1$, then $V(K)$ can be covered by at most d red d-dominating subgraphs and at most one blue d-dominating subgraph. Likewise, if $\omega(G[B]_{\text{red}}) \geq d + 1$. In particular, this implies $f(\omega, d) \leq d + 1$ for $\omega \geq d + 1$.

Proof. Suppose $\omega(G[R]_{\text{blue}}) \geq d + 1$ and let $X = \{x_1, \ldots, x_d, x_{d+1}\} \subseteq R$ be a set of order $d + 1$ which witnesses this fact; that is, for all distinct $x_i, x_j \in X$, (x_i, x_j) is blue. For $i \in [d]$ we define

$$W_i = \{v \in V(K) : (v, x_i) \text{ is red}\}.$$

Note that $x_i \in W_i$ and $K[W_i]$ is spanned by a red d-dominating subgraph for all $i \in [d]$.

Set $V' = X \cup (V(K) \setminus (\cup_{i \in [d]} W_i))$ and note that for all $v \in V' \setminus X$, $[v, X]$ is blue. To see that $G[V']_{\text{blue}}$ is d-dominating, let $S \subseteq V'$ with $1 \leq |S| \leq d$. Since $|X| > |S|$, there exists $x_i \in X \setminus S$ and by the properties mentioned above, every edge $[S, x_i]$ is blue. So there is one blue d-dominating subgraph which covers V', which together with the red d-dominating subgraphs $K[W_1], \ldots, K[W_d]$ gives the result.

When $\omega(G[B]_{\text{red}}) \geq d + 1$, the proof is the same by switching the colors. □

Now we are ready to prove our main result.

Proof of Theorem 1.2. Let $V(K) = R \cup B$ where R, B are the vertex sets of the red and blue loops, respectively. If $\omega(G[R]_{\text{blue}}) \geq d + 1$ or $\omega(G[B]_{\text{red}}) \geq d + 1$, then by Lemma 2.2, $R \cup B$ can be covered by at most $d + 1$ monochromatic d-dominating subgraphs. So suppose $\omega(G[R]_{\text{blue}}) \leq d$ and $\omega(G[B]_{\text{red}}) \leq d$. Now by Lemma 2.1, each of $K[R]$ and $K[B]$ can be covered by at most 4 monochromatic d-dominating subgraphs when $d = 2$, and by at most $\sum_{i=1}^{\omega} d^i \leq \sum_{i=1}^{d} d^i$ monochromatic d-dominating subgraphs when $d \geq 3$. □

3 | MOTIVATION: AN INFINITE ANALOGUE OF THE BURR-ERDŐS CONJECTURE

In this section, we provide the context for Problem 1.1 and the general solution provided by Theorem 1.2.

A graph G is d-degenerate if there is an ordering of the vertices $v_1, v_2, \dots$ such that for all $i \ge 1$, $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \le d$ (equivalently, every subgraph has a vertex of degree at most d). Burr and Erdős conjectured [3] that for all positive integers d, there exists $c_d > 0$ such that every 2-coloring of K_n contains a monochromatic copy of every d-degenerate graph on at most $c_d n$ vertices. This conjecture was recently confirmed by Lee [8].

The motivation for Problem 1.1 relates to the following conjecture also raised in [4, problem 1.5, conjecture 10.2] which is an infinite analogue of the Burr-Erdős conjecture.

Conjecture 3.1. For all positive integers d, there exists a real number $c_d > 0$ such that if G is a countably infinite d-degenerate graph with no finite dominating set, then in every 2-coloring of the edges of $K_{\mathbb{N}}$, there exists a monochromatic copy of G with vertex set $V \subseteq \mathbb{N}$ such that the upper density of V is at least c_d.

The case $d = 1$ was solved completely in [4] (regardless of whether G has a finite dominating set or not). For certain 2-colorings of $K_{\mathbb{N}}$, Theorem 1.2 implies a positive solution to Conjecture 3.1 for $d \ge 2$. As an example of such a 2-coloring, suppose that for some finite subset $F \subseteq \mathbb{N}$, we partition $\mathbb{N} \setminus F$ into (finitely or infinitely many) infinite sets $\mathcal{X} = \{X_1, \dots, X_n, \dots\}$. For all i, j, let $c_{ij} \in [2]$ and color the edges from X_i to X_j so that for all $v \in X_i$, $\{u \in X_j : \{u, v\}$ has color $c_{ij}\}$ is cofinite (by using the half graph coloring[1] when $c_{ij} \ne c_{ji}$ for instance). This last condition ensures that if there exist $X_{i_1}, \dots, X_{i_n}$ and X_j such that $c_{i_1 j} = \dots = c_{i_n j} =: c$, then every finite collection of vertices in $X_{i_1} \cup \dots \cup X_{i_n}$ has infinitely many common neighbors of color c in X_j.

The above coloring of $K_{\mathbb{N}}$ naturally corresponds to a 2-colored complete digraph in the following way: Let K be a 2-colored complete digraph on $\mathcal{X}$ where we color (X_i, X_j) with color c if for all $v \in X_i$, $\{u \in X_j : \{u, v\}$ has color $c\}$ is cofinite. Now by Theorem 1.2, K can be covered by $t \le f(d+1)$ monochromatic $(d+1)$-dominating subgraphs $G_1, \dots, G_t$. Since $\mathbb{N} \setminus F = \bigcup_{i \in [t]} (\bigcup_{X \in V(G_i)} X)$, there exists $i \in [t]$ such that $V_i := \bigcup_{X \in V(G_i)} X$ has upper density at least $1/f(d+1)$. Without loss of generality, suppose the edges of G_i are red. By the construction, V_i has the property that for all $S \subseteq V_i$ with $1 \le |S| \le d+1$, there is an infinite subset $W \subseteq V_i$ such that every edge in $E(S, W)$ is red. As shown in [4, proposition 6.1], if G is a graph satisfying the hypotheses of Conjecture 3.1, then there exists a red copy of G which spans V_i and thus has upper density at least $1/f(d+1)$.

4 | CONNECTION TO PARADOXICAL TOURNAMENTS

Our method of bounding $f(d)$ was to extend the problem and bound $f(\omega, d)$ for all $1 \le \omega \le d$. We proved that $f(1, 2) = 1$ and $f(1, d) \le d$ for all $d \ge 3$. Naturally, we wondered if the upper bound on $f(1, d)$ could be improved when $d \ge 3$, since any improvement on $f(1, d)$ would improve the general upper bound on $f(d)$. However, this cannot be done.

Theorem 4.1. $f(1, d) = d$ for all $d \ge 3$.

[1]Given a totally ordered set Z and disjoint $X, Y \subseteq Z$ the half graph coloring of the complete bipartite graph $K_{X,Y}$ is a 2-coloring of the edges of $K_{X,Y}$ where for all $i \in X$, $j \in Y$, $\{i, j\}$ is red if and only if $i \le j$.

We discovered that the lower bound of Theorem 4.1 would follow from the existence of certain d-dominated tournaments. The problem of the existence of d-dominated tournaments was proposed by Schütte and was first proved by Erdős [6] with the probabilistic method, then Graham and Spencer [7] gave an explicit construction using sufficiently large Paley tournaments. Babai [1] coined the term d-paradoxical tournament for what we refer to as d-dominated tournament. In this spirit, we say that a tournament is perfectly d-paradoxical if it is d-dominating, d-dominated, has no $(d+1)$-dominating subtournaments, and has no $(d+1)$-dominated subtournaments.

It follows from a result of Esther and George Szekeres [9] that the Paley tournaments QT_7, QT_{19} are perfectly 2-paradoxical and perfectly 3-paradoxical tournaments, respectively. It is an open question (which to the best of our knowledge has not been posed in the literature before now) whether every Paley tournament is perfectly d-paradoxical for some d. While this question remains open, Bukh [2], responding to our query, gave a beautiful (part-deterministic and part-probabilistic) construction of a perfectly d-paradoxical tournament for all $d \geq 2$.

Theorem 4.2 (Bukh [2]). *For all integers $d \geq 2$, there exists a perfectly d-paradoxical tournament.*

The interested reader can find the proof of Theorem 4.2 and the detailed derivation of Theorem 4.1 from Theorem 4.2 in [5].

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REFERENCES

1. L. Babai, Character sums, Weil's estimate and paradoxical tournaments, 2002. http://people.cs.uchicago.edu/~laci/reu02/paley.pdf
2. B. Bukh, Dominating sets in subtournaments of the Paley tournament. https://mathoverflow.net/q/383245
3. S. A. Burr and P. Erdős, On the magnitude of generalized Ramsey numbers for graphs, Colloq. Math. Soc. János Bolyai. 10 (1975), 215-240.
4. J. Corsten, L. DeBiasio, and P. McKenney, Density of monochromatic infinite subgraphs II, arXiv preprint arXiv:2007.14277, 2020.
5. L. DeBiasio and A. Gyárfás, Covering 2-colored complete digraphs by monochromatic d-dominating digraphs, arXiv preprint arXiv:2102.12794, 2021.
6. P. Erdős, On a problem in graph theory, Math. Gazette. 47 (1963), 220-223.
7. R. L. Graham and J. H. Spencer, A constructive solution to a tournament problem, Canadian Math. Bulletin. 14 (1971), 45-48.
8. C. Lee, Ramsey numbers of degenerate graphs, Ann. Math. 185 (2017), no. 3, 791-829.
9. E. Szekeres and G. Szekeres, On a problem of Schütte and Erdős, Math. Gazette. 49 (1965), 290-293.

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