Inverse Problems

PAPER

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IOP Publishing Inverse Problems
Inverse Problems **38** (2022) 065003 (34pp) https://doi.org/10.1088/1361-6420/ac5f75

Integrable nonlocal derivative nonlinear Schrödinger equations

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Received 13 November 2021, revised 14 March 2022
Accepted for publication 21 March 2022
Published 19 April 2022

Abstract

Integrable standard and nonlocal derivative nonlinear Schrödinger equations are investigated. The direct and inverse scattering are constructed for these equations; included are both the Riemann–Hilbert and Gel'fand–Levitan–Marchenko approaches and soliton solutions. As a typical application, it is shown how these derivative NLS equations can be obtained as asymptotic limits from a nonlinear Klein–Gordon equation.

Keywords: inverse scattering transform, Riemann–Hilbert problems, Gel'fand–Levitan–Marchenko equations, the derivative NLS equations, solitons

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear Schrödinger (NLS) equations are among the most physically important equations in mathematical physics. The one space one time integrable cubic NLS equation

$$\mathrm{i}q_t + q_{xx} \pm 2q^2q^* = 0,$$

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where $*$ represents complex conjugation, is a universal model arising in nonlinear dispersive waves cf [1]. Soon after the Korteweg–deVries (KdV) equation was integrated for rapidly decaying data [17], this NLS equation was also found to be integrable by the inverse scattering methods [23]. In 1974, KdV, NLS, modified KdV, sine-Gordon and, more generally, a class of nonlinear equations were integrable by a unified method, termed the inverse scattering transform (IST) [4]. Many new and physically significant equations were subsequently found to be integrable by these procedures, both continuous and discrete cf [3, 9, 11], and there have been extensive results inspired by IST [1, 2, 10]. Among these equations is the derivative NLS equation which arises in plasma physics [19], see equation (2.11) below. In [19], a Gel'fand–Levitan–Marchenko approach was employed to carry out the inverse scattering, the case of non-vanishing background was subsequently considered by Kawata and Inoue [20], and the associated 1-soliton solutions were attained accordingly [19, 20]. The 2-soliton solutions for zero and nonzero boundary conditions were reconstructed by Kawata *et al* [21]; later Chen and Lam utilized Riemann–Hilbert methods to explore the case of non-decaying data [14]. Subsequently, the multi-soliton solutions of the derivative NLS equation with vanishing and non-vanishing backgrounds were investigated [15, 26]. More recently, the double-pole solitons were formulated via IST [25]. Moreover, Liu, Perry and Sulem applied IST to study global existence for the derivative NLS equation [22]. Subsequently, the long-time asymptotics for the solution of the derivative NLS equation with generic initial data in a weighted Sobolev space was analyzed and the asymptotical stability for the soliton solutions was proven [18]. In addition, the well-posedness and regularity of the derivative NLS equation on the half line were discussed by Erdoğan *et al* [16]. Even though there was extensive research in the field of integrable systems/soliton theory, it was not until 2013–2017 that large classes of new nonlocal equations (of very simple form) were obtained and solved via AKNS procedure [5–7]. This included the PT symmetric NLS, the reverse space time (RST) NLS and reverse time (RT) NLS equations:

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) \pm 2q^2\left((x,t)q^*(-x,t) = 0, \quad \text{PT NLS},\right.$$

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) \pm 2q^2(x,t)q(-x,-t) = 0, \quad \text{RST NLS},$$

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) \pm 2q^2(x,t)q(x,-t) = 0, \quad \text{RT NLS}.$$

In this paper, we analyze the derivative NLS equation in the general case, the 'standard' derivative NLS equation (2.11), the nonlocal PT symmetric derivative NLS equation (2.13) and the RST derivative NLS equation (2.16); we remark that we do not find an equivalent integrable RT derivative NLS equation.

It should be pointed out that the derivative NLS type equations are extremely important. From mathematical viewpoint, they are integrable systems and hence amenable to IST. Thus, they have deep underlying mathematical structure, an infinite number of conserved quantities and soliton solutions. In addition, the nonlocal PT and RST derivative NLS equations are new. In this paper, the IST is employed to investigate the Cauchy problems for these novel nonlocal equations; we revisit the standard derivative NLS equation. The inverse scattering via both Riemann–Hilbert and Gel'fand–Levitan–Marchenko approaches and soliton solutions are formulated. The nonlocal derivative NLS systems are extremely simple in form. From physical intuition, one expects that simple equations should be derivable from physically related problems. Indeed that is what we find. Here the derivative NLS type equations, including the standard and two nonlocal cases, are obtained from a nonlinear Klein–Gordon type equation via multi-scale methods.

Note that different symmetry reductions between potentials q and r yield different spectral problems, see equation (2.1). For the standard derivative NLS equation, there is one-to-one correspondence between eigenvalues k_j^2 defined in the upper half plane and $\overline{k}_j^2$ in the lower half plane. However, such symmetries are not valid for either the PT or RST derivative NLS equations. Consequently, there are different types of solutions that can occur; e.g. they admit both singular and non-singular solutions; furthermore, the simplest soliton solution for the PT case is a 2-soliton (there is no pure 1-soliton solution).

The outline of this paper is as follows. In section 2, we find compatible linear systems associated with the general derivative NLS equations including the standard derivative NLS and two nonlocal derivative NLS equations: the PT symmetric and RST derivative NLS equations. Section 3 contains the direct scattering analysis, time dependence, symmetries and trace formulae. Unlike the standard case, trace formulae are necessary in order to carry out the complete IST in the nonlocal cases. Section 4 details the inverse scattering via the Riemann–Hilbert method and pure soliton solutions. The Gel'fand–Levitan–Marchenko approach is discussed in section 5. In section 6, we address the important question of how the standard and nonlocal systems arise in physically related systems. Here, by allowing for solutions to be complex, we show how the general 'q, r' derivative NLS system: (2.8) and (2.9) is derived as a quasi-monochromatic asymptotic limit from a nonlinear Klein–Gordon type equation. Since the general derivative NLS equation has reductions to the standard and nonlocal derivative NLS equations, they are all contained as asymptotic limits from this nonlinear Klein–Gordon type equation. This is consistent with the result in [8], where it was shown that there are quasi-monochromatic asymptotic reductions from nonlinear Klein–Gordon, KdV and water wave equations to the general 'q, r' NLS equations found in [4]; these 'q, r' NLS equations have symmetry reductions to the nonlocal NLS equations: PT NLS, RST NLS and RT NLS equations. We then conclude.

2. Compatible linear system: nonlocal derivative NLS equations

We begin with the linear scattering problem

$$\mathbf{v}_x = \mathsf{X}\mathbf{v} = (\mathrm{i}k^2 D + kQ(x,t))\mathbf{v}, \tag{2.1}$$

where $\mathbf{v} = \mathbf{v}(x,t)$ is a two-component vector: $\mathbf{v}(x,t) = (v_1(x,t), v_2(x,t))^{\mathrm{T}}$; X is a 2×2 matrix; k is a complex spectral parameter; $D = \mathrm{diag}(-1,1)$ and $Q(x,t)$ is an off diagonal matrix depending on two complex-valued potentials: $q(x,t)$, $r(x,t)$ that vanish rapidly as $|x| \to \infty$. Below we show that $q(x,t)$, $r(x,t)$ satisfy coupled nonlinear equations. More explicitly, the matrix X takes the form

$$\mathsf{X} = \begin{pmatrix} -\mathrm{i}k^2 & kq(x,t) \\ kr(x,t) & \mathrm{i}k^2 \end{pmatrix}. \tag{2.2}$$

Associated with the scattering problem (2.1), the time evolution equation of the eigenfunctions v_j, $j = 1, 2$, is given by

$$\mathbf{v}_t = \mathsf{T}\mathbf{v}, \tag{2.3}$$

where

$$\mathsf{T} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix}, \tag{2.4}$$

and the quantities A, B and C are scalar functions of $q(x,t), r(x,t)$ and the spectral parameter k. Depending on the choice of these functions, one finds an evolution equation for the potential functions $q(x,t)$ and $r(x,t)$ which, under a certain symmetry restriction, leads to a single evolution equation for either $q(x,t)$ or $r(x,t)$. The above formulation [19] is a generalization of the AKNS construction [3, 4, 11].

We are interested in the case where the quantities A, B and C are polynomials of degree four in the constant parameter k with coefficients depending on $q(x,t), r(x,t)$:

$$\mathsf{A} = -\mathrm{i}(2k^4 + \mathrm{i}k^2 q(x,t)r(x,t)), \tag{2.5}$$

$$\mathsf{B} = -\mathrm{i}(2\mathrm{i}k^3 q(x,t) - kq_x + \mathrm{i}krq^2), \tag{2.6}$$

$$\mathsf{C} = -\mathrm{i}(2\mathrm{i}k^3 r(x,t) + kr_x + \mathrm{i}kr^2 q). \tag{2.7}$$

The compatibility condition of system (2.1) and (2.3) leads to

$$q_t(x,t) = \mathrm{i}q_{xx}(x,t) + (q^2(x,t)r(x,t))_x, \tag{2.8}$$

$$r_t(x,t) = -\mathrm{i}r_{xx}(x,t) + (r^2(x,t)q(x,t))_x. \tag{2.9}$$

Below we give three symmetries of this 'q, r' system associated with the spectral/scattering problem (2.1) and (2.2). Under the symmetry reduction

$$\text{(i) } r(x,t) = \sigma q^*(x,t), \qquad \sigma = \pm 1, \tag{2.10}$$

the system (2.8) and (2.9) is compatible; this yields the standard derivative nonlinear Schrödinger (derivative NLS) equation:

$$q_t(x,t) = \mathrm{i}q_{xx}(x,t) + \sigma(q^2(x,t)q^*(x,t))_x, \tag{2.11}$$

which was found/analyzed in [19]. There are two more symmetry reductions which lead to integrable nonlocal nonlinear equations.

$$\text{(ii) } r(x,t) = \mathrm{i}\sigma q^*(-x,t), \qquad \sigma = \pm 1. \tag{2.12}$$

In this case, the compatibility of (2.8) and (2.9) yields the nonlocal PT derivative nonlinear Schrödinger (PT derivative NLS) equation:

$$q_t(x,t) = \mathrm{i}q_{xx}(x,t) + \mathrm{i}\sigma(q^2(x,t)q^*(-x,t))_x. \tag{2.13}$$

We note that when the equation is put in the form

$$q_t(x,t) = \mathrm{i}q_{xx}(x,t) + \mathrm{i}\sigma(V[q;x,t]q(x,t))_x, \qquad V[q;x,t] = q(x,t)q^*(-x,t), \tag{2.14}$$

then the potential $V[q;\ x,t]$ satisfies the PT-symmetry condition $V[q;\ x,t] = V^*[q;-x,t]$.

$$\text{(iii) } r(x,t) = \sigma q(-x,-t), \qquad \sigma = \pm 1. \tag{2.15}$$

In this case, the compatibility of (2.8) and (2.9) leads to the reverse space-time derivative nonlinear Schrödinger (RST derivative NLS) equation:

$$q_t(x,t) = \mathrm{i}q_{xx}(x,t) + \sigma(q^2(x,t)q(-x,-t))_x. \tag{2.16}$$

We further remark that unlike the standard NLS equation, the sign of $\sigma = \pm 1$ does not matter in the derivative NLS equations; it can be rescaled to unity. So it is sufficient to carry out the analysis for $\sigma = 1$ only. Indeed, it is due to the invariance $x \to -x$. The 'q, r' system (2.8) and (2.9) here does not admit pure RT (without reverse space) symmetry; this is different from the 'q, r' system in the AKNS case [7], i.e. there is no analog of the RT NLS equation mentioned in the introduction.

The 'q, r' system (2.8) and (2.9) has an infinite number of conserved quantities, among which the simplest one is

$$C_0 = \int_{-\infty}^{\infty} qr\,\mathrm{d}x \equiv \int qr\,\mathrm{d}x. \tag{2.17}$$

We also note that the above nonlocal equations which are nonlocal in space or nonlocal in both space and time are embedded into the local 'q, r' system. Namely, they satisfy the local system (2.8) and (2.9). Then the nonlocal equation (2.13) is obtained from (2.8) and (2.9) with the initial condition $r(x, t = 0) = \mathrm{i}\sigma q^*(-x, t = 0)$; similarly the nonlocal equation (2.15) is obtained from (2.8) and (2.9) with the initial condition $r(x, t = 0) = \sigma q(-x, t = 0)$.

3. Direct scattering, time dependence, symmetries, trace formulae

3.1. Direct scattering

We will assume that $q(x, t), r(x, t) \to 0$ rapidly as $|x| \to \infty$. The solutions to the scattering problem (2.1) and (2.2) are defined by their boundary conditions

$$\begin{aligned} &\phi(x,t,k) \sim \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-\mathrm{i}k^2x}, \qquad \overline{\phi}(x,t,k) \sim \begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}k^2x}, \quad \text{as } x\to-\infty,\\ &\psi(x,t,k) \sim \begin{pmatrix}0\\1\end{pmatrix}\mathrm{e}^{\mathrm{i}k^2x}, \qquad \overline{\psi}(x,t,k) \sim \begin{pmatrix}1\\0\end{pmatrix}\mathrm{e}^{-\mathrm{i}k^2x}, \quad \text{as } x\to+\infty. \end{aligned} \tag{3.1}$$

Note that the bar on the top of a quantity does not denote complex conjugation. In addition, the bounded eigenfunctions are defined as follows:

$$M(x,t,k) = \phi(x,t,k)\mathrm{e}^{\mathrm{i}k^2x}, \qquad \overline{M}(x,t,k) = \overline{\phi}(x,t,k)\mathrm{e}^{-\mathrm{i}k^2x}, \tag{3.2}$$

$$N(x,t,k) = \psi(x,t,k)\mathrm{e}^{-\mathrm{i}k^2x}, \qquad \overline{N}(x,t,k) = \overline{\psi}(x,t,k)\mathrm{e}^{\mathrm{i}k^2x}. \tag{3.3}$$

From the asymptotics (3.1), one has

$$W(\phi,\overline{\phi}) = \lim_{x\to-\infty} W(\phi(x,t,k),\overline{\phi}(x,t,k)) = 1, \tag{3.4}$$

$$W(\psi,\overline{\psi}) = \lim_{x\to+\infty} W(\psi(x,t,k),\overline{\psi}(x,t,k)) = -1, \tag{3.5}$$

where $W(u, v)$ is the Wronskian of the two solutions u, v of the scattering problem (2.1) and (2.2), i.e., $W(u, v) = u_1v_2 - v_1u_2$.

The functions $\phi(x, t, k)$, $\overline{\phi}(x, t, k)$ and $\psi(x, t, k)$, $\overline{\psi}(x, t, k)$ are linearly independent, hence we can write

$$\phi(x,t,k) = a(k,t)\overline{\psi}(x,t,k) + b(k,t)\psi(x,t,k) \tag{3.6}$$

and

$$\overline{\phi}(x,t,k) = \overline{a}(k,t)\psi(x,t,k) + \overline{b}(k,t)\overline{\psi}(x,t,k). \tag{3.7}$$

From (3.4) and (3.5), we deduce the following characterization equation

$$a(k,t)\overline{a}(k,t) - b(k,t)\overline{b}(k,t) = 1. \tag{3.8}$$

In addition, the scattering data: $a(k,t), b(k,t), \overline{a}(k,t), \overline{b}(k,t)$ are related to the Wronskian of the system via the relations below:

$$a(k,t) = W(\phi(x,t,k), \psi(x,t,k)), \tag{3.9}$$

$$\overline{a}(k,t) = W(\overline{\psi}(x,t,k), \overline{\phi}(x,t,k)), \tag{3.10}$$

and

$$b(k,t) = W(\overline{\psi}(x,t,k), \phi(x,t,k)), \tag{3.11}$$

$$\overline{b}(k,t) = W(\overline{\phi}(x,t,k), \psi(x,t,k)). \tag{3.12}$$

The functions ϕ, ψ are analytic in the upper half k^2-plane, and $\overline{\phi}, \overline{\psi}$ are analytic in the lower half k^2-plane. Equivalently, ϕ, ψ are analytic in quadrants I and III, and $\overline{\phi}, \overline{\psi}$ are analytic in quadrants II and IV. The proof makes use of Neumann series for $\Im k^2 > 0$ or $\Im k^2 < 0$, respectively (see lemma 2.1 in [9]).

As a result, $a(k)$ is analytic in quadrants I and III, and $\overline{a}(k)$ is analytic in quadrants II and IV. Further, the components

$$\phi_1, \psi_2, \overline{\psi}_1, \overline{\phi}_2 \text{ are } \textbf{even functions of } k; \quad \phi_2, \psi_1, \overline{\psi}_2, \overline{\phi}_1 \text{ are } \textbf{odd functions of } k. \tag{3.13}$$

Therefore, the scattering data

$$a, \overline{a} \text{ are } \textbf{even functions of } k; \quad b, \overline{b} \text{ are } \textbf{odd functions of } k. \tag{3.14}$$

This follows from transformations to standard scattering problems. For example, letting

$$\phi_1 = \tilde{m}_1\, \mathrm{e}^{s_1}\, \mathrm{e}^{-\mathrm{i}k^2 x}, \tag{3.15}$$

$$k\phi_2 = \left(\tilde{m}_2\, \mathrm{e}^{-s_1} + \frac{\mathrm{i}r}{2}\tilde{m}_1\, \mathrm{e}^{s_1}\right) \mathrm{e}^{-\mathrm{i}k^2 x}, \tag{3.16}$$

which transforms (2.1) and (2.2) with $\mathbf{v} = \phi$ to

$$\tilde{m}_{1,x} = \tilde{Q}_L \tilde{m}_2, \tag{3.17}$$

$$\tilde{m}_{2,x} = 2\mathrm{i}k^2 \tilde{m}_2 + \tilde{R}_L \tilde{m}_1, \tag{3.18}$$

where

$$s_1 = \frac{\mathrm{i}}{2}\int_{-\infty}^{x} qr\,\mathrm{d}x', \qquad \tilde{R}_L = -\frac{\mathrm{i}}{2}\left(r_x + \frac{\mathrm{i}}{2}qr^2\,\mathrm{e}^{2s_1}\right), \qquad \tilde{Q}_L = q\,\mathrm{e}^{-2s_1}.$$

Standard estimates [4, 11] show that $\tilde{m}_1, \tilde{m}_2$, and hence $\phi_1\,\mathrm{e}^{\mathrm{i}k^2x}, \phi_2\,\mathrm{e}^{\mathrm{i}k^2x}$ are analytic in the upper half k^2-plane. The above equations also show that $\tilde{m}_1, \phi_1$ are even functions of k, while $\tilde{m}_2, \phi_2$ are odd functions of k.

Following the methods of AKNS [4, 11], we find the following asymptotics as $k \to \infty$,

$$\tilde{m}_1 \sim 1 - \frac{1}{2\mathrm{i}k^2}\int_{-\infty}^{x} \tilde{Q}_L\tilde{R}_L\,\mathrm{d}x', \tag{3.19}$$

$$\tilde{m}_2 \sim -\frac{1}{2\mathrm{i}k^2}\tilde{R}_L. \tag{3.20}$$

In terms of ϕ, the key term for large k from (3.15) and (3.16) is

$$\phi \sim \begin{pmatrix} 1 \\ \frac{\mathrm{i}r}{2k} \end{pmatrix} \mathrm{e}^{s_1 - \mathrm{i}k^2x}. \tag{3.21}$$

Similar analysis can be employed for the other functions $\psi, \overline{\psi}, \overline{\phi}$ by considering the asymptotics as $k \to \infty$, which are given by

$$\overline{\phi} \sim \begin{pmatrix} \frac{-\mathrm{i}q}{2k} \\ 1 \end{pmatrix} \mathrm{e}^{-s_1 + \mathrm{i}k^2x}, \qquad \psi \sim \begin{pmatrix} \frac{-\mathrm{i}q}{2k} \\ 1 \end{pmatrix} \mathrm{e}^{s_2 + \mathrm{i}k^2x}, \qquad \overline{\psi} \sim \begin{pmatrix} 1 \\ \frac{\mathrm{i}r}{2k} \end{pmatrix} \mathrm{e}^{-s_2 - \mathrm{i}k^2x}, \tag{3.22}$$

where $s_2 = \frac{\mathrm{i}}{2}\int_x^{\infty} qr\,\mathrm{d}x'$. With the above results, we find the asymptotics of $a(k,t), \overline{a}(k,t)$ as $k \to \infty$:

$$a(k,t) = W(\phi, \psi) \sim \mathrm{e}^{s}, \qquad s = s_1 + s_2 = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} qr\,\mathrm{d}x, \tag{3.23}$$

$$\overline{a}(k,t) = -W(\overline{\phi}, \overline{\psi}) \sim \mathrm{e}^{-s}. \tag{3.24}$$

The zeros of $a(k,t), \overline{a}(k,t)$ are the eigenvalues, which are associated with decaying eigenfucntions; i.e. the bound states. These values are assumed simple and finite in number; they are $a(k_j, t) = 0, j = 1, 2, \ldots, J$ and $\overline{a}(\overline{k}_j, t) = 0,\ j = 1, 2, \ldots, \overline{J}$. Below we show that $a(k,t)$ and $\overline{a}(k,t)$ are time-independent so that the eigenvalues k_j and $\overline{k}_j$ are also time-independent. Moreover, at these points

$$\phi(x,t,k_j) = b(k_j,t)\psi(x,t,k_j), \qquad \overline{\phi}(x,t,\overline{k}_j) = \overline{b}(\overline{k}_j,t)\overline{\psi}(x,t,\overline{k}_j). \tag{3.25}$$

We simply write $b_j(t) = b(k_j,t), \overline{b}_j(t) = \overline{b}(\overline{k}_j,t)$.

3.2. Time dependence

The evolution of the data will be needed in order to obtain solutions of the derivative NLS equations. As $|x| \to \infty$, the coefficients of the time evolution of the eigenfunctions in (2.4) behave like

$$A \to A_\infty(k) = -2\mathrm{i}k^4, \quad B \to 0, \ C \to 0. \tag{3.26}$$

To account for the fact that the above eigenfunctions are time-independent, the time evolution equation (2.3) needs to be modified:

$$\phi_t(x,t,k) = \begin{pmatrix} A - A_\infty(k) & B \\ C & -A - A_\infty(k) \end{pmatrix} \phi(x,t,k). \tag{3.27}$$

Substituting the scattering equation (3.6) into the above equation yields

$$\begin{aligned} &a_t(k,t) = 0, \qquad b_t(k,t) = -2A_\infty(k)b(k,t). \\ &\qquad => a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)\mathrm{e}^{-2A_\infty(k)t} = b(k,0)\mathrm{e}^{4\mathrm{i}k^4 t}. \end{aligned} \tag{3.28}$$

Hence, the eigenvalues k_j are time-independent. Applying a similar analysis on $\overline{\phi}$ leads to

$$\begin{aligned} &\bar{a}_t(k,t) = 0, \qquad \bar{b}_t(k,t) = 2A_\infty(k)\bar{b}(k,t). \\ &\qquad => \bar{a}(k,t) = \bar{a}(k,0), \qquad \bar{b}(k,t) = \bar{b}(k,0)\mathrm{e}^{2A_\infty(k)t} = \bar{b}(k,0)\mathrm{e}^{-4\mathrm{i}k^4 t}. \end{aligned} \tag{3.29}$$

Therefore, the eigenvalues $\overline{k}_j$ are time-independent. Later we will also need the time dependence of $b_j(t) = b(k_j,t)$, $\overline{b}_j(t) = \overline{b}(\overline{k}_j,t)$. A similar procedure as above shows that they satisfy the same time dependence as $b(k,t)$ and $\overline{b}(k,t)$. For convenience, we assume sufficient decay on the initial data so that we can extend the data $b(k,t), \overline{b}(k,t)$ off the real axis.

3.3. Symmetries

In what follows, we use the notations $a(k) := a(k,0)$, $\overline{a}(k) := \overline{a}(k,0)$, $b(k) := b(k,0)$ and $\overline{b}(k) := \overline{b}(k,0)$.

Associated with the scattering problem, (2.1) and (2.2) admit three symmetries in the physical space: (i): (2.10), (ii): (2.12), (iii): (2.15). With each of them, below we provide the spectral symmetries. Later we use these symmetries to find solutions.

3.3.1. General case. If $(v_1(x,t,k), v_2(x,t,k))^{\mathrm{T}}$ satisfies (2.1), then $(v_1(x,t,-k), -v_2(x,t,-k))^{\mathrm{T}}$ also solves for (2.1). Taking into account the boundary conditions (3.1), one has

$$\phi(x,t,k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi(x,t,-k), \qquad \overline{\psi}(x,t,k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{\psi}(x,t,-k), \tag{3.30}$$

$$\overline{\phi}(x,t,k) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \overline{\phi}(x,t,-k), \qquad \psi(x,t,k) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \psi(x,t,-k). \tag{3.31}$$

Thus,

$$\begin{aligned} a(k,t) &= a(-k,t),\\ \overline{a}(k,t) &= \overline{a}(-k,t),\\ b(k,t) &= -b(-k,t),\\ \overline{b}(k,t) &= -\overline{b}(-k,t). \end{aligned} \tag{3.32}$$

It follows that if $k = k_j$ is a zero of $a(k,t)$, then so is $k = -k_j$. Similarly, if $k = \overline{k}_j$ is a zero of $\overline{a}(k,t)$, then $k = -\overline{k}_j$ is also a zero of $\overline{a}(k,t)$. Thus, the number of eigenvalues must be even, and the minimal number is two for the non-trivial case.

Remark 3.1. The above symmetry relations imply that $\phi_1, \psi_2, \overline{\psi}_1, \overline{\phi}_2$ are even functions of k, and $\phi_2, \psi_1, \overline{\psi}_2, \overline{\phi}_1$ are odd functions of k. Moreover, $a, \overline{a}$ are even functions of k and $b, \overline{b}$ are odd functions of k. These properties are consistent with (3.13) and (3.14).

3.3.2. Standard derivative NLS equation. The first physical space symmetry

$$\text{(i) } r(x,t) = \sigma q^*(x,t), \qquad \sigma = \pm 1$$

is the one connected with the standard derivative NLS equation (2.11).

In spectral space, we find the following symmetries associated with the eigenfunctions:

$$\overline{\psi}(x,t,k) = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \psi^*(x,t,k^*), \tag{3.33}$$

$$\overline{\phi}(x,t,k) = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} \phi^*(x,t,k^*). \tag{3.34}$$

These symmetries translate into the following symmetries in the scattering data

$$\overline{a}(k,t) = a^*(k^*,t), \qquad \overline{b}(k,t) = \sigma b^*(k^*,t), \qquad \overline{c}_j(t) = \sigma c_j^*(t), \tag{3.35}$$

where $c_j(t) = b(k_j,t)/a'(k_j,t), \overline{c}_j = \overline{b}(\overline{k}_j,\ t)/\overline{a}'(\overline{k}_j,t)$. We note that if k_j is a zero of $a(k,t)$, then $\overline{k}_j = k_j^*$ is also a zero of $\overline{a}(k,t)$, where the eigenvalue k_j is located either in quadrant I or III.

3.3.3. PT derivative NLS equation. The second physical space symmetry

$$\text{(ii) } r(x,t) = \mathrm{i}\sigma q^*(-x,t), \qquad \sigma = \pm 1$$

is the one connected with the PT derivative NLS equation (2.13).

In spectral space, we obtain the following symmetries associated with the eigenfunctions:

$$\phi(x,t,k) = \begin{pmatrix} 0 & 1 \\ \pm\sigma & 0 \end{pmatrix} \psi^*(-x,t,\pm\mathrm{i}k^*), \tag{3.36}$$

$$\overline{\phi}(x,t,k) = \begin{pmatrix} 0 & \pm\sigma \\ 1 & 0 \end{pmatrix} \overline{\psi}^*(-x,t,\pm\mathrm{i}k^*). \tag{3.37}$$

These symmetries yield the following symmetries in the scattering data

$$a(k,t) = a^*(\pm\mathrm{i}k^*,t), \qquad \overline{a}(k,t) = \overline{a}^*(\pm\mathrm{i}k^*,t), \qquad \overline{b}(k,t) = \mp\sigma b^*(\pm\mathrm{i}k^*,t). \tag{3.38}$$

We note that if k_j is a zero of $a(k,t)$, then so is $\pm\mathrm{i}k_j^*$, and similarly, if $\overline{k}_j$ is a zero of $\overline{a}(k,t)$, then so is $\pm\mathrm{i}\overline{k}_j^*$. In general, the eigenvalues come in quartets, where $\{\pm k_j, \pm\mathrm{i}k_j^*\}$ are located in quadrants I, III and $\{\pm\overline{k}_j, \pm\mathrm{i}\overline{k}_j^*\}$ are in quadrants II, IV.

The normalization constants are defined as $c_j(t) = b(k_j,t)/a'(k_j,t)$, $\overline{c}_j = \overline{b}(\overline{k}_j,t)/\overline{a}'(\overline{k}_j,t)$. In this case, we determine a symmetry for $b(k_j,t)$ separately. We use trace formulae to determine $a'(k_j,t)$ (similarly for $\overline{b}(\overline{k}_j,t)$ and $\overline{a}'(\overline{k}_j,t)$) [6, 7]. To obtain the symmetry for $b(k_j,t)$, we use (3.25) and the symmetry (3.36). Similarly, (3.25) and the symmetry (3.37) are applied to $\overline{b}(\overline{k}_j,t)$. This leads to

$$b(k_j,t)b^*(\pm\mathrm{i}k_j^*,t) = \pm\sigma, \qquad \overline{b}(\overline{k}_j,t)\overline{b}^*(\pm\mathrm{i}\overline{k}_j^*,t) = \pm\sigma. \tag{3.39}$$

Indeed, from (3.2), (3.3), (3.6), (3.7) and (3.36)–(3.38), we derive

$$M_1(x,t,k_j) = b(k_j,t)N_1(x,t,k_j)\mathrm{e}^{2\mathrm{i}k_j^2x} = \pm\sigma b(k_j,t)M_2^*(-x,t,\pm\mathrm{i}k_j^*)\mathrm{e}^{2\mathrm{i}k_j^2x}, \tag{3.40}$$

$$M_2(-x,t,\pm\mathrm{i}k_j^*) = b(\pm\mathrm{i}k_j^*,t)N_2(-x,t,\pm\mathrm{i}k_j^*)\mathrm{e}^{2\mathrm{i}k_j^{*2}x}. \tag{3.41}$$

It follows that

$$\begin{aligned} M_1(x,t,k_j) &= \pm\sigma b(k_j,t)b^*(\pm\mathrm{i}k_j^*,t)N_2^*(-x,t,\pm\mathrm{i}k_j) \\ &= \pm\sigma b(k_j,t)b^*(\pm\mathrm{i}k_j^*,t)M_1(x,t,k_j), \end{aligned} \tag{3.42}$$

which implies

$$b(k_j,t)b^*(\pm\mathrm{i}k_j^*,t) = \pm\sigma. \tag{3.43}$$

Similarly, one has

$$\overline{b}(\overline{k}_j,t)\overline{b}^*(\pm\mathrm{i}\overline{k}_j^*,t) = \pm\sigma. \tag{3.44}$$

Although, in principle, we can have two eigenvalues $\pm k_1$ on the rays $\pi/4$ and $5\pi/4$, this turns out not allowed by equation (3.38). The reason is as follows. When $k_1 = r\,\mathrm{e}^{\mathrm{i}\pi/4}, r > 0$, then $k_1 = \mathrm{i}k_1^*$, and for $J = 2$, we can take $k_2 = \mathrm{i}k_1^*$; similarly if $k_1 = r\,\mathrm{e}^{5\mathrm{i}\pi/4}, r > 0$, then again $k_1 = \mathrm{i}k_1^*$; hence (3.32) and (3.39) yield $b(k_j,t)b^*(\mathrm{i}k_j^*,t) = |b(k_j,t)|^2 = \sigma$; therefore, we must take $\sigma = 1$. On the other hand, if we take $\overline{k}_1 = r\,\mathrm{e}^{-\mathrm{i}\pi/4}$ or $\overline{k}_1 = r\,\mathrm{e}^{3\mathrm{i}\pi/4}$, then for $J = 2$, we need to take $\overline{k}_2 = -\mathrm{i}\overline{k}_1^* = \overline{k}_1$. However, in this case, the second symmetry condition yields $\overline{b}(\overline{k}_j,t)\overline{b}^*(-\mathrm{i}\overline{k}_j^*,t) = |\overline{b}(\overline{k}_j,t)|^2 = -\sigma$; hence in this case, we must take $\sigma = -1$, which contradicts what we found for $k_1 = r\,\mathrm{e}^{\mathrm{i}\pi/4}$ or $k_1 = r\,\mathrm{e}^{5\mathrm{i}\pi/4}$. Consequently, we have to consider quartets: $\pm k_1, \pm k_2 = \pm\mathrm{i}k_1^*$ off the rays $\pi/4, 5\pi/4$ and analogous situation for $\pm\overline{k}_1, \pm\overline{k}_2 = \mp\mathrm{i}\overline{k}_1^*$ off the rays $3\pi/4, -\pi/4$ in order to construct a solution. This case is discussed below.

Remark 3.2. $\{\pm k_j, \pm\mathrm{i}k_j^* : \Re k_j \cdot \Im k_j > 0 \text{ and } \Re k_j \neq \Im k_j\}_{j=1}^{J_1}$ is the zero set of $a(k)$. The number of eigenvalues is $J = 4J_1$, and the simplest non-trivial case is obtained when $J_1 = 1$. Similarly, $\{\pm\bar{k}_j, \pm\mathrm{i}\bar{k}_j^* : \Re\bar{k}_j \cdot \Im\bar{k}_j < 0 \text{ and } \Re\bar{k}_j \neq -\Im\bar{k}_j\}_{j=1}^{\bar{J}_1}$ is the zero set of $\overline{a}(k)$.

Figure 1 shows the locations of eigenvalues, i.e., zeros of $a(k)$ and $\overline{a}(k)$, respectively.

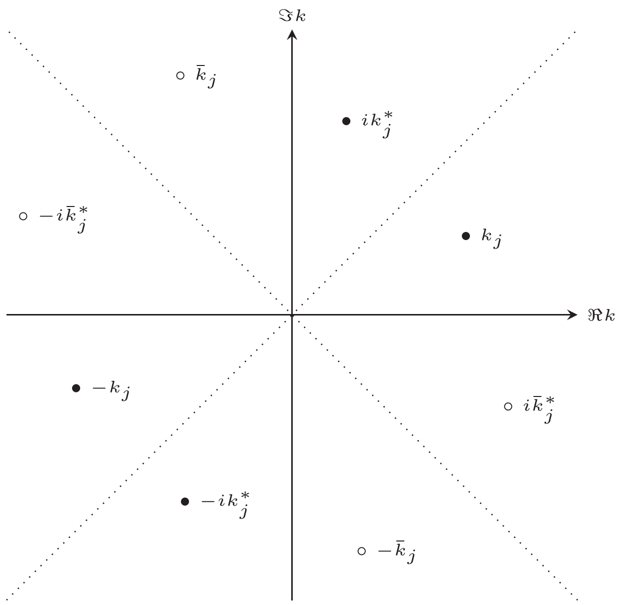

Figure 1. The sets of eigenvalues, where the solid/hollow dots are zeros of $a(k)/\overline{a}(k)$, respectively.

3.3.4. RST derivative NLS equation. The third physical space symmetry

$$\text{(iii)}\ r(x,t) = \sigma q(-x,-t), \qquad \sigma = \pm 1$$

is the one connected with the RST derivative NLS equation (2.16).

In spectral space, we find the following symmetries associated with the eigenfunctions:

$$\phi(x,t,k) = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \psi(-x,-t,-k), \tag{3.45}$$

$$\overline{\phi}(x,t,k) = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} \overline{\psi}(-x,-t,-k). \tag{3.46}$$

These symmetries translate into the following symmetries in the scattering data

$$a(k,t) = a(-k,-t), \qquad \overline{a}(k,t) = \overline{a}(-k,-t), \qquad \overline{b}(k,t) = -\sigma b(-k,-t). \tag{3.47}$$

As above, the normalization constants are $c_j(t) = b(k_j,t)/a'(k_j,t), \overline{c}_j(t) = \overline{b}(\overline{k}_j,t)/\overline{a}'(\overline{k}_j,t)$, and in this case, we determine the symmetry for $b(k_j,t)$ separately and use trace formulae to determine $a'(k_j,t)$ (similarly for $\overline{b}(\overline{k}_j,t), \overline{a}'(\overline{k}_j,t)$). Using the same procedure that we used to determine (3.38), we find

$$b(k_j,t)b(-k_j,-t) = \sigma, \qquad \overline{b}(\overline{k}_j,t)\overline{b}(-\overline{k}_j,-t) = \sigma. \tag{3.48}$$

By (3.32), one obtains

$$b(k_j,t)b(k_j,-t) = -\sigma, \qquad \overline{b}(\overline{k}_j,t)\overline{b}(\overline{k}_j,-t) = -\sigma, \tag{3.49}$$

which implies

$$\begin{aligned} b^2(k_j,0) &= -\sigma,\\ \overline{b}^2(\overline{k}_j,0) &= -\sigma,\\ b(k_j,0)b(-k_j,0) &= \sigma,\\ \overline{b}(\overline{k}_j,0)\overline{b}(-\overline{k}_j,0) &= \sigma. \end{aligned} \tag{3.50}$$

3.4. Trace formulae

Unlike the standard derivative NLS equation, in the nonlocal derivative NLS equations, the numerator and denominator of the normalization constants $c_j(t) = b(k_j,t)/a'(k_j,t)$, $\overline{c}_j = \overline{b}(\overline{k}_j,t)/\overline{a}'(\overline{k}_j,t)$ need to be separated; we use the trace formulae to evaluate derivatives $a'(k_j,t), \overline{a}'(\overline{k}_j,t)$. To find the trace formulae, we first define new scattering coefficients

$$\tilde{a}(k,t) = a(k,t)\mathrm{e}^{-s}, \qquad \tilde{\overline{a}}(k,t) = \overline{a}(k,t)\mathrm{e}^{s}.$$

We assume that $a(k)$ and $\overline{a}(k)$ have simple zeros $\{\pm k_j : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J_1}$ and $\{\pm\overline{k}_j : \Re\overline{k}_j \cdot \Im\overline{k}_j < 0\}_{j=1}^{\overline{J}_1}$, respectively. Indeed, from the symmetry relation (3.32), one has if $k_j(\overline{k}_j)$ is a zero of $a(k)(\overline{a}(k))$, then so is $-k_j(-\overline{k}_j)$.

3.4.1. General trace formulae. Letting $J_1 = \overline{J}_1$, we define

$$\alpha(k) = \tilde{a}(k)\cdot\prod_{j=1}^{J_1}\frac{(k-\overline{k}_j)(k+\overline{k}_j)}{(k-k_j)(k+k_j)}, \qquad \overline{\alpha}(k) = \tilde{\overline{a}}(k)\cdot\prod_{j=1}^{J_1}\frac{(k-k_j)(k+k_j)}{(k-\overline{k}_j)(k+\overline{k}_j)}. \tag{3.51}$$

Thus, $\alpha(k)$ ($\overline{\alpha}(k)$) is analytic in quadrants I and III (II and IV). Moreover, $\alpha(k), \overline{\alpha}(k) \to 1$ as $k \to \infty$ and have no zeros in their respective quadrants. Hence, we have

$$\log\,\alpha(k) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log\,\alpha(\xi)}{\xi-k}\,\mathrm{d}\xi, \qquad \frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log\,\overline{\alpha}(\xi)}{\xi-k}\mathrm{d}\xi = 0 \tag{3.52}$$

for $\Re k\cdot\Im k > 0$, and

$$\log\,\overline{\alpha}(k) = -\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log\,\overline{\alpha}(\xi)}{\xi-k}\,\mathrm{d}\xi, \qquad \frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log\,\alpha(\xi)}{\xi-k}\,\mathrm{d}\xi = 0 \tag{3.53}$$

for $\Re k\cdot\Im k < 0$, where $\Sigma = \Sigma_1\cup\Sigma_2$, $\Sigma_1 := \overrightarrow{(+\mathrm{i}\infty,0]}\cup\overrightarrow{[0,+\infty)}$ and $\Sigma_2 := \overrightarrow{(-\mathrm{i}\infty,0]}\cup\overrightarrow{[0,-\infty)}$. The contour Σ is depicted in figure 2.

Adding/subtracting the above equations in their quadrants respectively yields

$$\log\,\alpha(k) = \frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log\,\alpha(\xi)\overline{\alpha}(\xi)}{\xi-k}\,\mathrm{d}\xi, \quad \Re k\cdot\Im k > 0, \tag{3.54}$$

$$\log\,\overline{\alpha}(k) = -\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log\,\alpha(\xi)\overline{\alpha}(\xi)}{\xi-k}\,\mathrm{d}\xi, \quad \Re k\cdot\Im k < 0. \tag{3.55}$$

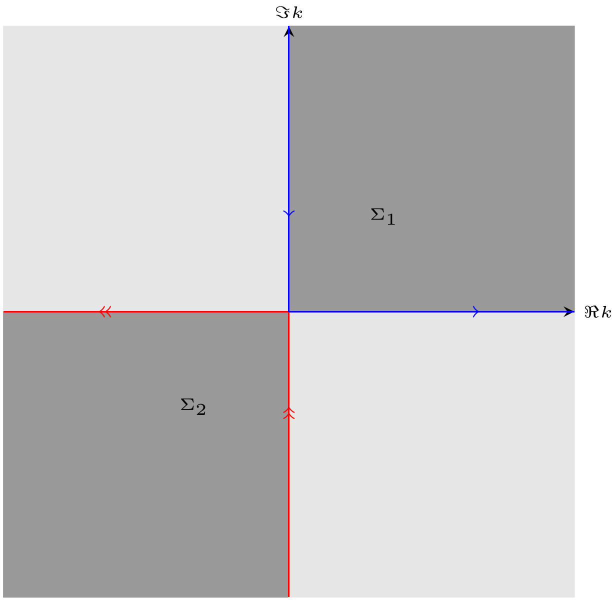

Figure 2. The contour $\Sigma = \Sigma_1 \cup \Sigma_2$, where the blue one is for Σ_1 and the red one is for Σ_2.

From (3.8) and (3.51), one obtains

$$\log \tilde{a}(k) = \sum_{j=1}^{J_1} \log\left(\frac{(k-k_j)(k+k_j)}{(k-\bar{k}_j)(k+\bar{k}_j)}\right) + \frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi - k}\,\mathrm{d}\xi \quad (3.56)$$

for $\Re k \cdot \Im k > 0$,

$$\log \tilde{\bar{a}}(k) = \sum_{j=1}^{J_1} \log\left(\frac{(k-\bar{k}_j)(k+\bar{k}_j)}{(k-k_j)(k+k_j)}\right) - \frac{1}{2\pi \mathrm{i}}\int_{\Sigma} \frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi - k}\,\mathrm{d}\xi \quad (3.57)$$

for $\Re k \cdot \Im k < 0$.

In order to reconstruct potentials, $\tilde{a}'(k_j)$, $\tilde{a}'(-k_j)$, $\tilde{\bar{a}}'(\bar{k}_j)$ and $\tilde{\bar{a}}'(-\bar{k}_j)$ are needed. These derivatives are found to be

$$\tilde{a}'(k_j) = \frac{\prod_{m\neq j}(k_j - k_m)\cdot\prod_{m=1}^{J_1}(k_j+k_m)}{\prod_{m=1}^{J_1}\left[(k_j-\bar{k}_m)\cdot(k_j+\bar{k}_m)\right]}\cdot \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi-k_j}\,\mathrm{d}\xi}, \quad (3.58)$$

$$\tilde{a}'(-k_j) = \frac{\prod_{m=1}^{J_1}(-k_j - k_m)\cdot\prod_{m\neq j}(-k_j+k_m)}{\prod_{m=1}^{J_1}\left[(-k_j-\bar{k}_m)\cdot(-k_j+\bar{k}_m)\right]}\cdot \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi+k_j}\,\mathrm{d}\xi}, \quad (3.59)$$

$$\tilde{\bar{a}}'(\bar{k}_j) = \frac{\prod_{m\neq j}(\bar{k}_j - \bar{k}_m)\cdot\prod_{m=1}^{J_1}(\bar{k}_j+\bar{k}_m)}{\prod_{m=1}^{J_1}\left[(\bar{k}_j-k_m)\cdot(\bar{k}_j+k_m)\right]}\cdot \mathrm{e}^{-\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi-\bar{k}_j}\,\mathrm{d}\xi}, \quad (3.60)$$

$$\tilde{\bar{a}}'(-\bar{k}_j) = \frac{\prod_{m=1}^{J_1}(-\bar{k}_j - \bar{k}_m)\cdot\prod_{m\neq j}(-\bar{k}_j+\bar{k}_m)}{\prod_{m=1}^{J_1}\left[(-\bar{k}_j-k_m)\cdot(-\bar{k}_j+k_m)\right]}\cdot \mathrm{e}^{-\frac{1}{2\pi\mathrm{i}}\int_{\Sigma}\frac{\log(1+b(\xi)\bar{b}(\xi))}{\xi+\bar{k}_j}\,\mathrm{d}\xi}. \quad (3.61)$$

In general, these four derivatives depend on the simple zeros $\{\pm k_j : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J_1}$ and $\{\pm \bar{k}_j : \Re \bar{k}_j \cdot \Im \bar{k}_j < 0\}_{j=1}^{J_1}$ as well as the scattering data $b(k)$ and $\overline{b}(k)$.

In particular, if $b(k) = 0$ or $\overline{b}(k) = 0$ on Σ, then it corresponds to the reflectionless potentials. Thus, these derivatives only depend on the above zeros; moreover, (3.56) and (3.57) imply $\tilde{a}(k) = \tilde{a}(-k)$ and $\tilde{\overline{a}}(k) = \tilde{\overline{a}}(-k)$. Combining (3.32), one has

$$\frac{b(k,t)}{\tilde{a}'(k,t)} = \frac{b(-k,t)}{\tilde{a}'(-k,t)}, \qquad \frac{\overline{b}(k,t)}{\tilde{\overline{a}}'(k,t)} = \frac{\overline{b}(-k,t)}{\tilde{\overline{a}}'(-k,t)}, \tag{3.62}$$

which will be applied in subsequent subsections.

3.4.2. Standard derivative NLS equation. By the symmetry relation $\overline{a}(k,t) = a^*(k^*,t)$, we have $\overline{k}_j = k_j^*$ and $\overline{J}_1 = J_1$. Thus, $\{\pm k_j^* : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J_1}$ are simple zeros of $\overline{a}(k)$. Combining (3.35) and section 3.4.1, it yields

$$\log\,\tilde{a}(k) = \sum_{j=1}^{J_1} \log\left(\frac{(k-k_j)(k+k_j)}{(k-k_j^*)(k+k_j^*)}\right) + \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1+\sigma b(\xi)b^*(\xi^*))}{\xi-k}\,\mathrm{d}\xi \tag{3.63}$$

for $\Re k \cdot \Im k > 0$,

$$\log\,\tilde{\overline{a}}(k) = \sum_{j=1}^{J_1} \log\left(\frac{(k-k_j^*)(k+k_j^*)}{(k-k_j)(k+k_j)}\right) - \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1+\sigma b(\xi)b^*(\xi^*))}{\xi-k}\,\mathrm{d}\xi \tag{3.64}$$

for $\Re k \cdot \Im k < 0$.

In order to solve the inverse problem, we need $\tilde{a}'(k_j)$, $\tilde{a}'(-k_j)$, $\tilde{\overline{a}}'(k_j^*)$ and $\tilde{\overline{a}}'(-k_j^*)$. In general, these derivatives are found below:

$$\tilde{a}'(k_j) = \frac{\tilde{a}(k)}{k-k_j}|_{k=k_j}, \qquad \tilde{a}'(-k_j) = \frac{\tilde{a}(k)}{k+k_j}|_{k=-k_j}, \tag{3.65}$$

$$\tilde{\overline{a}}'(k_j^*) = \frac{\tilde{\overline{a}}(k)}{k-k_j^*}|_{k=k_j^*}, \qquad \tilde{\overline{a}}'(-k_j^*) = \frac{\tilde{\overline{a}}(k)}{k+k_j^*}|_{k=-k_j^*}, \tag{3.66}$$

where

$$\tilde{a}(k) := \prod_{m=1}^{J_1} \frac{(k-k_m)\cdot(k+k_m)}{(k-k_m^*)\cdot(k+k_m^*)} \cdot \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1+\sigma b(\xi)b^*(\xi^*))}{\xi-k}\,\mathrm{d}\xi}, \tag{3.67}$$

$$\tilde{\overline{a}}(k) := \prod_{m=1}^{J_1} \frac{(k-k_m^*)\cdot(k+k_m^*)}{(k-k_m)\cdot(k+k_m)} \cdot \mathrm{e}^{-\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1+\sigma b(\xi)b^*(\xi^*))}{\xi-k}\,\mathrm{d}\xi}. \tag{3.68}$$

In general, these four derivatives depend on the simple zeros $\{\pm k_j : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J_1}$ as well as the scattering coefficient $b(k)$. In particular, if $b(k) = 0$ on Σ, then it corresponds to pure solitons. Thus, these derivatives only depend on $\{\pm k_j : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J_1}$.

3.4.3. PT derivative NLS equation. Under the symmetry reduction $r(x,t) = \mathrm{i}\sigma q^*(-x,t)$, (3.38) implies that $a(k)$ and $\overline{a}(k)$ have the simple zeros $\{\pm k_j, \pm \mathrm{i}k_j^* : \Re k_j \cdot \Im k_j > 0 \text{ and } \Re k_j \neq \Im k_j\}_{j=1}^{J_1}$ and $\{\pm \bar{k}_j, \pm \mathrm{i}\bar{k}_j^* : \Re \bar{k}_j \cdot \Im \bar{k}_j < 0 \text{ and } \Re \bar{k}_j \neq -\Im \bar{k}_j\}_{j=1}^{\overline{J}_1}$, respectively.

Letting $J_1 = \overline{J}_1$, we define

$$\begin{aligned}\alpha(k) &= \tilde{a}(k) \cdot \prod_{j=1}^{J_1} \frac{(k - \overline{k}_j)(k + \overline{k}_j)(k - \mathrm{i}\overline{k}_j^*)(k + \mathrm{i}\overline{k}_j^*)}{(k - k_j)(k + k_j)(k - \mathrm{i}k_j^*)(k + \mathrm{i}k_j^*)},\\ \overline{\alpha}(k) &= \tilde{\overline{a}}(k) \cdot \prod_{j=1}^{J_1} \frac{(k - k_j)(k + k_j)(k - \mathrm{i}k_j^*)(k + \mathrm{i}k_j^*)}{(k - \overline{k}_j)(k + \overline{k}_j)(k - \mathrm{i}\overline{k}_j^*)(k + \mathrm{i}\overline{k}_j^*)}.\end{aligned} \tag{3.69}$$

Thus, $\alpha(k)$ ($\overline{\alpha}(k)$) is analytic in quadrants I and III (II and IV). Moreover, $\alpha(k), \overline{\alpha}(k) \to 1$ as $k \to \infty$ and have no zeros in their respective quadrants. Hence, we have

$$\log\, \alpha(k) = \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \alpha(\xi)}{\xi - k}\, \mathrm{d}\xi, \qquad \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \overline{\alpha}(\xi)}{\xi - k}\, \mathrm{d}\xi = 0 \tag{3.70}$$

for $\Re k \cdot \Im k > 0$, and

$$\log\, \overline{\alpha}(k) = -\frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \overline{\alpha}(\xi)}{\xi - k}\, \mathrm{d}\xi, \qquad \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \alpha(\xi)}{\xi - k}\, \mathrm{d}\xi = 0 \tag{3.71}$$

for $\Re k \cdot \Im k < 0$. Adding/subtracting the above equations in their quadrants respectively yields

$$\log\, \alpha(k) = \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \alpha(\xi)\overline{\alpha}(\xi)}{\xi - k}\, \mathrm{d}\xi, \quad \Re k \cdot \Im k > 0, \tag{3.72}$$

$$\log\, \overline{\alpha}(k) = -\frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log\, \alpha(\xi)\overline{\alpha}(\xi)}{\xi - k}\, \mathrm{d}\xi, \qquad \Re k \cdot \Im k < 0. \tag{3.73}$$

From (3.8), (3.38) and (3.51), one obtains

$$\begin{aligned}\log\, \tilde{a}(k) = &\sum_{j=1}^{J_1} \log \left(\frac{(k - k_j)(k + k_j)(k - \mathrm{i}k_j^*)(k + \mathrm{i}k_j^*)}{(k - \overline{k}_j)(k + \overline{k}_j)(k - \mathrm{i}\overline{k}_j^*)(k + \mathrm{i}\overline{k}_j^*)} \right)\\ &+ \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log(1 \mp \sigma b(\xi) b^*(\pm \mathrm{i}\xi^*))}{\xi - k}\, \mathrm{d}\xi\end{aligned} \tag{3.74}$$

for $\Re k \cdot \Im k > 0$,

$$\begin{aligned}\log\, \tilde{\overline{a}}(k) = &\sum_{j=1}^{J_1} \log \left(\frac{(k - \overline{k}_j)(k + \overline{k}_j)(k - \mathrm{i}\overline{k}_j^*)(k + \mathrm{i}\overline{k}_j^*)}{(k - k_j)(k + k_j)(k - \mathrm{i}k_j^*)(k + \mathrm{i}k_j^*)} \right)\\ &- \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\log(1 \mp \sigma b(\xi) b^*(\pm \mathrm{i}\xi^*))}{\xi - k}\, \mathrm{d}\xi\end{aligned} \tag{3.75}$$

for $\Re k \cdot \Im k < 0$.

In order to recover the potentials, $\tilde{a}'(k_j)$, $\tilde{a}'(-k_j)$, $\tilde{a}'(\mathrm{i}k_j^*)$, $\tilde{a}'(-\mathrm{i}k_j^*)$, $\tilde{\bar{a}}'(\bar{k}_j)$, $\tilde{\bar{a}}'(-\bar{k}_j)$, $\tilde{\bar{a}}'(\mathrm{i}\bar{k}_j^*)$, $\tilde{\bar{a}}'(-\mathrm{i}\bar{k}_j^*)$ are needed. These derivatives are found as follows:

$$\tilde{a}'(k_j) = \frac{\tilde{a}(k)}{k-k_j}\Big|_{k=k_j}, \qquad \tilde{a}'(-k_j) = \frac{\tilde{a}(k)}{k+k_j}\Big|_{k=-k_j}, \tag{3.76}$$

$$\tilde{a}'(\mathrm{i}k_j^*) = \frac{\tilde{a}(k)}{k-\mathrm{i}k_j^*}\Big|_{k=\mathrm{i}k_j^*}, \qquad \tilde{a}'(-\mathrm{i}k_j^*) = \frac{\tilde{a}(k)}{k+\mathrm{i}k_j^*}\Big|_{k=-\mathrm{i}k_j^*}, \tag{3.77}$$

$$\tilde{\bar{a}}'(\bar{k}_j) = \frac{\tilde{\bar{a}}(k)}{k-\bar{k}_j}\Big|_{k=\bar{k}_j}, \qquad \tilde{\bar{a}}'(-\bar{k}_j) = \frac{\tilde{\bar{a}}(k)}{k+\bar{k}_j}\Big|_{k=-\bar{k}_j}, \tag{3.78}$$

$$\tilde{\bar{a}}'(\mathrm{i}\bar{k}_j^*) = \frac{\tilde{\bar{a}}(k)}{k-\mathrm{i}\bar{k}_j^*}\Big|_{k=\mathrm{i}\bar{k}_j^*}, \qquad \tilde{\bar{a}}'(-\mathrm{i}\bar{k}_j^*) = \frac{\tilde{\bar{a}}(z)}{k+\mathrm{i}\bar{k}_j^*}\Big|_{k=-\mathrm{i}\bar{k}_j^*}, \tag{3.79}$$

where

$$\begin{aligned}\tilde{a}(k) &:= \prod_{m=1}^{J_1} \frac{(k-k_m)\cdot(k+k_m)\cdot(k-\mathrm{i}k_m^*)(k+\mathrm{i}k_m^*)}{(k-\bar{k}_m)\cdot(k+\bar{k}_m)\cdot(k-\mathrm{i}\bar{k}_m^*)\cdot(k+\mathrm{i}\bar{k}_m^*)} \\ &\quad \times \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1\mp\sigma b(\xi)b^*(\pm\mathrm{i}\xi^*))}{\xi-k}\,\mathrm{d}\xi},\end{aligned} \tag{3.80}$$

$$\begin{aligned}\tilde{\bar{a}}(k) &:= \prod_{m=1}^{J_1} \frac{(k-\bar{k}_m)\cdot(k+\bar{k}_m)\cdot(k-\mathrm{i}\bar{k}_m^*)\cdot(k+\mathrm{i}\bar{k}_m^*)}{(k-k_m)\cdot(k+k_m)\cdot(k-\mathrm{i}k_m^*)\cdot(k+\mathrm{i}k_m^*)} \\ &\quad \times \mathrm{e}^{-\frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\log(1\mp\sigma b(\xi)b^*(\pm\mathrm{i}\xi^*))}{\xi-k}\,\mathrm{d}\xi}.\end{aligned} \tag{3.81}$$

In general, these derivatives depend on the simple zeros $\{\pm k_j, \pm\mathrm{i}k_j^* : \Re k_j \cdot \Im k_j > 0 \text{ and } \Re k_j \neq \Im k_j\}_{j=1}^{J_1}$, $\{\pm\bar{k}_j, \pm\mathrm{i}\bar{k}_j^* : \Re\bar{k}_j \cdot \Im\bar{k}_j < 0 \text{ and } \Re\bar{k}_j \neq -\Im\bar{k}_j\}_{j=1}^{J_1}$ and the scattering coefficient $b(k)$. In particular, if $b(k) = 0$ on Σ, then these derivatives only rely on the above simple zeros, which corresponds to the case of pure solitons.

3.4.4. RST derivative NLS equation. Under the symmetry reduction $r(x,t) = \sigma q(-x,-t)$, there are no more symmetries among eigenvalues, thus, the statement of trace formulae is the same as the general case (section 3.4.1).

4. Inverse scattering: Riemann–Hilbert approach

The inverse scattering problem constructs the potentials 'q, r' from suitable scattering data. To do this, we first determine a Riemann–Hilbert problem from the analytic properties of the eigenfunctions and then use the above large k formulae to determine q, r. Recall

$$\tilde{a}(k,t) = a(k,t)\mathrm{e}^{-s}, \qquad \tilde{\bar{a}}(k,t) = \bar{a}(k,t)\mathrm{e}^{s}.$$

Multiplying equation (3.6) by $\mathrm{e}^{\mathrm{i}k^2x-s_1}$ and subtracting the bounded term and poles yield

$$\Phi_+(k;x,t)-\Phi_-(k;x,t)=\tilde{\rho}(k,t)\mathrm{e}^{2\mathrm{i}k^2x}N(x,t,k)\mathrm{e}^{-s_1},\qquad \tilde{\rho}(k,t)=\frac{b(k,t)}{\tilde{a}(k,t)} \tag{4.1}$$

on $\Re k=0\cup\Im k=0$, where

$$\Phi_+(k;x,t)=\left\{\frac{M(x,t,k)\mathrm{e}^{-s_1}}{\tilde{a}(k,t)}-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2x}N(x,t,k_j)\mathrm{e}^{-s_1}}{k-k_j}\right\}, \tag{4.2}$$

$$\Phi_-(k;x,t)=\left\{\overline{N}(x,t,k)\mathrm{e}^{s_2}-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2x}N(x,t,k_j)\mathrm{e}^{-s_1}}{k-k_j}\right\}. \tag{4.3}$$

$\Phi_\pm(k;\ x,t)$ is analytic in the upper/lower half k^2-plane, and following the procedure outlined earlier, the time dependence of the 'normalization' constants are given by

$$\tilde{c}_j(t)=\frac{b_j(t)}{\tilde{a}'(k_j,t)}=\tilde{c}_j(0)\mathrm{e}^{4\mathrm{i}k_j^4t},\qquad \tilde{\bar{c}}_j(t)=\frac{\overline{b}_j(t)}{\tilde{\bar{a}}'(\overline{k}_j,t)}=\tilde{\bar{c}}_j(0)\mathrm{e}^{-4\mathrm{i}\overline{k}_j^4t}, \tag{4.4}$$

where $\tilde{c}_j(0):=\frac{b(k_j,0)}{\tilde{a}'(k_j,0)}$ and $\tilde{\bar{c}}_j(0):=\frac{\overline{b}(\overline{k}_j,0)}{\tilde{\bar{a}}'(\overline{k}_j,0)}$. Similarly, multiplying equation (3.7) by $\mathrm{e}^{-\mathrm{i}k^2x+s_1}$ and subtracting the bounded term and poles yield

$$\Psi_+(k;x,t)-\Psi_-(k;x,t)=-\tilde{\bar{\rho}}(k,t)\mathrm{e}^{-2\mathrm{i}k^2x}\bar{N}(x,t,k)\mathrm{e}^{s_1},\qquad \tilde{\bar{\rho}}(k,t)=\frac{\bar{b}(k,t)}{\tilde{\bar{a}}(k,t)}, \tag{4.5}$$

on $\Re k=0\cup\Im k=0$, where

$$\Psi_-(k;x,t)=\left\{\frac{\overline{M}(x,t,k)\mathrm{e}^{s_1}}{\tilde{\bar{a}}(k,t)}-\begin{pmatrix}0\\1\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{\bar{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2x}\overline{N}(x,t,\overline{k}_j)\mathrm{e}^{s_1}}{k-\overline{k}_j}\right\}, \tag{4.6}$$

$$\Psi_+(k;x,t)=\left\{N(x,t,k)\mathrm{e}^{-s_2}-\begin{pmatrix}0\\1\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{\bar{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2x}\overline{N}(x,t,\overline{k}_j)\mathrm{e}^{s_1}}{k-\overline{k}_j}\right\}, \tag{4.7}$$

and $\Psi_\pm(k;\ x,t)$ is analytic in the upper/lower half k^2-plane. For convenience, we define the projection operators

$$P_\pm f = \frac{1}{2\pi \mathrm{i}}\int_\Sigma \frac{f(\xi)}{\xi-(k\pm 0)}\,\mathrm{d}\xi, \tag{4.8}$$

where $k\pm 0$ represents that k slightly moves inside the $+/-$ regions of the cross in figure 2. The cross separates the plane into four regions with the usual four quadrants. The $+$ region consists of contours in the first and third quadrants: inside quadrants I and III with the arrows indicating the positive direction; the—region consists of analogous regions inside quadrants II and IV. If $f_\pm(k)$ is analytic in quadrants I and III (II and IV) and $f_\pm(k)\to 0$ as $|k|\to\infty$, then

$$P_\pm(f_\mp)(k)=0, \qquad P_\pm(f_\pm)(k)=\pm f_\pm(k).$$

Recall that Σ is illustrated in figure 2.

Taking the minus/plus projector of equations (4.1) and (4.5) respectively yields an integral/algebraic system of equations for $N(x,t,k),\overline{N}(x,t,k),N(x,t,k_j),\overline{N}(x,t,\overline{k}_j)$. Thus,

$$\begin{aligned}\overline{N}(x,t,k)\mathrm{e}^{s_2}-\begin{pmatrix}1\\0\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2x}N(x,t,k_j)\mathrm{e}^{-s_1}}{k-k_j}\\ =\frac{1}{2\pi\mathrm{i}}P_-\int_+\frac{\tilde{\rho}(\xi,t)\mathrm{e}^{2\mathrm{i}\xi^2x}N(x,\xi,t)\mathrm{e}^{-s_1}}{\xi-k}\,\mathrm{d}\xi,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}N(x,t,k)\mathrm{e}^{-s_2}-\begin{pmatrix}0\\1\end{pmatrix}-\sum_{j=1}^{J}\frac{\tilde{\overline{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2x}\overline{N}(x,t,\overline{k}_j)\mathrm{e}^{s_1}}{k-\overline{k}_j}\\ =-\frac{1}{2\pi\mathrm{i}}P_+\int_-\frac{\tilde{\overline{\rho}}(\xi,t)\mathrm{e}^{-2\mathrm{i}\xi^2x}\overline{N}(x,\xi,t)\mathrm{e}^{s_1}}{\xi-k}\,\mathrm{d}\xi,\end{aligned} \tag{4.10}$$

where values of k in the P_-/P_+ projector are taken just inside the $-/+$ region of the cross in figure 2. The system is closed when we evaluate equation (4.9) at $k=\overline{k}_m$ and evaluate equation (4.10) at $k=k_m$, $m=1,2,\ldots,J$.

By equating the $O(\frac{1}{k})$ terms from equations (3.22) and (4.9), we can determine the potentials $q(x,t),r(x,t)$, which are

$$\begin{aligned}q(x,t)=2\mathrm{i}\sum_{j=1}^{J}\tilde{\overline{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2x}\overline{N}_1(x,t,\overline{k}_j)\mathrm{e}^{s_1}\\ +\frac{1}{\pi}\int_\Sigma\tilde{\overline{\rho}}(\xi,t)\mathrm{e}^{-2\mathrm{i}\xi^2x}\overline{N}_1(x,\xi,t)\mathrm{e}^{s_1}\,\mathrm{d}\xi,\end{aligned} \tag{4.11}$$

$$\begin{aligned}r(x,t)=-2\mathrm{i}\sum_{j=1}^{J}\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2x}N_2(x,t,k_j)\mathrm{e}^{-s_1}\\ +\frac{1}{\pi}\int_\Sigma\tilde{\rho}(\xi,t)\mathrm{e}^{2\mathrm{i}\xi^2x}N_2(x,\xi,t)\mathrm{e}^{-s_1}\,\mathrm{d}\xi,\end{aligned} \tag{4.12}$$

where $\tilde{c}_j(t), \tilde{\bar{c}}_j(t)$ are given by equation (4.4). The structure of the equations for $N, \overline{N}$ implies that we can solve for $\overline{N}_1\,\mathrm{e}^{s_2}$ and $N_2\,\mathrm{e}^{-s_2}$ in terms of scattering data. Note that the product qr can be solved in terms of scattering data.

If the potentials decay rapidly at infinity such that $\tilde{\rho}$ and $\tilde{\bar{\rho}}$ can be analytically continued to include all poles $\{k_j : \Re k_j \cdot \Im k_j > 0\}_{j=1}^{J}$ and $\{\overline{k}_j : \Re \overline{k}_j \cdot \Im \overline{k}_j < 0\}_{j=1}^{J}$, respectively, then (4.9) and (4.10) can be written in reduced notation as

$$\overline{N}(x,t,k)\mathrm{e}^{s_2} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}_0} \frac{\tilde{\rho}(\xi,t)\mathrm{e}^{2\mathrm{i}\xi^2 x}N(x,\xi,t)\mathrm{e}^{-s_1}}{\xi - k}\,\mathrm{d}\xi, \tag{4.13}$$

$$N(x,t,k)\mathrm{e}^{-s_2} = \begin{pmatrix}0\\1\end{pmatrix} - \frac{1}{2\pi \mathrm{i}}\int_{\bar{\mathcal{C}}_0} \frac{\tilde{\bar{\rho}}(\xi,t)\mathrm{e}^{-2\mathrm{i}\xi^2 x}\overline{N}(x,\xi,t)\mathrm{e}^{s_1}}{\xi - k}\,\mathrm{d}\xi, \tag{4.14}$$

where the contours $\mathcal{C}_0 = \mathcal{C}_1 \cup \mathcal{C}_3$ and $\bar{\mathcal{C}}_0 = \mathcal{C}_2 \cup \mathcal{C}_4$; here $\mathcal{C}_1$ is a contour beginning from $+\mathrm{i}\infty + \epsilon$ continuing to $+\infty + \mathrm{i}\epsilon$ outside all poles inside quadrant I and $\mathcal{C}_3$ is a contour beginning at $-\mathrm{i}\infty - \epsilon$ continuing to $-\infty - \mathrm{i}\epsilon$ outside all poles inside quadrant III, $0 < \epsilon \ll 1$. The integral over $\mathcal{C}_0$ contains the continuous spectrum of P_+ plus the pole contributions. Similarly, $\mathcal{C}_2 \cup \mathcal{C}_4$ is taken so that it contains the continuous spectrum of P_- plus the pole contributions in quadrants II and IV.

Under the same hypothesis, (4.11) and (4.12) can be simplified as

$$q(x,t) = \frac{1}{\pi}\int_{\mathcal{C}_0} \tilde{\bar{\rho}}(\xi,t)\mathrm{e}^{-2\mathrm{i}\xi^2 x}\overline{N}_1(x,\xi,t)\mathrm{e}^{s_1}\,\mathrm{d}\xi, \tag{4.15}$$

$$r(x,t) = \frac{1}{\pi}\int_{\mathcal{C}_0} \tilde{\rho}(\xi,t)\mathrm{e}^{2\mathrm{i}\xi^2 x}N_2(x,\xi,t)\mathrm{e}^{-s_1}\,\mathrm{d}\xi. \tag{4.16}$$

4.1. Soliton solutions

The above system for $N, \overline{N}$ is reduced to an algebraic system when $\tilde{\rho}(k,t) = 0,\ \tilde{\bar{\rho}}(k,t) = 0$:

$$\overline{N}(x,t,k)\mathrm{e}^{s_2} - \begin{pmatrix}1\\0\end{pmatrix} - \sum_{j=1}^{J} \frac{\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2 x}N(x,t,k_j)\mathrm{e}^{-s_1}}{k - k_j} = 0, \tag{4.17}$$

$$N(x,t,k)\mathrm{e}^{-s_2} - \begin{pmatrix}0\\1\end{pmatrix} - \sum_{j=1}^{J} \frac{\tilde{\bar{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2 x}\overline{N}(x,t,\overline{k}_j)\mathrm{e}^{s_1}}{k - \overline{k}_j} = 0, \tag{4.18}$$

and we take $k = \overline{k}_m$ in equation (4.17) and $k = k_m$ in equation (4.18). The solution of this system yields the eigenfunctions/soliton solutions.

4.2. 1-soliton solution

The simplest case is a single soliton. From equation (3.14), eigenvalues always come in pairs $\pm k_1, \pm \overline{k}_1$, hence the simplest soliton occurs when $J = 2$. Since the components $N_1(x,t,k), \overline{N}_2(x,t,k)$ are odd in k, $N_2(x,t,k), \overline{N}_1(x,t,k)$ are even in k, $\tilde{c}_1, \tilde{\bar{c}}_1$ are even with respect to $\pm k_1, \pm \overline{k}_1$, we have

$$N_1(x,t,-k_1) = -N_1(x,t,k_1), \qquad N_2(x,t,-k_1) = N_2(x,t,k_1),$$

$$\overline{N}_1(x,t,-\overline{k}_1) = \overline{N}_1(x,t,\overline{k}_1), \qquad \overline{N}_2(x,t,-\overline{k}_1) = -\overline{N}_2(x,t,\overline{k}_1).$$

From equations (3.62) and (4.17), we get

$$\begin{pmatrix}\overline{N}_1(x,t,k)\\ \overline{N}_2(x,t,k)\end{pmatrix} \mathrm{e}^{s_2} = \begin{pmatrix}1\\0\end{pmatrix} + \tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}\left(\begin{pmatrix}N_1(x,t,k_1)\\ N_2(x,t,k_1)\end{pmatrix}\frac{1}{k-k_1} + \begin{pmatrix}N_1(x,t,-k_1)\\ N_2(x,t,-k_1)\end{pmatrix}\frac{1}{k+k_1}\right). \tag{4.19}$$

Hence, evaluating this equation at $k=\overline{k}_1$ yields

$$\begin{pmatrix}\overline{N}_1(x,t,\overline{k}_1)\\ \overline{N}_2(x,t,\overline{k}_1)\end{pmatrix} \mathrm{e}^{s_2} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{\tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}}{\overline{k}_1^2 - k_1^2}\begin{pmatrix}2k_1N_1(x,t,k_1)\\ 2\overline{k}_1N_2(x,t,k_1)\end{pmatrix}. \tag{4.20}$$

Similarly, from equation (4.17), we find

$$\begin{pmatrix}N_1(x,t,k_1)\\ N_2(x,t,k_1)\end{pmatrix} \mathrm{e}^{-s_2} = \begin{pmatrix}0\\1\end{pmatrix} + \frac{\tilde{\overline{c}}_1(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1}}{k_1^2 - \overline{k}_1^2}\begin{pmatrix}2k_1\overline{N}_1(x,t,\overline{k}_1)\\ 2\overline{k}_1\overline{N}_2(x,t,\overline{k}_1)\end{pmatrix}. \tag{4.21}$$

Taking the first component of each equation and solving for $\overline{N}_1(x,t,\overline{k}_1)$ yield

$$\overline{N}_1(x,t,\overline{k}_1) = \frac{\mathrm{e}^{-s_2}}{1+\frac{(2k_1)^2\tilde{c}_1(t)\tilde{\overline{c}}_1(t)\mathrm{e}^{2\mathrm{i}\left(k_1^2-\overline{k}_1^2\right)x}}{\left(k_1^2-\overline{k}_1^2\right)^2}}. \tag{4.22}$$

From (4.11), the solution corresponding to the above eigenfunction is given by

$$q(x,t) = \frac{4\mathrm{i}\tilde{\overline{c}}_1(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1-s_2}}{1+\frac{(2k_1)^2\tilde{c}_1(t)\tilde{\overline{c}}_1(t)\mathrm{e}^{2\mathrm{i}\left(k_1^2-\overline{k}_1^2\right)x}}{\left(k_1^2-\overline{k}_1^2\right)^2}}. \tag{4.23}$$

Taking the second component of each equation and solving for $N_2(x,t,k_1)$ yield

$$N_2(x,t,k_1) = \frac{\mathrm{e}^{s_2}}{1+\frac{(2\overline{k}_1)^2\tilde{c}_1(t)\tilde{\overline{c}}_1(t)\mathrm{e}^{2\mathrm{i}\left(k_1^2-\overline{k}_1^2\right)x}}{\left(k_1^2-\overline{k}_1^2\right)^2}}. \tag{4.24}$$

Using (4.11), the solution corresponding to the above eigenfunction is given by

$$r(x,t) = \frac{-4\mathrm{i}\tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{s_2-s_1}}{1+\frac{(2\overline{k}_1)^2\tilde{c}_1(t)\tilde{\overline{c}}_1(t)\mathrm{e}^{2\mathrm{i}\left(k_1^2-\overline{k}_1^2\right)x}}{\left(k_1^2-\overline{k}_1^2\right)^2}}. \tag{4.25}$$

Since

$$s_1 = \frac{\mathrm{i}}{2}\int_{-\infty}^{x} qr\,\mathrm{d}x', s_2 = \frac{\mathrm{i}}{2}\int_{x}^{\infty} qr\,\mathrm{d}x',$$

we have

$$s_1 - s_2 = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} qr\,\mathrm{d}x' - \mathrm{i}\int_{x}^{\infty} qr\,\mathrm{d}x'. \tag{4.26}$$

The first integral above is a constant of the motion. To calculate $q(x,t)$ or $r(x,t)$, we must calculate $s_1 - s_2$. We note that $qr(x,t)$ is independent of s_1, s_2, hence it is determined purely in terms of scattering data. Indeed, qr can be readily integrated. We see that it takes the form

$$qr(x,t) = \frac{-A\,\mathrm{e}^{\mathrm{i}\mu x}}{(1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x})(1 + B\overline{k}_1^2\mathrm{e}^{\mathrm{i}\mu x})},$$

where

$$A := (4\mathrm{i})^2\tilde{c}_1(t)\tilde{\overline{c}}_1(t), \qquad B := -\frac{A}{4(k_1^2 - \overline{k}_1^2)^2}, \qquad \mu := 2(k_1^2 - \overline{k}_1^2),$$

whereupon

$$-qr(x,t) = \frac{\alpha_1\,\mathrm{e}^{\mathrm{i}\mu x}}{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x}} + \frac{\alpha_2\,\mathrm{e}^{\mathrm{i}\mu x}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}, \quad \alpha_1 = \frac{Ak_1^2}{k_1^2 - \overline{k}_1^2}, \quad \alpha_2 = -\frac{A\overline{k}_1^2}{k_1^2 - \overline{k}_1^2},$$

hence

$$\begin{aligned}\int qr(x,t)\mathrm{d}x &= \frac{-A}{\mathrm{i}\mu B(k_1^2 - \overline{k}_1^2)}\log\left(\frac{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}\right)\\ &= -2\mathrm{i}\,\log\left(\frac{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}\right),\end{aligned}$$

$$\begin{aligned}s_1 - s_2 &= \log\left(\frac{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x'}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x'}}\right)\Bigg|_{-\infty}^{x} - \log\left(\frac{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x'}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x'}}\right)\Bigg|_{x}^{\infty}\\ &= 2\,\log\left(\frac{1 + Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}{1 + B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}\right).\end{aligned} \tag{4.27}$$

Note that

$$\tilde{c}_1(t) = \frac{b(k_1,0)\mathrm{e}^{4\mathrm{i}k_1^4 t}}{\tilde{a}'(k_1)}, \qquad \tilde{\overline{c}}_1(t) = \frac{\overline{b}(\overline{k}_1,0)\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}}{\tilde{\overline{a}}'(\overline{k}_1)}. \tag{4.28}$$

From the trace formulae (section 3.4), one deduces

$$\tilde{a}'(k_1) = \frac{2k_1}{k_1^2 - \overline{k}_1^2}, \qquad \tilde{\overline{a}}'(\overline{k}_1) = \frac{2\overline{k}_1}{\overline{k}_1^2 - k_1^2}. \tag{4.29}$$

Thus, $s_1 - s_2$ only depends on eigenvalues k_1, $\overline{k}_1$ and norming constants $b(k_1,0)$, $\overline{b}(\overline{k}_1,0)$. Moreover,

$$\tilde{c}_1(t) = \frac{(k_1^2 - \overline{k}_1^2)b_1\,\mathrm{e}^{4\mathrm{i}k_1^4 t}}{2k_1}, \qquad \tilde{\overline{c}}_1(t) = \frac{(\overline{k}_1^2 - k_1^2)\overline{b}_1\,\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}}{2\overline{k}_1}, \tag{4.30}$$

where $b_1 := b(k_1,0)$ and $\overline{b}_1 := \overline{b}(\overline{k}_1,0)$. Then (4.23) can be written as

$$q(x,t) = \frac{2\mathrm{i}(\overline{k}_1^2 - k_1^2)\overline{b}_1\,\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}\,\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1 - s_2}}{\overline{k}_1 - k_1 b_1\overline{b}_1\,\mathrm{e}^{4\mathrm{i}(k_1^4 - \overline{k}_1^4)t}\,\mathrm{e}^{2\mathrm{i}(k_1^2 - \overline{k}_1^2)x}}, \tag{4.31}$$

where $s_1 - s_2$ is given in (4.27). Specifically, (4.31) can be written in an explicit form:

$$q(x,t) = \frac{2\mathrm{i}(\overline{k}_1^2 - k_1^2)\overline{b}_1\,\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}\,\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\left(\frac{1+Bk_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}{1+B\overline{k}_1^2\,\mathrm{e}^{\mathrm{i}\mu x}}\right)^2}{\overline{k}_1 - k_1 b_1 \overline{b}_1\,\mathrm{e}^{4\mathrm{i}(k_1^4-\overline{k}_1^4)t}\,\mathrm{e}^{2\mathrm{i}(k_1^2-\overline{k}_1^2)x}}. \tag{4.32}$$

(4.32) is the general formula of a pure 1-soliton solution. Once the symmetry reduction is imposed, the above $q(x,t)$ will be specified. In general, the 1-soliton does not blow up if $\frac{\overline{k}_1}{k_1 b_1 \overline{b}_1} \neq \mathrm{e}^{4\mathrm{i}(k_1^4-\overline{k}_1^4)t}\,\mathrm{e}^{2\mathrm{i}(k_1^2-\overline{k}_1^2)x}$.

4.3. 2-soliton solution

From (3.14), one has eigenvalues always come in pairs $\pm k_j, \pm\overline{k}_j$, $j = 1, 2$. Thus, a pure 2-soliton solution occurs when $J = 4$. From (3.31), (3.62) and (4.17), one has

$$\begin{aligned}
\begin{pmatrix}\overline{N}_1(x,t,k)\\ \overline{N}_2(x,t,k)\end{pmatrix}\mathrm{e}^{s_2} &= \begin{pmatrix}1\\0\end{pmatrix} + \tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}\left(\begin{pmatrix}N_1(x,t,k_1)\\ N_2(x,t,k_1)\end{pmatrix}\frac{1}{k-k_1}\right.\\
&\quad\left. + \begin{pmatrix}N_1(x,t,-k_1)\\ N_2(x,t,-k_1)\end{pmatrix}\frac{1}{k+k_1}\right)\\
&\quad + \tilde{c}_2(t)\mathrm{e}^{2\mathrm{i}k_2^2 x}\,\mathrm{e}^{-s_1}\left(\begin{pmatrix}N_1(x,t,k_2)\\ N_2(x,t,k_2)\end{pmatrix}\frac{1}{k-k_2}\right.\\
&\quad\left. + \begin{pmatrix}N_1(x,t,-k_2)\\ N_2(x,t,-k_2)\end{pmatrix}\frac{1}{k+k_2}\right)\\
&= \begin{pmatrix}1\\0\end{pmatrix} + \tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}\left(\begin{pmatrix}N_1(x,t,k_1)\\ N_2(x,t,k_1)\end{pmatrix}\frac{1}{k-k_1}\right.\\
&\quad\left. + \begin{pmatrix}-N_1(x,t,k_1)\\ N_2(x,t,k_1)\end{pmatrix}\frac{1}{k+k_1}\right)\\
&\quad + \tilde{c}_2(t)\mathrm{e}^{2\mathrm{i}k_2^2 x}\,\mathrm{e}^{-s_1}\left(\begin{pmatrix}N_1(x,t,k_2)\\ N_2(x,t,k_2)\end{pmatrix}\frac{1}{k-k_2}\right.\\
&\quad\left. + \begin{pmatrix}-N_1(x,t,k_2)\\ N_2(x,t,k_2)\end{pmatrix}\frac{1}{k+k_2}\right)\\
&= \begin{pmatrix}1\\0\end{pmatrix} + \frac{2\tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}}{k^2-k_1^2}\begin{pmatrix}k_1 N_1(x,t,k_1)\\ kN_2(x,t,k_1)\end{pmatrix}\\
&\quad + \frac{2\tilde{c}_2(t)\mathrm{e}^{2\mathrm{i}k_2^2 x}\,\mathrm{e}^{-s_1}}{k^2-k_2^2}\begin{pmatrix}k_2 N_1(x,t,k_2)\\ kN_2(x,t,k_2)\end{pmatrix}.
\end{aligned} \tag{4.33}$$

Evaluating the above equation at $k = \overline{k}_1$ and $k = \overline{k}_2$ yields

$$\begin{aligned}
\begin{pmatrix}\overline{N}_1(x,t,\overline{k}_1)\\ \overline{N}_2(x,t,\overline{k}_1)\end{pmatrix}\mathrm{e}^{s_2} &= \begin{pmatrix}1\\0\end{pmatrix} + \frac{2\tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}}{\overline{k}_1^2-k_1^2}\begin{pmatrix}k_1 N_1(x,t,k_1)\\ \overline{k}_1 N_2(x,t,k_1)\end{pmatrix}\\
&\quad + \frac{2\tilde{c}_2(t)\mathrm{e}^{2\mathrm{i}k_2^2 x}\,\mathrm{e}^{-s_1}}{\overline{k}_1^2-k_2^2}\begin{pmatrix}k_2 N_1(x,t,k_2)\\ \overline{k}_1 N_2(x,t,k_2)\end{pmatrix}
\end{aligned} \tag{4.34}$$

and

$$\begin{pmatrix} \overline{N}_1(x,t,\overline{k}_2) \\ \overline{N}_2(x,t,\overline{k}_2) \end{pmatrix} \mathrm{e}^{s_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2\tilde{c}_1(t)\mathrm{e}^{2\mathrm{i}k_1^2 x}\,\mathrm{e}^{-s_1}}{\overline{k}_2^2 - k_1^2} \begin{pmatrix} k_1 N_1(x,t,k_1) \\ \overline{k}_2 N_2(x,t,k_1) \end{pmatrix} + \frac{2\tilde{c}_2(t)\mathrm{e}^{2\mathrm{i}k_2^2 x}\,\mathrm{e}^{-s_1}}{\overline{k}_2^2 - k_2^2} \begin{pmatrix} k_2 N_1(x,t,k_2) \\ \overline{k}_2 N_2(x,t,k_2) \end{pmatrix}. \tag{4.35}$$

Similarly, we deduce

$$\begin{pmatrix} N_1(x,t,k) \\ N_2(x,t,k) \end{pmatrix} \mathrm{e}^{-s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2\tilde{\tilde{c}}_1(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1}}{k^2 - \overline{k}_1^2} \begin{pmatrix} k\overline{N}_1(x,t,\overline{k}_1) \\ \overline{k}_1 \overline{N}_2(x,t,\overline{k}_1) \end{pmatrix} + \frac{2\tilde{\tilde{c}}_2(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_2^2 x}\,\mathrm{e}^{s_1}}{k^2 - \overline{k}_2^2} \begin{pmatrix} k\overline{N}_1(x,t,\overline{k}_2) \\ \overline{k}_2 \overline{N}_2(x,t,\overline{k}_2) \end{pmatrix}. \tag{4.36}$$

Evaluating this equation at $k = k_1$ and $k = k_2$ gives

$$\begin{pmatrix} N_1(x,t,k_1) \\ N_2(x,t,k_1) \end{pmatrix} \mathrm{e}^{-s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2\tilde{\tilde{c}}_1(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1}}{k_1^2 - \overline{k}_1^2} \begin{pmatrix} k_1\overline{N}_1(x,t,\overline{k}_1) \\ \overline{k}_1 \overline{N}_2(x,t,\overline{k}_1) \end{pmatrix} + \frac{2\tilde{\tilde{c}}_2(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_2^2 x}\,\mathrm{e}^{s_1}}{k_1^2 - \overline{k}_2^2} \begin{pmatrix} k_1\overline{N}_1(x,t,\overline{k}_2) \\ \overline{k}_2 \overline{N}_2(x,t,\overline{k}_2) \end{pmatrix} \tag{4.37}$$

and

$$\begin{pmatrix} N_1(x,t,k_2) \\ N_2(x,t,k_2) \end{pmatrix} \mathrm{e}^{-s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2\tilde{\tilde{c}}_1(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_1^2 x}\,\mathrm{e}^{s_1}}{k_2^2 - \overline{k}_1^2} \begin{pmatrix} k_2\overline{N}_1(x,t,\overline{k}_1) \\ \overline{k}_1 \overline{N}_2(x,t,\overline{k}_1) \end{pmatrix} + \frac{2\tilde{\tilde{c}}_2(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_2^2 x}\,\mathrm{e}^{s_1}}{k_2^2 - \overline{k}_2^2} \begin{pmatrix} k_2\overline{N}_1(x,t,\overline{k}_2) \\ \overline{k}_2 \overline{N}_2(x,t,\overline{k}_2) \end{pmatrix}. \tag{4.38}$$

From equations (4.11) and (4.12), the solutions are given by

$$q(x,t) = 2\mathrm{i}\sum_{j=1}^{2} \tilde{\tilde{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2 x}\overline{N}_1(x,t,\overline{k}_j)\mathrm{e}^{s_1} \tag{4.39}$$

and

$$r(x,t) = -2\mathrm{i}\sum_{j=1}^{2} \tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2 x}N_2(x,t,k_j)\mathrm{e}^{-s_1} \tag{4.40}$$

with $\tilde{c}_j(t), \tilde{\tilde{c}}_j(t),\ \ j = 1,2$ given by equation (4.4). Hence we need $\overline{N}_1(x,t,\overline{k}_j)$, $N_2(x,t,k_j)$, $j = 1,2$.

From (4.34), (4.35), (4.37) and (4.38), after some algebra, we find the solutions of the form

$$\overline{N}_1(x,t,\overline{k}_1) = \mathrm{e}^{-s_2}\cdot\frac{A_{12}+A_{22}}{D_A}, \qquad \overline{N}_1(x,t,\overline{k}_2) = \mathrm{e}^{-s_2}\cdot\frac{A_{21}+A_{11}}{D_A},$$
$$D_A = A_{11}A_{22} - A_{12}A_{21}, \tag{4.41}$$

$$N_2(x,t,k_1) = \mathrm{e}^{s_2} \cdot \frac{B_{12}+B_{22}}{D_B}, \qquad N_2(x,t,k_2) = \mathrm{e}^{s_2} \cdot \frac{B_{21}+B_{11}}{D_B},$$
$$D_B = B_{11}B_{22} - B_{12}B_{21}, \tag{4.42}$$

where

$$A_{11} = 1 - k_1^2 C_1(\overline{k}_1)\overline{C}_1(k_1) - k_2^2 C_2(\overline{k}_1)\overline{C}_1(k_2),$$
$$A_{12} = k_1^2 C_1(\overline{k}_1)\overline{C}_2(k_1) + k_2^2 C_2(\overline{k}_1)\overline{C}_2(k_2), \tag{4.43}$$

$$A_{21} = k_1^2 C_1(\overline{k}_2)\overline{C}_1(k_1) + k_2^2 C_2(\overline{k}_2)\overline{C}_1(k_2),$$
$$A_{22} = 1 - k_1^2 C_1(\overline{k}_2)\overline{C}_2(k_1) - k_2^2 C_2(\overline{k}_2)\overline{C}_2(k_2),$$

$$B_{11} = 1 - \overline{k}_1^2 \overline{C}_1(k_1) C_1(\overline{k}_1) - \overline{k}_2^2 \overline{C}_2(k_1) C_1(\overline{k}_2),$$
$$B_{12} = \overline{k}_1^2 \overline{C}_1(k_1) C_2(\overline{k}_1) + \overline{k}_2^2 \overline{C}_2(k_1) C_2(\overline{k}_2), \tag{4.44}$$

$$B_{21} = \overline{k}_1^2 \overline{C}_1(k_2) C_1(\overline{k}_1) + \overline{k}_2^2 \overline{C}_2(k_2) C_1(\overline{k}_2),$$
$$B_{22} = 1 - \overline{k}_1^2 \overline{C}_1(k_2) C_2(\overline{k}_1) - \overline{k}_2^2 \overline{C}_2(k_2) C_2(\overline{k}_2),$$

and

$$C_j(k) = \frac{2\tilde{c}_j(t)\mathrm{e}^{2\mathrm{i}k_j^2 x}}{k^2 - k_j^2}, \qquad \overline{C}_j(k) = \frac{2\tilde{\overline{c}}_j(t)\mathrm{e}^{-2\mathrm{i}\overline{k}_j^2 x}}{k^2 - \overline{k}_j^2}.$$

Recall that

$$\tilde{c}_1(t) = \frac{b_1\,\mathrm{e}^{4\mathrm{i}k_1^4 t}}{\tilde{a}'(k_1)}, \qquad \tilde{c}_2(t) = \frac{b_2\,\mathrm{e}^{4\mathrm{i}k_2^4 t}}{\tilde{a}'(k_2)}, \qquad \tilde{\overline{c}}_1(t) = \frac{\overline{b}_1\,\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}}{\tilde{\overline{a}}'(\overline{k}_1)}, \qquad \tilde{\overline{c}}_2(t) = \frac{\overline{b}_2\,\mathrm{e}^{-4\mathrm{i}\overline{k}_2^4 t}}{\tilde{\overline{a}}'(\overline{k}_2)}, \tag{4.45}$$

where $b_1 := b(k_1, 0)$, $b_2 := b(k_2, 0)$, $\overline{b}_1 := \overline{b}(\overline{k}_1, 0)$, $\overline{b}_2 := \overline{b}(\overline{k}_2, 0)$. From the trace formulae (section 3.4), we have

$$\tilde{a}'(k_1) = \frac{2k_1 \cdot (k_1^2 - k_2^2)}{\left(k_1^2 - \overline{k}_1^2\right)\left(k_1^2 - \overline{k}_2^2\right)}, \qquad \tilde{a}'(k_2) = \frac{2k_2 \cdot (k_2^2 - k_1^2)}{\left(k_2^2 - \overline{k}_1^2\right)\left(k_2^2 - \overline{k}_2^2\right)}, \tag{4.46}$$

$$\tilde{\overline{a}}'(\overline{k}_1) = \frac{2\overline{k}_1 \cdot (\overline{k}_1^2 - \overline{k}_2^2)}{\left(\overline{k}_1^2 - k_1^2\right)\left(\overline{k}_1^2 - k_2^2\right)}, \qquad \tilde{\overline{a}}'(\overline{k}_2) = \frac{2\overline{k}_2 \cdot (\overline{k}_2^2 - \overline{k}_1^2)}{\left(\overline{k}_2^2 - k_1^2\right)\left(\overline{k}_2^2 - k_2^2\right)}. \tag{4.47}$$

By inserting the above into (4.39) and (4.40), the general formula of the 2-soliton solutions is found.

Remark 4.1. (4.39) and (4.40) give the general formula of 2-soliton solutions, and a symmetry reduction between potentials q, r will induce a 2-soliton solution to certain derivative NLS equation. Note that the product qr is independent of s_1, s_2.

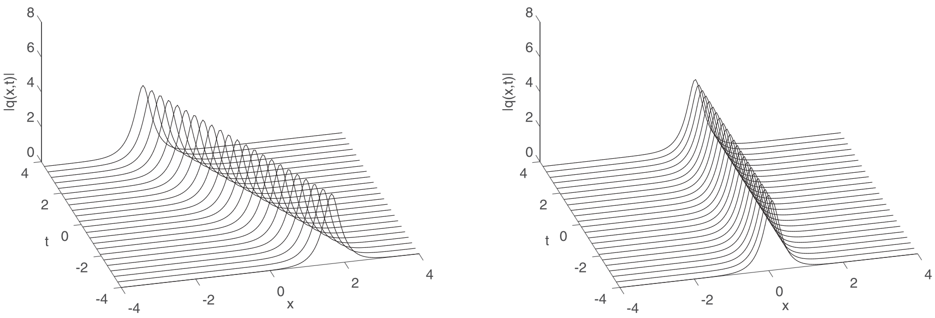

Figure 3. Typical 1-soliton solutions of the standard derivative NLS equation with $b_1 = 1 + \mathrm{i}$. Left: left moving soliton with $k_1 = -1 - 0.95\mathrm{i}$; right: standing soliton with $k_1 = 1 + \mathrm{i}$.

Remark 4.2. $q(x, t)$ and $r(x, t)$ are regular if $D_A \neq 0$ and $D_B \neq 0$, respectively.

Recall that σ can be rescaled to unity, in what follows, we only consider the case of $\sigma = 1$.

4.4. Standard derivative NLS equation

By the symmetry relation $\overline{a}(k) = a^*(k^*)$, we have $\overline{k}_j = k_j^*$ and $\overline{J}_1 = J_1$. In addition, from the general symmetry relations: $a(k) = a(-k)$ and $\overline{a}(k) = \overline{a}(-k)$, one has that $k = -k_j$ is also a zero of $a(k)$. Similarly, $k = -k_j^*$ is also a zero of $\overline{a}(k)$. Thus, $J = 2J_1$, it means that the number of eigenvalues is even, and the simplest soliton solution is a 1-soliton, which is obtained when $J_1 = 1$, i.e., $J = 2$. By (3.35), it implies $\overline{b}_1 = b_1^*$. Let $k_1 = \xi_1 + \mathrm{i}\eta_1$ with $\xi_1\eta_1 > 0$, then (4.32) gives

$$q(x,t) = \frac{8\xi_1\eta_1 b_1^*\,\mathrm{e}^{-4[4\xi_1\eta_1(\xi_1^2-\eta_1^2)+\mathrm{i}(\xi_1^4+\eta_1^4-6\xi_1^2\eta_1^2)]t}\mathrm{e}^{-2[2\xi_1\eta_1+\mathrm{i}(\xi_1^2-\eta_1^2)]x}}{(\xi_1 - \mathrm{i}\eta_1) - (\xi_1 + \mathrm{i}\eta_1)|b_1|^2\mathrm{e}^{-32\xi_1\eta_1(\xi_1^2-\eta_1^2)t}\mathrm{e}^{-8\xi_1\eta_1 x}} \cdot \left(\frac{1 - \frac{|b_1|^2(\xi_1^2-\eta_1^2+2\mathrm{i}\xi_1\eta_1)}{\xi_1^2+\eta_1^2}\mathrm{e}^{-8\xi_1\eta_1 x-32\xi_1\eta_1(\xi_1^2-\eta_1^2)t}}{1 - \frac{|b_1|^2(\xi_1^2-\eta_1^2-2\mathrm{i}\xi_1\eta_1)}{\xi_1^2+\eta_1^2}\mathrm{e}^{-8\xi_1\eta_1 x-32\xi_1\eta_1(\xi_1^2-\eta_1^2)t}} \right)^2. \tag{4.48}$$

This 1-soliton solution is always non-singular and the velocity is $4(\eta_1^2 - \xi_1^2)$. The minimal data we use for recovering the simplest reflectionless potential (1-soliton) contain the following two quantities: the eigenvalue $k_1 = \xi_1 + \mathrm{i}\eta_1$ and the norming constant b_1, where $\xi_1\eta_1 > 0$. In figure 3, we show typical profiles of left moving and standing waves.

Moreover, a 2-soliton solution is given by setting $J_1 = 2$, i.e., $J = 4$. From (3.35), one has $\overline{b}_j = b_j^*$, $j = 1, 2$. By (4.45), we obtain

$$\tilde{c}_1(t) = \frac{b_1\,\mathrm{e}^{4\mathrm{i}k_1^4 t}}{\tilde{a}'(k_1)}, \qquad \tilde{c}_2(t) = \frac{b_2\,\mathrm{e}^{4\mathrm{i}k_2^4 t}}{\tilde{a}'(k_2)}, \qquad \tilde{\overline{c}}_1(t) = \frac{b_1^*\,\mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}}{\tilde{\overline{a}}'(\overline{k}_1)}, \qquad \tilde{\overline{c}}_2(t) = \frac{b_2^*\,\mathrm{e}^{-4\mathrm{i}\overline{k}_2^4 t}}{\tilde{\overline{a}}'(\overline{k}_2)}. \tag{4.49}$$

Using the 2-soliton formulae derived earlier, from (4.39) and (4.40), we get a pure 2-soliton solution of the standard derivative NLS equation (2.11). The corresponding velocities are $4\left(\eta_j^2 - \xi_j^2\right)$, $j = 1, 2$.

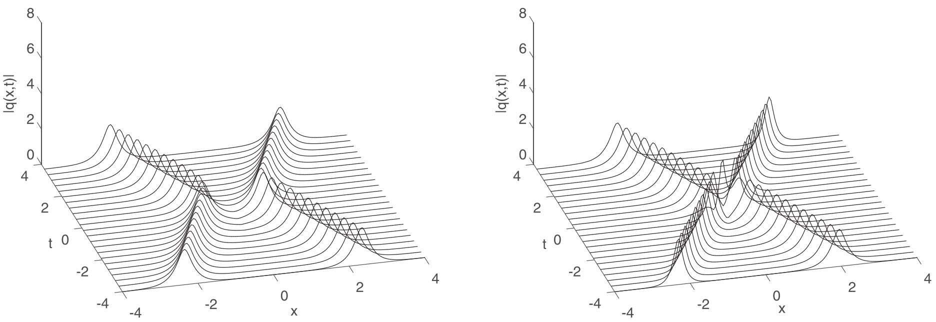

Figure 4. Typical 2-soliton solutions of the standard derivative NLS equation. Left: $k_1 = 1 + 1.05\mathrm{i}, k_2 = -1.05 - \mathrm{i}, b_1 = 1, b_2 = 0.9\mathrm{i}$; right: $k_1 = 1.3 + 1.35\mathrm{i}, k_2 = -1.05 - \mathrm{i}, b_1 = 1, b_2 = 0.9\mathrm{i}$.

The minimal data for reconstruction of pure 2-soliton solutions include two eigenvalues: k_1, k_2 and norming constants: b_1, b_2, where $\Re k_j \cdot \Im k_j > 0$, $j = 1, 2$.

Figure 4 describes 2-soliton interactions for the standard derivative NLS equation. Specifically, we find two solitons interact at around $t = 0$, and the left figure shows the two solitons are of the same magnitude, while the right one gives the collision of two solitons, whose magnitudes are different.

Remark 4.3. The solutions obtained above are consistent with the results in [19], but the methodology used here: Riemann–Hilbert problem is different.

4.5. PT derivative NLS equation

Recall that the eigenvalues come in quartets $\{\pm k_j, \pm \mathrm{i}k_j^* : \Re k_j \cdot \Im k_j > 0 \text{ and } \Re k_j \neq \Im k_j\}_{j=1}^{J_1}$, $\{\pm \overline{k}_j, \pm \mathrm{i}\overline{k}_j^* : \Re \overline{k}_j \cdot \Im \overline{k}_j < 0 \text{ and } \Re \overline{k}_j \neq -\Im \overline{k}_j\}_{j=1}^{\overline{J}_1}$, respectively. Let $J_1 = \overline{J}_1 = 1$, $k_2 = \mathrm{i}k_1^*$ and $\overline{k}_2 = \mathrm{i}\overline{k}_1^*$, thus, $J = 4$. In addition, (3.38) implies $b_2 = \frac{1}{b_1^*}$ and $\overline{b}_2 = \frac{1}{\overline{b}_1^*}$. Combining (4.45), one deduces

$$\tilde{c}_1(t) = \frac{b_1\, \mathrm{e}^{4\mathrm{i}k_1^4 t}}{\tilde{a}'(k_1)}, \qquad \tilde{c}_2(t) = \frac{\mathrm{e}^{4\mathrm{i}k_2^4 t}}{b_1^* \cdot \tilde{a}'(k_2)}, \qquad \tilde{\overline{c}}_1(t) = \frac{\overline{b}_1\, \mathrm{e}^{-4\mathrm{i}\overline{k}_1^4 t}}{\tilde{\overline{a}}'(\overline{k}_1)}, \qquad \tilde{\overline{c}}_2(t) = \frac{\mathrm{e}^{-4\mathrm{i}\overline{k}_2^4 t}}{\overline{b}_1^* \cdot \tilde{\overline{a}}'(\overline{k}_2)}.$$

By substituting all information into (4.39) and (4.40), a pure 2-soliton solution to the PT derivative NLS equation (2.13) is derived. This 2-soliton solution is non-singular if $D_A \neq 0$, i.e., $A_{11}A_{22} - A_{12}A_{21} = 1 - k_1^2 C_1(\mathrm{i}\overline{k}_1^*)\overline{C}_2(k_1) + k_1^{*2} C_2(\mathrm{i}\overline{k}_1^*)\overline{C}_2(\mathrm{i}k_1^*) - k_1^2 C_1(\mathrm{i}\overline{k}_1)\overline{C}_2(k_1) + k_1^{*2} C_2(\overline{k}_1)\overline{C}_1(\mathrm{i}k_1^*) \neq 0$. Thus, $\overline{k}_j = \pm k_j^*$ and $\overline{b}_j = \alpha_j b_j^*$ induce a regular 2-soliton with opposite velocities, i.e., $\pm 4(\eta_1^2 - \xi_1^2)$, where α_j is a real constant, $j = 1, 2$.

Note that $k_2 = \mathrm{i}k_1^*$, $\overline{k}_2 = \mathrm{i}\overline{k}_1^*$, and qr is independent of s_1 and s_2, thus, the minimal data needed for reconstructing the simplest pure soliton solution only include the eigenvalues: $k_1 = \xi_1 + \mathrm{i}\eta_1, \overline{k}_1 = \overline{\xi}_1 + \mathrm{i}\overline{\eta}_1$ and norming constants: $b_1, \overline{b}_1$, where $\xi_1 \cdot \eta_1 > 0, \overline{\xi}_1 \cdot \overline{\eta}_1 < 0, \xi_1 \neq \eta_1$ and $\overline{\xi}_1 \neq -\overline{\eta}_1$.

Figure 5 shows the simplest soliton solution which we call a 2-soliton collision for the PT derivative NLS equation. We see that the two solitons interact near $t = 0$. In addition, the left figure depicts the two solitons are of the same magnitude, however, the right one describes the interaction of two solitons with different amplitudes.

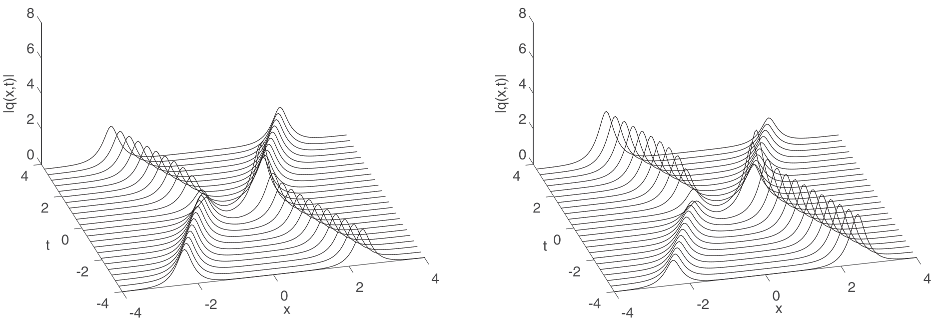

Figure 5. Typical 2-soliton solutions of the PT derivative NLS equation. Left: $k_1 = 1 + 1.05\mathrm{i}$, $\overline{k}_1 = 1 - 1.05\mathrm{i}$, $b_1 = 1$, $\overline{b}_1 = 1$; right: $k_1 = 1 + 1.05\mathrm{i}$, $\overline{k}_1 = 1 - 1.05\mathrm{i}$, $b_1 = 1$, $\overline{b}_1 = 0.5$.

4.6. RST derivative NLS equation

The simplest soliton solution is a 1-soliton, which is constructed when $J_1 = 1$, that is $J = 2$. Let $J_1 = 1$, $k_1 = \xi_1 + \mathrm{i}\eta_1$ with $\xi_1\eta_1 > 0$ and $\overline{k}_1 = \overline{\xi}_1 + \mathrm{i}\overline{\eta}_1$ with $\overline{\xi}_1\overline{\eta}_1 < 0$. From (3.50), one has $b_1^2 = -1$ and $\overline{b}_1^2 = -1$. Then $b_1 = \mathrm{i}\delta_1$ and $\overline{b}_1 = \mathrm{i}\delta_2$, where $\delta_m^2 = 1$, $m = 1, 2$. (4.32) reads

$$\begin{aligned} q(x,t) = \{ &- 2\delta_2[(\overline{\xi}_1^2 - \overline{\eta}_1^2 - \xi_1^2 + \eta_1^2) + 2\mathrm{i}(\overline{\xi}_1\overline{\eta}_1 - \xi_1\eta_1)] \\ &\times \mathrm{e}^{-4[\mathrm{i}(\overline{\xi}_1^4+\overline{\eta}_1^4-6\overline{\xi}_1^2\overline{\eta}_1^2)-4\overline{\xi}_1\overline{\eta}_1(\overline{\xi}_1^2-\overline{\eta}_1^2)]t-2[\mathrm{i}(\overline{\xi}_1^2-\overline{\eta}_1^2)-2\overline{\xi}_1\overline{\eta}_1]x} \\ &\times [(1 + ((\delta_1\delta_2(\xi_1 + \mathrm{i}\eta_1))/(\overline{\xi}_1 + \mathrm{i}\overline{\eta}_1)) \\ &\times \mathrm{e}^{4[\mathrm{i}(\xi_1^4+\eta_1^4-6\xi_1^2\eta_1^2-\overline{\xi}_1^4-\overline{\eta}_1^4+6\overline{\xi}_1^2\overline{\eta}_1^2)-4(\xi_1\eta_1(\xi_1^2-\eta_1^2)-\overline{\xi}_1\overline{\eta}_1(\overline{\xi}_1^2-\overline{\eta}_1^2))]t}) \\ &/ ((1 + ((\delta_1\delta_2(\overline{\xi}_1 + \mathrm{i}\overline{\eta}_1))/(\xi_1 + \mathrm{i}\eta_1)) \\ &\times \mathrm{e}^{4[\mathrm{i}(\xi_1^4+\eta_1^4-6\xi_1^2\eta_1^2-\overline{\xi}_1^4-\overline{\eta}_1^4+6\overline{\xi}_1^2\overline{\eta}_1^2)-4(\xi_1\eta_1(\xi_1^2-\eta_1^2)-\overline{\xi}_1\overline{\eta}_1(\overline{\xi}_1^2-\overline{\eta}_1^2))]t}))]^2 \} \\ &/ \Big\{(\overline{\xi}_1 + \mathrm{i}\overline{\eta}_1) + \delta_1\delta_2(\xi_1 + \mathrm{i}\eta_1) \\ &\times \mathrm{e}^{4[\mathrm{i}(\xi_1^4+\eta_1^4-6\xi_1^2\eta_1^2-\overline{\xi}_1^4-\overline{\eta}_1^4+6\overline{\xi}_1^2\overline{\eta}_1^2)-4(\xi_1\eta_1(\xi_1^2-\eta_1^2)-\overline{\xi}_1\overline{\eta}_1(\overline{\xi}_1^2-\overline{\eta}_1^2))]t-2[\mathrm{i}(\overline{\xi}_1^2-\overline{\eta}_1^2-\xi_1^2+\eta_1^2)-2(\overline{\xi}_1\overline{\eta}_1-\xi_1\eta_1)]x}\Big\}. \end{aligned} \tag{4.50}$$

Note that the above 1-soliton is regular if $\overline{k}_1 = \pm k_1^*$, whose velocity is $4(\eta_1^2 - \xi_1^2)$. The minimal data needed for reconstructing the simplest pure soliton solution ($J = 2$ and hence $J_1 = 1$) incorporate eigenvalues: $k_1 = \xi_1 + \mathrm{i}\eta_1$, $\overline{k}_1 = \overline{\xi}_1 + \mathrm{i}\overline{\eta}_1$ and the units δ_m, $m = 1, 2$, where $\delta_m^2 = 1$, $\xi_1\eta_1 > 0$ and $\overline{\xi}_1\overline{\eta}_1 < 0$. In general, we need the norming constants b_1 and $\overline{b}_1$, however, their values can be determined via the symmetries (3.50). In figure 6, we show typical profiles of right moving and standing solitons.

In addition, a 2-soliton solution is attained by setting $J_1 = 2$, that is $J = 4$. From (3.50), one derives $b_1 = \mathrm{i}\delta_1$, $b_2 = \mathrm{i}\delta_2$, $\overline{b}_1 = \mathrm{i}\delta_3$, $\overline{b}_2 = \mathrm{i}\delta_4$, where $\delta_m^2 = 1$, $m = 1, 2, 3, 4$.

Thus, once all information is put into (4.39) and (4.40), a pure 2-soliton solution to the RST derivative NLS equation (2.16) can be obtained. Note that the 2-soliton is non-singular if $\overline{k}_j = \pm k_j^*$, which leads to $D_A \neq 0$. The corresponding velocities are $4(\eta_j^2 - \xi_j^2)$, $j = 1, 2$.

The minimal data used to recover pure 2-soliton solutions include eigenvalues: $k_1, k_2, \overline{k}_1, \overline{k}_2$ and the units δ_m, $m = 1, 2, 3, 4$, where $\delta_m^2 = 1$, $\Re k_j \cdot \Im k_j > 0$ and $\Re \overline{k}_j \cdot \Im \overline{k}_j < 0$, $j = 1, 2$.

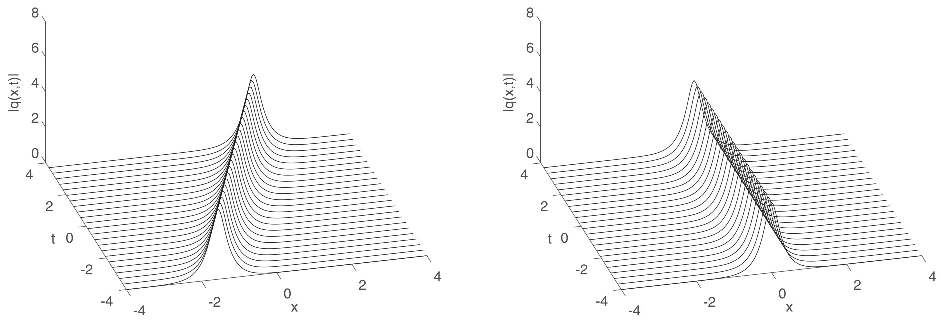

Figure 6. Typical 1-soliton solutions of the RST derivative NLS equation with $b_1 = \mathrm{i}$, $\bar{b}_1 = -\mathrm{i}$. Left: right moving soliton with $k_1 = 0.95 + \mathrm{i}$, $\bar{k}_1 = -0.95 + \mathrm{i}$; right: standing soliton with $k_1 = 1 + \mathrm{i}$, $\bar{k}_1 = 1 - \mathrm{i}$.

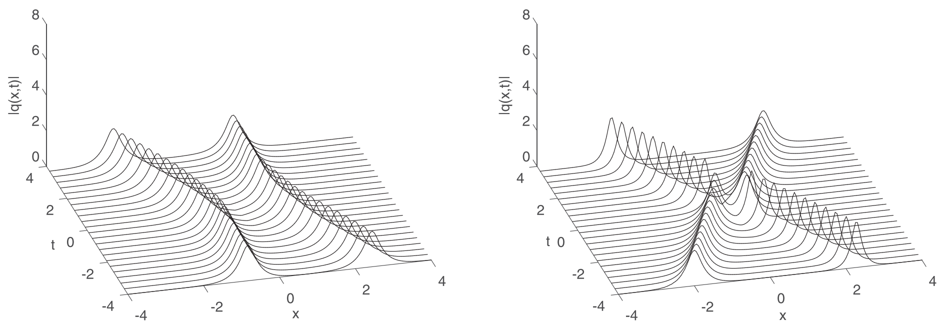

Figure 7. Typical 2-soliton solutions of the RST derivative NLS equation. Left: $k_1 = 1 + \mathrm{i}$, $\bar{k}_1 = 1 - \mathrm{i}$, $b_1 = \mathrm{i}$, $\bar{b}_1 = -\mathrm{i}$, $k_2 = 1 + 0.95\mathrm{i}$, $\bar{k}_2 = 1 - 0.95\mathrm{i}$, $b_2 = -\mathrm{i}$, $\bar{b}_2 = \mathrm{i}$; right: $k_1 = 1 + 1.05\mathrm{i}$, $\bar{k}_1 = 1 - 1.05\mathrm{i}$, $b_1 = \mathrm{i}$, $\bar{b}_1 = -\mathrm{i}$, $k_2 = -1.3 - 1.25\mathrm{i}$, $\bar{k}_2 = -1.3 + 1.25\mathrm{i}$, $b_2 = -\mathrm{i}$, $\bar{b}_2 = \mathrm{i}$.

Figure 7 depicts typical 2-soliton solutions for the RST derivative NLS equation. It is seen that two solitons interact near $t = 0$. In addition, the left figure gives the two solitons are of the same magnitude, while the right one describes the interaction of two solitons with different amplitudes.

5. Inverse scattering–Gel'fand–Levitan–Marchenko (GLM) equations

In this section, we reconstruct the potentials by developing the Gel'fand–Levitan–Marchenko equations instead of the Riemann–Hilbert approach. In fact, we assume that $N(x,t,k)\mathrm{e}^{-s_2}$ and $\overline{N}(x,t,k)\mathrm{e}^{s_2}$ have the following triangular forms

$$N(x,t,k)\mathrm{e}^{-s_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^{+\infty} \begin{pmatrix} kK^{(1)}(x,t,u)\mathrm{e}^{-2s_2} \\ K^{(2)}(x,t,u) \end{pmatrix} \times \mathrm{e}^{-\mathrm{i}k^2(x-u)}\,\mathrm{d}u, \quad u > x,\ \Im k^2 > 0, \tag{5.1}$$

$$\overline{N}(x,t,k)\mathrm{e}^{s_2} = \begin{pmatrix}1\\0\end{pmatrix} + \int_x^{+\infty} \begin{pmatrix}\overline{K}^{(1)}(x,t,u)\\ k\overline{K}^{(2)}(x,t,u)\mathrm{e}^{2s_2}\end{pmatrix} \times \mathrm{e}^{\mathrm{i}k^2(x-u)}\,\mathrm{d}u, \quad u > x,\ \Im k^2 < 0, \tag{5.2}$$

where $v^{(j)}$ denotes the jth component of the vector v. By inserting (5.1) and (5.2) into (4.13) and (4.14), one derives

$$\overline{K}^{(1)}(x,t,y) - \mathrm{i}\int_x^{+\infty} K^{(1)}(x,t,u)F'(u+y,t)\mathrm{d}u = 0 \tag{5.3}$$

and

$$K^{(1)}(x,t,y) + \overline{F}(x+y,t) + \int_x^{+\infty} \overline{K}^{(1)}(x,t,u)\overline{F}(u+y,t)\mathrm{d}u = 0, \tag{5.4}$$

where $F'(z,t) := \frac{\partial F(z,t)}{\partial z}$,

$$\begin{aligned} F(x,t) &= \frac{1}{2\pi}\int_{\mathcal{C}_0} \tilde{\rho}(\xi,t)\mathrm{e}^{\mathrm{i}\xi^2 x}\,\mathrm{e}^{s}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\rho}(\xi,t)\mathrm{e}^{\mathrm{i}\xi^2 x}\,\mathrm{e}^{s}\,\mathrm{d}\xi - \mathrm{i}\sum_{j=1}^{J}\tilde{c}_j(t)\mathrm{e}^{\mathrm{i}k_j^2 x}\,\mathrm{e}^{s} \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} \overline{F}(x,t) &= \frac{1}{2\pi}\int_{\overline{\mathcal{C}}_0} \tilde{\overline{\rho}}(\xi,t)\mathrm{e}^{-\mathrm{i}\xi^2 x}\,\mathrm{e}^{-s}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\overline{\rho}}(\xi,t)\mathrm{e}^{-\mathrm{i}\xi^2 x}\,\mathrm{e}^{-s}\,\mathrm{d}\xi + \mathrm{i}\sum_{j=1}^{\overline{J}}\tilde{\overline{c}}_j(t)\mathrm{e}^{-\mathrm{i}\overline{k}_j^2 x}\,\mathrm{e}^{-s}. \end{aligned} \tag{5.6}$$

It should be pointed out that (5.3) and (5.4) constitute the Gel'fand–Levitan–Marchenko (GLM) equations.

By substituting (5.1) and (5.2) into (4.15) and (4.16), the reconstruction of the potentials is obtained in terms of the kernels of the GLM equations, i.e.,

$$q(x,t) = -2K^{(1)}(x,t,x)\mathrm{e}^{-2s_2}, \qquad r(x,t) = -2\overline{K}^{(2)}(x,t,x)\mathrm{e}^{2s_2}. \tag{5.7}$$

Besides, $K^{(1)}$ and $K^{(2)}$ satisfy

$$(\partial_x - \partial_u)K^{(1)} = q\left(K^{(2)} - \frac{\mathrm{i}}{2}rK^{(1)}\mathrm{e}^{-2s_2}\right)\mathrm{e}^{2s_2}, \tag{5.8}$$

$$(\partial_x - \partial_u)\left(K^{(2)} - \frac{\mathrm{i}}{2}rK^{(1)}\mathrm{e}^{-2s_2}\right) = -\frac{\mathrm{i}}{2}K^{(1)}\mathrm{e}^{-s_2}\partial_x\left(r\,\mathrm{e}^{-s_2}\right) \tag{5.9}$$

subject to the boundary conditions: $q(x,t) = -2K^{(1)}(x,t,x)\mathrm{e}^{-2s_2}$ and $K^{(j)}(x,t,u) \to 0$ as $u \to +\infty$.

This is a Goursat problem. It can be proven that such a system is uniquely solved, from which it follows the existence of the integral representation (5.1). A similar conclusion holds for (5.2).

5.1. Standard derivative NLS equation

The symmetry reduction $r(x,t) = \sigma q^*(x,t)$ is amenable to the standard derivative NLS equation. The symmetries among scattering data and eigenfunctions yield

$$\overline{F}(x,t) = \sigma F^*(x,t) \tag{5.10}$$

and

$$\overline{K}(x,t,y) = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} K^*(x,t,y). \tag{5.11}$$

5.2. PT derivative NLS equation

The symmetry reduction $r(x,t) = \mathrm{i}\sigma q^*(-x,t)$ is subject to the PT derivative NLS equation. It turns out that

$$\overline{K}^{(2)}(x,t,y)\mathrm{e}^{2s_2} = \mathrm{i}\sigma K^{(1)*}(-x,t,y)\left(\mathrm{e}^{-2s_2}\right)^*. \tag{5.12}$$

5.3. RST derivative NLS equation

The symmetry reduction $r(x,t) = \sigma q(-x,-t)$ is the one connected with the RST derivative NLS equation. As a result,

$$\overline{K}^{(2)}(x,t,y)\mathrm{e}^{2s_2} = \sigma K^{(1)}(-x,-t,y)\mathrm{e}^{-2s_2}. \tag{5.13}$$

Remark 5.1. Unlike the standard derivative NLS equation, there is no symmetry relation between $F(x,t)$ and $\overline{F}(x,t)$ for the PT and RST cases.

6. Derivation of derivative NLS systems

The derivative NLS equation and its alternative forms can be derived in many applied fields, such as nonlinear optics and magneto-hydrodynamics [12, 13, 24]. Different from the standard NLS equation, the derivative NLS equation is not generic for any envelope dynamics. It is usually valid for the wave packets associated with special modes. In the following, we give a comprehensive derivation from a nonlinear Klein–Gordon type equation.

Consider the following general nonlinear Klein–Gordon type equation

$$u_{tt} - u_{xx} + u + \epsilon\left(\alpha f_1(u;\partial_t;\partial_x) + \beta f_2(u;\partial_t;\partial_x\right) = 0, \tag{6.1}$$

where f_1, f_2 are cubically nonlinear functionals. If only f_1 or f_2 are present, then we can derive the standard NLS equation. However, if α and β are chosen appropriately, then we can derive the derivative NLS equation. The following prototype illustrates the situation. The specific choices of f_1, f_2 are made so that α and β are related via a physically meaningful parameter (the phase speed of the underlying wave).

We illustrate the derivation with the following nonlinear Klein–Gordon type equation

$$u_{tt} - u_{xx} + u + \epsilon(\alpha\partial_t\partial_x + \beta\partial_x^2))\,u^3 = 0, \tag{6.2}$$

where $\alpha > 0$. In the weakly nonlinear regime, i.e., $|\epsilon| \ll 1$, we consider the effective dynamics of wave packets associated with a special plane wave solution by implementing the multi-scale method. Define

$$\theta = kx - \omega t, \qquad X = \epsilon x, \qquad T = \epsilon t, \qquad \tau = \epsilon^2 t,$$

where the linear dispersion relation satisfies

$$\omega^2 = k^2 + 1.$$

Note that we want to understand the effective dynamics at the time scale $O(1/\epsilon^2)$, so we introduce two slow times T and τ.

Expand the solution in an asymptotic form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots \tag{6.3}$$

with $u_j = u_j(\theta, X, T, \tau)$, $j = 1, 2, \ldots$, and substitute this into equation (6.2).

At the leading order, we have

$$\mathcal{L}u_0 \equiv (\omega^2 - k^2)\partial_\theta^2 u_0 + u_0 = 0. \tag{6.4}$$

Solving the leading equation yields

$$u_0 = A\,\mathrm{e}^{\mathrm{i}\theta} + B\,\mathrm{e}^{-\mathrm{i}\theta}, \tag{6.5}$$

where $A = A(X, T, \tau), B = B(X, T, \tau)$. Since in general $B \neq A^*$, the asymptotic limiting equations will be in the complex domain.

In order to obtain the most interested nonlinear dynamics, i.e., the nonlinearity is suitably balanced with the slow time and wide envelope scales, we expand the nonlinear terms first.

$$\begin{aligned}(\alpha\partial_t\partial_x + \beta\partial_x^2)u^3 = {}& \alpha(-\omega k\partial_\theta^2 + \epsilon(-\omega\partial\theta\partial_X + k\partial_\theta\partial_T) + O(\epsilon^2)) \\ & + \beta(k^2\partial_\theta^2 + 2\epsilon k\partial_\theta\partial_X)(u_0^3 + 3\epsilon u_0^2 u_1 + O(\epsilon^2)).\end{aligned} \tag{6.6}$$

Generically, cubic nonlinearity is obtained. In order to obtain the derivative cubic nonlinear terms, we consider a special wave number (phase velocity and α, β satisfy)

$$\alpha\omega k - k^2\beta = 0. \tag{6.7}$$

Namely,

$$\beta = \alpha\frac{\omega}{k}.$$

Then the leading nonlinear terms originally appear at $O(\epsilon)$ which just vanish if we choose this special wave number. Thus, the nonlinear effects appear at $O(\epsilon^2)$.

At $O(\epsilon)$, the equation is still linear

$$\begin{aligned}\mathcal{L}u_1 &= (2\omega\partial_\theta\partial_T + 2k\partial_\theta\partial_X)u_0 \\ &= 2\mathrm{i}(\omega A_T + kA_X)\mathrm{e}^{\mathrm{i}\theta} - 2\mathrm{i}(\omega B_T + kB_X)\mathrm{e}^{-\mathrm{i}\theta}.\end{aligned}$$

Removal of secular terms yields that

$$\omega A_T + kA_X = 0, \quad \omega B_T + kB_X = 0. \tag{6.8}$$

Taking $u_1 = 0$, we come to the balanced order $O(\epsilon^2)$

$$\mathcal{L}u_2 = -[(\partial_T^2 - \partial_X^2 - 2\mathrm{i}\omega\partial_\tau)A\,\mathrm{e}^{\mathrm{i}\theta} + (\partial_T^2 - \partial_X^2 + 2\mathrm{i}\omega\partial_\tau)B\,\mathrm{e}^{-\mathrm{i}\theta}] - \mathrm{NL2},$$

where the nonlinear term is

$$\begin{aligned}\mathrm{NL2} = {} & \left[3(\alpha(-\mathrm{i}\omega\partial_X + \mathrm{i}k\partial_T) + 2\beta k\mathrm{i}\partial_X)A^2B\right]\mathrm{e}^{\mathrm{i}\theta} \\ & - \left[3(\alpha(-\mathrm{i}\omega\partial_X + \mathrm{i}k\partial_T) + 2\beta k\mathrm{i}\partial_X)A^2B\right]\mathrm{e}^{-\mathrm{i}\theta} \\ & + [\ldots]\mathrm{e}^{3\mathrm{i}\theta} + [\ldots]\mathrm{e}^{-3\mathrm{i}\theta}.\end{aligned} \tag{6.9}$$

For the simplicity, we have omitted the coefficients of non-secular terms.

Recall the linear dispersion relation

$$\omega^2 = k^2 + 1.$$

Then

$$\omega'(k) = \frac{k}{\omega}, \quad \omega''(k) = \frac{1}{\omega}(1 - (\omega'(k))^2).$$

Introducing the moving coordinate $\xi = X - \omega'(k)T$ and $\tau = \epsilon T$ so that the terms in the equation (6.8) move to $O(\epsilon^2)$ and removing the secular terms yields

$$\begin{aligned}2\mathrm{i}\omega\partial_\tau A &= ((\omega'(k))^2 - 1)\partial_\xi^2 A + (-3\mathrm{i}\alpha\omega + 6\mathrm{i}\beta k - 3\mathrm{i}\alpha k\omega'(k))\partial_X A^2B, \\ 2\mathrm{i}\omega\partial_\tau B &= -((\omega'(k))^2 - 1)\partial_\xi^2 B + (-3\mathrm{i}\alpha\omega + 6\mathrm{i}\beta k - 3\mathrm{i}\alpha k\omega'(k))\partial_X B^2A.\end{aligned}$$

Using the dispersion relation, the above equations are rewritten as

$$\partial_\tau A = \mathrm{i}\frac{\omega''(k)}{2}\partial_\xi^2 A + \gamma\partial_\xi A^2B, \tag{6.10}$$

$$\partial_\tau B = -\mathrm{i}\frac{\omega''(k)}{2}\partial_\xi^2 B + \gamma\partial_\xi B^2A, \tag{6.11}$$

where

$$\gamma = \frac{-3\alpha\omega + 6\beta k - 3\alpha k\omega'(k)}{2\omega}.$$

Note that we have chosen the special k such that $\beta = \alpha C(k) = \alpha\frac{\omega}{k}$. Using the dispersion relation yields that

$$\gamma = \frac{3\alpha}{2\omega^2} > 0.$$

Introduce the following rescallings

$$\xi = \mu\tilde{\xi}, \qquad A = \frac{1}{\sqrt{\gamma\mu}}\tilde{A}, \qquad B = \frac{1}{\sqrt{\gamma\mu}}\tilde{B}$$

with $\mu = \left(\frac{|\omega''|}{2}\right)^{1/2}$. For the convenience, we drop the tildes and get

$$\partial_\tau A = \mathrm{i}s\partial_\xi^2 A + \partial_\xi A^2 B,$$
$$\partial_\tau B = -\mathrm{i}s\partial_\xi^2 B + \partial_\xi B^2 A, \tag{6.12}$$

where $s = \mathrm{sgn}(\omega''(k))$. If $s = 1$, the above system is the derivative q, r system given by the q, r equations (2.8) and (2.9). Otherwise, i.e., $s = -1$, interchanging A, B, we still get the same derivative q, r system. Since this q, r system reduces to the standard derivative NLS equation and the nonlocal derivative NLS equations they are too included in this asymptotic limiting procedure.

7. Conclusion

The derivative NLS equation is an important nonlinear dispersive equation, which arises in many different contexts. In this paper, we revisit the standard derivative NLS equation and investigate two nonlocal integrable derivative NLS equations via the IST. The direct problem is analyzed, and the inverse scattering is formulated in terms of the Riemann–Hilbert (RH) and Gel'fand–Levitan–Marchenko methods. Explicit soliton solutions are obtained. Finally, it is shown how these equations can be derived, e.g., from a nonlinear Klein–Gordon type equation.

Acknowledgments

Mark J Ablowitz was partially supported by NSF under Grant DMS-2005343. Xu-Dan Luo was partially supported by NSFC under Grant 12101590. Yi Zhu was partially supported by NSFC under Grant 11871299.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Ablowitz M J 2011 *Nonlinear Dispersive Waves, Asymptotic Analysis and Solitons* (Cambridge: Cambridge University Press)

[2] Ablowitz M J, Biondini G and Prinari B 2007 Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions *Inverse Problems* **23** 1711

[3] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* vol 149 (Cambridge: Cambridge University Press)

[4] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 The inverse scattering transform-Fourier analysis for nonlinear problems *Stud. Appl. Math.* **53** 249–315

[5] Ablowitz M J and Musslimani Z H 2013 Integrable nonlocal nonlinear Schrödinger equation *Phys. Rev. Lett.* **110** 064105

[6] Ablowitz M J and Musslimani Z H 2016 Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation *Nonlinearity* **29** 915

[7] Ablowitz M J and Musslimani Z H 2017 Integrable nonlocal nonlinear equations *Stud. Appl. Math.* **139** 7–59

[8] Ablowitz M J and Musslimani Z H 2019 Integrable nonlocal asymptotic reductions of physically significant nonlinear equations *J. Phys. A: Math. Theor.* **52** 15LT02

[9] Ablowitz M J, Prinari B and Trubatch A D 2004 *Discrete and Continuous Nonlinear Schrödinger Systems* vol 302 (Cambridge: Cambridge University Press)

[10] Ablowitz M J, Prinari B and Trubatch A D 2004 Soliton interactions in the vector NLS equation *Inverse Problems* **20** 1217

[11] Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (*SIAM Journal on Applied Mathematics*) (Philadelphia, PA: SIAM)

[12] Anderson D and Lisak M 1983 Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides *Phys. Rev.* A **27** 1393–8

[13] Champeaux S, Laveder D, Passot T and Sulem P L 1999 Remarks on the parallel propagation of small-amplitude dispersive Alfvénic waves *Nonlinear Processes in Geophysics* **6** 169–78

[14] Chen X-J and Lam W K 2004 Inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions *Phys. Rev.* E **69** 066604

[15] Chen X-J, Yang J and Lam W K 2006 *N*-soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions *J. Phys. A: Math. Gen.* **39** 3263

[16] Erdoğan M B, Gürel T B and Tzirakis N 2018 The derivative nonlinear Schrödinger equation on the half line *Ann. Inst. Henri Poincare* C **35** 1947–73

[17] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–deVries equation *Phys. Rev. Lett.* **19** 1095

[18] Jenkins R, Liu J, Perry P and Sulem C 2018 Soliton resolution for the derivative nonlinear Schrödinger equation *Commun. Math. Phys.* **363** 1003–49

[19] Kaup D J and Newell A C 1978 An exact solution for a derivative nonlinear Schrödinger equation *J. Math. Phys.* **19** 798–801

[20] Kawata T and Inoue H 1978 Exact solutions of the derivative nonlinear Schrödinger equation under the nonvanishing conditions *J. Phys. Soc. Japan* **44** 1968–76

[21] Kawata T, Kobayashi N and Inoue H 1979 Soliton solutions of the derivative nonlinear Schrödinger equation *J. Phys. Soc. Japan* **46** 1008–15

[22] Liu J, Perry P A and Sulem C 2016 Global existence for the derivative nonlinear Schrödinger equation by the method of inverse scattering *Commun. PDE* **41** 1692–760

[23] Shabat A B and Zakharov V E 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys - JETP* **34** 62

[24] Miki W and Sogo K 1983 Gauge transformations in soliton theory *J. Phys. Soc. Japan* **52** 394–8

[25] Zhang G and Yan Z 2020 The derivative nonlinear Schrödinger equation with zero/nonzero boundary conditions: inverse scattering transforms and *N*-double-pole solutions *J. Nonlinear Sci.* **30** 3089–127

[26] Zhou G-Q and Huang N-N 2007 An *N*-soliton solution to the DNLS equation based on revised inverse scattering transform *J. Phys. A: Math. Theor.* **40** 13607