SIAM J. MATH. ANAL.
Vol. 53, No. 6, pp. 6993–7023

ON THE FEASIBILITY OF EXTRAPOLATION OF THE COMPLEX ELECTROMAGNETIC PERMITTIVITY FUNCTION USING KRAMERS–KRONIG RELATIONS*

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Abstract. We study the degree of reliability of extrapolation of complex electromagnetic permittivity functions based on their analyticity properties. Given two analytic functions, representing extrapolants of the same experimental data, we examine how much they can differ at an extrapolation point outside of the experimentally accessible frequency band. We give a sharp upper bound on the worst-case extrapolation error in terms of a solution of an integral equation of Fredholm type. We conjecture and give numerical evidence that this bound exhibits a power law precision deterioration as one moves further away from the frequency band containing measurement data.

Key words. extrapolation, quantification, optimal error estimate, complex electromagnetic permittivity, least squares problem, Herglotz functions

AMS subject classifications. 30A10, 30A99, 30C20, 45B05, 49K21, 49J40, 42B30, 30H15

DOI. 10.1137/20M1369427

1. Introduction.

Properties of linear, time-invariant, causal systems are characterized by functions analytic in a complex half-plane. Examples include transfer functions of digital filters [25], complex impedance and admittance functions of electrical circuits [5], and complex magnetic permeability and complex dielectric permittivity functions [33, 20]. Arising from the world of real-valued fields, these functions also possess specific symmetries. The underlying mathematical structure is the Fourier (or Laplace) transforms of real-valued functions that vanish on negative semiaxis. More generally, the analyticity arises from the analyticity of resolvents of linear operators, while their symmetries reflect that these operators are very often real and self-adjoint.

In a typical situation we can measure the values of such analytic functions on a compact subset of the boundary of their half-plane of analyticity. The real and imaginary parts of such a function are not independent but are Hilbert transforms of one another. In the context of the complex dielectric permittivity this fact is expressed by the Kramers–Kronig relations [15, 31, 44, 29]. It is therefore tempting to use these relations in order to reconstruct the analytic functions from their measured values. Unfortunately, such a reconstruction problem is ill-posed (e.g., [37]), and one needs to place additional constraints on the set of admissible analytic functions for the extrapolation problem to be mathematically well-posed.

In this paper we propose a physically natural regularization that implies that the underlying analytic functions can be analytically continued into a larger complex half-plane. In that case, the idea is to exploit the fact that complex analytic functions possess a large degree of rigidity, being uniquely determined by values at any infinite set of points in any finite interval. This rigidity also implies that even very small measurement errors will produce data *mathematically* inconsistent with values of an

*Received by the editors September 25, 2020; accepted for publication (in revised form) September 28, 2021; published electronically December 16, 2021.
https://doi.org/10.1137/20M1369427
Funding: This work was supported by National Science Foundation grant DMS-2005538.
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analytic function. In such cases the least squares approach [14, 13, 7, 8] that treats all data points equally is the most natural one. In the first part of the paper we prove that the least squares problem has a unique solution that yields a mathematically stable extrapolant. We show that the minimizer must be a rational function and derive the necessary and sufficient conditions for its optimality.

Recent work [45, 16, 27, 26] shows that, surprisingly, the space of analytic functions is also "flexible" in the sense that the data can often be matched up to a given precision by two physically admissible functions that are very different away from the interval, where the data is available. The second part of the paper quantifies this phenomenon by giving an optimal upper bound on the possible discrepancy between any two approximate extrapolants. This is done by first reformulating the problem as a question about analytic functions, which we have already studied in [27, 26], but without the symmetry constraints. Incorporating symmetry into the methods of [27] is nontrivial, and we address this question next. Our conclusion is that the symmetry has a virtually negligible regularizing effect, as far as the optimal upper bound on the extrapolation uncertainty is concerned.

2. Preliminaries.

When an electromagnetic wave passes through material, the incident electric field $\boldsymbol{E}(\boldsymbol{x},t)$ interacts with charge carriers inside the matter. We assume that the induced polarization field $\boldsymbol{P}(\boldsymbol{x},t)$ depends on the incident electric field linearly and locally. This is expressed by the constitutive relation

$$\boldsymbol{P}(\boldsymbol{x},t)=\int_0^{+\infty}\boldsymbol{E}(\boldsymbol{x},t-s)a(s)ds, \tag{2.1}$$

indicating that the polarization field depends only on the past values of $\boldsymbol{E}(\boldsymbol{x},t)$. The function $a(t)$ is called the impulse response or a memory kernel, which is assumed to decay exponentially. Its decay rate, $a(t)\sim e^{-t/\tau_0}$, $t\to\infty$, indicates how fast the system "forgets" the past values of the incident field. The parameter $\tau_0>0$ is called the relaxation time, which can be measured for many materials.

Let

$$a_0(t)=\begin{cases}a(t), & t\ge 0,\\ 0, & t<0.\end{cases}$$

Then we can extend the integral in (2.1) to the entire real line and apply the Fourier transform to convert the convolution into a product:

$$\widehat{\boldsymbol{P}}(\boldsymbol{x},\omega)=\widehat{a}_0(\omega)\widehat{\boldsymbol{E}}(\boldsymbol{x},\omega),$$

where

$$\widehat{f}(\omega)=\int_{\mathbb{R}}f(x)e^{i\omega x}dx$$

is the Fourier transform. In physics, the function $\varepsilon(\omega)=\varepsilon_0+\widehat{a}_0(\omega)$ is called the complex dielectric permittivity of the material, where ε_0 is the dielectric permittivity of the vacuum. Mathematically, it is more convenient to study $\widehat{a}_0(\omega)$ rather than $\varepsilon(\omega)$. From now on, we will denote

$$f(\omega)=\widehat{a}_0(\omega)$$

and refer to it as the complex electromagnetic permittivity, in a convenient abuse of terminology. Let us recall the well-known analytic properties of isotropic complex electromagnetic permittivity as a function of frequency ω of the incident electromagnetic wave [33, 20]:

(a) $\overline{f(\omega)} = f(-\overline{\omega})$;
(b) $f(\omega)$ is analytic in the complex upper half-plane $\mathbb{H}_+ = \{\omega \in \mathbb{C} : \mathfrak{Im}\,\omega > 0\}$;
(c) $\mathfrak{Im}\,f(\omega) > 0$ for ω in the first quadrant $\mathfrak{Re}(\omega) > 0$, $\mathfrak{Im}(\omega) > 0$;
(d) $f(\omega) = -A\omega^{-2} + O(\omega^{-3})$, $A > 0$ as $\omega \to \infty$.

Property (a) expresses the fact that physical fields are real. Property (b) is the consequence of the causality principle, i.e., independence of $\boldsymbol{P}(\boldsymbol{x}, t)$ of the future values of $E(\boldsymbol{x}, \tau)$, $\tau > t$. Property (c) comes from the fact that the electromagnetic energy gets absorbed by the material as the electromagnetic wave passes through. Property (d) is called the plasma limit, where at very high frequencies the electrons in the medium may be regarded as free. Complex analytic functions with properties (a)–(d), and their variants, are ubiquitous in physics. The complex impedance of electrical circuits as a function of frequency has similar properties [23, 5, 10]. Yet another example is the dependence of effective moduli of composites on the moduli of its constituents [4, 38, 39]. These functions appear in areas as diverse as optimal design problems [34] and nuclear physics [36, 35, 6]. Typically[1] only the values of such a function on a real line can be measured. In the case of complex electromagnetic permittivity the measurements are usually made either on a finite interval or at a discrete set of frequencies. However, the requirements (a)–(d) do not place any analyticity requirements on $f(\omega)$, when ω is real (see [2, 24] for the boundary behavior of such functions). For example, the function

$$f(\omega) = \frac{1}{\omega_0^2 - \omega^2}, \quad \omega_0 > 0,$$

satisfies properties (a)–(d), but blows up at the frequency $\omega_0 > 0$. We exclude such examples by assuming that the memory kernel $a(t)$ decays exponentially with relaxation time $\tau_0 > 0$. In this case $f(\omega)$ will have an analytic extension into the larger half-plane

$$\mathbb{H}_h = \{\omega \in \mathbb{C} : \mathfrak{Im}\,\omega > -h\}, \tag{2.2}$$

where $h = 1/\tau_0 > 0$ (cf. [43]). In general, the analytic continuation of $f(\omega)$ need not have positive imaginary part when $\mathfrak{Im}(\omega) > -h$ and $\mathfrak{Re}(\omega) > 0$. For example, $f(\omega) = -\frac{\omega+i}{(\omega+3i)^3}$ satisfies conditions (a)–(d) and is analytic in $\mathbb{H}_3$, but $\mathfrak{Im}\,f(x - i\epsilon)$ takes negative values for any $\epsilon \in (0, 3)$ for some $x > 0$. We therefore make an additional regularizing assumption that positivity property (c) continues to hold in the larger half-plane $\mathbb{H}_h$. In fact, under the additional assumption that the Elmore delay [18] is positive, i.e., $-if'(0) > 0$, the positivity condition can be guaranteed in some possibly smaller half-plane $\mathbb{H}_{h'}$, $0 < h' \le h$ (see the appendix). Thus, the class of all physically admissible complex dielectric permittivity functions is narrowed in a natural way to the class $\mathcal{K}_h$, defined as follows.

Definition 2.1. *A complex analytic function $f : \mathbb{H}_h \to \mathbb{C}$ belongs to the class $\mathcal{K}_h$ if it has the following list of physically justified properties:*

(S) *symmetry:* $\overline{f(\omega)} = f(-\overline{\omega})$;
(P) *passivity:* $\mathfrak{Im}(f(\omega)) > 0$, *when* $\mathfrak{Im}(\omega) > -h$, $\quad \mathfrak{Re}(\omega) > 0$;
(L) *plasma limit:* $f(\omega) = -A\omega^{-2} + O(\omega^{-3})$, $A > 0$ *as* $\omega \to \infty$.

[1]In the context of viscoelastic composites, measurements corresponding to values of $f(\omega)$ in the upper half-plane are also possible.

Functions in the set $\mathcal{K}_h$ are closely related to an important class of functions called Stieltjes functions.

DEFINITION 2.2. *A nonconstant function analytic in the complex upper half-plane is said to be of Stieltjes class* $\mathfrak{S}$ *if its imaginary part is positive, and it is analytic on the negative real axis, where it takes real and nonnegative values. Such functions, together with all nonnegative constant functions, form the Stieltjes class* $\mathfrak{S}$.

It is well known that a Stieltjes function $F(z)$ is uniquely determined by a constant $\rho \geq 0$ and a Borel-regular positive measure σ by the representation

$$F(z) = \rho + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - z}, \qquad \int_0^\infty \frac{d\sigma(\lambda)}{\lambda + 1} < +\infty. \tag{2.3}$$

The measure σ is often referred to as the *spectral measure* [12, 39]. Let us show that function $f \in \mathcal{K}_h$ can be represented by

$$f(\omega) = F((\omega + ih)^2), \quad F \in \mathfrak{S}, \quad \rho = 0, \quad \int_0^\infty d\sigma(\lambda) = A < +\infty, \tag{2.4}$$

where σ is the spectral measure for $F(z)$.

For any $f \in \mathcal{K}_h$ consider the function $g(\zeta) = f(\zeta - ih)$ which is analytic in $\mathbb{H}_+$, $\overline{g(\zeta)} = g(-\overline{\zeta})$, $\mathfrak{Im} g > 0$ in the first quadrant, and $g(\zeta) \sim -A\zeta^{-2}$ as $\zeta \to \infty$ for some $A > 0$.

Unfolding the first quadrant in the ζ-plane into the upper half-plane in the z-plane via $z = \zeta^2$ we obtain a function $F(z) = g(\sqrt{z})$, which is analytic in $\mathbb{H}_+$ and has a positive imaginary part there. The symmetry of g implies that it is real on $i\mathbb{R}_{>0}$, but then F is real on $\mathbb{R}_{<0}$. Clearly, analyticity of g on $i\mathbb{R}_{>0}$ implies that of F on $\mathbb{R}_{<0}$. The plasma limit assumption implies that $F(-x) \geq 0$ for x large enough, which is enough to conclude that F is a Stieltjes function (see the proof of [32, Theorem A.4]). Thus, F admits the representation (2.3). But then the asymptotic relation $F(z) \sim -Az^{-1}$ as $z \to \infty$ implies that $\rho = 0$ and $\int_0^\infty d\sigma(\lambda) = A < \infty$. Thus, $f(\omega) = g(\omega + ih) = F((\omega + ih)^2)$. Conversely, if f is given by (2.4), then it is straightforward to check that it satisfies all the required properties of class $\mathcal{K}_h$.

3. Main results.

Let us assume that the experimentally measured data $f_{\exp}(\omega)$ is known on a band of frequencies $\Gamma = [0, B]$. The unavoidable random noise makes the measured values mathematically inconsistent with the analyticity of the complex dielectric permittivity function. The standard way of dealing with the noise is to use the "least squares" approach by looking for a function $f \in \mathcal{K}_h$ that is closest to the experimental data $f_{\exp}(\omega)$ in the L^2 norm on Γ. Thus, after rescaling the frequency interval Γ to the interval $[0, 1]$ we arrive at the following least squares problem:

$$\inf_{f \in \mathcal{K}_h} \|f - f_{\exp}\|_{L^2(0,1)}. \tag{3.1}$$

One approach [11, 12] is to ignore the positivity requirement, while retaining the spectral representation (2.4). The resulting problem constrains f to a vector space, but becomes ill-posed. It is then solved by Tikhonov regularization techniques. Unfortunately, such an approach cannot guarantee that the solution possesses the required positivity.

We will see in section 4 that the positivity property of functions in $\mathcal{K}_h$ plays a regularizing role, making the least squares problem (3.1) well-posed. So the solution to (3.1) exists, is unique, and lies in the closure $\mathscr{S}_h = \overline{\mathcal{K}_h}$ with respect to the standard

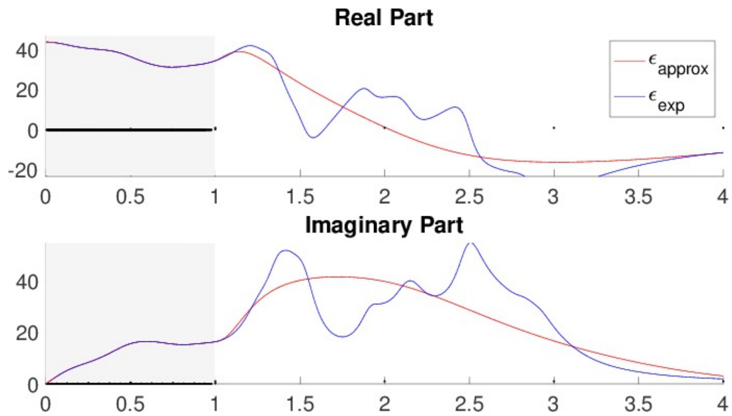


FIG. 1. *Apparent ill-posedness of the extrapolation process.*

topology[2] of the space $H(\mathbb{H}_h)$ of analytic functions on $\mathbb{H}_h$. We then characterize the set $\mathscr{S}_h$ and obtain stability of analytic continuation in the following sense: if $\{f_n\}, f \subset \mathscr{S}_h$ are such that $f_n \to f$ in $L^2(0,1)$, then $f_n \to f$ as $n \to \infty$ in $H(\mathbb{H}_h)$. In section 4.2 we study the properties of the minimizer of (3.1).

Even though we have established well-posedness and stability of the extrapolation problem, the above-mentioned results are not quantitative, since they do not give rates of convergence of the extrapolation errors. Figure 1 (corresponding to a small value of the natural regularization parameter) shows two perfectly admissible functions in $\mathcal{K}_h$ that are virtually indistinguishable on $[0,1]$, but separate almost immediately beyond the data window.

It suggests that the quantification of mathematical well-posedness is a matter of practical importance. While there is no shortage of proposed algorithms for extrapolation of experimental data in the vast literature on the subject, there is no mathematically rigorous quantitative analysis of uncertainty inherent in such extrapolation procedures. We therefore consider two different functions f and g in $\mathcal{K}_h$ that differ by less than a small fraction ϵ of their size on the frequency band $[0,1]$. Our goal is to estimate how much f and g can differ at a given point $\omega_0 > 1$. We begin by giving a precise formulation of this question. For any $\epsilon > 0$ we consider the set of pairs

$$U_h(\epsilon) = \left\{(f,g) \in \mathcal{K}_h : \frac{\|f-g\|_{L^2(0,1)}}{\max(\|\sigma_f\|, \|\sigma_g\|)} \le \epsilon\right\},$$

where σ_f and σ_g are the spectral measures in the representation (2.4) of f and g, respectively, and

$$\|\sigma_f\| := \int_0^\infty \frac{d\sigma_f(\lambda)}{\lambda+1} < +\infty$$

is finite interpreted as a "total norm" of f (it is the total variation of the measure $d\sigma_f/\lambda+1$). Our goal is to find an upper bound on the relative extrapolation error at the point ω_0,

$$\Delta_{\omega_0,h}(\epsilon) = \sup\left\{\frac{|f(\omega_0)-g(\omega_0)|}{\max(\|\sigma_f\|, \|\sigma_g\|)} : (f,g) \in U_h(\epsilon)\right\}. \tag{3.2}$$

[2]This is a metrizable topology of uniform convergence on compact subsets of $\mathbb{H}_h$.

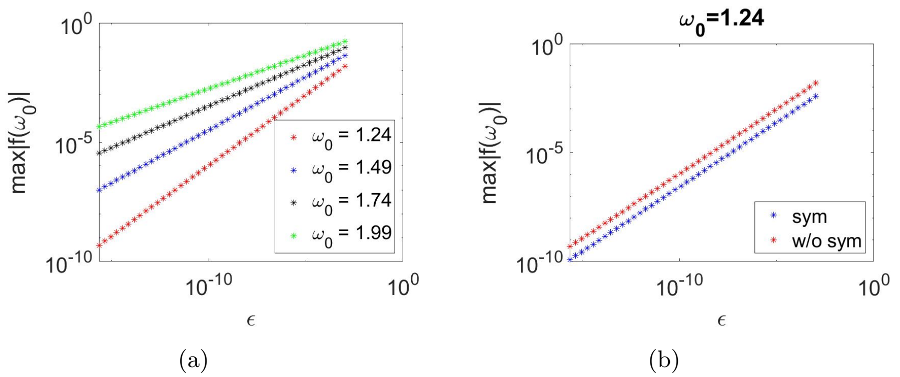


FIG. 2. *Numerical support for the power law transition principle.*

Two fundamental questions determine the reliability of the extrapolation procedures:

1. Is it true that $\Delta_{\omega_0,h}(\epsilon) \to 0$ as $\epsilon \to 0^+$?
2. What is the exact convergence rate of $\Delta_{\omega_0,h}(\epsilon)$ to 0?

The first insight is the realization that, in fact, these questions are about the difference $\phi = f - g$ rather than the pair (f, g). The difference ϕ has the same spectral representation (2.3), (2.4) as f and g, except the spectral measure is no longer positive. Our next observation is that the asymptotic behavior of $\Delta_{\omega_0,h}(\epsilon)$, as $\epsilon \to 0$, is insensitive to certain restrictions on the spectral measures σ, as long as the set of admissible measures is dense (in the weak-* topology) in the space of measures (2.3). For example, we may work only with absolutely continuous measures with densities in $L^2(0, +\infty)$, permitting us to use the theory of Hardy functions and Hilbert space methods to obtain the exact asymptotic behavior of $\Delta_{\omega_0,h}(\epsilon)$. The passage from pairs (f, g) to a single function $\phi = f - g$ is described in section 5.1. The analysis of the Hilbert space problem for the difference $\phi = f - g$ is in section 5.2, where it is shown that $\Delta_{\omega_0,h}(\epsilon) \lesssim \epsilon^\gamma$ for some $\gamma \in (0, 1)$, giving a positive answer to our first question. The answer to the second question is more nuanced, if we distinguish what we can prove rigorously and what we can conjecture based on the numerical and analytical evidence. The theory in section 5.2 permits numerical computation of the asymptotics of $\Delta_{\omega_0,h}(\epsilon)$ by relating it to a similar problem without the symmetry constraint (property (a) from section 2). Figure 2(a) shows that asymptotically $\Delta_{\omega_0,h}(\epsilon) \sim \epsilon^{\gamma(\omega_0,h)}$, while we also see from Figure 2(b) that the symmetry requirement does not change the value of the exponent $\gamma(\omega_0, h)$.

These results demonstrate the power law principle we have formulated in [26, 27], generalizing the Nevanlinna principle [13, 45]. It says that the largest value a bounded analytic function which is of order ϵ on a curve Γ inside its domain of analyticity can take at a point $\omega_0 \notin \Gamma$ decays as ϵ^γ, where the exponent $0 < \gamma < 1$ depends on the geometry of the domain, the curve Γ, and the point ω_0. Figure 3 shows how rapidly $\gamma(\omega_0, h)$ decays to 0, as ω_0 moves further away from Γ for several values of h. The larger the regularization parameter h is, the better behaved the extrapolation problem is.

In [27, 26] we have gained some insight into the mathematical structure of the maximizer function and the underlying mechanisms that cause the power law precision deterioration in problems without the symmetry constraint. Specifically, in the absence of symmetry the Hardy function $\phi(z)$ of unit norm maximizing $|\phi(\omega_0)|$ is a

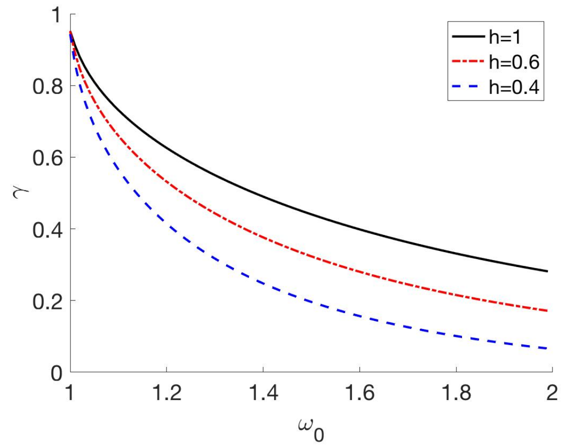


FIG. 3. *Power law exponent γ as a function of ω for several values of h.*

rescaled solution of a linear integral equation of Fredholm type,

$$\mathscr{K}_h u + \epsilon^2 u = p_{\omega_0}, \tag{3.3}$$

where

$$(\mathscr{K}_h u)(\omega) = \int_{-1}^{1} p_x(\omega) u(x) dx, \qquad p_{\omega_0}(\omega) = \frac{i}{2\pi(\omega - \overline{\omega_0} + 2ih)}. \tag{3.4}$$

The exponent $\gamma(\omega_0, h)$ can be computed from the unique solution $u_\epsilon = u_{\epsilon,\omega_0,h}$ of the integral equation:

$$\gamma(\omega_0, h) = 1 - \lim_{\epsilon \to 0^+} \frac{\ln \|u_\epsilon\|_{L^2(-1,1)}}{\ln(1/\epsilon)}. \tag{3.5}$$

The equality of the exponents for problems with and without symmetry shown in Figure 2(b) can be explained by the "quantitative asymmetry" of the solution u_ϵ:

$$\overline{\lim_{\epsilon \to 0}} \frac{|u_\epsilon(\omega_0)|}{|u_\epsilon(-\omega_0)|} < 1. \tag{3.6}$$

Indeed, the symmetrized solution $v_\epsilon(\omega) = u_\epsilon(\omega) + \overline{u_\epsilon(-\overline{\omega})}$ has the same order of magnitude at $\omega = \omega_0$ as $u_\epsilon(\omega_0)$, as $\epsilon \to 0$. While numerically (3.6) is seen to hold, we do not have a mathematical proof of this inequality. Nonetheless, the equality of the exponents for problems with and without symmetry is established in section 5.2.

Once the symmetry constraint is discarded, the problem reduces to the one that we have already studied in [27]. The insights from that study permit us to construct a "near-optimal" test function $\phi = f - g$ and give an analytic formula for an upper bound on $\gamma(\omega_0, h)$, which is tight for $h \geq 0.6$. To explain the construction of the near-optimal test function, consider the orthonormal eigenbasis $\{e_n : n \geq 1\} \subset L^2(-1, 1)$ of $\mathscr{K}_h$. We observe that by taking $u = e_n$ in (3.4) we obtain

$$(p_{\omega_0}, e_n)_{L^2} = \overline{(\mathscr{K}_h e_n)(\omega_0)} = \lambda_n \overline{e_n(\omega_0)},$$

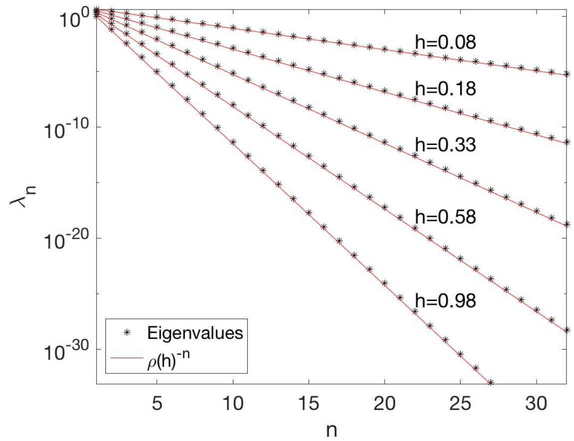


FIG. 4. *Comparison of the eigenvalues λ_n of $\mathscr{K}_h$ and $\rho(h)^{-n}$.*

where $\lambda_n > 0$ are the corresponding eigenvalues. Then the solution of (3.4) can be written as

$$u_\epsilon(\omega) = \sum_{n=1}^{\infty} \frac{\lambda_n \overline{e_n(\omega_0)} e_n(\omega)}{\lambda_n + \epsilon^2}.$$

The next idea comes from the upper bound on the decay of the eigenvalues λ_n from [3] and an identical asymptotics from [40]. Figure 4 shows that $\lambda_n \sim \rho^{-n}$, where ρ is the Riemann invariant of $G_h = \mathbb{C}_\infty \setminus ([-1, 1] \pm ih)$. The Riemann invariant of a doubly connected region is the unique value of $\rho > 1$ such that G_h is conformally equivalent to the annulus

$$A_\rho = \{z \in \mathbb{C} : \rho^{-1/2} < |z| < \rho^{1/2}\}.$$

If $\Psi : G_h \to A_\rho$ is the conformal isomorphism, then it maps $\Gamma_h = [-1, 1] + ih$ onto the circle $|z| = \rho^{-1/2}$ and the real line[3] is mapped to the unit circle. In the annulus A_ρ the same question we are studying in the upper half-plane can be analyzed completely (see [26] for details). In A_ρ the eigenfunctions of the corresponding integral operator are just functions z^n. Even though it is not true that the eigenfunctions of $\mathscr{K}_h$ are $\Psi(\omega)^n$, we can treat them as such, replacing $e_n(\omega)$ with $\widetilde{e}_n(\omega) = (\sqrt{\rho}\Psi(\omega))^n$ (so that $|\widetilde{e}_n(\omega)| = 1$ on Γ_h). This gives us the replacement

$$\widetilde{u}_\epsilon(\omega) = \sum_{n=1}^{\infty} \frac{\overline{\Psi(\omega_0)}^n \Psi(\omega)^n}{\rho^{-n} + \epsilon^2} \tag{3.7}$$

for the solution $u_\epsilon(\omega)$ of (3.4). Lemma 3.1 below shows that

$$\widetilde{u}_\epsilon(\omega_0) = \sum_{n=1}^{\infty} \frac{|\Psi(\omega_0)|^{2n}}{\rho^{-n} + \epsilon^2} \sim \epsilon^{-2\theta_0} P\left(\frac{2\ln(1/\epsilon)}{\ln \rho}\right),$$

[3]In order to explain the structure of the maximizer function it is convenient to work in a shifted plane $\mathbb{H}_h + ih$, so that the interval $[-1, 1]$ where frequencies are measured corresponds to Γ_h and the boundary of analyticity $\mathfrak{Im}\omega = -h$ shifts to the real line.

where

$$P(t) = \left(\frac{\rho}{|\Psi(\omega_0)|^2}\right)^t \sum_{k\in\mathbb{Z}} \frac{|\Psi(\omega_0)|^{2k}}{\rho^t + \rho^k}$$

is a smooth 1-periodic function of t, and

$$\theta_0 = 1 + \frac{2\ln|\Psi(\omega_0)|}{\ln\rho}.$$

The same lemma shows that when $\omega \in \Gamma_h$, then $|\Psi(\omega)| = \rho^{-1/2}$, and

$$|\widetilde{u}(\omega)| \sim \epsilon^{-2\theta_h}, \qquad \theta_h = \frac{1}{2} + \frac{\ln|\Psi(\omega_0)|}{\ln\rho},$$

while, when $\omega \in \mathbb{R}$, $|\Psi(\omega)| = 1$, and we have

$$|\widetilde{u}(\omega)| \sim \epsilon^{-2\theta_{\mathbb{R}}}, \qquad \theta_{\mathbb{R}} = 1 + \frac{\ln|\Psi(\omega_0)|}{\ln\rho}.$$

Then $M(\omega) = \epsilon^{2\theta_{\mathbb{R}}}\widetilde{u}(\omega)$ is $O(1)$ on $\mathbb{R}$, $O(\epsilon)$ on Γ_h, and $O(\epsilon^{\gamma_1})$ at ω_0, where

$$\gamma_1(\omega_0) = 2(\theta_{\mathbb{R}} - \theta_0) = -\frac{2\ln|\Psi(\omega_0)|}{\ln\rho}.$$

The explicit formula for the conformal isomorphism $\Psi : G_h \to A_\rho$ has been derived in [1, p. 138] in terms of elliptic functions and integrals, permitting us to compute an upper bound $\gamma_1(\omega_0)$ on the true exponent $\gamma(\omega_0)$. Figure 5 shows that $\gamma_1(\omega_0)$ is a very good approximation for γ when $h \geq 0.6$.

LEMMA 3.1. *Let $a \in \mathbb{C}$ and $b > 0$ be such that $0 < b < |a| < 1$. Let*

$$\phi(\eta) = \sum_{n=0}^{\infty} \frac{a^n}{\eta + b^n}. \tag{3.8}$$

Then the asymptotics of $\phi(\eta)$, as $\eta \to 0^+$, is surprisingly irregular, depending on the limit

$$t = \lim_{j\to\infty} \left\{\frac{\ln\eta_j}{\ln b}\right\}$$

along a sequence $\eta_j \to 0$, as $j \to \infty$, where $\{x\}$ denotes the fractional part of x. Specifically,

$$\phi(\eta_j) \sim \phi_0(t)\eta_j^{-\gamma},$$

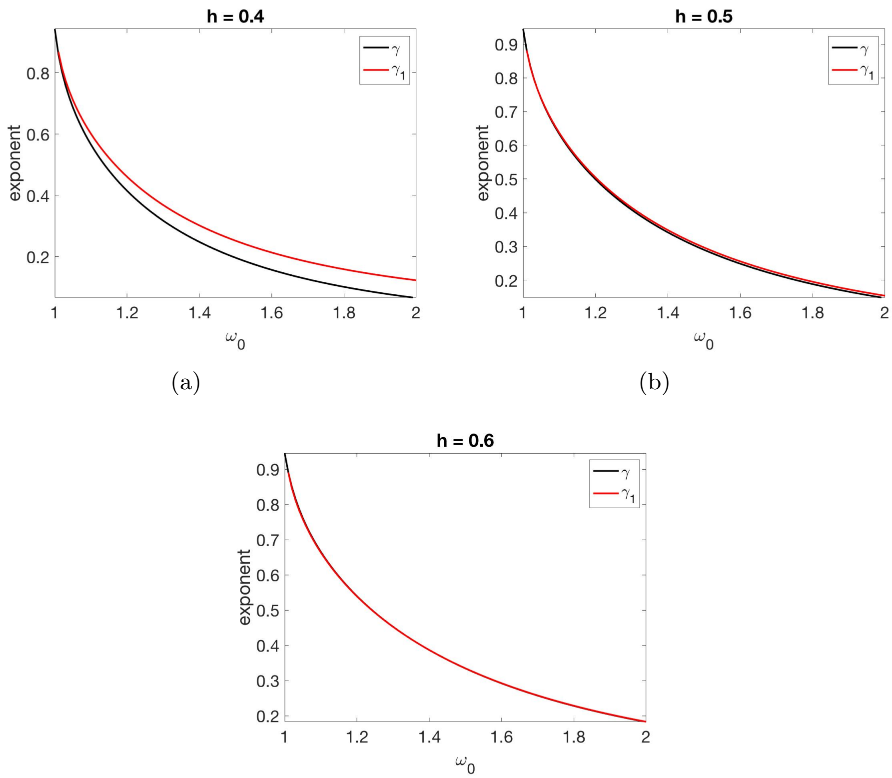
where

$$\phi_0(t) = \frac{b^t}{a^t} \sum_{k\in\mathbb{Z}} \frac{a^k}{b^t + b^k}$$

is a smooth 1-periodic function, and

$$\gamma = 1 - \frac{\ln a}{\ln b}.$$

In the formulas above, $a^t = e^{t\ln a}$ and $\ln$ can denote any analytic branch (independent of η) that agrees with the usual logarithm for positive real numbers.



(a) (b) (c)

FIG. 5. *Comparison of γ and γ_1.*

Proof. We first notice that, unlike $\phi(\eta)$, the function

$$\psi(\eta) = \sum_{n=1}^{\infty} \frac{a^{-n}}{\eta + b^{-n}}$$

is regular at $\eta = 0$. In fact, $\psi(0) = b/(a-b)$. We therefore define a new function

$$F(\eta) = \sum_{n\in\mathbb{Z}} \frac{a^n}{\eta + b^n} = \phi(\eta) + \psi(\eta),$$

which obviously satisfies

$$\lim_{j\to\infty} F(\eta_j)\eta_j^{\gamma} = \lim_{j\to\infty} \phi(\eta_j)\eta_j^{\gamma}$$

whenever $\eta_j \to 0^+$ and the limit on the right-hand side exists. Introducing the integer and fractional parts

$$N(\eta) = \left[\frac{\ln\eta}{\ln b}\right], \qquad \alpha(\eta) = \left\{\frac{\ln\eta}{\ln b}\right\},$$

we make a change of index of summation $k = n - N(\eta)$ and obtain, using

$$N(\eta) = \frac{\ln\eta}{\ln b} - \alpha(\eta),$$

after a short calculation, that

$$F(\eta)\eta^{\gamma} = \sum_{k\in\mathbb{Z}} \frac{a^{k-\alpha(\eta)}}{1+b^{k-\alpha(\eta)}} = \frac{b^{\alpha(\eta)}}{a^{\alpha(\eta)}} \sum_{k\in\mathbb{Z}} \frac{a^k}{b^{\alpha(\eta)}+b^k}.$$

The statement of the lemma is now apparent. ∎

In general, we have shown in [26, 27] that the exact exponent $\gamma(\omega_0, h)$ is determined by the exponential decay of the magnitudes $|e_n(\omega_0)|$ of the orthonormal eigenbasis e_n of the integral operator $\mathscr{K}_h$. Specifically, we have proved that if

$$\lambda_n \simeq e^{-\alpha(h)n}, \qquad |e_n(\omega_0)| \simeq e^{-\beta(\omega_0,h)n}, \tag{3.9}$$

then $0 < 2\beta(\omega_0, h) < \alpha(h)$, and

$$\gamma(\omega_0, h) = \frac{2\beta(\omega_0, h)}{\alpha(h)}. \tag{3.10}$$

The conjectured asymptotics $\lambda_n \sim \rho^{-n}$ of (squares of) singular values of the restriction operator $\mathscr{R}_h$ exactly coincides with the asymptotics of the restriction operators to smooth domains established in [40]. Unfortunately, the methods in [40] are not applicable, since the end-points of the interval $[-1, 1]$ can be regarded as corners of angle 0, violating the desired smoothness requirements. Nonetheless, Figure 4 indicates that the technical assumptions in [40] on the smoothness of domains could probably be significantly relaxed.

The eigenvalues λ_n are also connected to Kolmogorov n-widths [42], since they are squares of singular values of the restriction operator $\mathscr{R}_h : H^2(\mathbb{H}_h) \to L^2(-1, 1)$ (here H^2 is defined in (5.4)). Specifically (cf. [21, Theorem 6.1]), $\sqrt{\lambda_{n+1}}$ is the Kolmogorov n-width of the restriction to $L^2(-1, 1)$ of closed unit ball in $H^2(\mathbb{H}_h)$. The relation of the Kolmogorov n-widths of restrictions of various classes of analytic functions to corresponding Riemann invariants have been known in many cases [19, 46, 22].

4. The least squares problem.

4.1. Existence and uniqueness. We begin by examining the existence and uniqueness questions in the least squares problem (3.1). Let $f_n \in \mathcal{K}_h$ be a minimizing sequence in (3.1). Then it has to be bounded in the $L^2(0, 1)$ norm. We will show that this implies existence of a subsequence converging uniformly on compact subsets of $\mathbb{H}_h$ to an analytic function. In general, this limit does not need to be in $\mathcal{K}_h$, since it is not closed in $H(\mathbb{H}_h)$. We will, therefore, need to characterize the closure $\overline{\mathcal{K}_h}$ of $\mathcal{K}_h$.

We recall that a family of functions in $H(G)$ is called normal if every sequence has a convergent in the $H(G)$ subsequence. In other words, normal families of functions are exactly the precompact subsets in $H(G)$.

In fact, any family of Herglotz functions (i.e., analytic in the upper half-plane with nonnegative imaginary part) that is uniformly bounded at a single point is normal (cf. [17, Chap. II]). For our purposes, we consider a family of functions that is uniformly bounded in the $L^2(0, 1)$ norm.

THEOREM 4.1.
(i) *The closure of $\mathcal{K}_h$ in $H(\mathbb{H}_h)$ is $\mathscr{S}_h = \{f(\omega) = F((\omega + ih)^2) : F \in \mathfrak{S}\}$.*
(ii) *For any $M > 0$, the family of functions $\mathscr{S}_h^M = \{f \in \mathscr{S}_h : \|f\|_{L^2(0,1)} \leq M\}$ is normal.*

Proof. The proof is based on the representation (2.3), where we interpret the measure σ as an element of the Banach space $\mathscr{B}^*$ dual to

$$\mathscr{B} = \left\{\phi \in C([0,+\infty)) : \lim_{\lambda\to\infty} \lambda\phi(\lambda) = 0\right\},$$

with the norm

$$\|\phi\|_{\mathscr{B}} = \max_{\lambda\ge 0}(\lambda+1)|\phi(\lambda)|.$$

If we define the action of the measure σ on $\phi \in \mathscr{B}$ by

$$\langle\phi,\sigma\rangle = \int_0^\infty \phi(\lambda)d\sigma(\lambda),$$

then

$$\|\sigma\|_* = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda+1} \tag{4.1}$$

when the measure σ is nonnegative.

The conclusion of the theorem then follows easily from the fundamental estimate in the lemma below.

LEMMA 4.2. *There exist $c_h > 0$ and $C_h > 0$ depending only on h, such that for every $f \in \mathscr{S}_h$*

$$c_h\|f\|_{L^2(0,1)} \le \rho + \|\sigma\|_* \le C_h\|f\|_{L^2(0,1)},$$

where

$$\rho = \lim_{\omega\to\infty} f(\omega).$$

Proof. Let us start by proving the second inequality. Applying the Hölder inequality to the representation

$$f(\omega) = \rho + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda-(\omega+ih)^2} \tag{4.2}$$

we obtain

$$\|f\|_{L^2(0,1)} \ge \left(\int_0^1 |\mathfrak{Re}(f)|^2 d\omega\right)^{\frac{1}{2}} \ge \left|\int_0^1 \mathfrak{Re}(f)d\omega\right|.$$

Applying Fubini's theorem we then compute

$$\int_0^1 \mathfrak{Re}(f)d\omega = \rho + \int_0^1\int_0^\infty \mathfrak{Re}\left(\frac{1}{\lambda-(\omega+ih)^2}\right)d\sigma(\lambda)d\omega = \rho + \int_0^\infty \varphi(\sqrt{\lambda})\frac{d\sigma(\lambda)}{\lambda+1},$$

where

$$\varphi(x) = \frac{x^2+1}{4x}\ln\left(1+\frac{4x}{(x-1)^2+h^2}\right).$$

Note that $\varphi(x) > 0$ for $x > 0$, and because $\ln(1+x) \sim x$ as $x \to 0$ we get

$$\lim_{x\to 0}\varphi(x) = \frac{1}{1+h^2} > 0, \qquad \lim_{x\to\infty}\varphi(x) = 1 > 0.$$

Thus $\inf_{[0,\infty)} \varphi(x) = \mu_h > 0$, which implies the desired estimate with $C_h = 1/\mu_h$.

Let us now turn to the first inequality. Again, by Hölder's inequality

$$\begin{aligned}\frac{1}{2}\|f\|^2_{L^2(0,1)} - \rho^2 &\le \int_0^1 \left(\int_0^\infty \frac{d\sigma(\lambda)}{|\lambda - (\omega+ih)^2|}\right)^2 d\omega \\ &\le \int_0^\infty \frac{d\sigma(\lambda)}{\lambda+1} \cdot \int_0^1 \int_0^\infty \frac{\lambda+1}{|\lambda-(\omega+ih)^2|^2} d\sigma(\lambda) d\omega \\ &= \|\sigma\|_* \cdot \int_0^\infty \psi(\lambda) d\sigma(\lambda),\end{aligned}$$

where

$$\psi(\lambda) = \int_0^1 \frac{\lambda+1}{|\lambda-(\omega+ih)^2|^2} d\omega = \frac{\varphi(\sqrt{\lambda})}{\lambda+h^2} + \frac{\lambda+1}{4h(\lambda+h^2)}\left(\arctan\frac{\sqrt{\lambda}+1}{h} - \arctan\frac{\sqrt{\lambda}-1}{h}\right).$$

Note that $(\lambda+1)\psi(\lambda)$ is bounded in $[0,\infty)$, because φ is a bounded function and the difference of arctangents can be bounded by $\frac{2h}{\lambda-1}$ for $\lambda > 1$ by the mean value theorem. But then the desired inequality follows from the estimate

$$\int_0^\infty \psi(\lambda) d\sigma(\lambda) \le C_h \int_0^\infty \frac{d\sigma(\lambda)}{\lambda+1} = C_h \|\sigma\|_*. \qquad \square$$

Obviously $\mathcal{K}_h \subset \mathscr{S}_h$ and Theorem 4.1 follows from the next lemma.

LEMMA 4.3.
(i) $\mathscr{S}_h$ *is closed in* $H(\mathbb{H}_h)$.
(ii) $\mathscr{S}_h \subset \overline{\mathcal{K}_h}$.

Proof. (i) Let $\{f_n\} \subset \mathscr{S}_h$ be a sequence such that $f_n \to f$ in $H(\mathbb{H}_h)$. Then according to Lemma 4.2 the sequences $\{\rho_n\} \subset \mathbb{R}$ and $\{\sigma_n\} \subset \mathscr{B}^*$ are bounded. By the Banach–Alaoglu theorem the closed unit ball in $\mathscr{B}^*$ is compact in the weak-* topology. It is also sequentially compact because the Banach space $\mathscr{B}$ is separable. Thus, there exist subsequences (which we do not relabel) $\rho_n \to \rho$ and $\sigma_n \overset{*}{\rightharpoonup} \sigma$ weakly-* in $\mathscr{B}^*$. Let us write

$$f_n(\omega) = \rho_n + \|\sigma_n\|_* + \int_0^\infty G(\omega,\lambda) d\sigma_n(\lambda),$$

where

$$G(\omega,\lambda) = \frac{1}{\lambda-(\omega+ih)^2} - \frac{1}{\lambda+1} = \frac{1+(\omega+ih)^2}{(\lambda-(\omega+ih)^2)\,(\lambda+1)}.$$

It is now evident that $G(\omega,\cdot) \in \mathscr{B}$ for each fixed $\omega \in \mathbb{H}_h$. Upon extracting the convergent subsequence of the bounded sequence $\{\|\sigma_n\|_*\}$, with limit denoted by a, we obtain that

$$f(\omega) = \lim_{n\to\infty} f_n(\omega) = \rho + a + \int_0^\infty G(\omega,\lambda) d\sigma(\lambda) = \rho + a - \|\sigma\|_* + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda-(\omega+ih)^2}.$$

By lower semicontinuity of the norm $a \ge \|\sigma\|_*$, hence we conclude that $f \in \mathscr{S}_h$.

(ii) 1. Let us start by showing that for any constant $\rho \ge 0$, there exists $\{g_n\} \subset \mathcal{K}_h$ such that $g_n \to \rho$ uniformly on $[0,1]$ as $n \to \infty$. Indeed, define

$$g_n(\omega) = \rho \int_n^{n+1} \frac{\lambda d\lambda}{\lambda-(\omega+ih)^2}.$$

Clearly, $g_n \in \mathcal{K}_h$ and

$$g_n(\omega) - \rho = \rho(\omega + ih)^2 \int_n^{n+1} \frac{d\lambda}{\lambda - (\omega + ih)^2},$$

which approaches zero, as $n \to \infty$, uniformly on compact subsets of $\mathbb{H}_h$.

2. Now let $f \in \mathscr{S}_h$ and let ρ and σ be as in its definition. Consider the functions

$$h_n(\omega) = \int_0^n \frac{d\sigma(\lambda)}{\lambda - (\omega + ih)^2}.$$

Note that $h_n \in \mathcal{K}_h$, since its corresponding measure is $d\sigma_n = \chi_{(0,n)} d\sigma$ and

$$\int_0^\infty d\sigma_n(\lambda) = \int_0^n d\sigma(\lambda) \le (n+1) \int_0^n \frac{d\sigma(\lambda)}{\lambda + 1} < \infty.$$

Now

$$f(\omega) - h_n(\omega) = \rho + \int_n^\infty \frac{d\sigma(\lambda)}{\lambda - (\omega + ih)^2}$$

and by dominated convergence the above difference tends to ρ uniformly on compact subsets of $\mathbb{H}_h$. It remains to use the sequence $\{g_n\}$ from part 1 to get that $g_n + h_n$ is the desired sequence in $\mathcal{K}_h$ converging to f in $H(\mathbb{H}_h)$. ∎

To prove part (ii) of Theorem 4.1 we observe that for any compact subset $K \subset \mathbb{H}_h$ there exists a constant C_K so that

$$C_K = \sup_{\lambda \ge 0} \sup_{\omega \in K} \frac{\lambda + 1}{|\lambda - (\omega + ih)^2|} < +\infty.$$

Thus, for any $\omega \in K$ and $f \in \mathscr{L}_h$ we have from representation (4.2)

$$|f(\omega)| \le \rho + C_K \|\sigma\|_*.$$

Now, Lemma 4.2 implies that the family of functions $\mathscr{L}_h^M$ is locally equibounded. We conclude, by Montel's theorem, that $\mathscr{L}_h^M$ is a normal family of analytic functions. ∎

A corollary of Theorem 4.1 is stability of analytic continuation.

Corollary 4.4. *Let $\{f_n\}, f \subset \mathscr{S}_h$, be such that $f_n \to f$ in $L^2(0,1)$; then $f_n \to f$ as $n \to \infty$ in $H(\mathbb{H}_h)$.*

Proof. Indeed, if $f_n \to f$ in $L^2(0,1)$, then $\|f_n\|_{L^2(0,1)}$ is bounded. Then any converging subsequence $f_{n_k} \to g$ in $H(\mathbb{H}_h)$ must also converge to g in $L^2(0,1)$. But then $f = g$ on $(0,1)$. Since both f and g are analytic in $\mathbb{H}_h$, then $f = g$ everywhere. Since the set of limits of converging subsequences of f_n consists of a single element $\{f\}$, we conclude that $f_n \to f$ in $H(\mathbb{H}_h)$. ∎

Let us now return to the least squares problem (3.1).

Theorem 4.5. *For a given $f_{\exp} \in L^2(0,1)$, the least squares problem*

$$\mathfrak{E} = \mathfrak{E}(f_{\exp}) = \min_{f \in \mathscr{S}_h} \|f - f_{\exp}\|_{L^2(0,1)} \tag{4.3}$$

has a unique solution. Moreover,

$$\inf_{f \in \mathcal{K}_h} \|f - f_{\exp}\|_{L^2(0,1)} = \mathfrak{E}(f_{\exp}).$$

Proof. To prove existence, let $\{f_n\}_{n=1}^{\infty} \in \mathscr{S}_h$ be a minimizing sequence; then it is bounded in $L^2(0,1)$. Let us extract a weakly convergent subsequence, not relabeled, $f_n \rightharpoonup f_0$ in $L^2(0,1)$, as $n \to \infty$. The limiting function f_0 is in $\mathscr{S}_h$. By the convexity of the L^2 norm we have

$$\mathfrak{E} = \varliminf_{n\to\infty} \|f_n - f_{\text{exp}}\|_{L^2(0,1)} \geq \|f_0 - f_{\text{exp}}\|_{L^2(0,1)}.$$

Hence, f_0 is a minimizer. To prove that the infimum in (4.3) stays the same if we replace $\mathscr{S}_h$ by $\mathcal{K}_h$ we note that if $f_0 \in \mathscr{S}_h$ is a minimizer, then there exists a sequence $\{g_n\} \subset \mathcal{K}_h$ converging to f_0 strongly in $L^2(0,1)$.

To prove uniqueness, let f_1 and f_2 be two different solutions. Then $\|f_j - f_{\text{exp}}\|_{L^2(0,1)} = \mathfrak{E}$ for $j = 1,2$. Observe that the function $f_t = tf_1 + (1-t)f_2$ is also admissible, and therefore

$$\mathfrak{E} \leq \|f_t - f_{\text{exp}}\|_{L^2(0,1)} \leq t\|f_1 - f_{\text{exp}}\|_{L^2(0,1)} + (1-t)\|f_2 - f_{\text{exp}}\|_{L^2(0,1)} = \mathfrak{E};$$

thus $\|f_t - f_{\text{exp}}\|_{L^2(0,1)} = \mathfrak{E}$ for all $t \in [0,1]$. However,

$$\|f_t - f_{\text{exp}}\|^2_{L^2(0,1)} = t^2\|f_1 - f_2\|^2_{L^2(0,1)} + 2t\Re(f_1 - f_2, f_2 - f_{\text{exp}}) + \|f_2 - f_{\text{exp}}\|^2_{L^2(0,1)},$$

which cannot be constant, since the coefficient at t^2 is nonzero by our assumption $f_1 \neq f_2$. The obtained contradiction concludes the theorem. ∎

4.2. Properties of the minimizer. In this section we will prove that if the minimum in (4.3) is nonzero, then the minimizer must be a rational function in $\mathbb{C}$ with poles (and zeros) on the line $\mathfrak{Im}(\omega) = h$. We use the method of Caprini [7, 9] to prove the statement. The method for finding the necessary and sufficient conditions for a minimizer in (4.3) is based on our ability to compute the effect of the change of ρ and spectral measure σ in representation (2.3) on the value of the functional we want to minimize. Suppose that

$$f_*(\omega) = \rho_* + \int_0^\infty \frac{d\sigma_*(\lambda)}{\lambda - (\omega + ih)^2}$$

is the minimizer and

$$f(\omega) = \rho + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - (\omega + ih)^2} \tag{4.4}$$

is a competitor. The variation $\phi = f - f_*$ can then be written as

$$\phi(\omega) = \Delta\rho + \int_0^\infty \frac{d\nu(\lambda)}{\lambda - (\omega + ih)^2}, \qquad \nu = \sigma - \sigma_*, \quad \Delta\rho = \rho - \rho_*.$$

We then compute

$$\|f - f_{\text{exp}}\|^2_{L^2} - \|f_* - f_{\text{exp}}\|^2_{L^2} = \Delta\rho \lim_{t\to\infty} tC(t) + \int_0^\infty C(t)d\nu(t) + \|\phi\|^2_{L^2}, \tag{4.5}$$

where

$$C(t) = 2\mathfrak{Re} \int_0^1 \frac{f_*(\omega) - f_{\text{exp}}(\omega)}{t - (\omega - ih)^2} d\omega, \qquad t \geq 0, \tag{4.6}$$

is the Caprini function of $f_*(\omega)$.

THEOREM 4.6. *Suppose the infimum in* (3.1) *is nonzero; then the minimizer* $f_* \in \mathscr{S}_h$ *in* (4.3) *is given by*

$$f_*(\omega) = \rho_* + \sum_{j=1}^{N} \frac{\sigma_j}{t_j - (\omega + ih)^2} \tag{4.7}$$

for some $N \geq 0$, $\sigma_j > 0$, $0 \leq t_1 < t_2 < \cdots < t_N$, *and* $\rho_* \geq 0$. *Moreover,* f_*, *given by* (4.7), *is the minimizer if and only if its Caprini function* $C(t)$ *is nonnegative and vanishes at* $t = t_j$, $j = 1, \ldots, N$, *and "at infinity," in the sense that*

$$2\mathfrak{Re} \int_0^1 (f_{\exp}(\omega) - f_*(\omega))d\omega = \lim_{t\to\infty} tC(t) = 0, \tag{4.8}$$

provided $\rho_* > 0$.

Proof. If $\rho_* > 0$, then we can consider the competitor (4.4) with $\sigma = \sigma_*$. Formula (4.5) then implies that

$$\Delta\rho \lim_{t\to\infty} tC(t) + (\Delta\rho)^2 \geq 0,$$

where $\Delta\rho$ can be either positive or negative and can be chosen as small in absolute value as we want. This implies (4.8).

Next, suppose $t_0 \in [0, +\infty)$ is in the support of σ_*. For every $\epsilon > 0$ we define $I_\epsilon(t_0) = \{t \geq 0 : |t - t_0| < \epsilon\}$. Saying that t_0 is in the support of σ_* is equivalent to $\sigma_*(I_\epsilon(t_0)) > 0$ for all $\epsilon > 0$. Then there are two possibilities. Either

(i) $\lim_{\epsilon\to 0} \sigma_*(I_\epsilon(t_0)) = 0$, or

(ii) $\lim_{\epsilon\to 0} \sigma_*(I_\epsilon(t_0)) = \sigma_0 > 0$.

Let us first consider case (i). Then we construct a competitor measure

$$\sigma_\epsilon(\lambda) = \sigma_*(\lambda) - \sigma_*|_{I_\epsilon(t_0)} + \theta\sigma_*(I_\epsilon(t_0))\delta_{t_0}(\lambda), \quad \theta > 0,$$

where instead of the distributed mass of $I_\epsilon(t_0)$ we place a single point mass at t_0. We then define

$$f_\epsilon(\omega) = \rho_* + \int_0^\infty \frac{d\sigma_\epsilon(\lambda)}{\lambda - (\omega + ih)^2}. \tag{4.9}$$

Formula (4.5) then implies

$$\lim_{\epsilon\to 0} \frac{\|f_{\exp} - f_\epsilon\|^2_{L^2(0,1)} - \|f_{\exp} - f_*\|^2_{L^2(0,1)}}{\sigma_*(I_\epsilon(t_0))} = (\theta - 1)C(t_0).$$

If f_* is a minimizer, then we must have $(\theta - 1)C(t_0) \geq 0$ for all $\theta > 0$, which implies that $C(t_0) = 0$.

In case (ii) we have $\sigma_*(\{t_0\}) = \sigma_0 > 0$. Then for every $|\epsilon| < \sigma_0$ we construct a competitor measure

$$\sigma_\epsilon(\lambda) = \sigma_*(\lambda) + \epsilon\delta_{t_0}(\lambda), \quad |\epsilon| < \sigma_0,$$

as well as the corresponding f_ϵ, given by (4.9). We then compute

$$\lim_{\epsilon\to 0} \frac{\|f_{\exp} - f_\epsilon\|^2_{L^2(0,1)} - \|f_{\exp} - f_*\|^2_{L^2(0,1)}}{\epsilon} = C(t_0). \tag{4.10}$$

Since in this case ϵ can be both positive and negative we conclude that $C(t_0) = 0$.

Hence, we have shown that $C(t_0) = 0$ whenever $t_0 \in [0, +\infty)$ is in the support of the spectral measure σ of the minimizer f_*. It remains to observe that for any $t \in \mathbb{R}$

$$C(t) = \int_0^1 \frac{f_{\exp}(\omega) - f_*(\omega)}{t - (\omega - ih)^2} d\omega + \int_0^1 \frac{\overline{f_{\exp}(\omega)} - \overline{f_*(\omega)}}{t - (\omega + ih)^2} d\omega.$$

Thus, $C(t)$ is a restriction to the real line of a complex analytic function on the neighborhood of the real line in the complex t-plane. By assumption, $f_{\exp} \neq f_*$, and therefore $C(t)$ is not identically zero. In particular, the zeros of $C(t)$ cannot have an accumulation point on the real line. We can also see that the sequence of zeros of $C(t)$ cannot go to infinity by considering

$$B(s) = C\left(\frac{1}{s}\right) = s\int_0^1 \frac{f_{\exp}(\omega) - f_*(\omega)}{1 - s(\omega + ih)^2} d\omega + s\int_0^1 \frac{\overline{f_{\exp}(\omega)} - \overline{f_*(\omega)}}{1 - s(\omega - ih)^2} d\omega,$$

which is analytic in a neighborhood of 0, and hence cannot have a sequence of zeros $s_n \to 0$, as $n \to \infty$. We conclude that the support of the spectral measure of the minimizer f_* must be finite,

$$\sigma_*(\lambda) = \sum_{j=1}^{N} \sigma_j \delta_{t_j}(\lambda),$$

and the minimizer must be a rational function.

Now let us consider the competitor (4.4) defined by $\rho = \rho_*$ and $\sigma(\lambda) = \sigma_* + \epsilon\delta_{t_0}(\lambda)$, where $\epsilon > 0$ and $t_0 \notin \{t_1, \ldots, t_N\}$. Formula (4.5) then implies that

$$\epsilon C(t_0) + \epsilon^2 \|\phi_0\|^2_{L^2} \geq 0, \qquad \phi_0(\omega) = \frac{1}{t_0 - (\omega + ih)^2}$$

for all sufficiently small $\epsilon > 0$, which implies that $C(t) \geq 0$ for all $t \geq 0$. The necessity of the stated properties of the Caprini function $C(t)$ is now established.

Sufficiency is a direct consequence of formula (4.5), since we can write

$$\nu(\lambda) = \sigma(\lambda) - \sigma_*(\lambda) = \sum_{j=1}^{N} \Delta\sigma_j \delta_{t_j}(\lambda) + \widetilde{\nu}(\lambda),$$

where $\widetilde{\nu}(\lambda)$ is a positive Radon measure without any point masses at $\lambda = t_j$, $j = 1, \ldots, N$. We then compute, via formula (4.5), taking into account that $C(t) \geq 0$ for all $t \geq 0$ and $C(t_j) = 0$, that

$$\|f_* + \phi - f_{\exp}\|^2_{L^2} - \|f_* - f_{\exp}\|^2_{L^2} = \Delta\rho \lim_{t\to\infty} tC(t) + \int_0^\infty C(t)d\widetilde{\nu}(t) + \|\phi\|^2_{L^2} \geq 0,$$

since the first term on the right-hand side is either nonnegative, if $\rho_* = 0$, or zero, if $\rho_* > 0$. ∎

We observe that

if $t_j > 0$, then we must also have $C'(t_j) = 0$, since $t = t_j$ is a point of local minimum of $C(t)$. If we write formula (4.7) in the form

$$f_*(\omega) = \rho_* - \frac{\sigma_0}{(\omega + ih)^2} + \sum_{j=1}^{N} \frac{\sigma_j}{t_j - (\omega + ih)^2},$$
$$\rho_* \geq 0,\ \sigma_0 \geq 0,\ t_j > 0,\ \sigma_j > 0,\ j = 1, \ldots, N,$$

then we have exactly $2(N+1)$ equations for $2(N+1)$ unknowns ρ_*, σ_0, t_j, σ_j, $j=1,\dots,N$:

$$\rho_* \lim_{t\to\infty} tC(t)=0, \quad \sigma_0 C(0)=0, \quad C(t_j)=0, \quad C'(t_j)=0, \quad j=1,\dots,N.$$

Obviously, these equations do not imply that critical points t_j are local minima of $C(t)$, nor do they enforce the nonnegativity of $C(t)$. Taken together with their highly nonlinear dependence on t_j and an unknown value of N, their practical utility for finding f_* is dubious. Instead, Theorem 4.6 could be used to verify that a particular $f_*(\omega)$ is the minimizer of (3.1).

5. Worst case error analysis.

Notation. We write $A \lesssim B$ if there exists a constant c such that $A \le cB$, and likewise the notation $A \gtrsim B$ will be used. If both $A \lesssim B$ and $A \gtrsim B$ are satisfied, we will write $A \simeq B$. Throughout the paper all the implicit constants will be independent of ϵ. Let also

$$Sf(\omega) := \overline{f(-\overline{\omega})}. \tag{5.1}$$

In this section we analyze the quantity $\Delta_{\omega_0,h}(\epsilon)$, given by (3.2), and answer the two questions posed in section 3 about $\Delta_{\omega_0,h}(\epsilon)$ by showing that we can restate the questions entirely in terms of the difference $f-g$.

5.1. Reformulation of the problem.

To analyze $\Delta_{\omega_0,h}(\epsilon)$ we examine the difference $\phi = f-g$. First observe that ϕ also has an integral representation (2.4) with a signed measure $\sigma = \sigma_f - \sigma_g$. Now let $\sigma = \sigma^+ - \sigma^-$ be the unique Hahn decomposition of σ as a difference of two mutually orthogonal positive measures $\sigma^\pm$. Then we may write $\phi = \phi^+ - \phi^-$, where $\phi^\pm \in \mathcal{K}_h$ are given by

$$\phi^\pm(\omega) := \int_0^\infty \frac{d\sigma^\pm(\lambda)}{\lambda-(\omega+ih)^2}. \tag{5.2}$$

Thus, we expect that asymptotically $\Delta_{\omega_0,h}(\epsilon)$ and

$$\sup\left\{\frac{|\phi(\omega_0)|}{\max\|\sigma^\pm\|_*} : \phi\in\mathcal{K}_h-\mathcal{K}_h \quad \text{and} \quad \frac{\|\phi\|_{L^2(0,1)}}{\max\|\sigma^\pm\|_*}\le\epsilon\right\}, \tag{5.3}$$

must be equivalent. Here we have abbreviated $\max\|\sigma^\pm\|_* := \max(\|\sigma^+\|_*, \|\sigma^-\|_*)$. The next idea comes from the realization that the asymptotics of the worst possible error is not very sensitive to specific norms and spaces. The reason, as we have seen in [27] for a similar problem, is that the analytic function delivering the largest error at ω_0 is analytic in a larger half-space $\mathbb{H}_{2h}$ and is therefore bounded in a wide variety of norms. Our idea is therefore to prove asymptotic equivalence of $\Delta_{\omega_0,h}(\epsilon)$ to a quadratic optimization problem in a Hilbert space, permitting us to express the asymptotics of $\Delta_{\omega_0,h}(\epsilon)$ in terms of the solution of the integral equation (3.3).

Let us recall the definition of the Hardy class $H^2(\mathbb{H}_h)$:

$$H^2(\mathbb{H}_h) = \left\{ f \text{ is analytic in } \mathbb{H}_h : \sup_{y>-h}\|f\|_{L^2(\mathbb{R}+iy)}<\infty\right\}. \tag{5.4}$$

It is well known [30] that functions in H^2 have L^2 boundary data and that $\|f\|_{H^2(\mathbb{H}_h)} = \|f\|_{L^2(\mathbb{R}-ih)}$ defines a norm in H^2. We describe the relation between the Hardy space $H^2(\mathbb{H}_h)$ and $\mathcal{K}_h-\mathcal{K}_h$ more precisely in the following lemma.

LEMMA 5.1. *Let* $f \in H^2(\mathbb{H}_h)$ *with* $Sf = f$ *and* $\int_0^\infty x|\mathfrak{Im} f(x - ih)| < \infty$; *then* $f \in \mathcal{K}_h - \mathcal{K}_h$ *with*

$$d\sigma(\lambda) = \frac{1}{\pi}\mathfrak{Im} f(\sqrt{\lambda} - ih)d\lambda. \tag{5.5}$$

Moreover, $f^\pm \in \mathcal{K}_h$ *and*

$$\max \|\sigma_{f^\pm}\|_* \le \frac{1}{2\sqrt{\pi}}\|f\|_{H^2(\mathbb{H}_h)}. \tag{5.6}$$

Proof. We observe that it is enough to prove the lemma for $h = 0$ and then apply it to functions $f(\omega - ih) \in H^2(\mathbb{H}_+)$, where $f \in H^2(\mathbb{H}_h)$ and $\omega \in \mathbb{H}_+$.

For Hardy functions the following representation formula holds (cf. [30, p. 128]):

$$f(\omega) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathfrak{Im} f(x)}{x - \omega}dx, \qquad \omega \in \mathbb{H}_+. \tag{5.7}$$

Passing to limits in the symmetry relation $Sf(\omega) = f(\omega)$ as $\mathfrak{Im}\omega \downarrow 0$, and taking imaginary parts, we see that $-\mathfrak{Im} f(x) = \mathfrak{Im} f(-x)$. The formula (5.7) now gives

$$\pi f(\omega) = \int_0^\infty \frac{\mathfrak{Im} f(x)}{x - \omega}dx + \int_0^\infty \frac{\mathfrak{Im} f(-x)}{-x - \omega}dx = \int_0^\infty \frac{2x\mathfrak{Im} f(x)dx}{x^2 - \omega^2} = \int_0^\infty \frac{\mathfrak{Im} f(\sqrt{\lambda})d\lambda}{\lambda - \omega^2},$$

which implies (5.5).

Next, consider the functions

$$f^\pm(\omega) = \int_0^\infty \frac{d\sigma^\pm(\lambda)}{\lambda - \omega^2}, \qquad d\sigma^\pm(\lambda) = \frac{1}{\pi}\left(\mathfrak{Im} f\right)^\pm(\sqrt{\lambda})d\lambda,$$

where $(\mathfrak{Im} f)^\pm$ denote the positive and negative parts of the real valued function $\mathfrak{Im} f$. Then $f = f^+ - f^-$ and since $\int_0^\infty x|\mathfrak{Im} f(x)|dx < \infty$, the measures $\sigma^\pm$ are finite and so $f^\pm \in \mathcal{K}_0$.

Finally, we prove inequality (5.6). We compute

$$\|\sigma^\pm\|_* = \frac{2}{\pi}\int_0^\infty \frac{x(\mathfrak{Im} f)^\pm(x)}{1 + x^2}dx.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\|\sigma^\pm\|_* \le \frac{1}{\sqrt{\pi}}\|(\mathfrak{Im} f)^\pm\|_{L^2(0,+\infty)} \le \frac{1}{\sqrt{\pi}}\|\mathfrak{Im} f\|_{L^2(0,+\infty)} = \frac{1}{2\sqrt{\pi}}\|f\|_{H^2(\mathbb{H}_+)},$$

where we have used the symmetry and the fact that the real part of a Hardy function is the Hilbert transform of its imaginary part [30], and therefore

$$\|f\|^2_{H^2(\mathbb{H}_+)} = 2\|\mathfrak{Im} f\|^2_{L^2(\mathbb{R})} = 4\|\mathfrak{Im} f\|^2_{L^2(0,+\infty)}. \qquad \square$$

In order to complete the transition from $\mathcal{K}_h$ to Hardy spaces we need to replace the norm $\|\sigma\|_*$ in (5.3) with an equivalent Hilbert space norm. This is accomplished in our next lemma.

LEMMA 5.2. *Let* $h' \in (0, h)$; *then for any* $f \in \mathcal{K}_h$,

$$\|f\|_{h'} := \left\|\frac{f}{\omega + ih}\right\|_{H^2(\mathbb{H}_{h'})} \simeq \|\sigma\|_*, \tag{5.8}$$

where the implicit constants depend only on $h - h'$.

Proof. Since $\mathbb{H}_{h'} \subset \mathbb{H}_h$, it is clear that the function $f(\omega)/(\omega + ih)$ is analytic in $\mathbb{H}_{h'}$. Next, letting $\delta = h - h'$, using the integral representation (2.4) for f and Fubini's theorem, we compute

$$\|f\|_{h'}^2 = \int_{\mathbb{R}} \frac{1}{x^2+\delta^2} \int_0^\infty \int_0^\infty \frac{d\sigma(\lambda)d\sigma(t)}{[\lambda - (x+i\delta)^2][t-(x-i\delta)^2]} dx$$
$$= \int_0^\infty \int_0^\infty I(\lambda,t) \frac{d\sigma(\lambda)}{\lambda+1} \frac{d\sigma(t)}{t+1},$$

where

$$I(\lambda,t) = \frac{\pi(\lambda+1)(t+1)}{\delta(\lambda+4\delta^2)(t+4\delta^2)} \cdot \frac{(\lambda-t)^2 + 12\delta^2(\lambda+t) + 96\delta^4}{(\lambda-t)^2 + 8\delta^2(\lambda+t) + 16\delta^4}.$$

This concludes the proof, since it is clear that the function $I(\lambda, t)$ is bounded above and below by two positive constants depending only on δ. ∎

Now we are ready to give the desired Hilbert space reformulation of our problem. For any $h > 0$ we define

(5.9)
$$D_h(\epsilon) = \sup\left\{|f(\omega_0)| \ : f \in H^2(\mathbb{H}_h),\ Sf = f,\ \|f\|_{H^2(\mathbb{H}_h)} \le 1, \text{ and } \|f\|_{L^2(-1,1)} \le \epsilon\right\}.$$

Notice that for convenience we suppressed the dependence on ω_0 and also replaced the interval from $[0, 1]$ by a symmetric interval $[-1, 1]$, resulting in an equivalent formulation due to the symmetry $Sf = f$ of the functions in $\mathcal{K}_h$.

THEOREM 5.3 (equivalence of Δ and D). *For any* $h' \in (0, h)$

$$D_h(\epsilon) \lesssim \Delta_h(\epsilon) \lesssim D_{h'}(\epsilon) \tag{5.10}$$

as $\epsilon \to 0$, *where the implicit constants depend only on* h *and* h'.

Proof. We first observe that

$$\Delta_h(\epsilon) = \sup\{|f(\omega_0) - g(\omega_0)| : \{f, g\} \subset \mathcal{K}_h,\ \max\{\|\sigma_f\|_*, \|\sigma_g\|_*\} = 1,\ \|f - g\|_{L^2(-1,1)} \le \epsilon\}.$$

To prove the first inequality in (5.10), let $\{f, g\} \subset \mathcal{K}_h$ be such that

$$\max\{\|\sigma_f\|_*, \|\sigma_g\|_*\} = 1, \qquad \|f - g\|_{L^2(-1,1)} \le \epsilon.$$

Let

$$\phi(\omega) = \frac{i(f(\omega) - g(\omega))}{\omega + ih}.$$

Then $S\phi = \phi$. Moreover, by Lemma 5.2, for any $h' \in (0, h)$ we estimate

$$\|\phi\|_{H^2(\mathbb{H}_{h'})} = \|f - g\|_{h'} \le \|f\|_{h'} + \|g\|_{h'} \lesssim \|\sigma_f\|_* + \|\sigma_g\|_* \le 2.$$

We conclude that there exists a constant $c > 0$, depending only on h and h', such that $c\phi$ is admissible for $D_{h'}(\epsilon)$. Therefore,

$$D_{h'}(\epsilon) \ge c|\phi(\omega_0)| = \frac{c|f(\omega_0) - g(\omega_0)|}{|\omega_0 + ih|}.$$

Taking the supremum over all such pairs (f, g), we conclude that

$$\Delta_h(\epsilon) \le C D_{h'}(\epsilon)$$

for some constant $C > 0$, which depends on h and h', but not on ϵ.

To prove the other inequality, let $\phi \in H^2(\mathbb{H}_h)$ be admissible for $D_h(\epsilon)$. The idea is to construct a pair of functions $\{f, g\} \subset \mathcal{K}_h$ that are admissible for $\Delta_h(\epsilon)$. Since ϕ might not decay sufficiently fast at infinity to be in $\mathcal{K}_h - \mathcal{K}_h$ we modify it and define

$$\psi(\omega) = \frac{\phi(\omega)}{(\omega + ih)^2}.$$

This modification preserves the symmetry ($S\psi = \psi$) and ensures the required decay, so that Lemma 5.1 is applicable. So that $\psi^{\pm} \in \mathcal{K}_h$ and $\|\sigma_{\psi^{\pm}}\|_* \lesssim 1$. Now let $\psi_0(\omega) \in \mathcal{K}_h$ be such that $\|\sigma_{\psi_0}\|_* = 1$. We define

$$F(\omega) = \psi^+(\omega) + \psi_0(\omega), \qquad G(\omega) = \psi^-(\omega) + \psi_0(\omega).$$

We observe that there exists a constant $C > 0$, such that

$$1 = \|\sigma_{\psi_0}\|_* \le \|\sigma_F\|_* \le C, \qquad 1 = \|\sigma_{\psi_0}\|_* \le \|\sigma_G\|_* \le C.$$

Thus, the pair (f, g) given by

$$f(\omega) = \frac{F(\omega)}{M}, \quad g(\omega) = \frac{G(\omega)}{M}, \quad M = \max\{\|\sigma_F\|_*, \|\sigma_G\|_*\} \ge 1$$

is admissible for $\Delta_h(\epsilon)$. Thus,

$$\Delta_h(\epsilon) \ge |f(\omega_0) - g(\omega_0)| = \frac{|\phi(\omega_0)|}{(\omega_0^2 + h^2)M} \ge \frac{|\phi(\omega_0)|}{C}.$$

Taking the supremum over all admissible ϕ we obtain the remaining inequality in (5.10). ∎

5.2. The effect of the symmetry constraint.

Notation. Let $H^2 := H^2(\mathbb{H}_h)$, and let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and its induced norm in H^2.

The goal of this section is to analyze the asymptotics of the quantity $D_h(\epsilon)$, as $\epsilon \to 0$. Modulo symmetry $Sf = f$, this has already been done in [26]. Investigating the effect that symmetry may have on the asymptotics of $D_h(\epsilon)$ means relating it to

$$D_h^0(\epsilon) = \sup\left\{|f(\omega_0)| \ : f \in H^2, \ \|f\| \le 1, \ \text{and} \ \|f\|_{L^2(-1,1)} \le \epsilon\right\}. \tag{5.11}$$

The key feature of (5.11) is its invariance under multiplying f by a constant phase factor, which allowed us to replace the target functional $|f(\omega_0)|$ by a linear one, $\mathfrak{Re}\, f(\omega_0)$. Since multiplication by nonreal factors breaks the symmetry $Sf = f$, this reduction does not work for $D_h(\epsilon)$. Nevertheless, convexity of the target functional permits us to relate it to linear functionals if we observe that

$$|f(\omega_0)| = \max_{|\lambda|=1} \mathfrak{Re}(\overline{\lambda} f(\omega_0)).$$

Interchanging the order of maxima with respect to λ and f permits us to use our solution of (5.11) from [27] if we can eliminate the symmetry constraint. This is indeed possible. Following the ideas from the theory of reproducing kernel Hilbert spaces [41], we write the Cauchy integral formula as an inner product in H^2: $f(\omega_0) = (f, p_{\omega_0})$, where p_{ω_0} is given by (3.4). It is easy to check that for $f \in H^2$, satisfying the symmetry constraint we have

$$\mathfrak{Re}(\overline{\lambda}f(\omega_0)) = \mathfrak{Re}(f, \lambda p_{\omega_0}) = \mathfrak{Re}(f, q_{\omega_0.\lambda}), \qquad q_{\omega_0,\lambda} = \frac{\lambda p_{\omega_0} + S(\lambda p_{\omega_0})}{2}.$$

We can now discard the symmetry constraint. We claim that the maximizer function of the problem

$$D^0_{\lambda,h}(\epsilon) = \sup\left\{\mathfrak{Re}(f, q_{\omega_0,\lambda}) \;: f \in H^2,\ \|f\| \le 1,\ \text{and}\ \|f\|_{L^2(-1,1)} \le \epsilon\right\} \tag{5.12}$$

automatically has the required symmetry. Indeed, if $f \in H^2$ solves (5.12), we can decompose it into its symmetric and antisymmetric parts $f = f_s + f_a$, which are mutually real-orthogonal both in H^2 and $L^2(-1,1)$. In other words, they satisfy

$$\mathfrak{Re}(f_s, f_a) = \mathfrak{Re}(f_s, f_a)_{L^2(-1,1)} = 0.$$

Thus,

$$\|f\|^2 = \|f_s\|^2 + \|f_a\|^2 \ge \|f_s\|^2, \quad \|f\|^2_{L^2(-1,1)} = \|f_s\|^2_{L^2(-1,1)} + \|f_a\|^2_{L^2(-1,1)} \ge \|f_s\|^2_{L^2(-1,1)},$$

which implies that

$$\kappa = \max\left\{\|f_s\|, \frac{\|f_s\|_{L^2(-1,1)}}{\epsilon}\right\} \le 1.$$

Also, by the symmetry of $q_{\omega_0,\lambda}$ we find that

$$\mathfrak{Re}(f, q_{\omega_0,\lambda}) = \mathfrak{Re}(f_s, q_{\omega_0,\lambda}).$$

But then the function f_s/κ satisfies the constraints of (5.12) and strictly increases the value of the target functional unless $\kappa = 1$, or, equivalently, $f_a = 0$. Thus, if f is the maximizer, then it has to be symmetric.

According to Theorem 5.4 from the next section, the maximizer function $f^*_\epsilon(\omega)$ for (5.11) has the property that $f^*_\epsilon(\omega_0) = D^0_h(\epsilon) > 0$. Since removing the symmetry constraint increases the set of admissible functions, we have an obvious inequality:

$$D_h(\epsilon) \le f^*_\epsilon(\omega_0) = D^0_h(\epsilon). \tag{5.13}$$

Our foregoing discussion suggests that the function $v_{\lambda,\epsilon} = \lambda f^*_\epsilon$ must be a good candidate for the maximizer in $D^0_{\lambda,h}(\epsilon)$. Using it as a test function we get the inequality

$$D^0_{\lambda,h}(\epsilon) \ge \mathfrak{Re}(\lambda f^*_\epsilon, q_{\omega_0,\lambda}) = \frac{f^*_\epsilon(\omega_0)}{2} + \frac{1}{2}\mathfrak{Re}(\lambda^2(f^*_\epsilon, Sp_{\omega_0})).$$

We conclude that

$$D_h(\epsilon) = \max_{|\lambda|=1} D^0_{\lambda,h}(\epsilon) \ge \frac{f^*_\epsilon(\omega_0)}{2} + \frac{1}{2}|(f^*_\epsilon, Sp_{\omega_0})| \ge \frac{f^*_\epsilon(\omega_0)}{2} = \frac{1}{2}D^0_h(\epsilon).$$

Hence, we have shown that

$$\frac{1}{2}D^0_h(\epsilon) \le D_h(\epsilon) \le D^0_h(\epsilon). \tag{5.14}$$

5.3. Optimal bound for $D_h^0(\epsilon)$. Let us define

$$\gamma(\omega_0, h) = \gamma(h) = \varliminf_{\epsilon\to 0} \frac{\ln D^0_{\omega_0,h}(\epsilon)}{\ln \epsilon}. \tag{5.15}$$

Combining Theorem 5.3 and inequality (5.14) we see that $D_h^0(\epsilon) \lesssim \Delta_h(\epsilon) \lesssim D^0_{h'}(\epsilon)$ for any $h' \in (0, h)$ with implicit constants depending only on h and h'. This in particular implies

$$\gamma(\omega_0, h') \le \varliminf_{\epsilon\to 0} \frac{\ln \Delta_{\omega_0,h}(\epsilon)}{\ln \epsilon} \le \gamma(\omega_0, h) \qquad \forall h' \in (0, h). \tag{5.16}$$

It is clear that continuity of $\gamma(\omega_0, h)$ in h will imply that $\Delta_{\omega_0,h}(\epsilon)$ also has power law exponent $\gamma(\omega_0, h)$. Let us show that the same conclusion will follow under continuity of $\gamma(\omega_0, h)$ in ω_0 as well. Indeed, it is enough to show that

$$\gamma(\omega_0, h') \ge \gamma\left(\frac{h}{h'}\omega_0, h\right) \tag{5.17}$$

and combine this with (5.16). To prove inequality (5.17), let $f^*_{\epsilon,\omega_0,h'}(\omega)$ be the maximizer function for $D^0_{\omega_0,h'}(\epsilon)$ (cf. Theorem 5.4 below) and consider the function

$$g(z) = \sqrt{\frac{h'}{h}} f^*\left(\frac{h'}{h}z\right).$$

Note that $\|g\|_{H^2(\mathbb{H}_h)} = \|f^*\|_{H^2(\mathbb{H}_{h'})} = 1$ and $\|g\|_{L^2(-1,1)} \le \|f^*\|_{L^2(-1,1)} = \epsilon$. Therefore, g is an admissible function for $D^0_{\frac{h\omega_0}{h'},h'}(\epsilon)$, and hence

$$D^0_{\frac{h\omega_0}{h'},h'}(\epsilon) \ge g(\tfrac{h\omega_0}{h'}) = \sqrt{\frac{h'}{h}} f^*(\omega_0) = \sqrt{\frac{h'}{h}} D^0_{\omega_0,h'}(\epsilon),$$

which implies inequality (5.17). In particular, inequalities (5.16) and (5.17) imply that $\gamma(\omega_0, h)$ is a nonincreasing function of ω_0. Numerical computations of $\gamma(\omega_0, h)$ shown in Figure 3 indicate that $\gamma(\omega_0, h)$ is indeed a continuous function of ω_0. In Appendix A.2 we prove that $\gamma(\omega_0, h)$ is also a nondecreasing function of h, satisfying $\gamma(\omega_0, h) \in (0, 1)$ for any $h > 0$, and that $\lim_{h\to 0^+} \gamma(\omega_0, h) = 0$.

To find γ we derive an optimal bound for D^0_h. Consider the restriction operator $\mathscr{R} : H^2(\mathbb{H}_h) \to L^2(-1, 1)$ [40, 28]; then $\mathscr{K} = \mathscr{R}^*\mathscr{R}$ is a positive, compact, and self-adjoint integral operator defined by (3.4) (where we suppressed the h dependence from the notation). In particular, $\|f\|^2_{L^2(-1,1)} = (\mathscr{K}f, f)$. Multiplying f by a constant phase factor we can rewrite (5.11) as

$$\sup\left\{\mathfrak{Re}(f, p_{\omega_0}) \ : (f, f) \le 1 \text{ and } (\mathscr{K}f, f) \le \epsilon^2\right\}. \tag{5.18}$$

Theorem 5.4. *Let $\mathscr{K}$ and p_{ω_0} be given by* (3.4)*, and let $\eta = \eta(\epsilon, h, \omega_0) > 0$ be the unique solution of $\|(\mathscr{K} + \eta)^{-1}p_{\omega_0}\|_{L^2(-1,1)} = \epsilon\|(\mathscr{K} + \eta)^{-1}p_{\omega_0}\|$; then*

$$D^0_h(\epsilon) = \frac{u^*(\omega_0)}{\|u^*\|}, \tag{5.19}$$

where $u^* = u^*_{\epsilon,h,\omega_0}$ solves the integral equation $(\mathscr{K} + \eta)u^* = p_{\omega_0}$. In particular, the maximizer function is $f^* = u^*/\|u^*\|$.

We can actually express $D_h^0(\epsilon)$ only in terms of η.

LEMMA 5.5. *Let $\eta = \eta(\epsilon) > 0$ be as in Theorem 5.4; then*

$$D_h^0(\epsilon) = C \exp\left\{-\int_\epsilon^1 \frac{tdt}{t^2 + \eta(t)}\right\}, \tag{5.20}$$

where C is a constant independent of ϵ, namely, $C = D_h^0(1)$.

Proof. The definition of u^* implies $u^*(\omega_0) = (u^*, p_{\omega_0}) = (u^*, \mathscr{K}u^* + \eta u^*) = (u^*, \mathscr{K}u^*) + \eta(u^*, u^*)$, i.e.,

$$u^*(\omega_0) = \|u^*\|_{L^2(-1,1)}^2 + \eta\|u^*\|^2 = (\epsilon^2 + \eta)\|u^*\|^2, \tag{5.21}$$

where the last step follows from the definition of η. In particular we find that $D_h^0(\epsilon) = (\epsilon^2 + \eta)\|u^*\|$, and therefore it is enough to derive a formula for $\|u^*\|$ in terms of η. Let us write u_ϵ^* instead of u^* to show its dependence on ϵ. The key observation is the relation between $\partial_\epsilon u_\epsilon^*(\omega_0)$ and $\|u_\epsilon^*\|$, which we are going to use in (5.21) to deduce the desired formula. Let $\{e_n\}_{n=1}^\infty$ be the orthonormal basis of H^2 consisting of the eigenfunctions of $\mathscr{K}$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^\infty$. The integral equation for u_ϵ^* diagonalizes in this basis, and we find $(e_n, u_\epsilon^*) = e_n(\omega_0)/(\lambda_n + \eta(\epsilon))$. Therefore,

$$u_\epsilon^*(\omega_0) = \sum_{n=1}^\infty \frac{|e_n(\omega_0)|^2}{\lambda_n + \eta(\epsilon)}, \qquad \|u_\epsilon^*\|^2 = \sum_{n=1}^\infty \frac{|e_n(\omega_0)|^2}{(\lambda_n + \eta(\epsilon))^2}.$$

These formulas readily imply

$$\partial_\epsilon u_\epsilon^*(\omega_0) = -\eta'(\epsilon)\|u_\epsilon^*\|^2. \tag{5.22}$$

Differentiating (5.21) with respect to ϵ and using the relation (5.22) we find

$$(2\epsilon + \eta'(\epsilon))\|u_\epsilon^*\|^2 + 2\|u_\epsilon^*\|\left(\epsilon^2 + \eta(\epsilon)\right)\partial_\epsilon\|u_\epsilon^*\| = -\eta'(\epsilon)\|u_\epsilon^*\|^2,$$

which then gives

$$\frac{\partial_\epsilon\|u_\epsilon^*\|}{\|u_\epsilon^*\|} = -\frac{\epsilon + \eta'(\epsilon)}{\epsilon^2 + \eta(\epsilon)} = -\frac{2\epsilon + \eta'(\epsilon)}{\epsilon^2 + \eta(\epsilon)} + \frac{\epsilon}{\epsilon^2 + \eta(\epsilon)}. \tag{5.23}$$

Integrating (5.23) we find

$$\|u_\epsilon^*\| = \frac{C}{\epsilon^2 + \eta(\epsilon)} \exp\left\{-\int_\epsilon^1 \frac{tdt}{t^2 + \eta(t)}\right\}, \tag{5.24}$$

which concludes the proof. ∎

Combining (5.19) with (5.21) on one hand and using (5.20) on the other hand (where we change the variables in the integral), we obtain two different representations for the power law exponent:

$$\gamma(h) = \varliminf_{\epsilon\to 0} \frac{\ln\left((\epsilon + \frac{\eta}{\epsilon})\|u^*\|_{L^2(-1,1)}\right)}{\ln\epsilon} = \varliminf_{t\to+\infty} \frac{1}{t}\int_0^t \frac{dx}{1 + e^{2x}\eta(e^{-x})}. \tag{5.25}$$

Thus, understanding the asymptotic behavior of $\eta(\epsilon)$ as $\epsilon \to 0$ is crucial to unraveling the above formulas. Expanding the two norms in the eigenbasis of $\mathscr{K}$, we see that η solves

$$\Phi(\eta) := \frac{\sum_{n=1}^\infty \frac{\lambda_n|e_n(\omega_0)|^2}{(\lambda_n+\eta)^2}}{\sum_{n=1}^\infty \frac{|e_n(\omega_0)|^2}{(\lambda_n+\eta)^2}} = \epsilon^2. \tag{5.26}$$

This equation has a unique solution $\eta = \eta(\epsilon) > 0$, because $\Phi(\eta)$ is monotone increasing (since its derivative can be shown to be positive), $\Phi(+\infty) = (\mathscr{K} p_{\omega_0}, p_{\omega_0})/\|p_{\omega_0}\|^2$, and $\Phi(0^+) = 0$ (see [27] for technical details). Finding the asymptotics of $\eta(\epsilon)$ lies beyond the capabilities of classical asymptotic methods. Nevertheless, under the purported exponential decay (3.9) of eigenvalues and eigenfunctions (at the point ω_0) of $\mathscr{K}$ we proved in [27] that $\Phi(\eta) \simeq \eta$ with implicit constants independent of η, leading to $\eta(\epsilon) \simeq \epsilon^2$ with implicit constants independent of ϵ. Moreover, we also showed that $\|u^*\|_{L^2(-1,1)} \simeq \epsilon^{\frac{2\beta}{\alpha}-1}$, which then implies that the ratio inside the first $\underline{\lim}$ in (5.25) converges as $\epsilon \to 0$ and gives the formula $\gamma(h) = 2\beta/\alpha$.

On the other hand, substituting $\lambda_n, |e_n(\omega_0)|$ in (5.26) with their corresponding exponentials from (3.9), and applying (a version) of Lemma 3.1, we can approximate

$$\Phi(\eta) \approx \eta L\left(\ln\left(\frac{1}{\eta}\right)\right), \qquad L(\tau) = \frac{e^{\tau} \displaystyle\sum_{k\in\mathbb{Z}} \frac{e^{(\alpha+2\beta)k}}{(e^{\alpha k}+e^{-\tau})^2}}{\displaystyle\sum_{k\in\mathbb{Z}} \frac{e^{2\beta k}}{(e^{\alpha k}+e^{-\tau})^2}}. \tag{5.27}$$

Note that $L(\tau)$ is an elliptic function with periods α and $2\pi i$; furthermore, it has symmetries $\overline{L(\tau)} = L(\overline{\tau})$ and $L(2\beta - \tau) = L(\tau)$. Figure 6 shows the plot of L. Therefore, we expect $\epsilon^{-2}\eta(\epsilon)$ to be oscillatory and periodic as $\epsilon \to 0$; more precisely,

$$\epsilon^{-2}\eta(\epsilon) \sim \frac{1}{L(-2\ln\epsilon)}.$$

So the integral averages of the function $r(x) = (1+e^{2x}\eta(e^{-x}))^{-1}$ in the second formula of (5.25) converge to the integral (over one period) of its periodic approximation, namely,

$$\frac{2\beta}{\alpha} = \gamma(h) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t r(x)dx = \lim_{t\to+\infty}\int_0^1 r(tx)dx = \int_0^1 \frac{L(2x)}{1+L(2x)}dx.$$

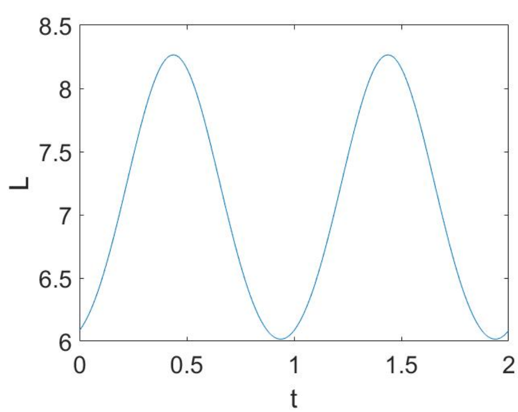


FIG. 6. *The graph of $L(t)$ for $\alpha = 4$ and $\beta = 1.75$.*

This insight into the asymptotic behavior of $\eta(\epsilon)$ allowed us to prove a bound that is optimal up to the constant 3/2, but which is accessible numerically. Namely, with $u = u_{\epsilon,h,\omega_0}$ denoting the solution of the integral equation $(\mathscr{K} + \epsilon^2)u = p_{\omega_0}$, in [27] we showed that

$$D_h^0(\epsilon) \le \frac{3}{2} u(\omega_0) \min\left\{ \frac{1}{\|u\|}, \frac{\epsilon}{\|u\|_{L^2(-1,1)}} \right\}.$$

We expect the two quantities under the above minimum to be comparable (this is just a restatement of $\eta(\epsilon) \simeq \epsilon^2$, which holds under (3.9), in fact it also holds under weaker conditions, as we observed in [27]), in which case the formula for $\gamma(h)$ given in (3.5) follows (compare with the first part of (5.25)).

The proof of Theorem 5.4 follows from [27] without much change. The only difference is that in the above formulation we presented the exact maximizer for D_h^0, versus the 3/2-maximizer presented in [27]. For the sake of completeness we give a short recap of the argument.

Proof of Theorem 5.4. For every f satisfying the two constraints of (5.18) and for every nonnegative numbers μ and ν ($\mu^2 + \nu^2 \neq 0$) we have the inequality

$$((\mu + \nu\mathscr{K})f, f) \le \mu + \nu\epsilon^2. \tag{5.28}$$

Applying convex duality to the quadratic functional on the left-hand side of (5.28) we get

$$\Re(f, p_{\omega_0}) - \frac{1}{2}\left((\mu + \nu\mathscr{K})^{-1} p_{\omega_0}, p_{\omega_0}\right) \le \frac{1}{2}((\mu + \nu\mathscr{K})f, f) \le \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \tag{5.29}$$

so that

$$\Re(f, p_{\omega_0}) \le \frac{1}{2}\left((\mu + \nu\mathscr{K})^{-1} p_{\omega_0}, p_{\omega_0}\right) + \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \tag{5.30}$$

which is valid for every f satisfying the constraints of (5.18) and all $\mu > 0$, $\nu \ge 0$. In order for the bound to be optimal we must have equality in (5.29), which holds if and only if $p_{\omega_0} = (\mu + \nu\mathscr{K})f$, giving the formula for optimal vector f:

$$f = (\mu + \nu\mathscr{K})^{-1} p_{\omega_0}. \tag{5.31}$$

The goal is to choose the Lagrange multipliers μ and ν so that the constraints in (5.18) are satisfied by f, given by (5.31). If $\nu = 0$, then $f = \frac{p_{\omega_0}}{\|p_{\omega_0}\|}$ does not depend on the small parameter ϵ, which leads to a contradiction, because the second constraint $(\mathscr{K} f, f) \le \epsilon^2$ is violated when ϵ is small enough. If $\mu = 0$, then $\mathscr{K} f = \frac{1}{\nu} p_{\omega_0}$. But this equation has no solution in H^2 since p_{ω_0} has a singularity at $\overline{\omega}_0 - 2ih$, while $\mathscr{K} f$ has an analytic extension to $\mathbb{C}\setminus[-1,1] - 2ih$.

Thus we are looking for $\mu > 0$, $\nu > 0$, so that equalities in (5.18) hold (these are the complementary slackness relations in Karush–Kuhn–Tucker conditions), i.e.,

$$\begin{cases} \left((\mu + \nu\mathscr{K})^{-1} p_{\omega_0}, (\mu + \nu\mathscr{K})^{-1} p_{\omega_0}\right) = 1, \\ \left(\mathscr{K}(\mu + \nu\mathscr{K})^{-1} p_{\omega_0}, (\mu + \nu\mathscr{K})^{-1} p_{\omega_0}\right) = \epsilon^2. \end{cases} \tag{5.32}$$

Let $\eta = \frac{\mu}{\nu}$; solving the first equation in (5.32) for ν we find $\nu = \|(\mathscr{K} + \eta)^{-1} p_{\omega_0}\|$. The second equation then reads

$$\Phi(\eta) := \frac{\left(\mathscr{K}(\mathscr{K} + \eta)^{-1} p_{\omega_0}, (\mathscr{K} + \eta)^{-1} p_{\omega_0}\right)}{\|(\mathscr{K} + \eta)^{-1} p_{\omega_0}\|^2} = \epsilon^2,$$

which has a unique solution $\eta = \eta(\epsilon) > 0$, because $\Phi(\eta)$ is monotone increasing (since its derivative can be shown to be positive), $\Phi(+\infty) = (\mathscr{K} p_{\omega_0}, p_{\omega_0})/\|p_{\omega_0}\|^2$, and $\Phi(0^+) = 0$ (see [27] for technical details). Setting $u^* = (\mathscr{K} + \eta)^{-1} p_{\omega_0}$, (5.30) reads

$$\Re(f, p_{\omega_0}) \leq \frac{(u^*, p_{\omega_0})}{2\|u^*\|} + \frac{\|u^*\|}{2}(\epsilon^2 + \eta) = \frac{u^*(\omega_0)}{\|u^*\|},$$

where in the last step we used (5.21). ∎

Appendix A.

A.1. Extension of positivity.

PROPOSITION A.1. *Let f be analytic in $\mathbb{H}_h$ with $Sf = f$, where $Sf(\omega) := \overline{f(-\overline{\omega})}$, and $f(\omega) \sim -A\omega^{-2}$ as $\omega \to \infty$ for some $A > 0$. In addition, assume $f'(0) \neq 0$. Then the following are equivalent:*

(i) $\mathfrak{Im}\, f(x) > 0$ *for all* $x > 0$*;*

(ii) $\exists\, h' \in (0, h)$ *such that* $\mathfrak{Im}\, f(x - ih') > 0$ *for all* $x > 0$.

Proof. The second item immediately implies the first one. Indeed, the symmetry $Sf = f$ implies that $\mathfrak{Im} f = 0$ on the imaginary axis. Let $\Omega = \{\omega : \mathfrak{Im}\,\omega > -h',\ \mathfrak{Re}\,\omega > 0\}$. Note that $\mathfrak{Im} f \geq 0$ on $\partial\Omega$; by the strong maximum principle $\min_{\overline{\Omega}} \mathfrak{Im} f$ cannot be attained in Ω, and hence we conclude that $\mathfrak{Im} f > 0$ in Ω. (Note that the assumption $f'(0) \neq 0$ was not used here.)

Let us now turn to the converse implication. Let $h_0 \in (0, h)$; then f is analytic in the closure $\overline{\mathbb{H}}_{h_0}$ and in particular is bounded inside the semidisc $D = \{\omega \in \mathbb{H}_{h_0} : |\omega + ih_0| \leq M\}$, where $M > 0$ is a large number that can be chosen such that $|f(\omega)| \leq 2A/|\omega|^2$ for all $\omega \notin D$. With these two inequalities, it is straightforward to show that $\int_{\mathbb{R}} |f(x + iy)|^2 dx$ is bounded uniformly for $y > -h_0$. Thus, $f \in H^2(\mathbb{H}_{h_0})$, and following the calculations in the proof of Lemma 5.1 leading from (5.5) to (5.7), we obtain the representation

$$f(\omega) = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - (\omega + ih_0)^2}, \qquad \omega \in \mathbb{H}_{h_0},$$

where $d\sigma(\lambda) = \frac{1}{\pi}\mathfrak{Im} f(\sqrt{\lambda} - ih_0)d\lambda$. Using this, it is easy to find that f must have the more precise asymptotics, as $\omega \to \infty$ in $\mathbb{H}_{h_0}$:

$$f(\omega) \sim A\left(-\frac{1}{\omega^2} + \frac{2ih_0}{\omega^3}\right), \qquad A = \int_0^\infty d\sigma(\lambda).$$

But then for any $t \in (0, h_0)$,

$$\mathfrak{Im}\, f(x - it) \sim \frac{2A(h_0 - t)}{x^3} > 0, \qquad x \to +\infty. \tag{A.1}$$

Assume, for the sake of contradiction, that for each $t \in (0, h_0)$ there exists $x_t > 0$, such that $\mathfrak{Im} f(x_t - it) \leq 0$. Clearly, (A.1) implies that x_t remains bounded as $t \to 0^+$. Let us now extract the convergent subsequence (without relabeling it) $x_t \to x_0 \geq 0$ as $t \to 0^+$, but then $\mathfrak{Im}\, f(x_0) \leq 0$. Assumption (i) implies that $x_0 = 0$. Let us show that in this case $f'(0) = 0$, which is assumed to not be the case. Since $\mathfrak{Im}\, f(x_t) > 0$ and $\mathfrak{Im} f(x_t - it) \leq 0$, by continuity we conclude that $\exists\, \theta_t \in (0, 1]$ such that $\mathfrak{Im} f(x_t - i\theta_t t) = 0$. The symmetry $Sf = f$ implies that $\mathfrak{Im} f(-i\theta_t t) = 0$, and therefore by the mean value theorem $\mathfrak{Im} f'(\tilde{x}_t - i\theta_t t) = 0$ for some $\tilde{x}_t \in (0, x_t)$. Taking limits as $t \to 0^+$ we obtain $\mathfrak{Im} f'(0) = 0$, but by symmetry $f'(0) \in i\mathbb{R}$, and hence $f'(0) = 0$. ∎

A.2. Power law bounds. Let $D_h^0(\epsilon)$ and $\gamma(h)$ be defined by (5.11) and (5.15), respectively. Note that $D_h^0(\epsilon)$ is nonincreasing in h. Indeed, $\mathbb{H}_{h_1} \subset \mathbb{H}_{h_2}$ for $h_1 \le h_2$ and so admissible functions for $D_{h_2}^0(\epsilon)$ are also admissible for $D_{h_1}^0(\epsilon)$, showing that $D_{h_2}^0(\epsilon) \le D_{h_1}^0(\epsilon)$. Now dividing by $\ln\epsilon < 0$ and taking $\underline{\lim}$ in ϵ we conclude that $\gamma(h)$ is nondecreasing.

Let us turn to deriving power law upper and lower bounds on $D_h^0(\epsilon)$. We are going to use the following two results from [27] and [26]. The first one is an analytic continuation from a boundary interval: for any $s \in \mathbb{H}_+$,

$$\sup\{|f(s)| : f \in H^2(\mathbb{H}_+),\ \|f\|_{H^2(\mathbb{H}_+)} \le 1,\ \text{and}\ \|f\|_{L^2(-1,1)} \le \delta\} \le C(s)\delta^{\alpha(s)}, \tag{A.2}$$

where $C(s)^{-2} = \frac{s_i}{9}\left(\arctan\frac{s_r+1}{s_i} - \arctan\frac{s_r-1}{s_i}\right)$ with $s = s_r + is_i$, and $\alpha(s) = -\frac{1}{\pi}\arg\frac{s+1}{s-1} \in (0,1)$ is the angular size of $[-1,1]$ as seen from s, measured in the units of π radians. Moreover, the bound is optimal in δ and the maximizer function attaining the bound (up to a constant independent of δ) in (A.2) is given by

$$G(\zeta) = \frac{\delta}{\zeta - \overline{s}} e^{\frac{i}{\pi}\ln\delta\ln\frac{1+\zeta}{1-\zeta}}, \qquad \zeta \in \mathbb{H}_+ \tag{A.3}$$

where ln denotes the principal branch of logarithm.

The second one is an analytic continuation from a circle. Namely, let $\Gamma \subset \mathbb{H}_+$ be a circle and $s \in \mathbb{H}_+$ a point lying outside of Γ; then

$$\sup\{|f(s)| : f \in H^2(\mathbb{H}_+),\ \|f\|_{H^2(\mathbb{H}_+)} \le 1,\ \text{and}\ \|f\|_{L^2(\Gamma)} \le \epsilon\} \simeq \epsilon^{\beta(s)}, \tag{A.4}$$

with implicit constants independent of ϵ and $\beta(s) = \frac{\ln|m(s)|}{\ln\rho}$, where m is the Möbius map transforming the upper half-plane into the unit disc and the circle Γ into a concentric circle of radius $\rho < 1$.

LEMMA A.2. *There exist $\gamma_0, \gamma_1 \in (0,1)$ (depending on ω_0, h) such that*

$$\epsilon^{\gamma_1} \lesssim D_h^0(\epsilon) \lesssim \epsilon^{\gamma_0}, \tag{A.5}$$

where the implicit constants depend only on h and ω_0. Moreover, $\gamma_1(h) \to 0$ as $h \to 0^+$.

Proof. The lower bound is obtained by introducing an ansatz function admissible for $D_h^0(\epsilon)$. Consider the function G in (A.3) with $s = ih$, then the ansatz function is going to be $f(\omega) = G(\omega + ih)$. Note that we can rewrite

$$G(\zeta) = \frac{\delta^{\alpha(\zeta)} e^{i\theta_\delta(\zeta)}}{\zeta + ih}, \qquad \theta_\delta(\zeta) = \frac{1}{\pi}\ln\delta\ln\left|\frac{1+\zeta}{1-\zeta}\right|.$$

It is now clear that

$$\|G\|_{L^2((-1,1)+ih)} \lesssim \delta^{\alpha_0}, \qquad \alpha_0 = \min_{x\in[-1,1]}\alpha(x+ih) = \frac{1}{\pi}\arctan\frac{2}{h} \in (0,1),$$

and $|G(\omega_0 + ih)| \gtrsim \delta^\alpha$, where $\alpha = \alpha(\omega_0 + ih) < \alpha_0$ (see Figure 7). Thus,

$$\|f\|_{H^2(\mathbb{H}_h)} \lesssim 1, \qquad \|f\|_{L^2(-1,1)} \lesssim \delta^{\alpha_0}, \qquad |f(\omega_0)| \gtrsim \delta^\alpha. \tag{A.6}$$

Letting $\epsilon = \delta^{\alpha_0}$ we see that cf is an admissible function for $D_h^0(\epsilon)$ for some constant $c > 0$ independent of δ, and hence

$$D_h^0(\epsilon) \ge c|f(\omega_0)| \gtrsim \delta^\alpha = \epsilon^{\gamma_1},$$

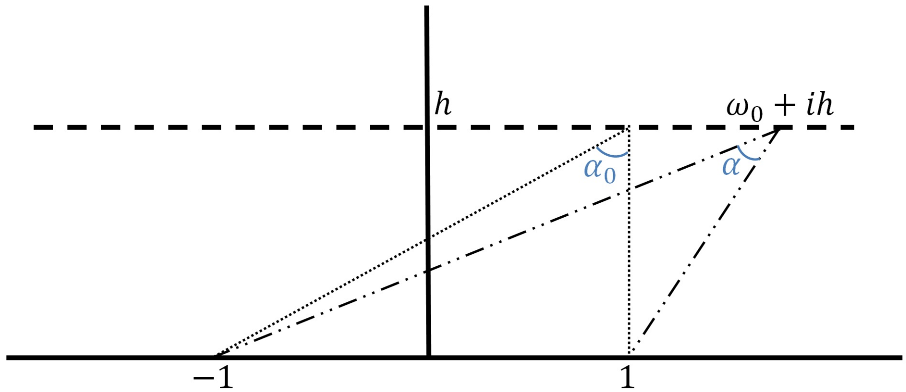


FIG. 7. *Comparison of angles.*

where $\gamma_1 = \gamma_1(h) = \alpha/\alpha_0 \in (0,1)$. It remains to notice that $\gamma_1(h) \to 0$ as $h \to 0^+$.

Let us now turn to the upper bound. Let f be an admissible function for $D_h^0(\epsilon)$; it is clear that f is also admissible for (A.2) with $\delta = \epsilon$. However, applying the estimate in (A.2) at the point $\omega_0 > 1$ doesn't give a useful bound, since $\alpha(\omega_0) = 0$. Instead let us apply (A.2) at the points s lying on the circle $\mathcal{C} = \{s \in \mathbb{H}_+ : |s - i| = \frac{1}{2}\}$. It is clear that the angle $\alpha(s)$ is the smallest at the top point of the circle, i.e., at $s_0 = \frac{3}{2}i$. Moreover, obviously the constant $C(s)$ in (A.2) is uniformly bounded for all $s \in \mathcal{C}$. Thus,

$$|f(s)| \lesssim \epsilon^{\beta_0} \qquad \forall s \in \mathcal{C}, \quad \text{where} \quad \beta_0 = \alpha(s_0) = \frac{1}{\pi} \arctan \frac{12}{5}$$

and the implicit constant is independent of s and ϵ. In particular, $\|f\|_{L^2(\mathcal{C})} \lesssim \epsilon^{\beta_0}$. Now we can apply (A.4) to the function $f(\cdot - ih)$ at the point $s = \omega_0 + ih$ and obtain

$$\text{(A.7)} \qquad |f(\omega_0)| \lesssim \epsilon^{\gamma_0}, \qquad \gamma_0 = \beta_0 \cdot \beta(\omega_0 + ih) = \beta_0 \frac{\ln |m(\omega_0 + ih)|}{\ln \rho},$$

where $m(z) = \frac{z - z_0}{z + z_0}$ with $z_0 = \frac{i}{2}\sqrt{4h^2 + 8h + 3}$ and $\rho = 2h + 2 - \sqrt{4h^2 + 8h + 3}$. Taking the supremum over f in (A.7) we conclude the proof of the upper bound. ☐

As an immediate corollary from Lemma A.2 we see that for any $h > 0$

$$\gamma(h) \in [\gamma_0(h), \gamma_1(h)] \subset (0,1)$$

and also $\gamma(h) \to 0$ as $h \to 0^+$.

Acknowledgment. We are grateful to Leslie Greengard for providing the quad precision FORTRAN code for solving the integral equation (3.3) for ϵ as low as 10^{-16} and for computing the eigenvalues of $\mathscr{K}_h$ as small as 10^{-32}.

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