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Powers of Hamiltonian cycles in multipartite graphs

Louis DeBiasio [a,1], Ryan R. Martin [b,2], Theodore Molla [c,*,3]

[a] Department of Mathematics, Miami University, United States of America
[b] Department of Mathematics, Iowa State University, United States of America
[c] Department of Mathematics and Statistics, University of South Florida, United States of America

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abstract

We prove that if G is a k-partite graph on n vertices in which all of the parts have order at most n/r and every vertex is adjacent to at least a $1 - 1/r + o(1)$ proportion of the vertices in every other part, then G contains the $(r-1)$-st power of a Hamiltonian cycle.

1. Introduction

For graphs G and H, we say that G has a perfect H-tiling if G contains $|V(G)|/|V(H)|$ vertex disjoint copies of H. For a positive integer r, the r-th power of H denoted H^r, is the graph on $V(H)$ where $uv \in E(H^r)$ if and only if the distance between u and v in H is at most r. We refer to the $(r-1)$-st power of a cycle as an $(r-1)$-cycle.

Hajnal and Szemerédi [5] proved that for all positive integers r and n, if r divides n and G is a graph on n vertices with $\delta(G) \geq \left(1 - \frac{1}{r}\right)n$, then G contains a perfect K_r-tiling. Komlós, Sárközy, and Szemerédi [13] proved that for all $r \geq 2$, there exists n_0 such that if G is a graph on $n \geq n_0$ vertices with $\delta(G) \geq \left(1 - \frac{1}{r}\right)n$, then G contains a Hamiltonian $(r-1)$-cycle. Note that if r divides n and G contains a Hamiltonian $(r-1)$-cycle, then G contains a perfect K_r-tiling, so the result of Komlós, Sárközy, and Szemerédi is stronger for fixed r and large n.

A graph G is a *k-partite graph with ordered partition* $P = (V_1, \ldots, V_k)$, if P is a partition of $V(G)$ and V_i is an independent set for every $i \in [k]$. For all $i \neq j \in [k]$, let

$$\delta_{ij}(G) = \frac{\min\{\deg_G(v, V_j) : v \in V_i\}}{|V_j|} \qquad \text{and} \qquad \delta_P(G) = \min_{i \neq j \in [k]} \delta_{ij}(G).$$

Fisher [4] conjectured an analogue of the Hajnal-Szemerédi theorem in balanced multipartite graphs; that is, if G is a balanced r-partite graph on n vertices with

* Corresponding author.
E-mail addresses: debiasld@miamioh.edu (L. DeBiasio), rymartin@iastate.edu (R.R. Martin), molla@usf.edu (T. Molla).

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$$\delta_P(G) \geq 1 - \frac{1}{r},$$

then G contains a perfect K_r-tiling. An earlier example of Catlin [1] provides a counterexample to Fisher's conjecture when r is odd, but Magyar and Martin [17] proved that for $r = 3$, Catlin's counterexample is the only one. Then Martin and Szemerédi [19] proved Fisher's conjecture for $r = 4$. After a relatively large gap in activity, Keevash and Mycroft [10] and independently Lo and Markström [15] proved that for all $\gamma > 0$ and $r \geq 2$, there exists n_0 such that for all $n \geq n_0$ in which r divides n, if G is a balanced r-partite graph on n vertices with

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma,$$

then G contains a perfect K_r-tiling. Later, an exact version was proved by Keevash and Mycroft [11] which again shows that Fisher's conjecture holds for sufficiently large n unless r is odd in which case Catlin's counterexample is the only one.

Our main result can be viewed as a strengthening of the asymptotic versions of all of the above results (both in the multipartite setting and in the ordinary setting).

Theorem 1.1. *For all $k \geq r \geq 2$ and all $0 < \gamma \leq \frac{1}{r}$, there exists n_0 such that for all $n \geq n_0$ the following holds. If G is a k-partite graph on n vertices with ordered partition $P = (V_1, \ldots, V_k)$ such that $|V_i| \leq n/r$ for all $i \in [k]$ and*

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma,$$

then G contains a Hamiltonian $(r-1)$-cycle.

Note that the condition $|V_i| \leq n/r$ for all $i \in [k]$ is necessary for the existence of a Hamiltonian $(r-1)$-cycle since the $(r-1)$-st power of a cycle on n vertices has independence number n/r. Also this result is seen to be asymptotically best possible by taking a complete k-partite graph with ordered partition $P = (V_1, \ldots, V_k)$ and letting $V_i \subseteq V_i$ for all $i \in [k]$ with $|V_i| = |V_i|/r + 1$ and deleting all edges inside $V_1 \cup \cdots \cup V_k$ to get a k-partite graph G with $\delta_P(G)$ just below $1 - \frac{1}{r}$ which has independence number larger than n/r and thus does not contain a Hamiltonian $(r-1)$-cycle.

2. Observations, definitions and tools

Observation 2.1. *It suffices to prove Theorem 1.1 in the cases where $r \leq k \leq 2r - 1$ and all of the parts have order at least $\frac{\gamma}{2r}n$.*

Proof. Suppose Theorem 1.1 is true provided $2 \leq r \leq k \leq 2r - 1$ and $|V_i| \geq \frac{\gamma}{2r}n$ for all $i \in [k]$. Now suppose for contradiction that there exists a counterexample to Theorem 1.1. Let k be minimal such that a counterexample exists. Let n_0 be the value coming from Theorem 1.1 when $k = k - 1$ and $\gamma = \frac{\gamma}{2r}$. Let G be a k-partite counterexample on $n \geq n_0$ vertices with ordered partition $P = (U_1, \ldots, U_k)$ where k is minimal.

We first claim that for all distinct $i, j \in [k]$, $|U_i| + |U_j| > n/r$. Suppose not and without loss of generality suppose that $i = k - 1$ and $j = k$; that is, suppose $|U_{k-1}| + |U_k| \leq n/r$. Let $V_i = U_i$ for all $i \in [k-2]$ and $V_{k-1} = U_{k-1} \cup U_k$ and let G be the $(k-1)$-partite graph with ordered partition $P = (V_1, \ldots, V_{k-1})$ obtained by deleting all edges between U_{k-1} and U_k. Since $\deg_G(v, V_{k-1}) \geq (1 - \frac{1}{r} + \gamma)|U_{k-1}| + (1 - \frac{1}{r} + \gamma)|U_k| = (1 - \frac{1}{r} + \gamma)|U_{k-1} \cup U_k|$ for all $v \in V(G) \setminus V_{k-1}$ we have

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma.$$

But now by minimality, $G \subseteq G$ has a Hamiltonian $(r-1)$-cycle contradicting the fact that G does not. Thus we may assume that $r \leq k \leq 2r - 1$ as otherwise the two smallest parts add up to at most n/r.

Now suppose G has a part of order less than $\gamma n = \frac{\gamma}{2r}n$; without loss of generality, suppose it is U_k. Because $|U_i| \leq n/r$ for every $i \in [k]$ and $_{i \in [k]} |U_i| = n$, the fact that $|U_k| \leq \gamma n < n/r$ implies that $k > r$. By the above, we may suppose that all other parts have order greater than $\frac{n}{r} - \gamma n$. Now partition U_k arbitrarily as $\{U_1, \ldots, U_{k-1}\}$ (allowing for empty sets in the partition) subject to $|U_i| + |U_i| \leq n/r$ for all $i \in [k-1]$. Let G be the $(k-1)$-partite graph with ordered partition $P = (V_1, \ldots, V_{k-1})$ where $V_i = U_i \cup U_i$ for all $i \in [k-1]$. Since

$$1 - \frac{1}{r} + \gamma \quad |U_i| \geq \quad 1 - \frac{1}{r} + \gamma \quad (|U_i| + \gamma n) \geq \quad 1 - \frac{1}{r} + \gamma \quad |V_i|,$$

we have

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma,$$

and thus by minimality and the choice of n_0, $G \subseteq G$ has a Hamiltonian $(r-1)$-cycle contradicting the fact that G does not.

The following simple fact is used implicitly throughout the paper.

Fact 2.2. *Let $\sigma > 0$ and G be a k-partite graph on n vertices with ordered partition $\mathcal{P} = (V_1, \ldots, V_k)$ such that every part has order at least σn. For every $U \subseteq V(G)$ such that $|U| \leq \sigma^2 n$, if $G = G - U$, then $\delta_{\mathcal{P}}(G) \geq \delta_{\mathcal{P}}(G) - \sigma$.*

Proof. For distinct $i, j \in [k]$ and every $v \in V(G) \cap V_i$, we have

$$\frac{\deg_G(v, V(G) \cap V_j)}{|V(G) \cap V_j|} \geq \frac{\deg_G(v, V(G) \cap V_j)}{|V_j|} \geq \frac{\deg_G(v, V_j)}{|V_j|} - \frac{|U|}{|V_j|} \geq \delta_{\mathcal{P}}(G) - \sigma.$$

Definition 2.3 *($(r-1)$-path/$(r-1)$-walk).* Let G be a graph and let $W = x_1, \ldots, x$ be an ordered sequence of vertices of G. The sequence W is an *$(r-1)$-walk of length* if every r consecutive vertices in W form a clique in G. If W is an $(r-1)$-walk of length , then it is an *$(r-1)$-path of length* if there are no repeated vertices in the sequence $x_1, \ldots, x$.

The following fact is immediate when one first observes that the number of $(r-1)$-walks of length that are not $(r-1)$-paths is at most $_2 \cdot n^{-1}$, and that, for every set $U \subseteq V(G)$, the total number of $(r-1)$-walks of length that contain a vertex from U is at most $\cdot |U| \cdot n^{-1}$. Throughout the remainder of the proof, we use the notation $a \quad b$ to indicate that there exists an increasing function $f(b)$ such that the result holds for every $a \leq f(b)$.

Fact 2.4. *Suppose $\frac{1}{n} \quad \sigma \quad \alpha, \frac{1}{\ }$ and let G be an n-vertex graph and $U \subseteq V(G)$ where $|U| \leq \sigma n$. If $\mathcal{W}$ is a collection of at least (αn) $(r-1)$-walks of length , then at least (σn) of the walks in $\mathcal{W}$ are $(r-1)$-paths that avoid the set U.*

To motivate the following definition, let us first comment that, at various times, we will need to connect disjoint $(r-1)$-paths to form longer $(r-1)$-paths. To highlight some issues that might arise in as simple a setting as possible, consider the case when $k = r = 3$ and let G be a balanced 3-partite graph with ordered partition (V_1, V_2, V_3) and let $P_1 = u_1, \ldots, u_6$ and $P_2 = w_1, \ldots, w_6$ be two disjoint 2-paths each on 6 vertices. Suppose that we would like to find a 2-path Q so that the sequence $P_1 Q P_2$ is itself a 2-path. This would be impossible if, say, $u_4 \in V_1$, $u_5 \in V_2$, and $u_6 \in V_3$ while $w_1 \in V_2$, $w_2 \in V_1$ and $w_3 \in V_3$. To see this, note that, in this setting, if $u_4 \in V_1$, $u_5 \in V_2$, and $u_6 \in V_3$ and $u_1, \ldots, u_{3p}$ is a 2-path, then for every $0 \leq i \leq p-1$ and $j \in [3]$, we must have that $u_{3i+j} \in V_j$. To deal with issues such as this, we will require that $(r-1)$-walks conform to the following definition.

Definition 2.5 *(Properly terminated).* Suppose that G is a k-partite graph with ordered partition $(V_1, \ldots, V_k)$ and let $W = v_1 v_2 \ldots v_p$ be an $(r-1)$-walk where $p \geq r$. We say that W is *properly terminated* if $v_i \in V_i$ and $v_{p-r+i} \in V_i$ for all $i \in [r]$. That is, W is properly terminated if its first r vertices traverse the sets $V_1, \ldots, V_r$ in order and its last r vertices traverse the sets $V_1, \ldots, V_r$ in order.

More generally, if $\mathcal{P} = (U_1, \ldots, U_r)$ is an ordered sequence of r disjoint sets, we say that the *initial r vertices of W respect the sequence $\mathcal{P}$* if $v_i \in U_i$ for every $i \in [r]$. Similarly, we say that the *final r vertices of W respect the sequence $\mathcal{P}$* if $v_{p-r+i} \in U_i$ for every $i \in [r]$. So, W is properly terminated if both the initial r vertices of W and the final r vertices of W respect the sequence $(V_1, \ldots, V_r)$.

Definition 2.6 *(Balanced).* Let $\mathcal{P}$ be a collection of disjoint sets. We say that $\mathcal{P}$ is *balanced* if every set in $\mathcal{P}$ has the same order.

If G is an r-partite graph with ordered partition $\mathcal{P} = (V_1, \ldots, V_r)$, we say that G is *balanced* if $\mathcal{P}$ is balanced and we say that a set $U \subseteq V(G)$ is *balanced* if $|U \cap V_i| = |U \cap V_j|$ for all $i, j \in [r]$.

A few times in the proof we will make use of a Chernoff bound on the concentration of binomial and hypergeometric distributions [8, Corollary 2.3 and Theorem 2.10]

Theorem 2.7 *(Chernoff bound). Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

3. Overview of the proof

We are attempting to prove that all sufficiently large k-partite graphs, in which all parts have at most n/r vertices, with proportional minimum degree at least $1 - \frac{1}{r} + \gamma$ have a Hamiltonian $(r-1)$-cycle. We are able to split the work into two tasks.

The first (and main) task is to prove the result in the case of *balanced* r-partite graphs. Lemma 3.1 below establishes that in a large balanced r-partite graph, and two properly terminated $(r-1)$-paths with the same ordering, K and K, there is a Hamiltonian $(r-1)$-path that starts with K and ends with K. If the graph is balanced and r-partite, then we simply

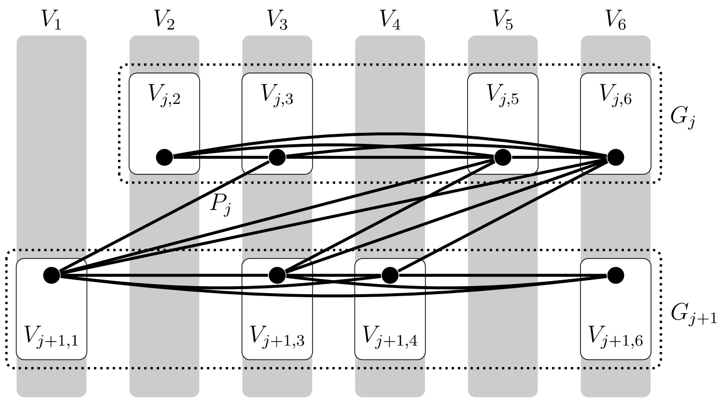

Fig. 1. An example for Lemma 3.2 in the case where $k = 6$, $r = 4$, $(i_{j,1}, i_{j,2}, i_{j,3}, i_{j,4}) = (2, 3, 5, 6)$, $(i_{j+1,1}, i_{j+1,2}, i_{j+1,3}, i_{j+1,4}) = (1, 3, 4, 6)$ and the 8-vertex 3-path, P_j.

apply this with $K = K$ and we are done. If not, then we use Lemma 3.2 below to partition the graph into balanced r-partite pieces and then stitch them together to create the $(r-1)$-cycle we require.

Lemma 3.1 *(Balanced case). For every $r \geq 2$ and $\gamma < \frac{1}{r}$, there exists n_0 such that for every $n \geq n_0$ the following holds. Let G be a balanced r-partite graph on n vertices with ordered partition $P = (V_1, \ldots, V_r)$ such that*

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma.$$

Suppose that K and K are r-cliques such that either $K = K$ or $K \cap K = \emptyset$ and let $v_i := V_i \cap K$ and $v_i := V_i \cap K$ for every $i \in [r]$. Then there is a Hamiltonian $(r-1)$-path P of $G - (K \cup K)$ such that $v_1, \ldots, v_r, P, v_1, \ldots, v_r$ is an $(r-1)$-walk in G.

The second task is to show that G can be partitioned into a small number of balanced r-partite graphs that each contain a Hamiltonian $(r-1)$-path and that these $(r-1)$-paths can be stitched together to form a Hamiltonian $(r-1)$-path of the original graph G. Lemma 3.2 below shows that the graph can be partitioned into balanced r-partite graphs $G_1, \ldots, G$, each with the appropriate minimum degree condition, together with short $(r-1)$-paths connecting G_i to G_{i+1} in sequence in such a way that every vertex is accounted for. Then applying Lemma 3.1 to each G_i we will construct the desired Hamiltonian $(r-1)$-cycle.

The technical issue for finding the partition is essentially numerical, requiring the sizes of the sets forming each G_i to be the same and to partition each vertex class. Once these constraints are achieved, we are able to meet the minimum degree condition by applying a Chernoff bound to show that a randomly chosen partition satisfying the numerical constraints will have the required degree condition with high probability.

Lemma 3.2 *(Partitioning and Sequencing). For all $r \geq 2$, $0 < \gamma \leq \frac{1}{r}$, and $r < k \leq 2r - 1$, there exist constants $0 < \frac{1}{n_0}$ β σ γ such that if G is a k-partite graph on $n \geq n_0$ vertices with ordered partition $P = (V_1, \ldots, V_k)$ in which $\gamma n \leq |V_k| \leq |V_{k-1}| \leq \cdots \leq |V_1| \leq \frac{n}{r}$ and*

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma,$$

then there exists an $(r-1)$-path P_0 with $|V(P_0)| \leq \beta n$ such that if $V_i = V_i \setminus V(P_0)$ for $i \in [k]$, then the following holds:

- *(A1) there exists a positive integer such that for all $i \in [k]$, there exists a partition $V_i = \{V(i, 1), \ldots, V(i,)\}$ (with $V(i, j)$ possibly empty) such that for all $j \in []$ there exists $1 \leq i_{j,1} < \cdots < i_{j,r} \leq k$ such that $|V(i_{j,1}, j)| = \cdots = |V(i_{j,r}, j)| \geq \beta n$ and if $i \in [k] \setminus \{i_{j,1}, \ldots, i_{j,r}\}$, then $V(i, j) = \emptyset$, and*
- *(A2) letting $P_j = (V(i_{j,1}, j), \ldots, V(i_{j,r}, j))$ and G_j be the natural r-partite graph induced by P_j, we have that $\delta_{P_j}(G_j) \geq 1 - \frac{1}{r} + \frac{\gamma}{2}$.*
- *(A3) We can prepend r vertices and append r vertices to P_0 to create an $(r-1)$-path P_0 such that the initial r vertices of P_0 respect the sequence P and the final r vertices of P_0 respect the sequence P_1.*
- *(A4) There exist vertex disjoint $(r-1)$-paths $P_1, \ldots, P_{-1}$ in $G - V(P_0)$ each on $2r$ vertices such that for all $j \in [\ -1]$ the initial r vertices of P_j respect the sequence P_j and the final r vertices of P_j respect the sequence P_{j+1} (Fig. 1).*

Lemma 3.1 and Lemma 3.2 together immediately imply Theorem 1.1.

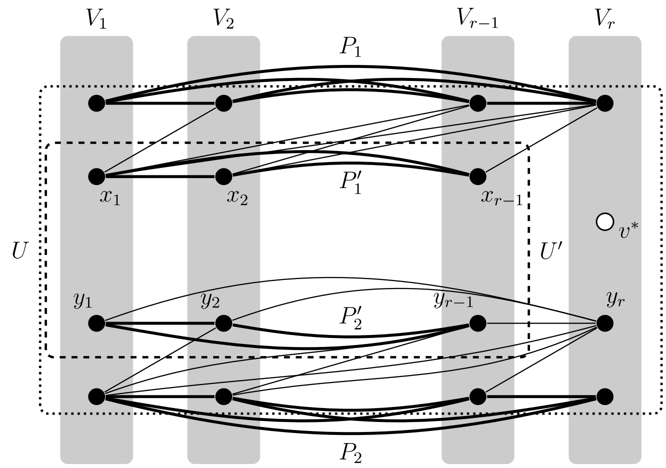

Fig. 2. Using induction to build the desired connection between P_1 and P_2 for Lemma 4.1.

Proof of Theorem 1.1. By Lemma 3.1 we can assume $k > r$ and by Observation 2.1 we can assume that $k \leq 2r - 1$ and every part of P has order at least γn where $\gamma \geq \gamma > 0$. Without loss of generality we can further assume that $\gamma n \leq |V_k| \leq |V_{k-1}| \leq \cdots \leq |V_1| \leq \frac{n}{k}$. Therefore, we can apply Lemma 3.2 to G (with γ playing the role of γ). Define $P = P_0$ and for each G_j, apply Lemma 3.1 to G_j with $K = P_{j-1} \cap G_j$ and $K = P_j \cap G_j$ to get a Hamiltonian $(r-1)$-path Q_i. Now $P_0 Q_1 \ldots Q$ is the desired Hamiltonian $(r-1)$-cycle.

In Section 4 we describe the three lemmas needed to prove Lemma 3.1. Then in Sections 5 to 8, we prove those lemmas. Finally in Section 9 we prove Lemma 3.2.

4. Statement of the principal lemmas

We prove Lemma 3.1 using the absorbing method of Rödl, Ruciński, and Szemerédi. As is typical with this method, we have connecting, absorbing, and covering lemmas.

Lemma 4.1 *(Connecting lemma). For every $r \geq 2$ and $0 < \nu \leq \frac{1}{r}$ there exists $\tau > 0$ such that the following holds for every n. Let G be an r-partite graph with ordered partition $P = (V_1, \ldots, V_r)$. Let $= r(2r-2)$. Suppose that $(U_1, \ldots, U_r)$ is a sequence of sets such that $U_i \subseteq V_i$ for $i \in [r]$, $U = {}_{i=1}^{r} U_i$, and*

$$\text{for every } i \in [r] \text{ and } v \in V \setminus V_i,\ |U_i| \geq \nu n \text{ and } \deg_G(v, U_i) \geq \left(1 - \frac{1}{r} + \nu\right)|U_i|. \tag{1}$$

Then for every pair of properly terminated $(r-1)$-walks P_1 and P_2 in G, there exist at least τn $(r-1)$-walks Q of length contained in $U_1 \cup \cdots \cup U_r$ such that $P_1 Q P_2$ is a properly terminated $(r-1)$-walk.

Lemma 4.2 *(Absorbing lemma). For $r \geq 2$, suppose that $\frac{1}{n}$ β $\gamma < \frac{1}{r}$ and let G be a balanced r-partite graph on n vertices with ordered partition $P = (V_1, \ldots, V_r)$ such that*

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma.$$

Then there exists a properly terminated $(r-1)$-path P_{abs} such that $|V(P_{abs})| \leq \beta n$, and, for every balanced set $Z \subseteq V(G) \setminus V(P_{abs})$ for which $|Z| \leq \beta^2 n$, there exists a Hamiltonian $(r-1)$-path of $G[V(P_{abs}) \cup Z]$ that begins with the same $(r-1)$ vertices as P_{abs} and ends with the same $(r-1)$ vertices as P_{abs}.

Lemma 4.3 *(Covering lemma). For $r \geq 2$, suppose that $\frac{1}{n}$ $\frac{1}{M_0}$ α $\gamma < \frac{1}{r}$ and let G be a balanced r-partite graph on n vertices with ordered partition $P = (V_1, \ldots, V_r)$ and*

$$\delta_P(G) \geq 1 - \frac{1}{r} + \gamma.$$

For some $M \leq M_0$, there exist vertex disjoint properly terminated $(r-1)$-paths $P_1, \ldots, P_M$ such that $W = V(G) \setminus {}_{i=1}^{M} V(P_i)$ is balanced and $|W| \leq \alpha n$.

Before proving these three lemmas, we first show how to use Lemmas 4.1, 4.2, and 4.3 to prove the balanced case of Theorem 1.1.

Proof of Lemma 3.1. We can select M_0, ν, α, and β so that

$$\frac{1}{n} \le \frac{1}{n_0} \quad \frac{1}{M_0} \quad \nu, \alpha \quad \beta \quad \gamma.$$

By Lemma 4.2, there exists a properly terminated-$(r-1)$ path P_{abs} disjoint from K and K such that

- $|P_{abs}| \le \beta n$; and
- for every balanced set $Z \subseteq V(G)$ such that $|Z| \le \beta^2 n$ there exists a Hamiltonian $(r-1)$-path of $G[V(P_{abs}) \cup Z]$ that starts and ends with the same $(r-1)$-vertices as P_{abs}.

Let $G = G - V(P_{abs}) \cup V(K) \cup V(K)$.

Uniformly at random select subsets $U_1, \dots, U_r$ such that for every $i \in [r]$, $U_i \subseteq V(G) \cap V_i$ and $|U_i| = \nu n$. By the Chernoff and union bounds, there exists an outcome such that (1) holds. Fix such an outcome and let $U = U_1 \cup \dots \cup U_r$ and let $G = G - V(U)$.

By Lemma 4.3, for some $M \le M_0$, there exist vertex disjoint properly terminated $(r-1)$-paths $P_1, \dots, P_M$ in G such that $W = V(G) \setminus \bigcup_{i=1}^{M} V(P_i)$ is balanced and $|W| \le \alpha n$. Since (1) holds, Fact 2.4 and Lemma 4.1 imply that we can find $m+2$ disjoint $(r-1)$-paths, each of length $= 2r(r-2)$, in $G[U]$ that connect

- K to P_{abs},
- P_{abs} to P_1,
- P_{i-1} to P_i, for $2 \le i \le M$; and
- P_M to K

to form a $(r-1)$-path P. Let $Z = |V(G) \setminus V(P)|$, and note that

$$|Z| \le |U| + |W| \le r \quad \nu n + \alpha n \le \beta^2 n.$$

Therefore, there exists a Hamiltonian $(r-1)$-path of $G[P_{abs} \cup Z]$ that starts and ends with the same $(r-1)$ vertices as P_{abs}. If $v_i = v_i$ for every $i \in [r]$, then we have constructed a Hamiltonian $(r-1)$-cycle. If $v_i = v_i$ for every $i \in [r]$, then we have constructed a Hamiltonian $(r-1)$-path that starts with K and ends with K .

5. Proof of the connecting lemma (Lemma 4.1)

Although we present the proof of Lemma 4.1 in full, it closely follows proofs of similar lemmas given in [7] and [6].

Definition 5.1. Let G be a graph on n vertices. For $U \subseteq V(G)$, we say that W is *(U, σ)-rich* if there are at least σn vertices $u \in U$ for which $N(u)$ contains W, otherwise W is called *(U, σ)-poor*.

The following simple observation and fact are critical for the inductive proof of the connecting lemma.

Observation 5.2. *For $r \ge 3$, let G be a graph on n vertices, let $P = (V_1, \dots, V_r)$ be an ordered partition of $V(G)$, and let $U_r \subseteq V_r$. Suppose that*

$$W = x_1, \dots, x_{r-1}, z_1^1, \dots, z_{r-1}^1, \dots, z_1^s, \dots, z_{r-1}^s, y_1, \dots, y_{r-1}$$

is an $(r-2)$-walk of length $(s+2)(r-1)$ such that $W \cap V_r = \emptyset$ that is (U_r, σ)-rich. Then, by the definition of (U_r, σ)-rich, there are at least $(\sigma n)^{s+1}$ tuples $(w^0, \dots, w^s)$ such that $\{w^0, \dots, w^s\} \subseteq U_r$ and $N(w^i)$ contains W for each $0 \le i \le s$. Therefore, for each such tuple

$$x_1, \dots, x_{r-1}, w^0, z_1^1, \dots, z_{r-1}^1, w^1, \dots, z_1^s, \dots, z_{r-1}^s, w^s, y_1, \dots, y_{r-1}$$

is an $(r-1)$-walk of length $(s+2)r-1$.

By double counting, the following fact formalizes the observation that most neighborhoods do not contain many poor paths for the simple reason that, by definition, poor paths are not contained in many neighborhoods,

Fact 5.3. *For $r \ge 3$, $p \ge 0$ and $\sigma > 0$ the following holds. If G is a graph on n vertices and $U \subseteq V(G)$, then there are at least $|U| - \sigma n$ vertices $u \in U$ such that only at most σn^p of the $(r-2)$-walks of length p contained in $N(u)$ are (U, σ^2)-poor.*

Proof. Let V_{poor} be the set of ordered $(p+1)$-tuples $(u, v_1, \ldots, v_p) \in V^{p+1}$ such that

- $u \in U$,
- $W = v_1, \ldots, v_p$ is a (U, σ^2)-poor $(r-2)$-walk, and
- $N(u)$ contains W.

Because the number of ordered p-tuples is at most n^p, we have that $|V_{poor}| \leq \sigma^2 n^{p+1}$ (cf. Definition 5.1). Let $U \subseteq U$ be the set of vertices $u \in U$ such that more than σn^p of the $(r-2)$-walks of length p contained in $N(u)$ are (U, σ^2)-poor. Then,

$$|U| \cdot \sigma n^p \leq |V_{poor}| \leq \sigma^2 n^{p+1}.$$

Therefore, $|U| \leq \sigma n$ and the conclusion follows.

Proof of Lemma 4.1 We will prove the lemma by induction on r. For the base case, note that when $r = 2$, we have $= 4$ and, by (1), the statement easily holds with $\tau = \nu^{5}/4$. To see this, note that we can select vertices $x_1, y_1 \in U_1$, and $y_2 \in U_2$ such that $P_1 x_1$ and $y_1 y_2 P_2$ are 1-paths. This can be done with $(1/2+\nu)|U_2|$ choices for y_2, $(1/2+\nu)|U_1|$ choices for y_1 and $(1/2+\nu)|U_1|$ choices for x_1 (recall that we only require $(r-1)$-walks). This gives at least $\frac{\nu n}{2}^{3}$ total selections and for every such selection we have

$$|N(x_1) \cap N(y_1) \cap U_2| \geq \deg_G(x_1, U_2) + \deg_G(y_1, U_2) - |U_2| \geq 2\nu|U_2| \geq 2\nu^2 n.$$

For the induction step, let $r \geq 3$ and suppose that the result holds for $r-1$. Let $s = 2(r-1)-2$, $q = (r-1)(2(r-1)-2) = (r-1)s$, and $p = q + 2(r-1)$, and note that

$$p + s + 2 = ((r-1)s + 2(r-1)) + s + 2 = r(s+2) = 2r(r-1) = . \tag{2}$$

Applying the induction hypothesis with $\nu/2$, $r-1$, and q playing the roles of ν, r, and respectively we get that there exists $\mu > 0$ (playing the role of τ) such that the following holds.

Claim 5.4. *If $U_i \subseteq U_i$ such that $|U_i| \geq \nu n/2$ for all $i \in [r-1]$, and*

$$\deg_G(v, U_i) \geq \quad 1 - \frac{1}{r-1} + \nu \quad |U_i| \text{ for all } v \in V \setminus V_i, \tag{3}$$

then for every pair of $(r-2)$-walks $x_1, \ldots, x_{r-1}$ and $y_1, \ldots, y_{r-1}$ such that $x_i, y_i \in U_i$ for all $i \in [r-1]$ there exist at least μn^q $(r-2)$-walks of length q contained in $U_1 \cup \ldots \cup U_{r-1}$ such that $x_1, \ldots, x_{r-1}, Q, y_1, \ldots, y_{r-1}$ is an $(r-2)$-walk.

Pick $\tau, \sigma > 0$ so that $\tau \quad \sigma \quad \mu, \nu$. First note that, by (1), there are at least $\frac{\nu n}{r} \geq \sigma n$ ways to select $y_r \in U_r$ so that $y_r P_2$ is an $(r-1)$-path. Next, because $|U_r| \geq \nu n > \sigma n$, Fact 5.3 implies that there exists $v^* \in U_r$ such that

$$\text{at most } \sigma n^p \text{ of the } (r-2)\text{-walks of length } p \text{ contained in } N \quad (v^*) \text{ are } (U_r, \sigma^2)\text{-poor.} \tag{4}$$

For every $i \in [r-1]$, let $U_i = N(v^*, U_i)$. Note that $|U_i| \geq \frac{r-1}{r}|U_i| \geq \nu n/2$ and for every $v \in V \setminus V_i$

$$\deg(v, U_i) \geq |U_i| - \quad \frac{1}{r} - \nu \quad |U_i| \geq |U_i| - \quad \frac{1}{r} - \nu \quad \frac{r}{r-1}|U_i| \geq \quad 1 - \frac{1}{r-1} + \nu \quad |U_i|.$$

Therefore, we can iteratively prepend vertices $y_{r-1}, \ldots, y_1$ to $y_r P_2$ and append vertices $x_1, \ldots, x_{r-1}$ to P_1 in at least $\frac{\nu^2 n}{2}^{2r-2}$ ways (Fig. 2) so that the following holds:

- $x_i, y_i \in U_i$ for $i \in [r-1]$; and
- both $P_1, x_1, \ldots, x_{r-1}$ and $y_1, \ldots, y_{r-1}, y_r, P_2$ are $(r-1)$-walks.

By Claim 5.4, the number of $(r-2)$-walks Q of length q contained in $U_1 \cup \cdots \cup U_{r-1}$ such that $x_1, \ldots, x_{r-1}, Q, y_1, \ldots, y_{r-1}$ is an $(r-2)$-path is at least μn^q.

Therefore, there are at least

$$\frac{\nu^2 n}{2}^{2r-2} \cdot \mu n^q = 2^{-2r+2} \cdot \nu^2 \cdot \mu n^p \geq 2\sigma n^p$$

$(r-1)$-walks

$$x_1, \ldots, x_{r-1}, Q, y_1, \ldots, y_{r-1} = x_1, \ldots, x_{r-1}, z_1^1, \ldots, z_{r-1}^1, \ldots, z_1^s, \ldots, z_{r-1}^s, y_1, \ldots, y_{r-1}$$

such that

- $N(v^*)$ contains $x_1, \ldots, x_{r-1}, Q, y_1, \ldots, y_{r-1}$;
- $x_1, \ldots, x_{r-1}, Q, y_1, \ldots, y_{r-1}$ is an $(r-2)$-walk of length p; and
- both $P_1, x_1, \ldots, x_{r-1}$ and $y_1, \ldots, y_r, P_2$ are $(r-1)$-walks.

By (4), only σn^p of these paths are (U_r, σ^2)-poor so at least σn^p of these paths are (U_r, σ^2)-rich. By Observation 5.2, for every such (U_r, σ^2)-rich walk, there are at least $\sigma^2 n^{s+1}$ ordered tuples $(w^0, \ldots, w^s)$ such that $\{w^0, \ldots, w^s\} \subseteq U_r$ and

$$x_1, x_2, \ldots, x_{r-1}, w^0, z^1_1, \ldots, z^1_{r-1}, w^1, \ldots, z^s_1, \ldots, z^s_{r-1}, w^s, y_1, \ldots, y_{r-1}$$

is an $(r-1)$-walk of length $p + s + 1 = \quad - 1$ (cf. (2)). Recalling that there were at least σn ways to select y_r gives us that the number of $(r-1)$-walks Q of length such that $P_1 Q P_2$ is an $(r-1)$-walk is at least $\sigma n \cdot \sigma n^p \cdot \sigma^2 n^{s+1} = \sigma^{2s+4} n \geq \tau n$.

6. Proof of the absorbing lemma (Lemma 4.2)

Definition 6.1. Let $2 \leq r \leq$, let G be an r-partite graph, and let X be a balanced subset of $V(G)$. A properly terminated $(r-1)$-path $a_1, \ldots, a$ in G is an *absorber of* X if there is an ordering of the vertices $\{a_1, \ldots, a\} \cup X$ that starts with the sequence $a_1, \ldots, a_{r-1}$ and ends with the sequence $a_{-r+1}, \ldots, a$ that is an $(r-1)$-path in G.

The proof of the absorbing lemma follows by a standard probabilistic argument after the proof of the Lemma 6.3 below. We will use the well known "supersaturation" result of Erdős [3] (see [20, Theorem 2.11]).

Theorem 6.2 *(Supersaturation). For all $r \geq 2$, $c > 0$, and positive integers $s_1, \ldots, s_r$, there exists n_0 and c such that if G is a r-partite r-uniform hypergraph with ordered partition $(V_1, \ldots, V_r)$ and at least $c\, n^r$ edges, then G contains at least $cn^{s_1+s_2+\cdots+s_r}$ complete r-partite graphs with s_i vertices in V_i for all $i \in [r]$.*

Lemma 6.3. *For all $r \geq 2$ and $\frac{1}{n}$ α α γ $\frac{1}{r}$ the following holds with $= 3r^2 - r$:*

Let G be a balanced r-partite graph on n vertices with ordered partition $(V_1, \ldots, V_r)$ such that $\delta_P(G) \geq 1 - \frac{1}{r} + \gamma$. If $X \subseteq V(G)$ is a balanced set of size r, then there are at least (αn) absorbers of X in G.

Proof. Let $x_1, \ldots, x_r$ be an ordering of X such that $x_i \in V_i$ for $i \in [r]$.

We first describe what an absorber of X will look like. Suppose

$$P = v^1_1 \cdots v^1_r v^2_1 \cdots v^2_r \cdots v^{r-1}_1 \cdots v^{r-1}_r v^r_1 \cdots v^r_r$$

is an $(r-1)$-path of order r^2 where $v^j_i \in V_i$ for all $i \in [r]$. For all $i, j \in [r]$, set $s^j_i = 2$ if $i = j$ and $s^j_i = 3$ otherwise.

Let P be the $(s^1_1, \ldots, s^1_r, \ldots, s^r_1, \ldots, s^r_r)$-blow up of P where D^j_i is the set corresponding to v^j_i. That is, replace each vertex v^j_i with a set D^j_i of order s^j_i, and if $\{v^j_i, v^j_i\}$ is an edge of P, add all edges between D^j_i and D^j_i .

We claim that, if we suppose that $D^i_1 \cup \cdots \cup D^i_{i-1} \cup D^i_{i+1} \cup \cdots \cup D^i_r \subseteq N(x_i)$, for all $i \in [r]$, then P contains an absorber of X. For all $i = j \in [r]$, label the vertices of D^j_i as a^j_i, b^j_i, c^j_i and label the vertices of D^i_i as a^i_i and c^i_i. Let

$$Q_1 = a^1_1 \cdots a^1_r x_1 b^1_2 \cdots b^1_r c^1_1 \cdots c^1_r a^2_1 \cdots a^2_r b^2_1 x_2 b^2_3 \cdots b^2_r c^2_1 \cdots c^2_r \cdots a^r_1 \cdots a^r_r b^r_1 \cdots b^r_{r-1} x_r c^r_1 \cdots c^r_r$$

and

$$Q_2 = a^1_1 \cdots a^1_r c^1_1 b^1_2 \cdots b^1_r a^2_1 c^1_2 \cdots c^1_r b^2_1 a^2_2 \cdots a^2_r c^2_1 c^2_2 b^2_3 \cdots b^2_r a^3_1 a^3_2 c^2_3 \cdots c^2_r \cdots b^r_1 \cdots b^r_{r-1} a^r_r c^r_1 \cdots c^r_r,$$

i.e., $Q_2 = T_1 \cdots T_r$ where

$$T_1 = a^1_1 \cdots a^1_r c^1_1 b^1_2 \cdots b^1_r a^2_1 c^1_2 \cdots c^1_r,$$
$$T_i = b^i_1 \cdots b^i_{i-1} a^i_i \cdots a^i_r c^i_1 \cdots c^i_i b^i_{i+1} \cdots b^i_r a^{i+1}_1 \cdots a^{i+1}_i c^i_{i+1} \cdots c^i_r \text{ for } 2 \leq i \leq r-1, \text{ and}$$
$$T_r = b^r_1 \cdots b^r_{r-1} a^r_r c^r_1 \cdots c^r_r.$$

Note that Q_1 and Q_2 are properly terminated $(r-1)$-paths which start with the same r vertices and end with the same r vertices, so P contains an absorber for X. See Fig. 3.

Example 6.4. In the case of $r = 3$, the 2-paths Q_1 and Q_2 are as follows:

$$Q_1 = a^1_1 a^1_2 a^1_3 x_1 b^1_2 b^1_3 c^1_1 c^1_2 c^1_3 a^2_1 a^2_2 a^2_3 b^2_1 x_2 b^2_3 c^2_1 c^2_2 c^2_3 a^3_1 a^3_2 a^3_3 b^3_1 b^3_2 x_3 c^3_1 c^3_2 c^3_3$$
$$Q_2 = a^1_1 a^1_2 a^1_3 c^1_1 b^1_2 b^1_3 a^2_1 c^1_2 c^1_3 b^2_1 a^2_2 a^2_3 c^2_1 c^2_2 b^2_3 a^3_1 a^3_2 c^2_3 b^3_1 b^3_2 a^3_3 c^3_1 c^3_2 c^3_3.$$

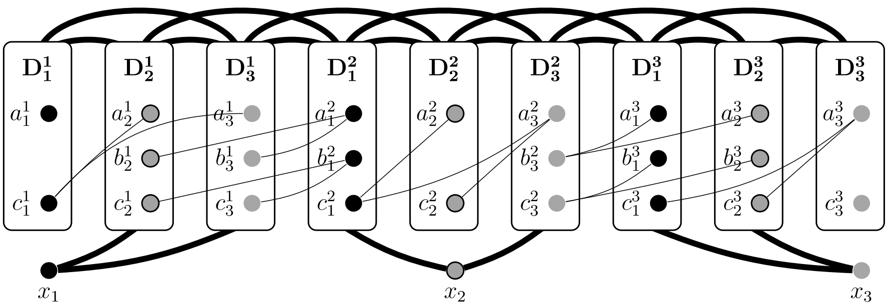

Fig. 3. An absorber for $X = \{x_1, x_2, x_3\}$ from Lemma 6.3. The edges between D_i^j's and between X and D_i^j are indicated by solid black lines. The edges of Q_1 are not shown. The edges of Q_2 that are not in Q_1 are shown.

Now we show that there are (n^{3r^2-r}) copies of P which contain the absorber of X as described above. By a Chernoff bound (Theorem 2.7), for all $i \in [r]$ there exists a partition $V_i = \{V_i^1, \dots, V_i^r\}$ such that for all $i, j \in [r]$ and all $v \in V(G) \setminus V_i$,

$$\deg\left(v, V_i^j\right) \geq \left(1 - \frac{1}{r} + \frac{\gamma}{2}\right)\left|V_i^j\right|.$$

One can see that constructing greedily (from the middle out), there are at least $(\frac{\gamma}{2}n)^{r^2}$ properly ordered $(r-1)$-paths $P = v_1 \dots v_{r^2}$ of order r^2 such that for all $i \in [r]$,

$$\{v_{ir+1}, \dots, v_{ir+i-1}, v_{ir+i+1}, \dots, v_{(i+1)r}\} \subseteq N(x_i).$$

Treating each such copy as an edge in an r^2-partite r^2-uniform hypergraph H with ordered partition $(V_1^1, \dots, V_r^1, \dots, V_1^r, \dots, V_r^r)$ and applying Theorem 6.2 to H, we have that there exists at least αn^{3r^2-r} copies of the $(s_1^1, \dots, s_r^1, \dots, s_1^r, \dots, s_r^r)$-blow up of P.

Proof of Lemma 4.2. Let α be such that $\frac{1}{n}$ α β, let $= 3r^2 - r$, and let $\mathcal{A}$ be the collection of all ordered sequences $(a_1, \dots, a)$ of vertices such that for every $i \in []$ and $j \in [r]$, if $a_i \in V_j$, then $i \equiv j \pmod r$. Let $\mathcal{X}$ be the collection of all balanced r-subsets of $V(G)$. For every $X \in \mathcal{X}$, let

$$\mathcal{A}_X = \{(a_1, \dots, a) \in \mathcal{A} : a_1, \dots, a \text{ is an absorber of } X\},$$

and note that, by Lemma 6.3, we have

$$|\mathcal{A}_X| \geq (\alpha n). \tag{5}$$

Now create a random set $\mathcal{A}_{ran}$ by select each sequence in $\mathcal{A}$ independently at random with probability $\rho = \beta^{1.1} n^{-+1}$, so since $|\mathcal{A}| = n$,

$$E|\mathcal{A}_{ran}| = \rho|\mathcal{A}| \leq \frac{\beta n}{4},$$

and, by (5), for every $X \in \mathcal{X}$,

$$E|\mathcal{A}_X \cap \mathcal{A}_{ran}| \geq \rho(\alpha n) \geq 4\beta^2 n.$$

So, by the Chernoff bound and the union bound, with high probability

$$|\mathcal{A}_{ran}| \leq \frac{\beta n}{3} \quad \text{and} \quad |\mathcal{A}_X \cap \mathcal{A}_{ran}| \geq 3\beta^2 n \quad \text{for every } X \in \mathcal{X}.$$

Let $\mathcal{A}_{rep}$ contain the pairs of tuples in $\mathcal{A}$ in which a vertex is repeated, i.e.,

$$\mathcal{A}_{rep} = \{\{S, T\} : S, T \in \mathcal{A}, S = T, \text{ and a vertex appears at least twice in sequence } S, T\}.$$

We can construct every pair in $\mathcal{A}_{rep}$ by selecting an arbitrary vertex, placing that vertex in 2 of the 2 possible entries, and then arbitrarily filling the remaining 2 − 2 entries, so

$$E|\mathcal{A}_{rep} \cap \mathcal{A}_{ran}| = \rho^2|\mathcal{A}_{rep}| \leq \rho^2 \cdot n \cdot \binom{2}{2} \cdot n^{2-2} \leq \beta^2 n.$$

By the Markov bound, with probability 1/2, we have that $|\mathcal{A}_{rep}| \leq 2\beta^2 n$. Therefore, there must exist some random outcome $\mathcal{A}_{ran}$ such that if we remove every pair in $\mathcal{A}_{rep} \cap \mathcal{A}_{ran}$ and every sequence that is not absorbing for some $X \in \mathcal{X}$ to form $\mathcal{A}$ then we have that

- $|\mathcal{A}| \le \beta n/(3)$;
- $|\mathcal{A} \cap \mathcal{A}_X| \ge \beta^2 n$ for every $X \in \mathcal{X}$;
- the sequences in $\mathcal{A}$ are pairwise vertex-disjoint; and
- for every $P \in \mathcal{A}$, P is an absorber for some $X \in \mathcal{X}$, so P is an $(r-1)$-path.

Lemma 4.1 (with $(V_1, \ldots, V_r)$ and $\mathcal{Y}$ playing the roles of $(U_1, \ldots, U_r)$ and $\mathcal{V}$, respectively) and Fact 2.4 together imply that we can connect the $(r-1)$-paths in $\mathcal{A}$ (in an arbitrary order) with paths of length $r(2r-2) < 2$ to form the desired absorbing $(r-1)$-path P_{abs}. We have that $|V(P_{abs})| = |\mathcal{A}| + r(2r-2)(|\mathcal{A}|-1) < 3|\mathcal{A}| \le \beta n$.

Let $Z \subseteq V(G) \setminus V(P_{abs})$ be a balanced set where $|Z| \le \beta^2 n$. We can partition Z into balanced r-subsets so that each part is in $\mathcal{X}$. Since there are at most $|Z|/r < \beta^2 n$ parts in such a partition, we can greedily match each part X to some path $P \in \mathcal{A} \cap \mathcal{A}_X$. Since P is an absorber of X, we can construct the desired Hamiltonian $(r-1)$-path of $G[V(P_{abs}) \cup Z]$.

7. The regularity lemma

We now review Szemerédi's well-known regularity lemma [21].

Definition 7.1. In a graph G, for each pair of disjoint non-empty sets $A, B \subseteq V(G)$ we write $G[A, B]$ for the bipartite subgraph of G with vertex classes A and B and whose edges are all edges of G with one endvertex in A and the other in B, and denote the *density* of $G[A, B]$ by $d_G(A, B) = \frac{e(G[A,B])}{|A||B|}$.

We say that $G[A, B]$ is *(d, ε)-regular* if $d_G(X, Y) = d \pm \varepsilon$ for every $X \subseteq A$ and $Y \subseteq B$ with $|X| \ge \varepsilon|A|$ and $|Y| \ge \varepsilon|B|$, and we write that $G[A, B]$ is *$(\ge d, \varepsilon)$-regular* to mean that $G[A, B]$ is (d', ε)-regular for some $d' \ge d$.

Also, we say that $G[A, B]$ is *(d, ε)-super-regular* if $G[A, B]$ is $(\ge d, \varepsilon)$-regular, every vertex of A has at least $(d-\varepsilon)|B|$ neighbors in B, and every vertex of B has at least $(d-\varepsilon)|A|$ neighbors in A.

The following results are well-known elementary consequences of the definitions.

Lemma 7.2 *(Slicing Lemma). For every $d, \varepsilon, \beta > 0$, if $G[A, B]$ is (d, ε)-regular, and $X \subseteq A$ and $Y \subseteq B$ have sizes $|X| \ge \beta|A|$ and $|Y| \ge \beta|B|$, then $G[X, Y]$ is $(d, \varepsilon/\beta)$-regular.*

Lemma 7.3. *For every $d, \varepsilon > 0$ with $\varepsilon < \frac{1}{2}$, if $G[A, B]$ is $(\ge d, \varepsilon)$-regular, then there are sets $X \subseteq A$ and $Y \subseteq B$ with sizes $|X| \ge (1-\varepsilon)|A|$, and $|Y| \ge (1-\varepsilon)|B|$ such that $G[X, Y]$ is $(d, 2\varepsilon)$-super-regular.*

Definition 7.4. Let G be a graph on n vertices and suppose that $\mathcal{C}$ is a collection of disjoint subsets of $V(G)$. Define the *$(G, \mathcal{C}, d, \varepsilon)$-cluster graph* to be the graph with vertex set $\mathcal{C}$ in which distinct $A, B \in \mathcal{C}$ form an edge if $G[A, B]$ is $(\ge d, \varepsilon)$-regular.

Definition 7.5. Let $\mathcal{P} = (V_1, \ldots, V_r)$ be an ordered partition of $V(G)$. We say that a collection $\mathcal{C}$ of vertex disjoint subsets of $V(G)$ *respects* $\mathcal{P}$ if for every $C \in \mathcal{C}$ we have $C \subseteq V_i$ for some $i \in [r]$. If $\mathcal{C}$ respects $\mathcal{P}$, we let $\mathcal{P}(\mathcal{C})$ be the partition $(\mathcal{C}_1, \ldots, \mathcal{C}_r)$ of $\mathcal{C}$ in which every $C \in \mathcal{C}$ is in $\mathcal{C}_i$ when $C \subseteq V_i$.

We now state the standard degree form of the regularity lemma.

Lemma 7.6 *(Degree Form of Szemerédi's Regularity Lemma). For every $\varepsilon > 0$ and $0 < d < 1$ and integers r and N_0 there exists N_1 such that the following holds. If G is an r-partite graph on n vertices with ordered partition $\mathcal{P}$, then there exists a partition $U_0, \ldots, U_N$ of $V(G)$ and a spanning subgraph R of G such that the following holds:*

- *$N_0 \le N \le N_1$;*
- *$|U_0| \le \varepsilon n$;*
- *$|U_1| = \cdots = |U_N|$;*
- *the collection $U_1, \ldots, U_N$ respects the partition $(V_1, \ldots, V_r)$;*
- *$\deg_R(v) \ge \deg_G(v) - (d+\varepsilon)n$ for every $v \in V(G)$;*
- *$|E(R[U_i])| = 0$ for every $1 \le i \le N$; and*
- *for every $1 \le i < j \le N$, the graph $R[U_i, U_j]$ either $(\ge d, \varepsilon)$-regular or has no edges.*

From the degree form of the regularity lemma, it is easy to show that we have Lemma 7.7 below. Since the proof is standard, we only provide a sketch.

Lemma 7.7. *Suppose that*

$$\frac{1}{n} \ll \frac{1}{N_1} \ll \varepsilon \ll d \ll \eta, \frac{1}{N_0}, \frac{1}{r}.$$

Let G be a balanced r-partite graph on n vertices with ordered partition $\mathcal{P}$. Then there exists $\mathcal{C}$, which is a collection of vertex disjoint subsets of $V(G)$ and R a spanning subgraph of G such that

(R1) $N_0 \le |\mathcal{C}| \le N_1$;
(R2) $\mathcal{C}$ covers all but at most εn vertices of G;
(R3) every element in $\mathcal{C}$ has the same order;
(R4) $\mathcal{C}$ respects the partition $\mathcal{P}$ and the partition $\mathcal{P}(\mathcal{C}) = (\mathcal{C}_1, \ldots, \mathcal{C}_r)$ is balanced;
(R5) for every $v \in V(G)$, we have $\deg_R(v) \ge \deg_G(v) - (d + \varepsilon)n$;
(R6) for every $U \in \mathcal{C}$, we have $E_R(U) = \emptyset$, and for every pair of distinct $A, B \in \mathcal{C}$, either $E(R[A, B]) = \emptyset$ or $R[A, B]$ is $(\ge d, \varepsilon)$-regular; and
(R7) if $\mathcal{G}$ is the $(G, \mathcal{C}, d, \varepsilon)$-cluster graph, then $\delta_{\mathcal{P}(\mathcal{C})}(\mathcal{G}) \ge \delta_{\mathcal{P}}(G) - \eta$.

Proof sketch. Pick ε and d such that $\frac{1}{N_1}$ ε ε d d. Lemma 7.6 implies that there exists a spanning subgraph R of G and $U_0, U_1, \ldots, U_N$ a collection of vertex disjoint subsets of $V(G)$ such that the conclusions of Lemma 7.6 hold with ε, d, r and $2N_0$ playing the roles of ε, d, r and N_0. In particular, we have that $N \ge 2N_0$ and $U_1, \ldots, U_N$ covers all but at most εn of the vertices of G. Therefore, by removing a small fraction of the sets from the collection $U_1, \ldots, U_N$ we can create $\mathcal{C}$ a collection of vertex disjoint subsets of $V(G)$ such that (R1) - (R6) all hold.

To see that (R7) holds as well, let $\mathcal{P}(\mathcal{C}) = (\mathcal{C}_1, \ldots, \mathcal{C}_r)$ and let $i, j \in [r]$ such that $i = j$. For every $C \in \mathcal{C}_i$ and $v \in C$, (R2), (R3), (R5), and (R6) imply that

$$\frac{\deg_{\mathcal{G}}(C, \mathcal{C}_j)}{|\mathcal{C}_j|} \ge \frac{\deg_R(v, V(\mathcal{C}_j))}{|C||\mathcal{C}_j|} \ge \frac{\deg_R(v, V_j) - \varepsilon n}{n/r}$$
$$\ge \frac{\deg_G(v, V_j) - (d + \varepsilon)n - \varepsilon n}{n/r} \ge \delta_{\mathcal{P}}(G) - \eta.$$

We make the following definition to help describe the version of the well-known blow-up lemma that we will need.

Definition 7.8. For a graph R and $\mathcal{C}$ be a collection of vertex disjoint subsets of $V(R)$, we let $K(\mathcal{C}, R)$ be the graph on $V(\mathcal{C})$ such that for every distinct $x, y \in V(R)$ the graph $K(\mathcal{C}, R)$ has the edge $\{x, y\}$ if and only if x and y are in distinct sets $A, B \in \mathcal{C}$ and $E(R[A, B]) = \emptyset$.

For a subgraph H of $K(\mathcal{C}, R)$, a *copy of H in R that respects* $\mathcal{C}$ is an injective function $f : V(H) \to V(R)$ such that $\{x, y\} \in E(H)$ implies $\{f(x), f(y)\} \in E(R)$ and, for every $v \in V(H)$ and $C \in \mathcal{C}$, $v \in C$ implies $f(v) \in C$.

Lemma 7.9 *(Blow-up Lemma [12]). Suppose that* $\frac{1}{m}$ ε $d, \frac{1}{D}$. *Let G be a graph on n vertices; let* $\mathcal{C}$ *be a collection of vertex disjoint subsets of* $V(G)$ *each of size m; and let R be a spanning subgraph of G such that for every* $U \in \mathcal{C}$, *we have* $E_R(U) = \emptyset$, *and for every pair of distinct* $A, B \in \mathcal{C}$, *either* $E(R[A, B]) = \emptyset$ *or* $R[A, B]$ *is* $(\ge d, \varepsilon)$*-super-regular. If* $H \subseteq K(\mathcal{C}, R)$ *and* $(H) \le D$, *then there exists a copy of H in R that respects* $\mathcal{C}$.

8. Proof of the covering lemma (Lemma 4.3)

Definition 8.1. Let G be a graph and let $\mathcal{K}$ be the copies of K_r in G. A *fractional* K_r*-tiling of a graph* G is a weight function $w : E(\mathcal{K}) \to \mathbb{R}_{\ge 0}$ in which, for every $v \in V(G)$, the sum of the weights on the copies of K_r that contain v is at most one. That is, we have that

$$\{w(K) : K \in \mathcal{K} \text{ and } K \text{ contains } v\} \le 1 \qquad \text{for every } v \in V(G).$$

The *size* of w is $\{w(K) : K \in \mathcal{K}\}$, and we say that w is *perfect* if the size of w is exactly $|V(G)|/r$. Note that w is perfect if and only if

$$\{w(K) : K \in \mathcal{K} \text{ and } K \text{ contains } v\} = 1 \qquad \text{for every } v \in V(G).$$

We will use the following lemma which can be found as a corollary to [18, Lemma 2.2]. (See also, [14,10].)

Lemma 8.2. *If G is a balanced r-partite graph on n vertices with partition* $\mathcal{P}$ *and* $\delta_{\mathcal{P}}(G) \ge 1 - \frac{1}{r}$, *then G has a perfect fractional* K_r*-tiling.*

Remark 8.3. Here we could have shortened our proof by using existing results on perfect K_r-tilings in multipartite graphs (see [10,15]). We chose to only use the above lemma on perfect fractional K_r-tilings, which is relatively short, to make this paper more self-contained.

The following lemma is a consequence of Lemma 7.2 (The Slicing Lemma), Lemma 7.3, and Lemma 7.9 (The Blow-up Lemma).

Lemma 8.4. *Let $\frac{1}{m} \ll \varepsilon \ll d \ll \alpha < \frac{1}{r}$, let G be an r-partite graph with ordered partition $(V_1, \dots, V_r)$ and for $i \in [r]$, let C_i be an m-subset of V_i. Suppose that the sets $C_1, \dots, C_r$ are pairwise $(\geq d, \varepsilon)$-regular and for every $i \in [r]$, we have $C_i' \subseteq C_i$. If z is a positive integer such that $|C_i \setminus C_i'| + z \leq (1 - \alpha)m$ for every $i \in [r]$, then there exists P a properly terminated $(r-1)$-path in $G[C_1' \cup \dots \cup C_r']$ such that for every $i \in [r]$ the path P intersects C_i' in exactly z vertices.*

Proof. Note that the conditions imply that $|C_i'| \geq \alpha m + z$ for every $i \in [r]$. So, Lemma 7.2 (the Slicing Lemma), implies that the sets $C_1', \dots, C_r'$ are pairwise $(\geq d, \varepsilon^{2/3})$-regular. By applying Lemma 7.3 $\binom{r}{2}$ times, we can construct $C_i'' \subseteq C_i'$ for $i \in [r]$ such that $|C_i''| \geq z$ and the sets $C_1'', \dots, C_r''$ are pairwise $(d, \varepsilon^{1/3})$-super-regular. Lemma 7.9 (the Blow-up Lemma) then implies the existence of the desired $(r-1)$-path P.

Proof of Lemma 4.3. Select constants N_0, M_0, ε, α, η and d so that

$$\frac{1}{n} \ll \frac{1}{M_0} \ll \frac{1}{N_1} \ll \varepsilon \ll \alpha \ll \alpha \ll d \ll \eta \ll \gamma < \frac{1}{r}.$$

Lemma 7.7 implies the existence of a collection $\mathcal{C}$ of disjoint subsets of $V(G)$ such that

- $|\mathcal{C}| \leq N_1$
- $\mathcal{C}$ covers all but at most εn of the vertices in $V(G)$;
- there exists m such that for every $C \in \mathcal{C}$ we have $|C| = m$;
- $\mathcal{C}$ respects $\mathcal{P}$ and if we let $\mathcal{P}' = \mathcal{P}(\mathcal{C})$ and $\mathcal{G} = (G, \mathcal{C}, d, \varepsilon)$, then $\mathcal{P}'$ is balanced and

$$\delta_{\mathcal{P}'}(\mathcal{G}) \geq 1 - \frac{1}{r} + \frac{\gamma}{2}.$$

Lemma 8.2 implies that there exists a perfect fractional K_r-tiling of $\mathcal{G}$, and let $K^1, \dots, K^M$ be an arbitrary ordering of the copies of K_r in $\mathcal{G}$ that receive positive weight in such a fractional K_r-tiling. Note that $M \leq \binom{N_1}{r}$ and that there are positive weights $w_1, \dots, w_M$ such that for every $C \in \mathcal{C}$,

$$\sum_{i=1}^{M} \{w_i : K^i \text{ contains the cluster } C\} = 1,$$

and $\sum_{i=1}^{M} w_i = |\mathcal{C}|/r \geq (1-\varepsilon)n/(mr)$. For each $i \in [M]$, let $z_i = \lfloor (1-\alpha) w_i m \rfloor$ and note that

$$\sum_{i=1}^{M} z_i = \sum_{i=1}^{M} \lfloor (1-\alpha) w_i m \rfloor \geq (1-\alpha)|\mathcal{C}|\frac{m}{r} - M \geq (1-\alpha)(1-\varepsilon)\frac{n}{r} - M \geq (1-\alpha)\frac{n}{r}.$$

We can now prove the lemma by constructing disjoint properly terminated $(r-1)$-paths $P_1, \dots, P_M$ such that for each $i \in [M]$, the $(r-1)$-path P_i has length exactly $r z_i$ because then $\sum_{i=1}^{m} |V(P_i)| \geq (1-\alpha)n$.

To see that such a construction is possible, assume that, for some $t \in [M]$, we have constructed $t-1$ disjoint properly terminated $(r-1)$-paths $P_1, \dots, P_{t-1}$ such that for every $j \in [t-1]$ the path P_j is contained in the clusters of K^j and for every cluster C contained in K^j the $(r-1)$-path P_j intersects C in exactly z_j vertices.

Let $C_1, \dots, C_r$ be the clusters in K^t. We can assume that $C_i \subseteq V_i$ for $i \in [r]$ since the partition $\mathcal{C}$ respects the partition $\mathcal{P}$ and the clusters $C_1, \dots, C_r$ are pairwise $(\geq d, \varepsilon)$-regular. For $i \in [r]$, let $C_i' \subseteq C_i$ be the vertices in C_i that do not intersect one of the previously constructed paths $P_1, \dots, P_{t-1}$. Recall that for $i \in [r]$, we have that K^t contains the cluster C_i, so

$$\begin{aligned}
|C_i \setminus C_i'| + z_t &= \left(\sum_{j=1}^{t-1} \{z_j : K^j \text{ contains the cluster } C_i\}\right) + z_t \\
&\leq \sum_{j=1}^{M} \{z_j : K^j \text{ contains the cluster } C_i\} \\
&\leq \sum_{j=1}^{M} \{(1-\alpha) w_j m : K^j \text{ contains the cluster } C_i\} = (1-\alpha)m.
\end{aligned}$$

Therefore, Lemma 8.4 implies that there exists an $(r-1)$-path P_t contained in $G[C_1' \cup \dots \cup C_r']$ such that, for $i \in [r]$, the path P_t intersects C_i' in exactly z_t vertices.

9. Proof of the partitioning and sequencing lemma (Lemma 3.2)

Before we begin the proof, we give some further terminology and observations regarding properly ordered paths.

For $1 \le k' \le k$, we say that an $(r-1)$-path $v_1, \dots, v_{k'}$ is *increasing* if for every $1 \le i < i' \le k'$ we have that $v_i \in V_j$ and $v_{i'} \in V_{j'}$ with $j < j'$, i.e., a path is increasing if it traverses the sets $V_1, \dots, V_k$ in order (though it might skip any number of the sets). All of the paths that we will construct can be partitioned into subpaths on either r or $r+1$ vertices that are increasing. We call such an $(r-1)$-path *properly ordered*. We now give a more formal definition.

Definition 9.1 *(Properly ordered/ j-th subsequence).* Let $P = v_1 v_2 \cdots v_p$ be an $(r-1)$-path and let $f : [p] \to [k]$ be such that $v_{f(i)} \in V_j$. We say that P is *properly ordered* if there exists $0 = p_0, p_1, \dots, p_q = p$ such that for all $i \in [q]$, $r \le p_i - p_{i-1} \le r+1$ and $f(p_{i-1}+1) < \cdots < f(p_i)$. For $j \in [q]$, let $v_{p_{j-1}+1}, \dots, v_{p_j}$ be the *j-th subsequence* of P.

Given a properly ordered path $P = v_{p_0+1} \cdots v_{p_1} v_{p_1+1} \cdots v_{p_2} \cdots v_{p_{q-1}+1} \cdots v_{p_q}$, we will say that the j-th subsequence, $v_{p_{j-1}+1}, \dots, v_{p_j}$, *has type* $z \in \mathbb{Z}^k$ if for $i \in [k]$, we have $z_i = 1$ when one of the vertices in the subsequence is in the part V_i and $z_i = 0$ otherwise. From the definition of properly ordered, this means that $v_{p_{j-1}+1}, \dots, v_{p_j}$ has type $z \in \mathbb{Z}^k$ if $z_i = |\{v_{p_{j-1}+1}, \dots, v_{p_j}\} \cap V_i|$ for every $i \in [k]$.

It is clear that we need the parts which contain every r consecutive vertices in P to be distinct. Given a properly ordered $(r-1)$-path, we will have this critical property if and only if the following condition is met for every $j \in [q-1]$, and $i \in \{p_{j-1}+1, \dots, p_j\}$, and $i' \in \{p_j+1, \dots, p_{j+1}\}$:

$$\text{If } v_i \text{ and } v_{i'} \text{ are contained in the same part, then } i' - i \ge r. \tag{6}$$

We can restate this observation in the following way: The parts which contain every r consecutive vertices in P are distinct if and only if for every $j \in [q-1]$ when we let z be the type of the j-th subsequence and z' be the type of the $(j+1)$-th subsequence we have the following:

$$\text{For every } i \in [k], \text{ if } z_i = z'_i = 1, \text{ then } \sum_{v=i+1}^{k} z_v + \sum_{v=1}^{i} z'_v \ge r. \tag{7}$$

Note that if the j-th and $(j+1)$-th subsequences of P both contain exactly r vertices (so, $\sum_{v=1}^{k} z_v = \sum_{v=1}^{k} z'_v = r$), then (7) can be restated as the following:

$$\text{For every } i \in [k], \text{ if } z_i = z'_i = 1, \text{ then } \sum_{v=1}^{i} z'_v = r - \sum_{v=i+1}^{k} z_v \le \sum_{v=i}^{i} z_v. \tag{8}$$

If the ordered pair (z, z') satisfies (7), then we say that (z, z') is *valid*.

Proof of Lemma 3.2. Let $2 \le r < k \le 2r-1$ and let β and σ be constants such that

$$\frac{1}{n} \ll \beta \ll \sigma \ll \gamma \le \frac{1}{r}. \tag{9}$$

Let G be an n-vertex k-partite graph with ordered partition $\mathcal{P} = (V_1, \dots, V_k)$ of $V = V(G)$ such that

$$\gamma n \le |V_k| \le |V_{k-1}| \le \cdots \le |V_1| \le \frac{n}{r}, \tag{10}$$

and

$$\delta_{\mathcal{P}}(G) \ge 1 - \frac{1}{r} + \gamma. \tag{11}$$

If $|V_1| \ge \frac{n}{r} - 2\sigma n$, we define $1 \le s \le k$ to be the largest integer such that $|V_s| \ge \frac{n}{r} - 2\sigma n$; otherwise, we set $s = 0$.

We start by greedily building a path P_0 such that when $V' = V \setminus V(P_0)$ and $V'_i = V_i \setminus V(P_0)$ for every $i \in [k]$, the following holds:

(T1) $|V'|$ is divisible by r;
(T2) $|V'_i| = |V'|/r$ for every $i \in [s]$;
(T3) $|V'_i| \ge \sigma n$ for every $i \in \{s+1, \dots, k\}$;
(T4) $|V'_i| \le |V'|/r - \sigma n$ for every $i \in \{s+1, \dots, k\}$;
(T5) $|V'| \ge (1 - 3r^2\sigma)n$; and
(T6) P_0 is properly ordered and properly terminated.

Let $z^{(0)}$ be the $(0,1)$-vector in $\mathbb{Z}^k$ in which the first $(r+1)$ entries are one and the remaining $k-r-1$ entries are zero. For $j \in [r+1]$, let $z^{(j)}$ be $z^{(0)}$ minus the j-th standard basis vector, i.e., all of the last $k-r-1$ entries of $z^{(j)}$ are zero and all of the first $(r+1)$ entries of $z^{(j)}$ are one except for the j-th entry, which is zero. Using (7) and (8) it is not hard to verify that the following holds for every $j, j' \in [r+1]$:

(V1) $(z^{(0)}, z^{(j)})$ is valid;
(V2) $(z^{(j)}, z^{(j')})$ is valid when $j' \le j+1$; and
(V3) $(z^{(j)}, z^{(j')})$ is not valid when $j' \ge j+2$.

Let $0 \le c_0 < r$ be such that $n - c_0$ is divisible by r and for $i \in [s]$, let

$$c_i = \frac{n - c_0}{r} - |V_i|. \tag{12}$$

Note, by (10) and the definition of s, we have that $\frac{n}{r} - 2\sigma n \le |V_s| \le \cdots \le |V_1| \le \frac{n}{r}$, so

$$2\sigma n \ge c_s \ge c_{s-1} \ge \cdots \ge c_1 \ge 0. \tag{13}$$

The sequences of vectors

$$c_0 z^{(0)}, z^{(r+1)}, z^{(r)}, z^{(r-1)}, \ldots, z^{(s+1)}, c_s z^{(s)}, c_{s-1} z^{(s-1)}, \ldots, c_1 z^{(1)}, z^{(r+1)},$$

will serve as our template for P_0.
That is, we greedily build P_0 so that

- the first c_0 subsequences are of type $z^{(0)}$ (these are the only subsequences that have $(r+1)$ instead of r vertices);
- the next $(r-s+1)$ subsequences have types $z^{(r+1)}, z^{(r)}, z^{(r-1)}, \ldots, z^{(s+1)}$, respectively;
- the next c_s subsequences are of type $z^{(s)}$, followed by c_{s-1} subsequences of type $z^{(s-1)}, \ldots,$ followed by c_1 subsequences of type $z^{(1)}$; and
- the last subsequence is of type $z^{(r+1)}$.

Note that it is possible to build P_0 in this way by (11), (13), (V1) and (V2) (To see that (13) is critical here, note that, by (V3), we need that if $t \in [s]$ is such that $c_t = 0$, then $c_{t-1} = c_{t-2} = \cdots = c_1 = 0$.) Define

$$q = c_0 + (r - s + 1) + \sum_{j=1}^{s} c_j + 1 \tag{14}$$

and note that q is the number of subsequences in P_0.

Claim 9.2. *The P_0 constructed as described above satisfies conditions (T1)–(T6).*

Proof of Claim 9.2.
(T6): The construction of P_0 requires P_0 to be properly ordered and properly terminated (even when $c_0 = 0$).
(T1): Recall that each subsequence has r vertices except the first c_0, which have $r+1$. By (14), the number of vertices in P_0 is

$$p = c_0(r+1) + (r-s+1)r + \sum_{j=1}^{s} c_j r + r = c_0 + qr. \tag{15}$$

So, since $n - c_0$ is divisible by r, we have that $|V'| = n - p$ is divisible by r.
(T5): By (13), (14), (15) and the fact that $s \le r$ and $c_0 < r$, we have that

$$p = c_0 + qr = c_0(r+1) + (r-s+2)r + r\sum_{j=1}^{s} c_j \le 3\sigma r^2 n. \tag{16}$$

(T3): By (10), for all $i \in \{s+1, \ldots, k\}$,

$$|V'_i| = |V_i \setminus V(P_0)| \ge \gamma n - 3\sigma r^2 n \ge \sigma n.$$

(T2): By (12) and (15), for all $i \in [s]$,

$$|V'_i| = |V_i| - q + c_i = |V_i| - q + \frac{n - c_0}{r} - |V_i| = \frac{n - c_0}{r} - \frac{p - c_0}{r} = \frac{n - p}{r} = \frac{|V'|}{r}.$$

(T4): Consider two cases: If $s+1 \le i \le r$, then, because every subsequence of P_0 except exactly one intersects V_i, we have

$$|V_i| = |V_i| - q + 1 = |V_i| - \frac{p-c_0}{r} + 1 < \frac{n}{r} - 2\sigma n - \frac{p}{r} + 2 = \frac{|V|}{r} - 2\sigma n + 2 < \frac{|V|}{r} - \sigma n.$$

If $r+1 \le i \le k$, (10) implies that $|V_i| \le n/i \le n/(r+1)$, so with (16),

$$|V_i| \le |V_i| \le \frac{n}{r+1} = \frac{n}{r} - \frac{n}{r(r+1)} \le \frac{n}{r} - 3\sigma rn - \sigma n \le \frac{n}{r} - \frac{p}{r} - \sigma n = \frac{|V|}{r} - \sigma n.$$

This concludes the proof of Claim 9.2.

Now we consider (A1). We stress that the issue here is largely numerical, which explains the general nature of the next two claims. Claim 9.3 provides the template and Claim 9.4 shows that V can be partitioned according to the template so that (A1) holds. The purpose of partitioning according to this specific template is to set things up so that (A3) and (A4) will be able to be satisfied in the end.

Let Z be the set of $(0,1)$-vectors in $\mathbb{Z}^k$ such that the first s entries are one and exactly $r-s$ of the remaining $k-s$ entries are one (so, for every $z \in Z$ exactly r of the k entries of z are one and the remaining $k-r$ entries are zero). Note that $= \binom{k-s}{r-s}$ is the order of Z.

Claim 9.3. *There exists a $k \times$ $(0,1)$-matrix $A = [a_{i,j}]$ such that the columns of A are the vectors in Z where the columns of A are ordered so that*

- *the first column is* $(\underbrace{1,\dots,1}_{r \text{ times}}, \underbrace{0,\dots,0}_{k-r \text{ times}})^T$;
- *the last column is* $(\underbrace{1,\dots,1}_{s \text{ times}}, \underbrace{0,\dots,0}_{k-r \text{ times}}, \underbrace{1,\dots,1}_{r-s \text{ times}})^T$; *and*
- *for every $j \in [\ -1]$ and $i \in [k]$,*

$$\text{if } a_{i,j} = a_{i,j+1} = 1, \text{ then } \sum_{v=1}^{i} a_{v,j} \le \sum_{v=1}^{i} a_{v,j+1} \quad (\text{cf. } (8)). \tag{17}$$

Proof of Claim 9.3. The proof is by induction on $k-s$. Note that if either $k=r$ or $r=s$, then the claim is trivially true. In particular, this establishes the base case since $k-s=0$ implies $k=r=s$. Now suppose that $k>r>s$. Let Z be the vectors in Z in which the $(s+1)$-th entry is one and let $Z = Z \setminus Z$. Let $= |Z| = \binom{k-s-1}{r-s-1}$ and $= |Z| = \binom{k-s-1}{r-s}$. By the induction hypothesis (with k, r, and $s+1$ playing the roles of k, r, and s, respectively), we can populate the first columns of A with the vectors in Z so that the first column is $(\underbrace{1,\dots,1}_{r \text{ times}}, \underbrace{0,\dots,0}_{k-r \text{ times}})^T$, the -th column is $(\underbrace{1,\dots,1}_{s+1 \text{ times}}, \underbrace{0,\dots,0}_{k-r \text{ times}}, \underbrace{1,\dots,1}_{r-s-1 \text{ times}})^T$, and (17) holds for $j \in [\ -1]$. Similarly, by the induction hypothesis (with $k-s-1$, $r-s$, and 0 playing the roles of k, r, and s, respectively), we can populate the remaining columns of A with Z so that the $(\ +1)$-th column is $(\underbrace{1,\dots,1}_{s \text{ times}}, 0, \underbrace{1,\dots,1}_{r-s \text{ times}}, \underbrace{0,\dots,0}_{k-r-1 \text{ times}})^T$, the last column is $(\underbrace{1,\dots,1}_{s \text{ times}}, \underbrace{0,\dots,0}_{k-r \text{ times}}, \underbrace{1,\dots,1}_{r-s \text{ times}})^T$, and (17) holds for $+1 \le j \le \ -1$. The claim then follows because (17) holds when $j =$.

Let A be the matrix guaranteed by Claim 9.3.

Claim 9.4. *Let $b = (|V_1|, |V_2|, \dots, |V_k|)^T$. There exists $x \in \mathbb{Z}$ such that $x_j \ge \beta n$ for every $j \in [\,]$ and such that $Ax = b$.*

Proof of Claim 9.4. We will iteratively construct a sequence of vectors $x^{(0)}, x^{(1)}, \dots, x^{(T)} \in \mathbb{Z}$ such that $x = x^{(T)}$ meets the conditions of the claim. For $t \ge 0$, define $b^{(t)} = b - Ax^{(t)}$; $n^{(t)} = \sum_{i=1}^{k} b_i^{(t)}$; and the following properties:

(P1) $n^{(t)} \ge 0$ is divisible by r;
(P2) $b_i^{(t)} = n^{(t)}/r$ for every $1 \le i \le s$;
(P3) $0 \le b_i^{(t)} \le n^{(t)}/r$ for every $s+1 \le i \le k$; and
(P4) $x_j^{(t)} \ge \beta n$ for every $j \in [\,]$.

To begin the construction, we let $m = \beta n$ and $x_j^{(0)} = m$ for every $j \in [\,]$. Clearly, we have that (P4) holds for $t=0$. First note that $n^{(0)} = |V| - r\ m$ so, by (T1), we have that (P1) holds for $t=0$. By (T2), we also have that

$$b_i^{(0)} = |V|/r - m = n^{(0)}/r \qquad \text{for every } i \in [s],$$

so (P2) holds for $t = 0$. By (T3), we have $b_i^{(0)} \geq b_i - m \geq \sigma n - m > 0$ for every $s + 1 \leq i \leq k$, and with (T4) we have that

$$b_i^{(0)} \leq |V|/r - \sigma n \leq |V|/r - m = n^{(0)}/r \qquad \text{for every } s + 1 \leq i \leq k.$$

Therefore, (P3) also holds for $t = 0$.

Now assume (P1), (P2), (P3), and (P4) hold for some $t \geq 0$. If $b_i^{(t)} = 0$ for every $i \in [k]$, then $Ax^{(t)} = b$, so with (P4), we can let $t = T$ and end the construction, because $x = x^{(t)} = x^{(T)}$ meets the conditions of the claim. Otherwise, let $I = \{i \in [k] : b_i^{(t)} = n^{(t)}/r\}$. Note that (P2) implies that $[s] \subseteq I$ and by (P3) we have that $b_i^{(t)} \leq n^{(t)}/r - 1$ for every $i \in [k] \setminus I$. We clearly have that $|I| \leq r$ and, by (P1), (P2) and (P3), there exists $I \subseteq I \subseteq [k]$ such that $|I| = r$ and $b_i^{(t)} > 0$ for every $i \in I$. Now let $j^{(t)}$ be the column of A such that $a_{i,j^{(t)}} = 1$ if and only if $i \in I$. If we then let

$$x_j^{(t+1)} = \begin{cases} x_j^{(t)} + 1 & \text{if } j = j^{(t)} \\ x_j^{(t)} & \text{otherwise} \end{cases}$$

it is clear that (P1), (P2), (P3), and (P4) all hold with t set to $t + 1$.

Now we use the preceding claims to show that (A1) and (A2) hold. Let $i \in [k]$ and recall that, since $Ax = b$, we have $\sum_{j=1} a_{i,j} \cdot x_j = b_i = |V_i|$. Therefore, for every $i \in [k]$, we can uniformly at random select a partition of V_i into parts $V(i,1), \ldots, V(i,)$ so that for every $j \in []$, we have $|V(i,j)| = a_{i,j} \cdot x_j$. (Note that we are allowing parts to be empty in these partitions).

Let $j \in []$, and note that since exactly r entries in the j-th column of A are 1, there exists a unique sequence $1 \leq i_{j,1} < \cdots < i_{j,r} \leq k$ such that $|V(i_{j,1}, j)| = \cdots = |V(i_{j,r}, j)| = x_j \geq \beta n$, so (A1) holds.

Let $P_j = (V(i_{j,1}, j), \ldots, V(i_{j,r}, j))$ and $G_j = G[V(i_{j,1}, j) \cup \cdots \cup V(i_{j,r}, j)]$. Therefore, (11), (T5), and the Chernoff and union bounds imply that, with high probability, there exists an outcome where for every $j \in []$, $h \in [r]$, and $v \in V \setminus V_{i_{h,j}}$ we have that

$$\deg(v, V(i_{j,h}, j)) \geq \left(1 - \frac{1}{r} + \frac{\gamma}{2}\right) |V(i_{j,h}, j)|. \tag{18}$$

Fix such an outcome. Note that $V(P_0), V(G_1), \ldots, V(G)$ is a partition of $V(G)$, and for every $j \in []$, G_j is a balanced r-partite graph with ordered partition P_j such that each part has order at least βn and, with (18) we have

$$\delta_{P_j}(G_j) \geq 1 - \frac{1}{r} + \frac{\gamma}{2}, \tag{19}$$

i.e., (P2) holds.

To see that (A4) holds, note that, by the ordering of the columns of A (cf. (17)) and (18), we can greedily construct $- 1$ vertex disjoint $(r-1)$-paths $P_1, \ldots, P_{-1}$ each on exactly $2r$ vertices so that, for every $j \in [- 1]$, the initial r vertices of P_j respect the sequence P_j and the final r vertices of P_j respect the sequence P_{j+1}.

Finally, we now show that (A3) holds. To see this, first note that, by Claim 9.3, if we let z be the last column of A and z be the first column of A, then by (8), we have that (z, z) is valid. Therefore, since (T6) implies that the $(r-1)$-path P_0 is properly terminated, we can use (18) to greedily prepend r vertices to P_0 to create an $(r-1)$-path in which the initial r vertices respect the sequence P while avoiding the path P_{-1}. Again because P_0 is properly terminated, (18) implies that we can greedily append r vertices to this path to create an $(r-1)$-path P_0 so that the final r vertices of P_0 respect the sequence P_1 and so that P_0 avoids P_1. This completes the proof.

10. Conclusion

10.1. Exact version

The main open problem which remains is to prove an exact version of Theorem 1.1. Note that it is possible that in the unbalanced case, there are extra variants of Catlin's example.

10.2. Total degree version

Another direction is to consider minimum total degree conditions for perfect K_r-tilings and Hamiltonian $(r-1)$-paths. In this direction, Johansson, Johansson, and Markström [9] proved that if G is a balanced 3-partite graph on n vertices with $\delta(G) \geq n/2$, then G has a perfect K_3-tiling. Later, Lo and Sanhueza-Matamala [16] proved that if G is a balanced r-partite graph on n vertices with $\delta(G) \geq \left(1 - \frac{3}{2r} + o(1)\right) n$, then G has a perfect K_r-tiling, which is asymptotically best possible.

It would be interesting to study the unbalanced version of this result and extend it to Hamiltonian $(r-1)$-cycles. This was done for $r = 2$ in [2], but the degree condition is quite complicated (in some sense necessarily so, since it is asymptotically tight in all cases) and thus determining an asymptotically tight minimum degree condition for perfect K_r-tilings in all valid k-partite graphs seems challenging.

As a start, we conjecture the following sufficient condition for perfect K_r-tilings (which will be asymptotically necessary in certain cases).

Conjecture 10.1. *Let $k \geq r \geq 2$ and $\gamma > 0$. If G is a k-partite graph with all parts at most n/r and $\delta(V_i) \geq \left(1 - \frac{1}{2r} + \gamma\right)\left(n - |V_i|\right)$ for all $i \in [k]$, then G has a perfect K_r-tiling.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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