Discussiones Mathematicae
Graph Theory xx (*xxxx*) *1–12*
doi:10.7151/dmgt.2390

A NOTE ABOUT MONOCHROMATIC COMPONENTS IN GRAPHS OF LARGE MINIMUM DEGREE

Louis DeBiasio [1]

Department of Mathematics
Miami University
Oxford, OH 45056, USA

e-mail: debiasld@miamioh.edu

and

Robert A. Krueger

Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA

e-mail: rak5@illinois.edu

Abstract

For all positive integers $r \geq 3$ and n such that $r^2 - r$ divides n and an affine plane of order r exists, we construct an r-edge colored graph on n vertices with minimum degree $(1-\frac{r-2}{r^2-r})n-2$ such that the largest monochromatic component has order less than $\frac{n}{r-1}$. This generalizes an example of Guggiari and Scott and, independently, Rahimi for $r = 3$ and thus disproves a conjecture of Gyárfás and S´ark¨ozy for all integers $r \geq 3$ such that an affine plane of order r exists.

Keywords: Ramsey theory, fractional matchings, block designs.

2010 Mathematics Subject Classification: 05D10, 05D15, 05B05.

1. Introduction

An affine plane of order q is a q-uniform hypergraph on q^2 vertices (called points), with $q(q + 1)$ edges (called lines) such that each pair of vertices is contained in exactly one edge.It is well known that an affine plane of order q exists whenever

[1] Research supported in part by NSF grant DMS-1954170.

q is a prime power (and it is unknown whether there exists an affine plane of non-prime power order). Given an affine plane G of order q, there exists a q+1-coloring of the edges of G such that every color class (called a parallel class) consists of a collection of q vertex disjoint edges of order q, every vertex is contained in exactly one edge of each color, and the union of the q + 1 edges incident to a given vertex is all of V (G).

Let $H = (\{x_1, \ldots, x_t\}, E)$ be a hypergraph which has a proper edge coloring with r colors (that is, every color class induces a matching). Let $\alpha = (\alpha_1, \ldots, \alpha_t) \in R^t$ be such that $\sum_{i=1}^{t} \alpha_i = 1$ and $\alpha_i > 0$ for all $i \in [t]$. For a positive integer n, let G be a graph on n vertices obtained by replacing each $x_i \in V(H)$ with a set X_i of order $\lceil \alpha_i n \rceil$ or $\lfloor \alpha_i n \rfloor$; for all $u \in X_i$, $v \in X_j$, let uv be an edge of G if and only if there exists $e \in E$ such that $\{x_i, x_j\} \subseteq e$, and color uv using the color which appears on e (if there are multiple such edges, choose a color arbitrarily from one such edge). We call G an α-weighted blow-up of H, and if $\alpha_i = \frac{1}{t}$ for all $i \in [t]$, we call G a uniform blow-up of H.

Given a graph G and a positive integer r, let $mc_r(G)$ be the largest integer m such that in every r-edge-coloring of G, there exists a monochromatic component (i.e., a maximal connected subgraph) of order at least m. For the rest of the paper, when we speak of an r-coloring of G, we mean an r-coloring of the edges of G.

Gyárfás [6] proved

$$mc_r(K_n) \geq \frac{n}{r-1}$$

and this is best possible when $(r-1)^2$ divides n and an affine plane of order $r-1$ exists. To see this, let K_n be a uniform blow-up of the affine plane of order $r-1$. Since every pair of distinct points from the affine plane is contained in exactly one edge the r-coloring of K_n is well defined, and since each line of the affine plane has order $r-1$ and there are $(r-1)^2$ points, the size of the largest monochromatic component in K_n is $(r-1)\frac{n}{(r-1)^2} = \frac{n}{r-1}$.

Gyárfás and Sárközy [7] raised the following interesting question: for a graph G on n vertices, how large does the minimum degree of G, denoted $\delta(G)$, need to be so that $mc_r(G) \geq \frac{n}{r-1}$? As noted in [8], the answer is $n-1$ for $r = 2$ because there is a 2-coloring of any non-complete graph on n vertices such that the largest monochromatic component has order at most $n-1$. So it was perhaps surprising that for all $r \geq 3$, they showed there exists $\varepsilon_r > 0$ such that if G is a graph on n vertices with n sufficiently large and $\delta(G) \geq (1-\varepsilon_r)n$, then $mc_r(G) \geq \frac{n}{r-1}$. The bounds on ε_r given in [7] were later improved in [3] as follows: for $r = 3$, $\delta(G) \geq 7n/8$ suffices and for $r \geq 4$, $\delta(G) \geq \left(1 - \frac{1}{3072(r-1)^5}\right)n$ suffices.

Gyárfás and Sárközy [7] also gave the following natural construction whenever an affine plane of order r exists and r^2 divides n. Repeat the construction given above, but instead of an affine plane of order $r-1$, take a uniform blow-up

of an affine plane of order r with one parallel class removed. This gives an r-colored graph on n vertices with minimum degree $1-\frac{r-1}{r^2}n-1$ where the largest monochromatic component has order $\frac{n}{r} < \frac{n}{r-1}$. They conjectured that the bound arising from this construction is tight.

Conjecture 1 (Gyárfás, Sárközy [7]). Let n and $r \geq 3$ be positive integers. If G is a graph on n vertices such that $\delta(G) \geq 1 - \frac{r-1}{r^2} n$, then $mc_r(G) \geq \frac{n}{r-1}$.

Recently, Guggiari and Scott, and independently Rahimi, disproved this conjecture for r = 3. The combination of their results gives the best possible minimum degree condition.

Theorem 2 (Guggiari, Scott [5], Rahimi [11]). Let G be a graph on n vertices. If $\delta(G) \geq \frac{5}{6}n - 1$, then $mc_3(G) \geq \frac{n}{2}$. Moreover, for every n, there exists a graph G on n vertices with $\delta(G) = \frac{5}{6}n - 2$ such that $mc_3(G) < \frac{n}{2}$.

Note that the 3-colorings of graphs with $\delta(G) = \frac{5}{6}n - 2$ given by Guggiari and Scott and Rahimi have largest monochromatic components of order just under $\frac{n}{2}$. This is in contrast to the example of Gyárfás and Sárközy above, where the largest monochromatic components have order $\frac{n}{3}$.

The purpose of this note is to generalize the lower bound construction of Guggiari and Scott and Rahimi which disproves Conjecture 1 whenever an affine plane of order r exists. (We note that Guggiari and Scott independently generalized this construction in a later version of [5].)

Theorem 3. Let n and r be integers such that $r \geq 3$ and $n \geq r(r-1)((r-1)(r-2)+1)$. If $(r^2 - r) \mid n$ and an affine plane of order r exists, then there exists a graph G on n vertices with

$$\delta(G) = 1 - \frac{r-2}{r^2-r} n - 2 = 1 - \frac{r-1}{r^2} + \frac{1}{r^2(r-1)} n - 2$$

such that $mc_r(G) < \frac{n}{r-1}$.

The construction is based on a blow-up of the following hypergraph H_r which is derived from an affine plane of order r.

Definition 4 H_r. Let $r \geq 3$ such that an affine plane of order r exists. Let $G_r = (V, L)$ be an affine plane of order r. Let $\{L_1, \ldots, L_{r+1}\}$ be the partition of L into parallel classes. Label the vertices of G_r as $v_{i,j}$ with $i, j \in [r]$ so that $L_1 = \{\{v_{i,1}, \ldots, v_{i,r}\} : i \in [r]\}$ and $L_{r+1} = \{\{v_{1,i}, \ldots, v_{r,i}\} : i \in [r]\}$ (in Figure 1, L_1 is represented by the rows and L_{r+1} by the columns). Let $S = \{v_{r,i} : i \in [r-1]\} \cup \{v_{r-1,r}\}$.

Let H_r be the hypergraph obtained from G_r by deleting the lines from L_{r+1} and the vertices of S from each of the remaining edges; i.e., let $H_r = (V \setminus S, E)$ where $E = \{e \setminus S : e \in L\}$.

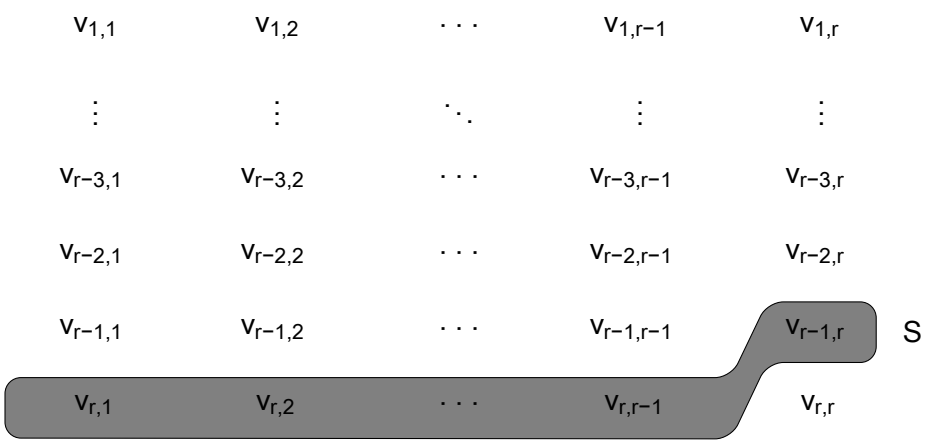

Figure 1. The hypergraph H_r.

Given a hypergraph $H = (V, E)$, the rank of H, denoted $r(H)$, is $\max\{|e| : e \in E\}$ and the proportional rank of H is $\frac{r(H)}{|V|}$. The edge chromatic number of H is the minimum number of colors needed to color the edges of H so that each color class forms a matching. Given a vertex $v \in V$, let $N[v] = \{u : \exists e \in E, \{u, v\} \subseteq e\}$; in other words, $N[v]$ is the set of all vertices (including v) which are contained in an edge with v. Let $\delta^*(H) = \min\{|N[v]| : v \in V\}$.

Note the following properties of $H_r = (V, E)$.

(P1) the edge chromatic number of H_r is r,

(P2) the proportional rank of H_r is $\frac{r}{r^2-r} = \frac{1}{r-1}$,

(P3) $\delta^*(H_r) = |V| - (r-2) = 1 - \frac{r-2}{r^2-r}|V|$.

Roughly speaking, we prove Theorem 3 by taking a uniform blow-up of H_r (which has monochromatic components of order $\frac{n}{r-1}$) and then slightly "perturbing" the sizes of the blown-up sets so that all the monochromatic components have order less than $\frac{n}{r-1}$. This raises the more general question of when such a perturbation is possible, which we address in Section 2.

As is elaborated in Section 3.1, the choice of vertices S to delete in the definition of H_r is to ensure that a uniform blow-up of H_r is "perturbable." As an example of a hypergraph which is not "perturbable," let H_3' be obtained from an affine plane of order 3, by deleting one parallel class and deleting the vertices from one of the remaining edges (say $v_{3,1}, v_{3,2}, v_{3,3}$ as in Figure 1). The edge chromatic number of H_3' is 3, the proportional rank is 1/2, and $\delta^*(H_3') = 5 = 1 - \frac{1}{6}|V|$. By taking a uniform blow-up of H_3' we obtain a 3-colored graph G with $\delta(G) = \frac{5n}{6} - 1$, and every monochromatic component has order at most $n/2$. However, no matter how we change the sizes of the blown-up sets, one of the monochromatic components will have order at least $n/2$.

2. Perturbable Hypergraphs

It is possible to skip directly to Section 3.2 to see the proof of Theorem 3; however, to understand where the construction comes from we need to take a slight detour.

The standard simplex of $\mathbb{R}^n$ is the set of vectors $(w_1, \ldots, w_n)$ such that $w_i \geq 0$ for all $i \in [n]$ and $\sum_{i=1}^{n} w_i = 1$. A weight assignment on a hypergraph $H = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is a function $w : V \to R$ such that $(w(v_1), \ldots, w(v_n))$ is in the standard simplex of R^n. For all $S \subseteq V$ let the weight of S, denoted $w(S)$, be $\sum_{v \in S} w(v)$. We say that $w : V \to R$ given by $w(v) = \frac{1}{|V|}$ for all $v \in V$ is the uniform weight assignment.

Definition 5 (Perturbation, perturbable). A perturbation on a hypergraph $H = (V, E)$ is a function $p : V \to R$ such that $\sum_{i=1}^{n} p(v_i) = 0$ and for all $e \in E$, $p(e) = \sum_{v \in e} p(v) < 0$. We say H is perturbable if a perturbation on H exists.

Observe that if w is a positive weight assignment on H (meaning $w(v) > 0$ for all $v \in V$) and p is a perturbation on H, then $w + \varepsilon p$ is also a weight assignment on H for sufficiently small $\varepsilon > 0$ (say $\varepsilon < \min\left\{\frac{w(v)}{p(v)} : v \in V, p(v) \neq 0\right\}$). Since $p(e) < 0$ for every $e \in E$, the weight assignment $w + \varepsilon p$ is strictly smaller than w on every edge of H. Thus, if a perturbation on H exists, then we can alter any weight assignment by at most ε at each vertex (for ε sufficiently small) and strictly decrease the weights on the edges.

Theorem 6 gives an equivalent condition for the existence of a perturbation, but first we must recall the following definitions. Given a hypergraph $H = (V, E)$, a fractional matching is a function $m : E \to [0, 1]$ such that for all $v \in V$, $\sum_{e \ni v} m(e) \leq 1$, and a fractional vertex cover is a function $t : V \to [0, 1]$ such that for all $e \in E$, $\sum_{v \in e} t(v) \geq 1$. A fractional matching is called perfect if we have equality for all $v \in V$. We let

$$\nu^*(H) = \max\left\{\sum_{e \in E} m(e) : m \text{ is a fractional matching on } H\right\}$$

and

$$\tau^*(H) = \min\left\{\sum_{v \in V} t(v) : t \text{ is a fractional vertex cover on } H\right\}.$$

It is well known consequence of the duality theorem in linear programming that $\tau^*(H) = \nu^*(H)$ for all hypergraphs. When it is clear from context we just write τ^* and ν^* for $\tau^*(H)$ and $\nu^*(H)$ respectively. Note that if H is k-uniform and has a perfect fractional matching, then $\nu^*(H) = \frac{n}{k}$.

Theorem 6. Let $H = (V, E)$ be a hypergraph. H is perturbable if and only if H does not have a perfect fractionalmatching.

Proof. Let $n := |V|$ and $e := |E|$ and let A be the n-by-e incidence matrix of H (with rows indexed by vertices and columns by edges). Let 1 be the n-dimensional vector of all 1's. Note that in this language, a perfect fractional matching m is a solution to the system $Am = 1$, $m \geq 0$, and a perturbation p is a solution to the system $A^T p < 0$, $1^T p = 0$.

Recall Farkas' Lemma (see [9]), which states for an n-by-e matrix A and n-dimensional vector b, there is no $m \geq 0$ such that $Am = b$ if and only if there exists w such that $A^T w \leq 0$ and $b^T w > 0$.

We claim that for the given A, the solvability of $A^T w \leq 0$, $1^T w > 0$ is equivalent to the solvability of $A^T p < 0$, $1^T p = 0$. So by Farkas' lemma, the result will follow by establishing this claim.

First suppose there exists w such that $A^T w \leq 0$ and $1^T w > 0$. Letting $p = w - \frac{1^T w}{n} 1$, we have $1^T p = 0$ and $p < w$. Since A has only nonnegative entries, and it has at least one positive entry in each column, we have $A^T p < A^T w \leq 0$, so p is a perturbation for H.

For the other direction, suppose there exists p such that $A^T p < 0$ and $1^T p = 0$. Let $\alpha > 0$ be the absolute value of the largest entry of $A^T p$ (smallest in absolute value), and let $w = p + \frac{\alpha}{n} 1$. Then $1^T w = \alpha > 0$ and the largest entry of $A^T w = A^T p + \frac{\alpha}{n} A^T 1$ is $-\alpha + \frac{\alpha}{n} n \leq 0$, since the largest entry of $A^T 1$ is the rank of H which is at most n. ■

3. When an Affine Plane of Order r Exists

Given a hypergraph $H = (V, E)$ and a weight assignment w, the top-level of H, denoted $\overline{H}$, is the hypergraph (V, E') where $E' \subseteq E$ is the set of edges of maximum weight.

3.1. Rough construction

Let $H_r = (V, E)$ be the hypergraph from Definition 4. We first show that under the uniform weight assignment, the top-level of $H_r = (V, E)$ is perturbable. This together with properties (P1), (P2), and (P3), imply that for all sufficiently large n we can use H_r to define a graph G on n vertices with $\delta(G) \geq 1 - \frac{r-2}{r^2-r} n - O(1)$ and $mc_r(G) < \frac{n}{r-1}$.

Observation 7. Let $r \geq 3$ be an integer such that an affine plane of order r exists. Under the uniform weight assignment, the top-level of $H_r = (V, E)$ is perturbable.

Proof. Given the uniform weight assignment on H_r, let $\overline{H_r} = (V, E')$ be the top-level of H_r (which is just the edges of order r in this case). Let $t : V \to R$ given by $t(v) = \frac{1}{r-1}$ if $v \in \{v_{r,r}\} \cup \{v_{i,j} : i \in [r-2], j \in [r-1]\}$ and $t(v) = 0$

otherwise (see Figure 2). We first claim that t is a fractional vertex cover of $\overline{H_r}$. Indeed, every edge of $\dot{E}$ either comes from L_1 or contains $v_{r,r}$, and consequently intersects $\{v_{r-1,i} : i \in [r-1]\} \cup \{v_{i,r} : i \in [r-2]\}$ in at most one vertex. So we have

$$v^* = \tau^* \leq \sum_{v \in V} t(v) = (r-1)(r-2)\left(\frac{1}{r-1} + \frac{1}{r-1}\right) = r - 2 + \frac{1}{r-1} < r - 1 = \frac{|V|}{r},$$

and thus $\overline{H_r}$ does not have a perfect fractional matching. By Theorem 6, $\overline{H_r}$ is perturbable. ■

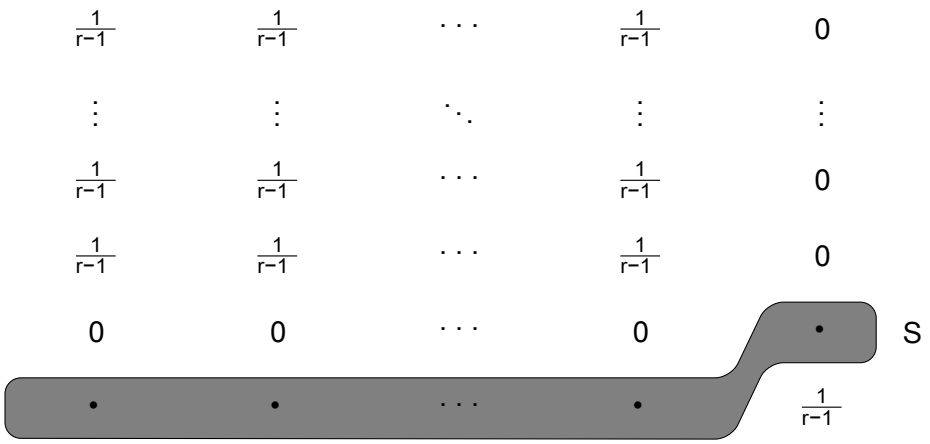

Figure 2. The fractional vertex cover of $\overline{H_r}$.

One may wonder if other choices of S in the definition of H_r would yield a perturbable hypergraph satisfying properties (P1), (P2), and (P3). An exhaustive search shows that there are no other choices of S (up to isomorphism) for r = 3 and r = 4, but there are other choices for say r = 5. While it would be interesting to characterize the possible choices of S, doing so would not improve the given construction.

3.2. Fine tuning

Theorem 8. Let n, r, c be integers such that $r \geq 3$, $c \geq 1$, and $n \geq r(r-1)((r-1)(r-2)+1)c$. If $(r^2 - r) \mid n$ and an affine plane of order r exists, then there exists a graph G on n vertices with $\delta(G) = 1 - \frac{r-2}{r^2-r} n - c - 1$ such that $mc_r(G) \leq \frac{n}{r-1} - c$.

Note that the main case of interest is when c = 1, but phrasing the result in general as we do shows that by lowering the minimum degree further, one can

further decrease the size of the largest monochromatic component. Also note that our construction only addresses the case $(r^2 - r) \mid n$ for simplicity. It is possible that in the case when $r^2 - r$ does not divide n, by slightly modifying this construction (as was done in [5] for the case $r = 3$), one can construct a graph G with $\delta(G) = \left(1 - \frac{r-2}{r^2-r}\right) n - 2$ such that $mc_r(G) \leq \frac{n}{r-1} - 1$.

Proof. Let $H_r = (V, E)$ be the hypergraph from Definition 4 and let G be a uniform blow-up of H_r where $v \in V$ becomes X_v in G (with $|X_v| = \frac{n}{r^2-r}$).

Let $A = \{v_{r-1,i} : i \in [r-1]\} \cup \{v_{r-2,r}\}$. We now adjust the size of each X_v as follows

$$|X_v| := \begin{cases} \frac{n}{r^2-r} - c, & \text{if } v \in V(H_r) \setminus A, \\ \frac{n}{r^2-r} + (r-2)c, & \text{if } v \in A. \end{cases}$$

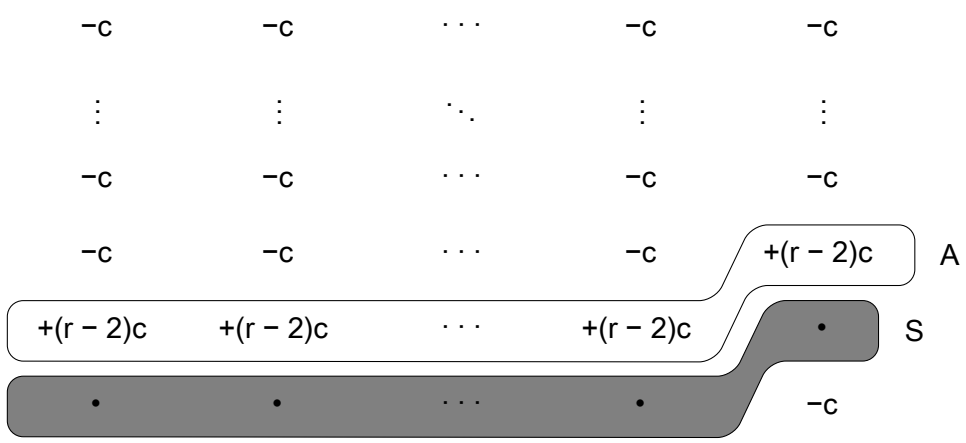

Figure 3. The adjustment of the sizes of the sets in a uniform blow-up of H_r. Each number corresponds to a vertex of H_r, with the rows corresponding to L_1 and the columns corresponding to L_{r+1}. The number at a vertex v is the amount in which we adjusted the size of X_v, i.e., $|X_v| - \frac{n}{r^2-r}$.

First note that $\sum_{v \in V} |X_v| = n$ (since each column sums to 0 in Figure 3).

Now we check the minimum degree condition. Let $v \in V(H_r)$, let $\ell_v \in L_{r+1}$ such that $v \in \ell_v$, and let $u \in X_v$. We have (see Figure 3 in which each vertex is adjacent to everything except the distinct members of its own column)

$$d(u) = (n-1) - \sum_{w \in \ell_v \setminus \{v\}} |X_w|$$

$$= \begin{cases} (n-1) - (r-3)\left(\frac{n}{r^2-r} - c\right) - \left(\frac{n}{r^2-r} + (r-2)c\right), & \text{if } v \in V(H_r) \setminus A, \\ (n-1) - (r-2)\left(\frac{n}{r^2-r} - c\right), & \text{if } v \in A, \end{cases}$$

and thus $\delta(G) = (n-1) - (r-2)\left(\frac{n}{r^2-r} - c\right) = \left(1 - \frac{r-2}{r^2-r}\right) n - c - 1$.

Finally we check that every monochromatic component of G, each of which corresponds to an edge ℓ from H_r, has order at most $\frac{n}{r-1} - c$. Since $c \le \frac{n}{r(r-1)((r-1)(r-2)+1)}$, we have $(r-1)\left(\frac{n}{r^2-r}\right) + (r-2)c \le \frac{n}{r-1} - c$ and thus we need only consider the edges ℓ of H_r of order r; that is, when $\ell \cap S = \emptyset$. Since every edge from H_r of order r intersects A in at most one vertex, the order of the largest monochromatic component in G will be at most

$$(r-1)\left(\frac{n}{r^2-r} - c\right) + \frac{n}{r^2-r} + (r-2)c = \frac{n}{r-1} - c.$$

■

4. When an Affine Plane of Order r Does Not Exist

It is known that there is no affine plane of order 6, so r = 6 is the first case for which the construction of the previous section does not apply. An example of a graph G with $\delta(G) = \frac{5n}{7} - 1$ such that $mc_6(G) < \frac{n}{5}$ (in fact, $mc_6(G) \le \frac{n}{7} < \frac{n}{5}$) is a uniform blow-up of an affine plane of order 7 with two parallel classes removed.

Problem 9. Construct an example of a graph G with large minimum degree such that $mc_6(G) < \frac{n}{5}$. In particular, for some $\alpha > 0$ and all n, construct a graph G on n vertices with $\delta(G) \ge \left(\frac{5}{7} + \alpha\right) n$ such that $mc_6(G) < \frac{n}{5}$.

In light of Section 2, it would suffice to construct a hypergraph H = (V, E) with edge chromatic number 6, proportional rank at most $\frac{1}{5}$, $\delta^*(H) > \frac{5}{7}|V|$ such that if the proportional rank of H is equal to $\frac{1}{5}$, then the top-level of H with respect to the uniform weight assignment (the edges of maximum rank) has no perfect fractional matching.

In general, when an affine plane of order r does not exist, trying to produce an example of an r-colored graph G with large minimum degree for which $mc_r(G) < \frac{n}{r-1}$ leads us back to the original problem for complete graphs. The purpose of this section is mostly to collect what is known in one place and make a few observations. These observations have consequences for the original problem for complete graphs and may be useful for extending our construction in the case when an affine plane of order r does not exist.

Recall that Gy´arf´as [6] proved $mc_r(K_n) \ge \frac{n}{r-1}$ and this is best possible when $(r-1)^2$ divides n and an affine plane of order r − 1 exists. For all r such that affine plane of order r − 1 does not exist, the problem of determining $mc_r(K_n)$ (even asymptotically) is still open. A hypergraph H is r-partite if V (H) can be partitioned into r parts so that every edge intersects every part in at most one vertex; H is intersecting if every pair of edges in E(H) has non-empty intersection; and we let v_r^* be the largest fractional matching over all r-partite intersecting hypergraphs. A result of F¨uredi [4, Theorem 3.3] implies

that $\frac{1}{n}mc_r(K_n) \to \frac{1}{v_r^*}$ as $n \to \infty$; thus determining $mc_r(K_n)$ asymptotically is equivalent to determining the value of v_r^*. As for the value of v_r^*, Füredi proved [4, Theorem 2.1] that $v_r^* \leq r-1-\frac{1}{r-1}$ unless a truncated projective plane of order $r-1$ exists (equivalently, an affine plane of order $r-1$ exists), in which case $v_r^* = r-1$. Combining these two results shows that one can improve Gyárfás' lower bound on $mc_r(K_n)$ whenever there is no affine plane of order $r-1$. (Note that the upper bound comes from the construction mentioned in the introduction.)

Theorem 10 (Füredi [4]). Let $r \geq 3$ be an integer, let q be the largest integer at most $r-1$ such that there exists an affine plane of order q, and let $n \geq q^2$ be an integer. If an affine plane of order $r-1$ does not exist, then $\frac{n}{r-1-\frac{1}{r-1}} \leq mc_r(K_n) \leq \frac{n}{q}$.

Since an affine plane is a hypergraph in which every pair of distinct vertices is contained in exactly one edge and the edges of the hypergraph can be decomposed into perfect matchings (and thus has the smallest possible edge chromatic number), a natural place to look for examples which improve the upper bound (when an affine plane of order $r-1$ does not exist) are resolvable balanced incomplete block designs.

A $(v, k, 1)$-resolvable balanced incomplete block design, a $(v, k, 1)$-RBIBD for short, is a k-uniform hypergraph H on v vertices such that each pair of vertices is contained in exactly one edge and the edges of H can be decomposed into $\frac{\binom{v}{2}/\binom{k}{2}}{v/k} = \frac{v-1}{k-1}$ perfect matchings. A necessary condition for the existence of a $(v, k, 1)$-RBIBD is that $v \equiv k \bmod k(k-1)$. Ray-Chaudhuri and Wilson [12] proved that for all $k \geq 3$ there exists a constant $C(k)$ such that if $v \geq C(k)$ and $v \equiv k \bmod k(k-1)$, then a $(v, k, 1)$-RBIBD exists. Later Chang [2] proved that $C(k) = \exp(\exp(k^{12k^2}))$ suffices. There are some other sporadic results for small k (see [1]), but in general, the existence of $(v, k, 1)$-RBIBDs is open.

Note that an affine plane of order k is a $(k^2, k, 1)$-RBIBD and by the necessary condition above, k^2 is the smallest v for which a non-trivial $(v, k, 1)$-RBIBD exists. Because of this, we parameterize v in terms of k and a non-negative integer t, and speak of $(k^2 + tk(k-1), k, 1)$-RBIBDs.

Given a hypergraph H, let $v(H)$ be the number of vertices in H and recall that $r(H)$ is the rank of H.

Fact 11. Let $k \geq 2$, $t \geq 0$, and $n \geq k^2 + tk(k-1)$ be integers such that $k^2 + tk(k-1)$ divides n. If there exists a $(k^2 + tk(k-1), k, 1)$-RBIBD, then there is a $((t+1)k+1)$-coloring of K_n such that every monochromatic component has order at most $\frac{n}{(t+1)k-t}$.

In particular, when $t = 0$ this means that if there exists an affine plane of order k, then there exists $(k+1)$-coloring of K_n such that every monochromatic component has order at most $\frac{n}{k}$.

Proof. This follows from the fact that the proportional rank of a $(k^2 + tk(k-1), k, 1)$-RBIBD is

$$\frac{k}{k^2 + tk(k-1)} = \frac{1}{(t+1)k - t}$$

and a $(k^2 + tk(k-1), k, 1)$-RBIBD has

$$\frac{\frac{k^2+tk(k-1)}{2} \;/\; \frac{k}{2}}{(k^2 + tk(k-1))/k} = \frac{k^2 + tk(k-1) - 1}{k-1} = (t+1)k + 1$$

parallel classes.Taking a uniform blow-up gives the desired conclusion. ■

The point of Fact 11 is that, for instance when r = 23, Theorem 10 implies that $\frac{n}{22-1/22} \le mc_{23}(K_n) \le \frac{n}{19}$ (if no affine plane of order 20, 21, or 22 exists). But by Fact 11, if a (231, 11, 1)-RBIBD exists (k = 11, t = 1), then $\frac{n}{22-1/22} \le mc_{23}(K_n) \le \frac{n}{21}$.

Also note that for r = 7, Theorem 10 implies that $\frac{6n}{35} \le mc_7(K_n) \le \frac{7n}{35} = \frac{n}{5}$. It is well known that a (15, 3, 1)-RBIBD exists; this is the original Kirkman schoolgirls problem (in fact, four out of the 80 Steiner triple systems on 15 vertices are resolvable – see [10]). So Fact 11 implies that there are at least four other examples which show that $mc_7(K_n) \le \frac{n}{5}$. Note that in light of the discussion before Theorem 10, improving the upper bound on $mc_7(K_n)$ is equivalent to finding a 7-partite intersecting hypergraph with fractional matching number greater than 5.

5. Conclusion

The main open problem is to prove an analogue of Theorem 2 for r ≥ 4 colors (the lack of additional evidence prevents us from calling it a conjecture). Note that the following is true for r = 2, 3.

Problem 12. Let n and r ≥ 2 be positive integers. Prove that if G is a graph on n vertices with $\delta(G) \ge 1 - \frac{r-2}{r^2-r} n - 1$ and an affine plane of order r exists, then $mc_r(G) \ge \frac{n}{r-1}$.

When r = 7, Theorem 3 says that when 42|n, there exists a graph G on n vertices with $\delta(G) = 1 - \frac{5}{42} n - 2$, such that $mc_7(G) < \frac{n}{6}$. However, Theorem 10 says $mc_7(K_n) \ge \frac{6n}{35} > \frac{n}{6}$ (and it is even conceivable that $mc_7(K_n) = \frac{n}{5}$).

So we can ask a modified version of the original question of Gyárfás and Sárközy which is different whenever an affine plane of order r − 1 does not exist.

Problem 13. If G is a graph on n vertices, how large does the minimum degree of G need to be so that $mc_r(G) = mc_r(K_n)$?

References

[1] R.J.R. Abel, G. Ge and J. Yin, Resolvable and Near-Resolvable Designs, in:Handbook of Combinatorial Designs, C. Colbourn and J. Dinitz, (Ed(s)), (Chapmann & Hall/CRC, Boca Raton, 1996).

[2] Y. Chang, The existence of resolvable BIBD with λ = 1, Acta Math. Appl. Sin. 16 (2000) 373–385.
doi:10.1007/BF02671127

[3] L. DeBiasio, R.A. Krueger and G.N. S´ark¨ozy, Large monochromatic components in multicolored bipartite graphs, J. Graph Theory 94 (2020) 117–130.
doi:10.1002/jgt.22510

[4] Z. Füredi, Covering the complete graph by partitions, Discrete Math. 75 (1989) 217–226.
doi:10.1016/0012-365X(89)90088-5

[5] H. Guggiari and A. Scott, Monochromatic components in edge-coloured graphs with large minimum degree.
arXiv:1909.09178v1 (2019)

[6] A. Gy´arf´as, Partition coverings and blocking sets in hypergraphs, Commun. Comput. Autom. Inst. Hungar. Acad. Sci. 71 (1977), in Hungarian.

[7] A. Gy´arf´as and G.N. S´ark¨ozy, Large monochromatic components in edge colored graphs with a minimum degree condition, Electron. J. Combin. 24 (2017) #P3.54.
doi:10.37236/7049

[8] A. Gy´arf´as and G.N. S´ark¨ozy, Star versus two stripes Ramsey numbers and a conjecture of Schelp, Combin. Probab. Comput. 21 (2012) 179–186.
doi:10.1017/S0963548311000599

[9] L. Lov´asz and M.D. Plummer, Matching Theory (AMS Chelsea Publishing, Providence, 2009).
doi:10.1090/chel/367

[10] R.A. Mathon, K.T. Phelps and A. Rosa, Small Steiner triple systems and their properties, Ars Combin. 15 (1983) 3–110.

[11] Z. Rahimi, Large monochromatic components in 3-colored non-complete graphs, J. Combin. Theory Ser. A 175 (2020) 105256.
doi:10.1016/j.jcta.2020.105256

[12] D.K. Ray-Chaudhuri and R.M. Wilson, The existence of resolvable block designs, in: A Survey of Combinatorial Theory (North-Holland, 1973) 361–375.
doi:10.1016/B978-0-7204-2262-7.50035-1

Received 24 July 2020
Revised 11 December 2020
Accepted 11 December 2020