Asymptotic spreading of interacting species with multiple fronts II: Exponentially decaying initial data

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Abstract

This is part two of our study on the spreading properties of the Lotka-Volterra competition-diffusion systems with a stable coexistence state. We focus on the case when the initial data are exponential decaying. By establishing a comparison principle for Hamilton-Jacobi equations, we are able to apply the Hamilton-Jacobi approach for Fisher-KPP equation due to Freidlin, Evans and Souganidis. As a result, the exact formulas of spreading speeds and their dependence on initial data are derived. Our results indicate that sometimes the spreading speed of the slower species is nonlocally determined. Connections of our results with the traveling profile due to Tang and Fife, as well as the more recent spreading result of Girardin and Lam, will be discussed.

Contents

1 Introduction

For monotone dynamical systems, the pioneering work of Weinberger et al. [55, 57] (see also [43]) relates the spreading speed of the population to the minimal speed of (monostable) traveling wave solutions. Their result can be applied to the diffusive Lotka-Volterra competition system. Suitably non-dimensionalized, the system is given by

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1-u-av), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t v - d\partial_{xx} v = rv(1-bu-v), & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), & \text{on } \mathbb{R}, \\ v(0,x) = v_0(x), & \text{on } \mathbb{R}, \end{cases} \tag{1.1}$$

with $a, b \in (0,1)$. It is clear that (1.1) admits a trivial equilibrium $(0,0)$, two semi-trivial equilibria $(1,0)$ and $(0,1)$, and further a linearly stable equilibrium

$$(k_1, k_2) = \left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right).$$

Theorem 1.1 (Lewis et al. [36])**.** *Let (u,v) be the solution of (1.1) with initial data*

$$u(0,x) = \rho_1(x), \quad v(0,x) = 1-\rho_2(x),$$

where $0 \le \rho_i < 1$ $(i=1,2)$ are compactly supported functions in $\mathbb{R}$. Then there exists some $c_{\mathrm{LLW}} \in [2\sqrt{1-a}, 2]$ such that

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{|x|<ct} (|u(t,x)-k_1| + |v(t,x)-k_2|) = 0 & \text{for each } c < c_{\mathrm{LLW}}, \\ \lim\limits_{t\to\infty} \sup\limits_{|x|>ct} (|u(t,x)| + |v(t,x)-1|) = 0 & \text{for each } \ c > c_{\mathrm{LLW}}. \end{cases}$$

In this case, we say that u spreads at speed c_{LLW}.

Remark 1.2. *If the initial data $(u,v)(0,x)$ is a compact perturbation of $(1,0)$, then there exists $\tilde{c}_{\mathrm{LLW}} \in [2\sqrt{dr(1-b)}, 2\sqrt{dr}]$ such that the species v spreads at speed $\tilde{c}_{\mathrm{LLW}}$.*

It is shown in [38,39] that the spreading speed c_{LLW} (resp. $\tilde{c}_{\mathrm{LLW}}$) is identical to the minimum wave speed of traveling wave solution connecting the pair of equilibria (k_1,k_2) and $(0,1)$ (resp. $(1,0)$). It is crucial for the theory that the pair of equilibria forms an ordered pair of equilibria (regarding the comparability of steady states in the theory of monotone semi-flows, see [49]).

For the weak competitive diffusive system (1.1), Tang and Fife [50] proved an additional class of traveling wave solutions connecting the positive equilibrium (k_1,k_2) with the trivial equilibrium $(0,0)$. In this case, the equilibria $(0,0)$ and (k_1,k_2) are un-ordered, and hence the existence of traveling wave, due to Tang and Fife [50], does not directly follow from the monotone dynamical systems framework due to Weinberger et al. [56,57] (see also [20,39]).

A natural question is whether the speed traveling wave solutions due to Tang and Fife, which connect (k_1,k_2) to $(0,0)$, determine the spreading speed of the populations in the Cauchy problem (1.1), provided the initial data (u_0, v_0) has the same asymptotics at $x = \pm\infty$ as the traveling wave solution? What happens for more general exponentially decaying initial data? Does the two species spread with different speeds?

In this paper, we continue our investigation in [41] on the spreading properties of solutions of the Cauchy problem (1.1). We are interested in determining the spreading speeds of each of the populations u and v, for the class of initial data (u_0, v_0) satisfying $(u_0, v_0)(-\infty) = (1, 0)$, $(u_0, v_0)(\infty) = (0, 0)$ and such that $u_0 \to 0$ exponentially at ∞ with rate $\lambda_u > 0$; $v_0 \to 0$ decays exponentially at ∞ (resp. $-\infty$) with rate $\lambda_v^+ > 0$ (resp. $\lambda_v^- > 0$).

We introduce the Hamilton-Jacobi approach to study the spreading of two-interacting species into an open habitat, and resolve a conjecture by Shigesada [48, Ch. 7]. Inspired by the pioneering work of Freidlin [22] and of Evans and Souganidis [17] on the Fisher-KPP equation, we shall derive, via the thin-front limit, a couple of Hamilton-Jacobi equations for which solutions have to be understood in the viscosity sense. In our previous work [41], we considered the Cauchy problem (1.1) endowed with compactly supported initial data, and used the dynamics programming approach to show the uniqueness of the limiting Hamilton-Jacobi equations, and to evaluate the solution by determining the path that minimizes certain action functional. In contrast to our previous paper, we will tackle the Cauchy problem with exponentially decaying initial data using entirely PDE arguments. For this purpose, we establish a general comparison principle for discontinuous viscosity solutions associated with piecewise Lipschitz Hamiltonians, the latter arising naturally in the spreading of multiple species. The proof of the comparison result is based on combining the ideas due to Ishii [33] and Tourin [51]. With this comparison principle at our disposal, we are able to obtain large-deviation type estimates of the solutions (u, v) to the Cauchy problem (1.1) by explicit construction of simple piecewise linear super- and sub-solutions.

1.1 Known results of a single population

We first recall some classical asymptotic spreading results concerning the single Fisher-KPP equation:

$$\begin{cases} \partial_t \phi - \tilde{d}\partial_{xx}\phi = \tilde{r}\phi(1-\phi), & \text{in } (0,\infty)\times\mathbb{R}, \\ \phi(0,x) = \phi_0(x), & \text{on } \mathbb{R}, \end{cases} \tag{1.2}$$

where $\tilde{d}, \tilde{r}$ are positive constants. If the initial data is a Heaviside function, supported on $(-\infty, 0]$, it is shown [3,21,35] that the population, whose density is given by $\phi(t,x)$ has the spreading speed $c^* = 2\sqrt{\tilde{d}\tilde{r}}$, i.e.,

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x<ct} |\phi(t,x) - 1| = 0 & \text{for all } c < c^*, \\ \lim\limits_{t\to\infty} \sup\limits_{x>ct} |\phi(t,x)| = 0 & \text{for all } c > c^*. \end{cases}$$

In addition, the spreading speed c^* coincides with the minimal speed of the traveling wave solutions to (1.2) in this case. If we broaden the scope of initial data ϕ_0 to include the class of exponentially decaying data, then the asymptotic behavior of the solution to (1.2) is sensitive to the rate of decay of ϕ_0 at $x = \pm\infty$ (see e.g. [30, pp.42]), which is the leading edge of the front. This is related to the fact that 0 is a saddle for (1.2), see [9,16,34,44,47].

Precisely, denoting $\lambda^* = \sqrt{\tilde{r}/\tilde{d}}$. It is proved [34,44] that:

(i) When the initial data $\phi_0(x)$ decays faster than $\exp\{-\lambda^* x\}$ at $x = \infty$, then the spreading speed $c^* = 2\sqrt{\tilde{d}\tilde{r}}$;

(ii) When the initial data $\phi_0(x)$ is the form of $\exp\{-(\lambda + o(1))x\}$ at $x = \infty$ with $\lambda < \lambda^*$, then the population has the spreading speed $c(\lambda) = \tilde{d}\lambda + \frac{\tilde{r}}{\lambda}$ which is strictly greater than $2\sqrt{\tilde{d}\tilde{r}}$.

For recent developments in asymptotic spreading of a single population in heterogeneous environments, we refer to [5,7,19] for the one-dimensional case, and to [6,8,45,56] for higher-dimensional case.

1.2 Known results of multiple populations

For close to three decades, researchers have been trying to extend these results to reaction-diffusion systems describing two or more interacting populations.

Motivated by the northward spreading of several tree species into the newly de-glaciated North American continent at the end of the last ice age, Shigesada et al. [48, Ch. 7] formulated the question of spreading of two or more competing species into an open habitat, i.e., one that is unoccupied by either species. In case of two competing species, it is conjectured that for large time, the solution behaves like stacked traveling fronts, i.e., it exhibits two transition layers moving at two different speeds $c_1 > c_2$, connecting three homogeneous equilibrium states $(0,0)$, E_1 and E_2. Here E_1 is the semi-trivial equilibrium where the faster species is present, and E_2 is either the other semi-trivial equilibrium or the coexistence equilibrium (if the latter exists). While it is not difficult to see that the spreading speed c_1 of the faster species can be predicted by the underlying single equation (since the slower species is essentially absent at the leading edge of the front), the determination of the second speed remained open over a decade. Lin and Li [40] first worked on the spreading properties of (1.1) in the weak competition case $0 < a, b < 1$ with compactly supported initial condition (u_0, v_0) and obtained estimates for the spreading speed c_2 of the slower species. For the strong competition case $a, b > 1$, Carrère [10] determined both of the spreading speeds, where c_2 is determined by the unique speed of traveling wave solutions connecting the semi-trivial steady state $(1,0)$ and $(0,1)$. The predator-prey system was considered by Ducrot et al. [15]. For cooperative systems with equal diffusion coefficients, the existence of stacked fronts for cooperative systems was also studied by [31]. In these cases, the spreading speeds of each individual species can be determined locally and is not influenced by the presence of other invasion fronts.

However, the second speed c_2 can in general be influenced by the first front with speed c_1, as demonstrated by the work of Holzer and Scheel [29] which applies in particular to (1.1) for the case $a = 0$ and $b > 0$. They showed that the second speed c_2 can be determined by the linear instability of the zero solution of a single equation with space-time inhomogeneous coefficient. For coupled systems, the case $0 < a < 1 < b$ was treated in a recently appeared paper of Girardin and the third author [26]. By deriving an explicit formula for c_2, it is observed that c_2 can sometimes be strictly greater than the minimal speed of traveling wave connecting E_1 and E_2, and that it depends on the first speed c_1 in a non-increasing manner. The proof in [26] is based on a delicate construction of (piecewise smooth) super- and sub-solutions for the parabolic system. In our previous paper [41], we showed that in the weak competition case $0 < a, b < 1$ the formula for c_2 is exactly the same as the one in [26] but with a novel strategy of proof based on obtaining large deviation estimates via analyzing the Hamilton-Jacobi equations obtained in the thin-front limit. We also mention that coupled parabolic systems were also treated in [18, 23] based on the large deviations approach, but in these papers all components spread with a single spreading speed.

1.3 Main results

In this paper, we study the spreading of two competing species into an open habitat with exponentially decaying (in space) initial data, with attention to how the spreading speeds are influenced by the exponential rates of decay at infinity.

For a function $g : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, we say that $g(x) \sim e^{-\lambda x}$ at ∞ if

$$0 < \liminf_{x\to\infty} e^{\lambda x} g(x) \leq \limsup_{x\to\infty} e^{\lambda x} g(x) < \infty.$$

Definition for $g(x) \sim e^{\lambda x}$ at $-\infty$ is similar. We now state our hypothesis for the initial data (u_0, v_0).

$$(\mathrm{H}_\lambda)\begin{cases} \text{The initial value } (u_0, v_0) \in C(\mathbb{R}; [0,1])^2 \text{ is strictly positive on } \mathbb{R}, \\ \text{ and there exist positive constants } \theta_0, \lambda_u, \lambda_v^+, \lambda_v^- \text{ such that} \\ u_0(x) \geq \theta_0 \quad \text{in } (-\infty, 0], \quad u_0(x) \sim e^{-\lambda_u x} \ \text{ at } \ \infty, \\ v_0(x) \sim e^{\lambda_v^- x} \ \text{ at } \ -\infty, \quad \text{and} \quad v_0(x) \sim e^{-\lambda_v^+ x} \ \text{ at } \ \infty. \end{cases}.$$

We denote

$$\begin{cases} \sigma_1 = d(\lambda_v^+ \wedge \sqrt{\frac{r}{d}}) + \dfrac{r}{\lambda_v^+ \wedge \sqrt{\frac{r}{d}}}, \quad \sigma_2 = (\lambda_u \wedge 1) + \dfrac{1}{\lambda_u \wedge 1}, \\ \sigma_3 = d(\lambda_v^- \wedge \sqrt{\dfrac{r(1-b)}{d}}) + \dfrac{r(1-b)}{\lambda_v^- \wedge \sqrt{\frac{r(1-b)}{d}}}, \end{cases} \tag{1.3}$$

where $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$. Here the quantity σ_1 (resp. σ_2) denotes the spreading speed of v (resp. u) in the absence of the competitor [34,44]. Without loss of generality, we assume $\sigma_1 \geq \sigma_2$ throughout this paper. This amounts to fixing the choice of v to be the faster spreading species.

Our main result is stated as follows.

Theorem 1.3. *Assume $\sigma_1 > \sigma_2$. Let (u, v) be the solution of (1.1) such that the initial data satisfies (H_λ). Then there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_3 < 0 < c_2 < c_1$, and for each small $\eta > 0$, the following spreading results hold:*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x>(c_1+\eta)t} (|u(t,x)| + |v(t,x)|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{(c_2+\eta)t<x<(c_1-\eta)t} (|u(t,x)| + |v(t,x) - 1|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{(c_3+\eta)t<x<(c_2-\eta)t} (|u(t,x) - k_1| + |v(t,x) - k_2|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{x<(c_3-\eta)t} (|u(t,x) - 1| + |v(t,x)|) = 0. \end{cases} \tag{1.4}$$

Precisely, the spreading speeds $c_3 < 0 < c_2 < c_1$ can be determined as follows:

$$c_1 = \sigma_1, \quad c_2 = \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\}, \quad c_3 = -\max\{\tilde{c}_{\mathrm{LLW}}, \sigma_3\}, \tag{1.5}$$

where c_{LLW} (resp. $\tilde{c}_{\mathrm{LLW}}$) is given in Theorem 1.1 (resp. Remark 1.2), and

$$\hat{c}_{\mathrm{nlp}} = \begin{cases} \frac{\sigma_1}{2} - \sqrt{a} + \frac{1-a}{\frac{\sigma_1}{2} - \sqrt{a}}, & \text{if } \sigma_1 < 2\lambda_u \text{ and } \sigma_1 \leq 2(\sqrt{a} + \sqrt{1-a}), \\ \tilde{\lambda}_{\mathrm{nlp}} + \frac{1-a}{\tilde{\lambda}_{\mathrm{nlp}}}, & \text{if } \sigma_1 \geq 2\lambda_u \text{ and } \tilde{\lambda}_{\mathrm{nlp}} \leq \sqrt{1-a}, \\ 2\sqrt{1-a}, & \text{otherwise}, \end{cases} \tag{1.6}$$

with the quantity $\tilde{\lambda}_{\mathrm{nlp}}$ being given by

$$\tilde{\lambda}_{\mathrm{nlp}} = \frac{1}{2}\left[\sigma_1 - \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}\right]. \tag{1.7}$$

To visualize the spreading result (1.4) visually, we consider the scaling

$$(\hat{u}, \hat{v})(t, x) = \lim_{\epsilon\to 0} (u, v)\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) \qquad \text{for } (t, x) \in (0, \infty) \times \mathbb{R},$$

whose asymptotic behaviors can be given in Figure 1.

Note that while the spreading speed c_1 of the faster species v is entirely determined by λ_v^+ (the exponential decay of v_0 at $x \approx \infty$), and is unaffected by the slower species u, the corresponding speed c_2 of species u depends upon σ_1 and λ_u (the exponential decay of u_0 at $x \approx \infty$). In particular, when $\lambda_v^+ \geq \sqrt{\frac{r}{d}}$ and $\lambda_u > \frac{\sigma_1}{2}$, i.e., $v_0(x)$ and $u_0(x)$ decay fast enough, the speeds c_1 and c_2 are the same as that of the case of compactly supported initial data (see [41, Theorem 1.2]).

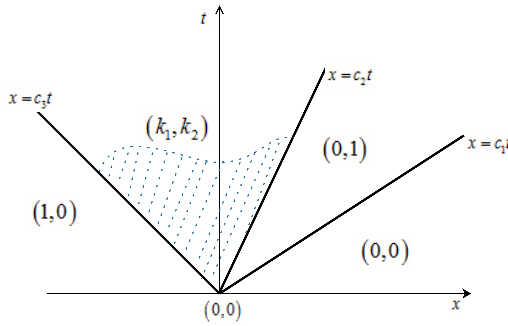

Figure 1: The asymptotic behaviors of $(\hat{u},\hat{v})$.

Remark 1.4. *We point out that the speed c_2 in Theorem 1.3 is non-increasing in both σ_1 and λ_u, which follows from the following observations:* (i) $\tilde{\lambda}_{\rm nlp}$ *given by* (1.7) *is non-decreasing in both σ_1 and λ_u;* (ii) $s+\frac{1-a}{s}$ *is non-increasing in* $(0,\sqrt{1-a}]$. *This fact makes intuitive sense:* (i) *a higher σ_1 means the region dominated by species v, which is roughly $\{(t,x): c_2t<x<\sigma_1 t\}$, is larger and thus rendering it more difficult for species u to invade;* (ii) *a higher λ_u means there are less population at the front to pull the invasion wave, which also makes it difficult for species u to invade.*

Fix σ_1, $\lambda_u>0$ and $0<a<1$, such that $\sigma_1>\sigma_2$ holds. We shall see that the quantity $\hat{c}_{\rm nlp}$ in (1.6) can be equivalently defined by

$$\{(t,x):\overline{w}_2(t,x)=0\}=\{(t,x):t>0 \text{ and } x\leq \hat{c}_{\rm nlp}t\},$$

where $\overline{w}_2(t,x)$ is the unique viscosity solution of the Hamilton-Jacobi equation

$$\begin{cases} \min\{\partial_t w+|\partial_x w|^2+1-a\chi_{\{x<\sigma_1 t\}},w\}=0, & \text{in } (0,\infty)\times\mathbb{R},\\ w(0,x)=\lambda_u\max\{x,0\}, & \text{on } \mathbb{R}. \end{cases} \tag{1.8}$$

Here χ_S is the indicator function of the set $S\in(0,\infty)\times\mathbb{R}$.

A further point of interest is the involvement of $(0,0)$ and (k_1,k_2) in co-invasion process of (1.1), which happens only in the weak competition case $0<a,b<1$. In this case, the equilibrium states $(0,0)$ and (k_1,k_2) are un-ordered, and hence the existence of traveling wave, due to Tang and Fife [50], cannot be established by monotone dynamical systems framework due to Weinberger et al. [57] (see also [20,39]). We will see that the invasion front (k_1,k_2) into $(0,0)$ is indeed realized in (1.1) for initial data with certain values of exponential decay rates λ_u,λ_v^+ at infinity, namely, when $\sigma_1=\sigma_2$.

Theorem 1.5. *Assume $\sigma_1=\sigma_2$. Let (u,v) be the solution of* (1.1) *such that* *the initial data satisfies* (H_λ). *Then for each small* $\eta > 0$, *it holds that*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x>(\sigma_1+\eta)t} (|u(t,x)| + |v(t,x)|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{(c_3+\eta)t<x<(\sigma_1-\eta)t} (|u(t,x)-k_1| + |v(t,x)-k_2|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{x<(c_3-\eta)t} (|u(t,x)-1| + |v(t,x)|) = 0, \end{cases} \tag{1.9}$$

where $c_3 = -\max\{\tilde{c}_{\mathrm{LLW}}, \sigma_3\}$ *and that* $\tilde{c}_{\mathrm{LLW}}$ *is given in Remark 1.2.*

For initial data with general exponential decay rates, Theorem 1.3 demonstrates that there are two separate monostable fronts where each of the two species invades with distinct speeds. Moreover, if the parameters of (1.1) changes in such a way that $|\sigma_1 - \sigma_2| \to 0$, the distance of the two fronts tends to zero. Therefore, the invasion front of (k_1, k_2) transitioning directly into $(0,0)$, due to Tang and Fife, is in fact the special case when these two monostable fronts coincide (Theorem 1.5).

Remark 1.6. *As in [17, 41], our approach can be applied to the spreading problem of competing species in higher dimensions under minor modifications. However, we choose to focus here on the one-dimensional case to keep our exposition simple, and close to the original formulation of the conjecture in [48, Chapter 7].*

1.4 Outline of main ideas

To determine c_1, c_2, c_3, we introduce large deviation approach and construct appropriate viscosity super- and sub-solutions for certain Hamilton-Jacobi equations, and then apply the comparison principle (Theorem A.1) to obtain the desired estimations. We outline the main steps leading to the determination of the nonlocally pulled spreading speed c_2, as stated Theorem 1.3, and remark that c_1, c_3 can be obtained by a similar even simpler argument as c_2.

1. To estimate c_2 from below, we consider the transformation $w_2^\epsilon(t,x) = -\epsilon \log u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$ and show that the half-relaxed limits

   $$w_{2,*}(t,x) = \liminf_{\substack{\epsilon\to 0\\(t',x')\to(t,x)}} w_2^\epsilon(t',x') \quad \text{and} \quad w_2^*(t,x) = \limsup_{\substack{\epsilon\to 0\\(t',x')\to(t,x)}} w_2^\epsilon(t',x')$$

   exist, upon establishing uniform bounds in C_{loc} (see Lemma 3.2). By constructing viscosity super-solution $\overline{w}_2$, which satisfies

   $$\{(t,x) : \overline{w}_2(t,x) = 0\} = \{(t,x) : t > 0 \text{ and } x \le \hat{c}_{\mathrm{nlp}} t\},$$

   and using the comparison principle (Theorem A.1), we can show that $w_2^* \le \overline{w}_2$, and thus $w_2^\epsilon \to 0$ locally uniformly in $\{(t,x) : x < \hat{c}_{\mathrm{nlp}} t\}$. One can then apply the arguments in [17, Section 4] to show that

   $$\liminf_{\epsilon\to 0} u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) > 0 \quad \text{in } \{(t,x) : t > 0 \text{ and } x < \hat{c}_{\mathrm{nlp}} t\}.$$

This implies that $c_2 \geq \hat{c}_{\rm nlp}$ (see Lemma 3.8).

2. To estimate c_2 from above, we construct viscosity sub-solution $\underline{w}_2$ and apply Theorem A.1 to estimate w_2 from below, see Proposition 4.2. This enables us to obtain a large deviation estimate of u. Namely, for each small $\delta > 0$, let $\hat{c}_\delta = \sigma_1 - \delta$, we have

$$u(t, \hat{c}_\delta t) \leq \exp\left(-[\hat{\mu}_\delta + o(1)]t\right) \qquad \text{for } t \gg 1,$$

where $\hat{\mu}_\delta = \overline{w}_2(1, \hat{c}_\delta) = \overline{w}_2(1, \sigma_1 - \delta)$. Now, recalling that (u, v) is a solution to (1.1) restricted to the domain $\{(t, x) : 0 \leq x \leq \hat{c}_\delta t\}$, with boundary condition satisfying

$$\lim_{t\to\infty}(u, v)(t, 0) = (k_1, k_2) \quad \text{ and } \quad \lim_{t\to\infty}(u, v)(t, \hat{c}_\delta t) = (0, 1),$$

we may apply Lemma B.2 in Appendix to show that $\hat{c}_\delta$ and $\hat{\mu}_\delta$ completely controls the spreading speed c_2 of u from above.

The rest of the paper is organized as follows: In Section 2, we give upper estimates c_i for $i = 1$, 2, 3 and $c_2 \geq c_{\rm LLW}$. In Section 3, we give lower estimates of c_1, c_2. The approximate asymptotic expressions of u and v are established in Section 4, where we also determine c_2, c_3. In Section 5, we discuss the relation of our results with the invasion mode due to Tang and Fife [50]. In Section 6, we discuss the relation of our result with that of [26] due to Girardin and the last author. In Section 7, we prove an extension which is associated to the spreading speeds of the three-species competition systems. We conclude the article with the Appendix. Therein we give the comparison principle of Hamilton-Jacobi equation with piecewise Lipschitz continuous Hamiltonian and two other useful lemmas.

This paper concerns the Cauchy problem of a system of reaction-diffusion equations modeling two competing species. For the spreading of two species into an open habitat, we refer to [37] for an integro-difference competition model, and to [14] for a competition model with free-boundaries. See also [27, 42, 53, 54, 58] for other related results in free-boundary problems. We also note that in those works the spreading speeds are always locally determined and thus do not interact.

2 Estimating the maximal and minimal speeds

The concepts of maximal and minimal spreading speeds are introduced in [28, Definition 1.2] for a single species; see also [24, 41]. In our setting, we define

$$\begin{cases} \overline{c}_1 = \inf\{c > 0 \mid \limsup\limits_{t\to\infty} \sup\limits_{x>ct} v(t,x) = 0\}, \\ \underline{c}_1 = \sup\{c > 0 \mid \liminf\limits_{t\to\infty} \inf\limits_{ct-1<x<ct} v(t,x) > 0\}, \\ \overline{c}_2 = \inf\{c > 0 \mid \limsup\limits_{t\to\infty} \sup\limits_{x>ct} u(t,x) = 0\}, \\ \underline{c}_2 = \sup\{c > 0 \mid \liminf\limits_{t\to\infty} \inf\limits_{ct-1<x<ct} u(t,x) > 0\}, \\ \overline{c}_3 = \inf\{c < 0 \mid \liminf\limits_{t\to\infty} \inf\limits_{ct<x<ct+1} v(t,x) > 0\}, \\ \underline{c}_3 = \sup\{c < 0 \mid \limsup\limits_{t\to\infty} \sup\limits_{x<ct} v(t,x) = 0\}, \end{cases} \tag{2.1}$$

where $\overline{c}_1$ and $\underline{c}_1$ (resp. $\overline{c}_2$ and $\underline{c}_2$) are the maximal and minimal rightward spreading speeds of species v (resp. species u), whereas $-\underline{c}_3$ and $-\overline{c}_3$ are the maximal and minimal leftward spreading speeds of v, respectively.

In this section, for initial data satisfying (H_λ), we will give some estimates of the maximal and minimal spreading speeds. The main result of this section can be precisely stated as follows.

Proposition 2.1. *Let (u,v) be a solution of (1.1) with initial data satisfying (H_λ). Then the spreading speeds defined in (2.1) satisfy*

(i) $\overline{c}_i \leq \sigma_i$ *for* $i = 1, 2$ *and* $\overline{c}_3 \leq -\sigma_3$*;*

(ii) $\underline{c}_2 \geq c_{\mathrm{LLW}}$*, and* $\overline{c}_3 \leq -\tilde{c}_{\mathrm{LLW}}$*,*

where $\sigma_1, \sigma_2, \sigma_3$ are defined in (1.3) and c_{LLW}, $\tilde{c}_{\mathrm{LLW}}$ are given respectively in Theorem 1.1 and Remark 1.2. Furthermore, we have

$$\lim_{t\to\infty} (|u(t,0) - k_1| + |v(t,0) - k_2|) = 0. \tag{2.2}$$

Proof. We will complete the proof in the following order: (1) $\overline{c}_2 \leq \sigma_2$, (2) $\overline{c}_1 \leq \sigma_1$, (3) $\overline{c}_3 \leq -\sigma_3$, (4) $\overline{c}_3 \leq -\tilde{c}_{\mathrm{LLW}}$, (5) $\underline{c}_2 \geq c_{\mathrm{LLW}}$, (6) (2.2) holds.

Step 1. We show assertions (1), (2) and (3).

Observe that for some $M > 0$ the function

$$\overline{u}(t,x) := \min\{1, M\exp(-\min\{\lambda_u, 1\}(x - \sigma_2 t))\}$$

is a weak super-solution to the single KPP-type equation

$$\partial_t \overline{u} - \partial_{xx}\overline{u} = \overline{u}(1 - \overline{u}) \quad \text{in } (0,\infty)\times\mathbb{R},$$

of which $u(t,x)$ is clearly a sub-solution. By choosing the constant $M > 0$ so large that $u_0(x) \leq \overline{u}(0,x)$ in $\mathbb{R}$, it follows by comparison that

$$u(t,x) \leq \overline{u}(t,x) = \min\{1, M\exp(-\min\{\lambda_u, 1\}(x - \sigma_2 t)\} \tag{2.3}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}$. In particular,

$$\lim_{t\to\infty} \sup_{x>(\sigma_2+\eta)t} |u(t,x)| = 0 \qquad \text{for each } \eta > 0. \tag{2.4}$$

This proves $\overline{c}_2 \le \sigma_2$, i.e., assertion (1) holds.

Similarly, we deduce assertion (2) by comparison with

$$\bar{v}(t,x) := \min\{1, M\exp(-\min\{\lambda_v^+, \sqrt{r/d}\}(x-\sigma_1 t))\}$$

which is the solution of

$$\begin{cases} \partial_t \overline{v} - d\partial_{xx}\overline{v} = r\overline{v}(1-\overline{v}), & \text{in } (0,\infty)\times\mathbb{R}, \\ \overline{v}(0,x) = \min(1, Me^{-\min\{\lambda_v^+, \sqrt{\frac{r}{d}}\}x}), & x \in \mathbb{R}. \end{cases}$$

To prove assertion (3), let $\tilde{v}(t,x) = v(t,-x)$, we turn to consider another single KPP-type equation

$$\begin{cases} \partial_t \underline{v} - d\partial_{xx}\underline{v} = r\underline{v}(1-b-\underline{v}), & \text{in } (0,\infty)\times\mathbb{R}, \\ \underline{v}(0,x) = v_0(-x), & x \in \mathbb{R}. \end{cases}$$

Again the scalar comparison principle implies $v(t,-x) = \tilde{v}(t,x) \ge \underline{v}$. By the results in [34] or [44], we have

$$\liminf_{t\to\infty} \inf_{(-\sigma_3+\eta)t<x\le 0} v \ge \liminf_{t\to\infty} \inf_{|x|<(\sigma_3-\eta)t} \tilde{v} \ge \frac{1-b}{2}, \tag{2.5}$$

which means $\overline{c}_3 \le -\sigma_3$.

Step 2. We show assertions (4) and (5).

Given any non-trivial, compactly supported function $\tilde{v}_0$ such that $0 \le \tilde{v}_0 \le v_0$. Then

$$(u_0(x), v_0(x)) \preceq (1, \tilde{v}_0(x)) \qquad \text{in } \mathbb{R}.$$

Let $(\tilde{u}_{\mathrm{LLW}}, \tilde{v}_{\mathrm{LLW}})$ be the solution to (1.1) with initial value $(1, \tilde{v}_0(x))$. Then Theorem 1.1 and Remark 1.2 guarantee the existence of $\tilde{c}_{\mathrm{LLW}} \ge 2\sqrt{dr(1-b)}$, such that

$$\liminf_{t\to\infty} \inf_{|x|<|c|t} \tilde{v}_{\mathrm{LLW}}(t,x) > 0 \qquad \text{for each } c \in (-\tilde{c}_{\mathrm{LLW}}, 0).$$

By the comparison principle for (1.1), we have $(u,v) \preceq (\tilde{u}_{\mathrm{LLW}}, \tilde{v}_{\mathrm{LLW}})$ for all $(t,x) \in (0,\infty)\times\mathbb{R}$, which yields, for each $c \in (-\tilde{c}_{\mathrm{LLW}}, 0)$,

$$\liminf_{t\to\infty} \inf_{ct<x<ct+1} v(t,x) \ge \liminf_{t\to\infty} \inf_{ct<x<ct+1} \tilde{v}_{\mathrm{LLW}}(t,x) > 0.$$

This proves $\overline{c}_3 \le -\tilde{c}_{\mathrm{LLW}}$ and thus assertion (4) holds.

Similarly, we can get show assertion (5), i.e., $\underline{c}_2 \ge c_{\mathrm{LLW}}$. By comparing (u,v) with the solution $(u_{\mathrm{LLW}}, v_{\mathrm{LLW}})$ of (1.1) with initial condition $(\tilde{u}_0, 1)$, for some compactly supported $\tilde{u}_0$ satisfying $0 \le \tilde{u}_0 \le u_0$, and then using Theorem 1.1. In this way, we get

$$\liminf_{t\to\infty} \inf_{|x|<ct} u \ge \liminf_{t\to\infty} \inf_{|x|<ct} u_{\mathrm{LLW}} > 0 \qquad \text{for each } c \in (0, c_{\mathrm{LLW}}). \tag{2.6}$$

Step 3. We show assertion (6). In view of (2.5) and (2.6), one can deduce (2.2) from items (a) and (c) of Lemma B.1. $\square$

3 Estimating $\overline{c}_1$ and $\overline{c}_2$ from below

We assume $\sigma_1 > \sigma_2$ throughout this section. In this section, we estimate $\underline{c}_1$ and $\underline{c}_2$ from below via the large deviation approach and applying Theorem A.1. To this end, we introduce a small parameter ϵ via the following scaling

$$u^\epsilon(t,x) = u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) \quad \text{and} \quad v^\epsilon(t,x) = v\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right). \tag{3.1}$$

Under the new scaling, we rewrite the equation of u^ϵ and v^ϵ in (1.1) as

$$\begin{cases} \partial_t u^\epsilon = \epsilon\partial_{xx}u^\epsilon + \frac{u^\epsilon}{\epsilon}(1-u^\epsilon-av^\epsilon), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t v^\epsilon = \epsilon d\partial_{xx}v^\epsilon + r\frac{v^\epsilon}{\epsilon}(1-bu^\epsilon-v^\epsilon), & \text{in } (0,\infty)\times\mathbb{R}, \\ u^\epsilon(0,x) = u_0(\frac{x}{\epsilon}), & \text{on } \mathbb{R}, \\ v^\epsilon(0,x) = v_0(\frac{x}{\epsilon}), & \text{on } \mathbb{R}. \end{cases} \tag{3.2}$$

To obtain the asymptotic behaviors of v^ϵ and u^ϵ as $\epsilon \to 0$, the idea is to consider the WKB ansatz w_1^ϵ and w_2^ϵ, which are given respectively by

$$w_1^\epsilon(t,x) = -\epsilon\log v^\epsilon(t,x), \quad w_2^\epsilon(t,x) = -\epsilon\log u^\epsilon(t,x), \tag{3.3}$$

and satisfy, respectively, the equations

$$\begin{cases} \partial_t w^\epsilon - \epsilon d\partial_{xx}w^\epsilon + d|\partial_x w^\epsilon|^2 + r(1-bu^\epsilon-v^\epsilon) = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ w^\epsilon(0,x) = -\epsilon\log v^\epsilon(0,x), & \text{on } \mathbb{R}, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \partial_t w^\epsilon - \epsilon\partial_{xx}w^\epsilon + |\partial_x w^\epsilon|^2 + 1-u^\epsilon-av^\epsilon = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ w^\epsilon(0,x) = -\epsilon\log u^\epsilon(0,x), & \text{on } \mathbb{R}. \end{cases} \tag{3.5}$$

Lemma 3.1. *Let G be an open set in $(0,\infty)\times\mathbb{R}$ and K, K' be compact sets such that $K \subset \operatorname{Int} K' \subset K' \subset G$.*

(a) *If $w_2^\epsilon \to 0$ uniformly in K' as $\epsilon \to 0$, then*

$$\liminf_{\epsilon\to 0}\inf_{K} u^\epsilon \geq 1 - a\limsup_{\epsilon\to 0}\sup_{K'} v^\epsilon; \tag{3.6}$$

(b) *If $w_1^\epsilon \to 0$ uniformly in K' as $\epsilon \to 0$, then*

$$\liminf_{\epsilon\to 0}\inf_{K} v^\epsilon \geq 1 - b\limsup_{\epsilon\to 0}\sup_{K'} u^\epsilon. \tag{3.7}$$

Proof. We first prove (a) by adapting the arguments from [17, Section 4]. Let K, K' and G be given as above.

Fix an arbitrary $(t_0, x_0) \in K$ and define the test function

$$\rho(t,x) = |x-x_0|^2 + (t-t_0)^2.$$

Since (i) $(t_0, x_0) \in K \subset \operatorname{Int} K'$ and (ii) $w_2^\epsilon \to 0$ uniformly in K', the function $w_2^\epsilon - \rho$ attains global maximum over K' at $(t_\epsilon, x_\epsilon) \in \operatorname{Int} K'$ such that $(t_\epsilon, x_\epsilon) \to (t_0, x_0)$ as $\epsilon \to 0$. Furthermore, $\partial_t \rho(t_\epsilon, x_\epsilon), \partial_x \rho(t_\epsilon, x_\epsilon) \to 0$, so that at the point (t_ϵ, x_ϵ),

$$o(1) = \partial_t \rho - \epsilon \partial_{xx} \rho + |\partial_x \rho|^2 \leq \partial_t w_2^\epsilon - \epsilon \partial_{xx} w_2^\epsilon + |\partial_x w_2^\epsilon|^2 \leq u^\epsilon - 1 + a \limsup_{\epsilon \to 0} \sup_{K'} v^\epsilon.$$

This yields

$$u^\epsilon(t_\epsilon, x_\epsilon) \geq 1 - a \limsup_{\epsilon \to 0} \sup_{K'} v^\epsilon + o(1).$$

Since $w_2^\epsilon - \rho$ attains maximum over K' at (t_ϵ, x_ϵ), we have in particular

$$w_2^\epsilon(t_\epsilon, x_\epsilon) \geq (w_2^\epsilon - \rho)(t_\epsilon, x_\epsilon) \geq (w_2^\epsilon - \rho)(t_0, x_0) = w_2^\epsilon(t_0, x_0).$$

Recalling $u^\epsilon(t_0, x_0) = e^{-\epsilon w_2^\epsilon(t_0, x_0)}$ and $u^\epsilon(t_\epsilon, x_\epsilon) = e^{-\epsilon w_2^\epsilon(t_\epsilon, x_\epsilon)}$, we therefore have

$$u^\epsilon(t_0, x_0) \geq u^\epsilon(t_\epsilon, x_\epsilon) \geq 1 - a \limsup_{\epsilon \to 0} \sup_{K'} v^\epsilon + o(1).$$

Since this argument is uniform for $(t_0, x_0) \in K$ (depends only on K, K' and G), we deduce assertion (a). The proof for (b) is analogous. □

Next, we will pass to the (upper and lower) limits using the half-relaxed limit method, which is due to Barles and Perthame [4]. Define

$$w_1^*(t, x) = \limsup_{\substack{\epsilon \to 0 \\ (t', x') \to (t, x)}} w_1^\epsilon(t', x'),$$

$$w_2^*(t, x) = \limsup_{\substack{\epsilon \to 0 \\ (t', x') \to (t, x)}} w_2^\epsilon(t', x') \quad \text{and} \quad w_{2,*}(t, x) = \liminf_{\substack{\epsilon \to 0 \\ (t', x') \to (t, x)}} w_2^\epsilon(t', x').$$

That the above are well defined is due to the following lemma:

Lemma 3.2. *Let w_1^ϵ and w_2^ϵ be the solutions to (3.4) and (3.5), respectively. Then there exits some $Q > 0$, independent of ϵ small, such that*

$$\max\{\lambda_v^+ x_+ + \lambda_v^- x_- - Q(t+\epsilon), 0\} \leq w_1^\epsilon(t, x) \leq \lambda_v^+ x_+ + \lambda_v^- x_- + Q(t+\epsilon), \tag{3.8a}$$

$$\max\{\lambda_u x_+ - Q(t + \epsilon), 0\} \leq w_2^\epsilon(t, x) \leq \lambda_u x_+ + Q(t + \epsilon), \tag{3.8b}$$

$$0 \leq w_1^\epsilon(t, x) \leq Q(\lambda_v^+ x_+ + \lambda_v^- x_- + \epsilon), \tag{3.8c}$$

$$0 \leq w_2^\epsilon(t, x) \leq Q(\lambda_u x_+ + \epsilon), \tag{3.8d}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}$, where $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$.

Proof. We only prove (3.8a) and the estimations (3.8b)-(3.8d) follow from a quite similar argument. Since $v^\epsilon \leq 1$, we have $w_1^\epsilon \geq 0$ by definition. By (H_λ), there exist positive constants C_1 and C_2 such that

$$C_2 e^{-(\lambda_v^+ x_+ + \lambda_v^- x_-)} \leq v(0, x) \leq C_1 e^{-(\lambda_v^+ x_+ + \lambda_v^- x_-)} \quad \text{for } x \in \mathbb{R}.$$

By definition (3.3), we have

$$\lambda_v^+ x_+ + \lambda_v^- x_- - \epsilon \log C_1 \le w_1^\epsilon(0,x) \le \lambda_v^+ x_+ + \lambda_v^- x_- - \epsilon \log C_2. \tag{3.9}$$

Define

$$\overline{z}_1^\epsilon = \lambda_v^+ x + Q(t+\epsilon).$$

We shall choose large Q independent of ϵ such that

$$w_1^\epsilon(t,x) \le \overline{z}_1^\epsilon \qquad \text{in } [0,\infty)\times[0,\infty). \tag{3.10}$$

To this end, observe that $\overline{z}_1^\epsilon$ is a (classical) super-solution of (3.4) in $(0,\infty)\times(0,\infty)$ provided $Q \ge r$. By (2.2) in Proposition 2.1, we find $-\log v(t,0)$ is uniformly bounded in $[0,\infty)$ (since $v(0,x)>0$ in $\mathbb{R}$), so that we may choose

$$Q = \max\left\{ \sup_{t\in[0,\infty)} [-\log v(t,0)],\, |\log C_2|,\, r \right\}, \tag{3.11}$$

such that

$$w_1^\epsilon(t,0) \le \overline{z}_1^\epsilon(t,0) \quad \text{ for all } t\ge 0, \quad w_1^\epsilon(0,x) \le \overline{z}_1^\epsilon(0,x) \quad \text{ for all } x \ge 0,$$

where the last inequality is due to (3.9). By comparison, (3.10) thus holds.

By a similar argument, we can verify

$$\overline{z}_2^\epsilon = -\lambda_v^- x + Q(t+\epsilon)$$

is a super-solution of (3.4) in $(0,\infty)\times(-\infty,0)$, so that

$$w_1^\epsilon(t,x) \le \overline{z}_2^\epsilon(t,x) \qquad \text{in } [0,\infty)\times(-\infty,0], \tag{3.12}$$

where Q is defined by (3.11). Combining with (3.10) and (3.12) gives the desired upper bound of w_1^ϵ.

To obtain the lower bound of w_1^ϵ, we may define functions

$$\underline{z}_1^\epsilon = \lambda_v^+ x - Q(t+\epsilon) \qquad \text{and} \qquad \underline{z}_2^\epsilon = -\lambda_v^+ x - Q(t+\epsilon).$$

By the same arguments as before, we can check

$$w_1^\epsilon(t,x) \ge \underline{z}_1^\epsilon \ \text{ in } [0,\infty)\times\mathbb{R} \ \text{ and } \ w_1^\epsilon(t,x) \ge \underline{z}_2^\epsilon \ \text{ in } [0,\infty)\times\mathbb{R},$$

by choosing $Q = \max\left\{ |\log C_1|,\, d(\lambda_v^+)^2 + d(\lambda_v^-)^2 + r \right\}$. This completes the proof of (3.8a). ☐

Remark 3.3. *According to Lemma 3.2, by letting $t=0$ and then $\epsilon \to 0$ in (3.8a) and (3.8b), we deduce that*

$$w_1^*(0,x) = \begin{cases} \lambda_v^+ x, & \textit{for } x\in[0,\infty), \\ \lambda_v^- x, & \textit{for } x \in (-\infty,0], \end{cases}$$

and

$$w_2^*(0,x) = w_{2,*}(0,x) = \begin{cases} \lambda_u x, & \textit{for } x\in[0,\infty), \\ 0, & \textit{for } x \in (-\infty,0]. \end{cases}$$

Similarly, by setting $x=0$ and then $\epsilon\to 0$ in (3.8c) and (3.8d), we have

$$w_1^*(t,0) = w_2^*(t,0) = w_{2,*}(t,0) = 0 \ \textit{ for } t \ge 0.$$

3.1 Estimating $\underline{c}_1$ from below

By Proposition 2.1, $\overline{c}_2 \leq \sigma_2$, so we deduce

$$0 \leq \limsup_{\substack{\epsilon \to 0 \\ (t',x') \to (t,x)}} u^\epsilon(t',x') \leq \chi_{\{x \leq \sigma_2 t\}}. \tag{3.13}$$

Lemma 3.4. *Let* (u, v) *be a solution of* (1.1) *with initial data satisfying* (H_λ). *Then*

(a) w_1^* *is a viscosity sub-solution of*

$$\begin{cases} \min\{\partial_t w + d|\partial_x w|^2 + r(1 - b\chi_{\{x \leq \sigma_2 t\}}), w\} = 0, & \text{in } (0,\infty) \times (0,\infty), \\ w(0,x) = \lambda_v^+ x, & \text{on } [0,\infty), \\ w(t,0) = 0, & \text{on } [0,\infty); \end{cases} \tag{3.14}$$

(b) w_1^* *is a viscosity sub-solution of*

$$\begin{cases} \min\{\partial_t w + d|\partial_x w|^2 + r(1 - b\chi_{\{x \leq \sigma_2 t\}}), w\} = 0, & \text{in } (0,\infty) \times (-\infty,0), \\ w(0,x) = -\lambda_v^- x, & \text{on } (-\infty,0], \\ w(t,0) = 0, & \text{on } [0,\infty), \end{cases} \tag{3.15}$$

where σ_2 *is defined by* (1.3) *and* λ_v^-, $\lambda_v^+ \in (0,\infty)$ *are given in* (H_λ).

Proof. First, observe that w_1^* is upper semicontinuous (usc) by construction. By Remark 3.3, the initial and boundary conditions of (3.14) and (3.15) are satisfied.

It remains to show that w_1^* is a viscosity sub-solution of $\min\{\partial_t w + d|\partial_x w|^2 + r(1 - b\chi_{\{x\leq\sigma_2 t\}}), w\} = 0$ in the domain $(0,\infty) \times \mathbb{R}$. According to definition of viscosity sub-solution of Hamilton-Jacobi equation, (see Appendix A), let $\varphi \in C^\infty((0,\infty)\times\mathbb{R})$ and let (t_0, x_0) be a strict local maximum point of $w_1^* - \varphi$ such that $w_1^*(t_0,x_0) > 0$. By passing to a sequence $\epsilon = \epsilon_k$ if necessary, $w_1^\epsilon - \varphi$ has a local maximum point at (t_ϵ, x_ϵ) such that $w_1^\epsilon(t_\epsilon, x_\epsilon) \to w_1^*(t_0,x_0)$ and $(t_\epsilon, x_\epsilon) \to (t_0, x_0)$ uniformly as $\epsilon \to 0$. At the point (t_ϵ, x_ϵ), we have

$$\begin{aligned} \epsilon d \partial_{xx}\varphi \geq \epsilon d\partial_{xx} w_1^\epsilon &= \partial_t w_1^\epsilon + d|\partial_x w_1^\epsilon|^2 + r(1 - bu^\epsilon - e^{-\frac{w_1^\epsilon}{\epsilon}}) \\ &= \partial_t \varphi + d|\partial_x\varphi|^2 + r(1 - bu^\epsilon - e^{-\frac{w_1^\epsilon}{\epsilon}}). \end{aligned}$$

By the fact that $e^{-w_1^\epsilon(t_\epsilon,x_\epsilon)/\epsilon} \to 0$ (as $w_1^\epsilon(t_\epsilon,x_\epsilon) \to w_1^*(t_0,x_0) > 0$), we may pass to the limit $\epsilon = \epsilon_k \to 0$ so that

$$0 \geq \partial_t\varphi(t_0,x_0) + d|\partial_x\varphi(t_0,x_0)|^2 + r(1 - b\chi_{\{(t,x):\, x\leq\sigma_2 t\}(t_0,x_0)} - 0).$$

Hence w_1^* is a viscosity sub-solution of (3.14) and (3.15). ☐

Lemma 3.5. *Let* (u, v) *be a solution of* (1.1) *with initial data satisfying* (H_λ). *Then*

$$\underline{c}_1 \geq \sigma_1,$$

where σ_1 *is defined by* (1.3).

Proof. Define the function $\overline{w}_1 : [0, \infty) \times [0, \infty) \to [0, \infty)$ by

$$\overline{w}_1(t, x) = \begin{cases} \lambda_v^+(x - (d\lambda_v^+ + \frac{r}{\lambda_v^+})t), & \text{for } \frac{x}{t} > d\lambda_v^+ + \frac{r}{\lambda_v^+}, \\ 0, & \text{for } 0 \leq \frac{x}{t} \leq d\lambda_v^+ + \frac{r}{\lambda_v^+}, \end{cases}$$

when $\lambda_v^+ \leq \sqrt{\frac{r}{d}}$, and by

$$\overline{w}_1(t, x) = \begin{cases} \lambda_v^+(x - (d\lambda_v^+ + \frac{r}{\lambda_v^+})t), & \text{for } \frac{x}{t} > 2d\lambda_v^+, \\ \frac{t}{4d}(\frac{x^2}{t^2} - 4dr), & \text{for } 2\sqrt{dr} < \frac{x}{t} \leq 2d\lambda_v^+, \\ 0, & \text{for } 0 \leq \frac{x}{t} \leq 2\sqrt{dr}, \end{cases}$$

when $\lambda_v^+ > \sqrt{\frac{r}{d}}$.

By construction, $\overline{w}_1$ is continuous in $[0, \infty) \times [0, \infty)$. Next, we claim that the continuous $\overline{w}_1$ is a viscosity super-solution of (3.14). We will check the latter case of $\lambda_v^+ > \sqrt{\frac{r}{d}}$ as the former case can be verified analogously. Under the condition $\lambda_v^+ > \sqrt{\frac{r}{d}}$, we have $\sigma_1 = 2\sqrt{dr}$. According to definition of viscosity super-solution of Hamilton-Jacobi equation (see Appendix A), let $\varphi \in C^\infty((0, \infty) \times \mathbb{R})$ and let (t_0, x_0) be a strict local minimum point of $\overline{w}_1 - \varphi$.

If $x_0/t_0 \neq 2\sqrt{dr}$, then $\overline{w}_1$ is a classical solution of (3.14).

If $x_0/t_0 = 2\sqrt{dr}$, then $\overline{w}_1(t_0, x_0) = 0$ by definition. Moreover,

$$-\varphi(t, 2\sqrt{dr}t) = (\overline{w}_1 - \varphi)(t, 2\sqrt{dr}t) \geq (\overline{w}_1 - \varphi)(t_0, x_0) = -\varphi(t_0, x_0) \quad \text{for } t \approx t_0,$$

and we must have $\partial_t\varphi(t_0, x_0) + 2\sqrt{dr}\partial_x\varphi(t_0, x_0)) = 0$, and hence

$$\begin{aligned} &\partial_t\varphi(t_0, x_0) + d|\partial_x\varphi(t_0, x_0)|^2 + r(1 - b\chi_{\{(t,x):\, x \leq \sigma_2 t\}}(t_0, x_0)) \\ &= -2\sqrt{dr}\partial_x\varphi(t_0, x_0) + d|\partial_x\varphi(t_0, x_0)|^2 + r \\ &= \left(\sqrt{d}\partial_x\varphi(t_0, x_0) - \sqrt{r}\right)^2 \geq 0, \end{aligned}$$

where the first equality follows from the fact that $x_0/t_0 = 2\sqrt{dr} = \sigma_1 > \sigma_2$.

By Remark 3.3 and the expression of $\overline{w}_1$, we have

$$\overline{w}_1(t, x) = \lambda_v^+ x = w_1^*(t, x) \ \text{ on } \partial[(0, \infty) \times (0, \infty)].$$

And recalling Lemma 3.4(a), $\overline{w}_1$ and w_1^* is a pair of viscosity super and sub-solutions of (3.14). Then, we may apply Theorem A.1 to get

$$0 \leq w_1^* \leq \overline{w}_1 \qquad \text{in } [0, \infty) \times [0, \infty),$$

which implies that

$$\{(t,x): w_1^*(t,x)=0\} \supset \{(t,x): \overline{w}_1(t,x)=0\} = \{(t,x): 0 \le x \le \sigma_1 t\}.$$

Letting $\epsilon \to 0$, we arrive at

$$w_1^\epsilon(t,x) = -\epsilon \log v^\epsilon(t,x) \to 0 \text{ locally uniformly on } \{(t,x): 0 \le x < \sigma_1 t\}.$$

Hence for each small $\eta > 0$, by choosing the compact sets $K = \{(1,x): \eta \le x \le \sigma_1 - \eta\}$ and $K' = \{(1,x): \frac{\eta}{2} \le x \le \sigma_1 - \frac{\eta}{2}\}$, we may apply Lemma 3.1(b) to deduce that

$$\liminf_{t\to\infty} \inf_{\eta t < x < (\sigma_1 - \eta)t} v(t,x) = \liminf_{\epsilon \to 0} \inf_K v^\epsilon(t,x) \ge \frac{1-b}{2} > 0. \tag{3.16}$$

This implies $\underline{c}_1 \ge \sigma_1$. □

Corollary 3.6. *Let $\sigma_1 > \sigma_2$ and let (u,v) be a solution of (1.1) with initial data satisfying* (H_λ)*. Then for each small $\eta > 0$,*

$$\lim_{t\to\infty} \sup_{x > (\sigma_1+\eta)t} (|u| + |v|) = 0, \tag{3.17a}$$

$$\lim_{t\to\infty} \sup_{(\overline{c}_2+\eta)t < x < (\sigma_1-\eta)t} (|u| + |v-1|) = 0, \tag{3.17b}$$

where σ_1 is defined by (1.3)*.*

Proof. By definition, $\underline{c}_1 \le \overline{c}_1$. It follows from Proposition 2.1 and Lemma 3.5 that $\sigma_1 \le \underline{c}_1 \le \overline{c}_1 \le \sigma_1$. Hence, $\underline{c}_1 = \overline{c}_1 = \sigma_1$. By Proposition 2.1(i), $\overline{c}_2 \le \sigma_2 < \sigma_1$, so that (3.17a) holds. In view of (3.16) and definition of $\overline{c}_2$, we have, for each small $\eta > 0$,

$$\liminf_{t\to\infty} \inf_{\eta t < x < (\sigma_1-\eta)t} v(t,x) > 0, \quad \text{and} \quad \lim_{t\to\infty} \sup_{x > (\overline{c}_2+\eta)t} u = 0.$$

We may then apply Lemma B.1(d) to deduce (3.17b). □

3.2 Estimating $\underline{c}_2$ from below

By Corollary 3.6, we have

$$\chi_{\{\sigma_2 t < x < \sigma_1 t\}} \le \liminf_{\substack{\epsilon \to 0 \\ (t',x') \to (t,x)}} v^\epsilon(t',x') \le \limsup_{\substack{\epsilon \to 0 \\ (t',x') \to (t,x)}} v^\epsilon(t',x') \le \chi_{\{x \le \sigma_1 t\}}. \tag{3.18}$$

Lemma 3.7. *Let (u,v) be a solution of (1.1) with initial data satisfying* (H_λ)*. Then, w_2^* is a viscosity sub-solution of*

$$\begin{cases} \min\{\partial_t w + |\partial_x w|^2 + 1 - a\chi_{\{x \le \sigma_1 t\}}, w\} = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ w(0,x) = \lambda_u \max\{x, 0\}, & \text{on } \mathbb{R}, \end{cases} \tag{3.19}$$

where σ_1 is defined by (1.3) *and $\lambda_u > 0$ is given in* (H_λ)*.*

Proof. The proof is analogous to Lemma 3.4 and we omit the details. □

Lemma 3.8. *Let* (u, v) *be a solution of* (1.1) *with initial data satisfying* (H_λ). *Then*

$$\underline{c}_2 \geq \hat{c}_{\text{nlp}},$$

where $\hat{c}_{\text{nlp}}$ *is defined in Theorem 1.3.*

Proof. According to definition of $\hat{c}_{\text{nlp}}$ in Theorem 1.3, we consider the three cases separately: (a) $\sigma_1 < 2\lambda_u$ and $\sigma_1 < 2(\sqrt{a} + \sqrt{1-a})$; (b) $\sigma_1 \geq 2\lambda_u$ and $\tilde{\lambda}_{\text{nlp}} \leq \sqrt{1-a}$; (c) otherwise.

First, we claim that we are done in case (c). Since in that case $\hat{c}_{\text{nlp}} = 2\sqrt{1-a}$, and, according to Proposition 2.1(ii), $\underline{c}_2 \geq c_{\text{LLW}}$ where $c_{\text{LLW}} \geq 2\sqrt{1-a}$ by Theorem 1.1. Thus

$$\underline{c}_2 \geq c_{\text{LLW}} \geq 2\sqrt{1-a} = \hat{c}_{\text{nlp}}.$$

It remains to consider cases (a) and (b). We start by defining

$$\bar{c}_{\text{nlp}} = \frac{\sigma_1}{2} - \sqrt{a} + \frac{1-a}{\frac{\sigma_1}{2} - \sqrt{a}}$$

and

$$\tilde{c}_{\text{nlp}} = \tilde{\lambda}_{\text{nlp}} + \frac{1-a}{\tilde{\lambda}_{\text{nlp}}} \quad \text{where} \quad \tilde{\lambda}_{\text{nlp}} = \frac{1}{2}\left(\sigma_1 - \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}\right). \tag{3.20}$$

Suppose case (a) holds, then $\hat{c}_{\text{nlp}} = \bar{c}_{\text{nlp}}$. Define $\overline{w}_2$ by

$$\overline{w}_2(t,x) = \begin{cases} \lambda_u(x - (\lambda_u + \frac{1}{\lambda_u})t), & \text{for } \frac{x}{t} \geq 2\lambda_u, \\ \frac{t}{4}(\frac{x^2}{t^2} - 4), & \text{for } \sigma_1 \leq \frac{x}{t} < 2\lambda_u, \\ (\frac{\sigma_1}{2} - \sqrt{a})(x - \bar{c}_{\text{nlp}}t), & \text{for } \bar{c}_{\text{nlp}} < \frac{x}{t} < \sigma_1, \\ 0, & \text{for } \frac{x}{t} \leq \bar{c}_{\text{nlp}}. \end{cases}$$

By construction, $\overline{w}_2$ is continuous in $[0,\infty) \times \mathbb{R}$. We claim that continuous $\overline{w}_2$ is a viscosity super-solution of (3.19). (Actually, it is the unique viscosity solution of (3.19), but we do not need this fact.) Indeed, $\overline{w}_2$ is a classical solution for (3.19) whenever $\frac{x}{t} \notin \{\sigma_1, \bar{c}_{\text{nlp}}\}$. Now, it remains to consider the case when $\overline{w}_2 - \varphi$ attains a strict local minimum at (t_0, x_0) for $\forall \varphi \in C^\infty(0,\infty) \times \mathbb{R})$, when $\frac{x_0}{t_0} = \sigma_1$ or $\bar{c}_{\text{nlp}}$. In case $\frac{x_0}{t_0} = \sigma_1$, $(\overline{w}_2 - \varphi)(t, \sigma_1 t) \geq (\overline{w}_2 - \varphi)(t_0, x_0)$ for all $t \approx t_0$, so that $\partial_t\varphi(t_0,x_0) + \sigma_1\partial_x\varphi(t_0,x_0) = \frac{\sigma_1^2}{4} - 1$. Hence, at (t_0, x_0), (note that $(-a\chi_{\{x \leq \sigma_1 t\}})^* = -a\chi_{\{x < \sigma_1 t\}}$)

$$\begin{aligned} \partial_t\varphi + |\partial_x\varphi|^2 + 1 - a\chi_{\{x<\sigma_1 t\}} &= \frac{\sigma_1^2}{4} - 1 - \sigma_1\partial_x\varphi + |\partial_x\varphi|^2 + 1 \\ &= \left(\partial_x\varphi - \frac{\sigma_1}{2}\right)^2 \geq 0. \end{aligned}$$

On the other hand, if $\frac{x_0}{t_0} = \bar{c}_{\rm nlp}$, then $\nabla\varphi(t_0,x_0)\cdot(1,\bar{c}_{\rm nlp}) = 0$, and

$$0 \leq \nabla\varphi(t_0,x_0)\cdot(-\bar{c}_{\rm nlp}t,1) \leq \nabla[(\frac{\sigma_1}{2}-\sqrt{a})(x-\bar{c}_{\rm nlp}t)]\cdot(-\bar{c}_{\rm nlp},1),$$

which means $\partial_t\varphi(t_0,x_0) = -\bar{c}_{\rm nlp}\partial_x\varphi$ and $0 \leq \partial_x\varphi(t_0,x_0) \leq \frac{\sigma_1}{2}-\sqrt{a}$, whence

$$\begin{aligned}\partial_t\varphi + |\partial_x\varphi|^2 + 1 - a\chi_{\{x<\sigma_1 t\}} &= -\bar{c}_{\rm nlp}\partial_x\varphi + |\partial_x\varphi|^2 + 1 - a \\ &= \left(\partial_x\varphi - \frac{1-a}{\frac{\sigma_1}{2}-\sqrt{a}}\right)\left(\partial_x\varphi - \frac{\sigma_1}{2}+\sqrt{a}\right) \geq 0\end{aligned}$$

at (t_0,x_0). The last inequality holds because $\partial_x\varphi \leq \frac{\sigma_1}{2}-\sqrt{a} \leq \sqrt{1-a} \leq \frac{1-a}{\frac{\sigma_1}{2}-\sqrt{a}}$. Hence, $\overline{w}_2$ is a viscosity super-solution of (3.19).

By Remark 3.3 and the express of $\overline{w}_2$, we have

$$\overline{w}_2(0,x) = \lambda_u\max\{x,0\} = w_2^*(0,x) \qquad \text{for } x\in\mathbb{R}.$$

And recalling that w_2^* is a viscosity sub-solution of (3.19), we may deduce by Theorem A.1 that

$$0 \leq w_2^* \leq \overline{w}_2 \qquad \text{in } [0,\infty)\times\mathbb{R}. \tag{3.21}$$

Now,

$$\{(t,x): w_2^*(t,x)=0\} \supset \{(t,x): \overline{w}_2(t,x)=0\} = \{(t,x): x\leq \hat{c}_{\rm nlp}t\}.$$

Hence,

$$w_2^\epsilon(t,x) = -\epsilon\log u^\epsilon(t,x) \to 0 \qquad \text{locally uniformly on } \{(t,x):\ x<\hat{c}_{\rm nlp}t\}.$$

Hence for each small $\eta>0$, by choosing the compact sets $K=\{(1,x):\eta\leq x\leq \hat{c}_{\rm nlp}-\eta\}$ and $K'=\{(1,x):\frac{\eta}{2}\leq x\leq\hat{c}_{\rm nlp}-\frac{\eta}{2}\}$, we may apply Lemma 3.1(a) to get

$$\liminf_{t\to\infty}\inf_{\eta t\leq x\leq(\hat{c}_{\rm nlp}-\eta)t} u(t,x) = \liminf_{\epsilon\to 0}\inf_K u^\epsilon(t,x) \geq \frac{1-a}{2} > 0,$$

which implies $\underline{c}_2 \geq \hat{c}_{\rm nlp}$.

Finally, for case (b), then we have $\hat{c}_{\rm nlp} = \tilde{c}_{\rm nlp}$. We define

$$\overline{w}_2(t,x) = \begin{cases} \lambda_u(x-(\lambda_u+\frac{1}{\lambda_u})t), & \text{for } \frac{x}{t}\geq\sigma_1, \\ \tilde{\lambda}_{\rm nlp}(x-\tilde{c}_{\rm nlp}t), & \text{for } \tilde{c}_{\rm nlp}<\frac{x}{t}<\sigma_1, \\ 0, & \text{for } \frac{x}{t}\leq\tilde{c}_{\rm nlp}. \end{cases}$$

Then one can verify that $\overline{w}_2$ is likewise a viscosity super-solution of (3.19), so that one can repeat the arguments for case (a) to show, again, that $\underline{c}_2 \geq \hat{c}_{\rm nlp}$. ☐

4 Estimating $\bar{c}_2$ from above and $\underline{c}_3$ from below

We assume $\sigma_1>\sigma_2$ throughout this section. It remains to show

$$\bar{c}_2 \leq \max\{c_{\rm LLW},\hat{c}_{\rm nlp}\} \quad \text{and} \quad \underline{c}_3 \geq -\max\{\tilde{c}_{\rm LLW},\sigma_3\}.$$

4.1 Estimating $\overline{c}_2$ from above

For $\delta \geq 0$, we will construct an exponent $\hat{\mu}_\delta$ depending continuously on δ such that

$$u(t, (\sigma_1 - \delta)t) \leq \exp\left(-(\hat{\mu}_\delta + o(1))t\right) \quad \text{for } t \gg 1,$$

so that we may apply Lemma B.2(a) to estimate $\overline{c}_2$ from above.

Lemma 4.1. *Let (u, v) be a solution of (1.1) with initial data satisfying* (H_λ). *Then $w_{2,*}$ is a viscosity super-solution of*

$$\begin{cases} \min\{\partial_t w + |\partial_x w|^2 + 1 - a\chi_{\{\sigma_2 t < x < \sigma_1 t\}}, w\} = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ w(0,x) = \lambda_u \max\{x, 0\}, & \text{on } \mathbb{R}, \end{cases} \tag{4.1}$$

where σ_1 and σ_2 are defined in (1.3).

Proof. It follows from standard arguments as in Lemma 3.4. □

Proposition 4.2. *Let (u, v) be a solution of (1.1) with initial data satisfying* (H_λ). *Then*

$$\overline{c}_2 \leq \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\},$$

where c_{LLW} and $\hat{c}_{\mathrm{nlp}}$ are defined respectively in Theorem 1.1 and 1.3.

Proof. **Step 1.** Define $\underline{w}_2 : [0,\infty) \times \mathbb{R}$ by

$$\underline{w}_2(t,x) = \begin{cases} \lambda_u(x - (\lambda_u + \frac{1}{\lambda_u})t), & \text{for } \frac{x}{t} \geq 2\lambda_u, \\ \frac{t}{4}(\frac{x^2}{t^2} - 4), & \text{for } 2 \leq \frac{x}{t} < 2\lambda_u, \\ 0, & \text{for } \frac{x}{t} < 2, \end{cases} \tag{4.2}$$

in case $\lambda_u > 1$, and by

$$\underline{w}_2(t,x) = \lambda_u \max\left\{x - (\lambda_u + \frac{1}{\lambda_u})t, 0\right\},$$

in case $\lambda_u \leq 1$. Then it is straightforward to verify that $\underline{w}_2$ is a viscosity sub-solution of (4.1). Since, $w_{2,*}(0,x) = \lambda_u \max\{x, 0\} = \underline{w}_2(0,x)$ in $\mathbb{R}$ (by Remark 3.3), we may apply Theorem A.1 to deduce

$$w_{2,*}(t,x) \geq \underline{w}_2(t,x) \quad \text{for } [0,\infty) \times \mathbb{R}. \tag{4.3}$$

Step 2. To show that, for each $\hat{c} \geq 0$,

$$u(t, \hat{c}t) \leq \exp\{-(\underline{w}_2(1,\hat{c}) + o(1))t\} \quad \text{for } t \gg 1. \tag{4.4}$$

And that $\underline{w}_2(1,\sigma_1)$ is given by

$$\underline{w}_2(1,\sigma_1) = \begin{cases} (\frac{\sigma_1}{2} - \sqrt{a})(\sigma_1 - \bar{c}_{\mathrm{nlp}}), & \text{for } \sigma_1 < 2\lambda_u, \\ \tilde{\lambda}_{\mathrm{nlp}}(\sigma_1 - \tilde{c}_{\mathrm{nlp}}), & \text{for } \sigma_1 \geq 2\lambda_u, \end{cases} \tag{4.5}$$

and $\bar{c}_{\text{nlp}}$, $\tilde{c}_{\text{nlp}}$, $\tilde{\lambda}_{\text{nlp}}$ are all given in Lemma 3.8.

By definition of $w_{2,*}$ and $w_2^\epsilon(t,x) = -\epsilon \log u^\epsilon(t,x)$, for each small $\epsilon > 0$, by applying Step 1, we have

$$-\epsilon \log u\left(\frac{1}{\epsilon}, \frac{\hat{c}}{\epsilon}\right) \geq w_{2,*}(1,\hat{c}) + o(1) \geq \underline{w}_2(1,\hat{c}) + o(1)$$

$$\iff u\left(\frac{1}{\epsilon}, \frac{\hat{c}}{\epsilon}\right) \leq \exp\left(-\frac{\underline{w}_2(1,\hat{c}) + o(1)}{\epsilon}\right),$$

which implies (4.4). By the formula of $\underline{w}_2$, we can show

(i) For $\sigma_1 < 2\lambda_u$, we substitute $(t,x) = (1,\sigma_1)$ in (4.2) to obtain

$$\underline{w}_2(1,\sigma_1) = \frac{1}{4}(\sigma_1^2 - 4) = (\frac{\sigma_1}{2} - \sqrt{a})(\sigma_1 - \bar{c}_{\text{nlp}}), \tag{4.6}$$

where $\bar{c}_{\text{nlp}} = \frac{\sigma_1}{2} - \sqrt{a} + \frac{1-a}{\frac{\sigma_1}{2} - \sqrt{a}}$;

(ii) For $\sigma_1 \geq 2\lambda_u$, we substitute $(t,x) = (1,\sigma_1)$ in (4.2) to obtain

$$\underline{w}_2(1,\sigma_1) = \lambda_u \left(\sigma_1 - (\lambda_u + \frac{1}{\lambda_u})\right). \tag{4.7}$$

Recalling the definition of $\tilde{\lambda}_{\text{nlp}}$ in (3.20), we have

$$\tilde{\lambda}_{\text{nlp}} - \lambda_u = \frac{1}{2}\left[(\sigma_1 - 2\lambda_u) - \sqrt{\sigma_1 - 2\lambda_u)^2 + 4a}\right],$$

so that

$$(\tilde{\lambda}_{\text{nlp}} - \lambda_u)^2 - (\sigma_1 - 2\lambda_u)(\tilde{\lambda}_{\text{nlp}} - \lambda_u) - a = 0. \tag{4.8}$$

Hence, (4.7) becomes

$$\underline{w}_2(1,\sigma_1) = \lambda_u \left(\sigma_1 - (\lambda_u + \frac{1}{\lambda_u})\right) = \tilde{\lambda}_{\text{nlp}}(\sigma_1 - \tilde{c}_{\text{nlp}}), \tag{4.9}$$

where $\tilde{c}_{\text{nlp}}, \tilde{\lambda}_{\text{nlp}}$ are as in (3.20).

This implies (4.5) holds, which completes Step 2.

Step 3. To show $\overline{c}_2 \leq \max\{c_{\text{LLW}}, \hat{c}_{\text{nlp}}\}$.

It follows from Proposition 2.1 and Corollary 3.6 that for $\hat{c} \in (\sigma_2, \sigma_1)$,

$$\lim_{t\to\infty}(u,v)(t,0) = (k_1,k_2) \quad \text{and} \quad \lim_{t\to\infty}(u,v)(t,\hat{c}t) = (0,1).$$

By Step 2 and observation $\lambda_{\text{LLW}} c_{\text{LLW}} = \lambda_{\text{LLW}}^2 + 1 - a$, then we apply Lemma B.2(a) in Appendix to conclude that for $\hat{c} \in (\sigma_2, \sigma_1)$,

$$\overline{c}_2 \leq c_{\hat{c},\underline{w}_2(1,\hat{c})} = \begin{cases} c_{\text{LLW}}, & \text{if } \underline{w}_2(1,\hat{c}) \geq -\lambda_{\text{LLW}}^2 + \lambda_{\text{LLW}}\hat{c} - (1-a), \\ \hat{c} - \frac{2\underline{w}_2(1,\hat{c})}{\hat{c} - \sqrt{\hat{c}^2 - 4(\underline{w}_2(1,\hat{c}) + 1 - a)}}, & \text{if } \underline{w}_2(1,\hat{c}) < -\lambda_{\text{LLW}}^2 + \lambda_{\text{LLW}}\hat{c} - (1-a). \end{cases} \tag{4.10}$$

Letting $\hat{c} \nearrow \sigma_1$, (4.10) can be expressed as (denote $\hat{\mu} = \underline{w}_2(1, \sigma_1)$)

$$\overline{c}_2 \le c_{\sigma_1,\hat{\mu}} = \begin{cases} c_{\rm LLW}, & \text{if } \hat{\mu} \ge -\lambda_{\rm LLW}^2 + \lambda_{\rm LLW}\sigma_1 - (1-a), \\ \sigma_1 - \frac{2\hat{\mu}}{\sigma_1 - \sqrt{\sigma_1^2 - 4(\hat{\mu}+1-a)}}, & \text{if } \hat{\mu} < -\lambda_{\rm LLW}^2 + \lambda_{\rm LLW}\sigma_1 - (1-a). \end{cases} \tag{4.11}$$

It remains to verify $c_{\sigma_1,\hat{\mu}} = \max\{c_{\rm LLW}, \hat{c}_{\rm nlp}\}$, where $\hat{c}_{\rm nlp} = \hat{\lambda}_{\rm nlp} + \frac{1-a}{\hat{\lambda}_{\rm nlp}}$ and

$$\hat{\lambda}_{\rm nlp} = \begin{cases} \frac{\sigma_1}{2} - \sqrt{a}, & \text{if } \sigma_1 < 2\lambda_u \text{ and } \sigma_1 \le 2(\sqrt{a} + \sqrt{1-a}), \\ \tilde{\lambda}_{\rm nlp}, & \text{if } \sigma_1 \ge 2\lambda_u \text{ and } \tilde{\lambda}_{\rm nlp} \le \sqrt{1-a}, \\ \sqrt{1-a}, & \text{otherwise}, \end{cases} \tag{4.12}$$

and $\tilde{\lambda}_{\rm nlp}$ is given in Lemma 3.8. Note that

$$\hat{\lambda}_{\rm nlp} = \min\{\lambda_{\hat{\mu}}, \sqrt{1-a}\}, \quad \text{where} \quad \lambda_{\hat{\mu}} := \begin{cases} \frac{\sigma_1}{2} - \sqrt{a}, & \text{for } \sigma_1 < 2\lambda_u, \\ \tilde{\lambda}_{\rm nlp}, & \text{for } \sigma_1 \ge 2\lambda_u. \end{cases} \tag{4.13}$$

By (4.6) and (4.9), $\hat{\mu} = \underline{w}_2(1, \sigma_1)$ can be expressed as

$$\hat{\mu} = G(\lambda_{\hat{\mu}}), \quad \text{where} \quad G(\lambda) := -\lambda^2 + \sigma_1\lambda - (1-a) \tag{4.14}$$

and $\lambda_{\hat{\mu}}$ is as defined in (4.13). Note that $G(\lambda)$ is strictly increasing on $[0, \frac{\sigma_1}{2}]$.

We note for later purposes that (4.14) is a quadratic equation in $\lambda_{\hat{\mu}}$, so that

$$\lambda_{\hat{\mu}} = \frac{\sigma_1 - \sqrt{\sigma_1^2 - 4(\hat{\mu}+1-a)}}{2}. \tag{4.15}$$

Since $\lambda_{\rm LLW} \in (0, \sqrt{1-a}]$, we divide our discussion into two cases: (i) $\lambda_{\hat{\mu}} < \lambda_{\rm LLW}$; (ii) $\lambda_{\rm LLW} \le \lambda_{\hat{\mu}}$.

(i) Case $\lambda_{\hat{\mu}} < \lambda_{\rm LLW}$. (Recall that $\lambda_{\rm LLW} \le \sqrt{1-a}$.)

By (4.13), $\hat{\lambda}_{\rm nlp} = \lambda_{\hat{\mu}} < \lambda_{\rm LLW}$, whence it follows from the observation

$$\hat{c}_{\rm nlp} = \hat{\lambda}_{\rm nlp} + \frac{1-a}{\hat{\lambda}_{\rm nlp}} \quad \text{and} \quad c_{\rm LLW} = \lambda_{\rm LLW} + \frac{1-a}{\lambda_{\rm LLW}}, \tag{4.16}$$

and the monotonicity of $s + \frac{1-a}{s}$ in $(0, \sqrt{1-a}]$ that $\hat{c}_{\rm nlp} \ge c_{\rm LLW}$. It remains to show that $c_{\sigma_1,\hat{\mu}} = \hat{c}_{\rm nlp}$.

Now, by monotonicity of G, we have

$$\hat{\mu} = G(\lambda_{\hat{\mu}}) < G(\lambda_{\rm LLW}) = -\lambda_{\rm LLW}^2 + \lambda_{\rm LLW}\sigma_1 - (1-a).$$

By (4.11), we have $c_{\sigma_1,\hat{\mu}} = \sigma_1 - \frac{2\hat{\mu}}{\sigma_1 - \sqrt{\sigma_1^2 - 4(\hat{\mu}+1-a)}}$. Hence,

$$c_{\sigma_1,\hat{\mu}} = \sigma_1 - \frac{\hat{\mu}}{\lambda_{\hat{\mu}}} = \lambda_{\hat{\mu}} + \frac{1-a}{\lambda_{\hat{\mu}}} = \hat{c}_{\rm nlp},$$

where the first and second equalities follow from (4.15) and (4.14), respectively.

(ii) Case $\lambda_{\rm LLW} \leq \lambda_{\hat\mu}$.

By (4.13),

$$\hat\lambda_{\rm nlp} = \min\{\lambda_{\hat\mu}, \sqrt{1-a}\} \geq \min\{\lambda_{\rm LLW}, \sqrt{1-a}\} = \lambda_{\rm LLW}.$$

It follows from (4.16) that $\hat c_{\rm nlp} \leq c_{\rm LLW}$. It remains to show that $\hat\mu \geq G(\lambda_{\rm LLW})$, so that $c_{\sigma_1,\hat\mu} = c_{\rm LLW} = \max\{c_{\rm LLW}, \hat c_{\rm nlp}\}$. Indeed, one can check that $\lambda_{\rm LLW} \leq \lambda_{\hat\mu} \leq \sigma_1/2$, and we deduce

$$\hat\mu = G(\lambda_{\hat\mu}) \geq G(\lambda_{\rm LLW}),$$

by the monotonicity of G in $[0, \sigma_1/2]$.

The proof of Proposition 4.2 is now complete. $\square$

4.2 Estimating $\underline{c}_3$ from below

For convenience, let $\tilde u(t,x) = u(t,-x)$, $\tilde v(t,x) = v(t,-x)$, and define

$$\tilde u^\epsilon(t,x) = \tilde u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \ \tilde v^\epsilon(t,x) = \tilde v\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \ w_3^\epsilon = -\epsilon \log \tilde v^\epsilon(t,x) \ \text{ in } [0,\infty)\times\mathbb{R}.$$

Again we pass to the half-relaxed limit:

$$w_{3,*}(t,x) = \liminf_{\substack{\epsilon \to 0 \\ (t',x') \to (t,x)}} w_3^\epsilon(t',x').$$

Lemma 4.3. *Let $(\tilde u, \tilde v)$ be a solution of (1.1) such that $x \to (\tilde u(0,-x), \tilde v(0,-x))$ satisfies* (H_λ). *Then, for each small $\eta > 0$,*

$$\lim_{t\to\infty} \sup_{x > (d\lambda_v^- + \frac{r}{\lambda_v^-} + \eta)t} (|\tilde u(t,x) - 1| + |\tilde v(t,x)|) = 0. \tag{4.17}$$

Proof. Let $v_{\rm KPP}$ be the solution of

$$\begin{cases} \partial_t v_{\rm KPP} - d\partial_{xx} v_{\rm KPP} = r v_{\rm KPP}(1 - v_{\rm KPP}), & \text{in } (0,\infty)\times\mathbb{R}, \\ v_{\rm KPP} = \min\{1, Ce^{-\lambda_v^- x}\}, & \text{on } x \in \mathbb{R}. \end{cases}$$

By choosing C to be sufficiently large, we may apply comparison principle to get $0 \leq \tilde v \leq v_{\rm KPP}$. Therefore, for each $\eta > 0$,

$$\lim_{t\to\infty} \sup_{x > (d\lambda_v^- + \frac{r}{\lambda_v^-} + \eta)t} |\tilde v(t,x)| = 0 \ \text{ for each } \eta > 0. \tag{4.18}$$

Let $u_{\rm KPP}$ be the solution of

$$\begin{cases} \partial_t u_{\rm KPP} - \partial_{xx} u_{\rm KPP} = u_{\rm KPP}(1 - a - u_{\rm KPP}), & \text{in } (0,\infty)\times\mathbb{R}, \\ u_{\rm KPP}(0,x) = u_0(x), & \text{on } x \in \mathbb{R}. \end{cases}$$

Again the scalar comparison principle implies $u \geq u_{\mathrm{KPP}}$. By the results in [34] or [44], we have, for each small $\eta > 0$,

$$\lim_{t\to\infty} \inf_{x > -(2\sqrt{1-a}+\eta)t} \tilde{u}(t,x) = \lim_{t\to\infty} \inf_{x<(2\sqrt{1-a}-\eta)t} u(t,x) \geq \frac{1-a}{2}. \tag{4.19}$$

By small $\eta>0$, we have (4.18) and (4.19) hold, thus we may apply Lemma B.1(b) to deduce (4.17). ☐

In view of Lemma 4.3, we obtain

$$\chi_{\{x>(d\lambda_v^- + \frac{r}{\lambda_v^-})t\}} \leq \liminf_{\substack{\epsilon\to 0\\ (t',x')\to(t,x)}} \tilde{u}^\epsilon(t',x') \leq \limsup_{\substack{\epsilon\to 0\\ (t',x')\to(t,x)}} \tilde{u}^\epsilon(t',x') \leq 1. \tag{4.20}$$

Lemma 4.4. *Let $(\tilde{u},\tilde{v})$ be a solution of (1.1) such that $x \to (\tilde{u}(0,-x),\tilde{v}(0,-x))$ satisfies* (H_λ). *Then, $w_{3,*}$ is a viscosity super-solution of*

$$\begin{cases} \min\{\partial_t w + d|\partial_x w|^2 + r(1 - b\chi_{\{x>(d\lambda_v^- + \frac{r}{\lambda_v^-})t\}}), w\} = 0 & \text{in } (0,\infty)\times(0,\infty),\\ w(0,x) = \lambda_v^- x, & \text{on } [0,\infty),\\ w(t,0) = 0, & \text{for } \ t>0. \end{cases} \tag{4.21}$$

Proof. The proof is similar to proof of Lemma 3.4(b) and is omitted. ☐

Proposition 4.5. *Let (u,v) be a solution of (1.1) with initial data satisfying* (H_λ). *Then*

$$\underline{c}_3 \geq -\max\{\tilde{c}_{\mathrm{LLW}}, \sigma_3\}.$$

where $\tilde{c}_{\mathrm{LLW}}$ and σ_3 are defined in Remark 1.2 and (1.3), respectively.

Proof. **Step 1.** To show

$$w_{3,*}(t,x) \geq \underline{w}_3(t,x) \quad \text{ for } [0,\infty)\times[0,\infty), \tag{4.22}$$

where $\underline{w}_3 : [0,\infty)\times[0,\infty)$ is defined by

$$\underline{w}_3(t,x) = \lambda_v^- \max\left\{x - (d\lambda_v^- + \frac{r}{\lambda_v^-})t, 0\right\}.$$

As in Step 1 of Proposition 4.2, one can verify that $\underline{w}_3$ is a viscosity sub-solution of (4.21). By the expression of $\underline{w}_3$, Remark 3.3 and $w_{1,*}(t,-x) = w_{3,*}(t,x)$, we have $\underline{w}_3(t,x) = \lambda_v^- \max\{x,0\} = w_{3,*}(t,x)$ on $\partial[(0,\infty)\times(0,\infty)]$. Hence we apply Theorem A.1 to obtain (4.22).

Step 2. To show for each $\hat{c} \geq 0$, we have

$$\tilde{v}(t,\hat{c}t) \leq \exp\{(\underline{w}_3(1,\hat{c}) + o(1))t\} \quad \text{ for } t \gg 1. \tag{4.23}$$

This can be done as in Step 2 of Proposition 4.2.

Step 3. To show $\underline{c}_3 \geq -\max\{\tilde{c}_{\mathrm{LLW}}, \sigma_3\}$.

Fix $\hat{c} > (d\lambda_v^- + \frac{r}{\lambda_v^-})$. By Proposition 2.1 and Lemma 4.3, we arrive at

$$\lim_{t\to\infty}(\tilde{u},\tilde{v})(t,0) = \lim_{t\to\infty}(u,v)(t,0) = (k_1,k_2) \text{ and } \lim_{t\to\infty}(\tilde{u},\tilde{v})(t,\hat{c}t) = (1,0). \tag{4.24}$$

This verifies condition (i) of Lemma B.2(b). Next, by Step 2, we have

$$\tilde{v}(t,\hat{c}t) \leq \exp\{-(\hat{\mu}_2 + o(1))t\} \quad \text{ for } t \gg 1,$$

where

$$\hat{\mu}_2 = \underline{w}_3(1,\hat{c}) = \lambda_v^-(\hat{c} - (d\lambda_v^- + \frac{r}{\lambda_v^-})). \tag{4.25}$$

We note for later purposes that $\hat{\mu}_2$ is a quadratic expression in λ_v^-, so that

$$\hat{\mu}_2 = \lambda_v^-\hat{c} - d(\lambda_v^-)^2 - r, \quad \text{ and } \quad \lambda_v^- = \frac{\hat{c} - \sqrt{\hat{c}^2 - 4d(\hat{\mu}_2 + r)}}{2d}. \tag{4.26}$$

We may then apply Lemma B.2(b) to conclude

$$-\underline{c}_3 \leq \tilde{c}_{\hat{c},\hat{\mu}_2} = \begin{cases} \tilde{c}_{\rm LLW}, & \text{if } \hat{\mu}_2 \geq \tilde{\lambda}_{\rm LLW}(\hat{c} - \tilde{c}_{\rm LLW}), \\ \hat{c} - \frac{2d\hat{\mu}_2}{\hat{c} - \sqrt{\hat{c}^2 - 4d[\hat{\mu}_2 + r(1-b)]}}, & \text{if } 0 < \hat{\mu}_2 < \tilde{\lambda}_{\rm LLW}(\hat{c} - \tilde{c}_{\rm LLW}). \end{cases} \tag{4.27}$$

To complete the proof, we need to verify

$$\limsup_{\hat{c}\to\infty} \tilde{c}_{\hat{c},\hat{\mu}_2} \leq \max\left\{\tilde{c}_{\rm LLW}, \sigma_3\right\}.$$

Since $0 = -d\tilde{\lambda}_{\rm LLW}^2 + \tilde{\lambda}_{\rm LLW}\tilde{c}_{\rm LLW} - r(1-b)$, then

$$\begin{aligned} \hat{\mu}_2 - \tilde{\lambda}_{\rm LLW}(\hat{c} - \tilde{c}_{\rm LLW}) &= \hat{\mu}_2 - (-d\tilde{\lambda}_{\rm LLW}^2 + \tilde{\lambda}_{\rm LLW}\hat{c} - r(1-b)) \\ &= \left(\lambda_v^- - \tilde{\lambda}_{\rm LLW}\right)\left[\hat{c} - d(\lambda_v^- + \tilde{\lambda}_{\rm LLW})\right] - rb, \end{aligned} \tag{4.28}$$

where (4.26) is used for the last inequality.

(i) For the case $\lambda_v^- > \tilde{\lambda}_{\rm LLW}$, we take $\hat{c} \to \infty$ in (4.28) to get

$$\hat{\mu}_2 \geq \tilde{\lambda}_{\rm LLW}\left(\hat{c} - \tilde{c}_{\rm LLW}\right),$$

so that by (4.27), $-\underline{c}_3 \leq \tilde{c}_{\rm LLW} \leq \max\{\tilde{c}_{\rm LLW}, \sigma_3\}$;

(ii) For the case $\lambda_v^- \leq \tilde{\lambda}_{\rm LLW}$, we have $\lambda_v^- \leq \tilde{\lambda}_{\rm LLW} \leq \sqrt{\frac{r(1-b)}{d}}$ and

$$\sigma_3 = d\lambda_v^- + \frac{r(1-b)}{\lambda_v^-} \geq d\tilde{\lambda}_{\rm LLW} + \frac{r(1-b)}{\tilde{\lambda}_{\rm LLW}} = \tilde{c}_{\rm LLW}.$$

we have

$$0 < \hat{\mu}_2 < \tilde{\lambda}_{\rm LLW}\left(\hat{c} - \tilde{c}_{\rm LLW}\right).$$

Denote $\lambda_{\hat{c},\hat{\mu}_2} = \frac{\hat{c}-\sqrt{\hat{c}^2-4d[\hat{\mu}_2+r(1-b)]}}{2d}$. Then

$$d(\lambda_{\hat{c},\hat{\mu}_2})^2 - \hat{c}\lambda_{\hat{c},\hat{\mu}_2} + \hat{\mu}_2 + r(1-b) = 0, \tag{4.29}$$

and $\lambda_{\hat{c},\hat{\mu}_2} \leq \lambda_v^-$ (by comparing with the second part of (4.26)). Hence, we arrive at

$$-\underline{c}_3 \leq c_{\hat{c},\hat{\mu}_2} = \hat{c} - \frac{\hat{\mu}_2}{\lambda_{\hat{c},\hat{\mu}_2}} = d\lambda_{\hat{c},\hat{\mu}_2} + \frac{r(1-b)}{\lambda_{\hat{c},\hat{\mu}_2}}. \tag{4.30}$$

Next, we claim that

$$\lim_{\hat{c}\to\infty} \lambda_{\hat{c},\hat{\mu}_2} = \lambda_v^-. \tag{4.31}$$

To this end, subtract the first part of (4.26) from (4.29) to get

$$d(\lambda_{\hat{c},\hat{\mu}_2})^2 - \hat{c}(\lambda_{\hat{c},\hat{\mu}_2} - \lambda_v^-) - d(\lambda_v^-)^2 - rb = 0.$$

Dividing the above by $\hat{c}$ and letting $\hat{c} \to \infty$, we obtain (4.31).

By (4.31), we can take $\hat{c} \to \infty$ in (4.30) to get $-\underline{c}_3 \leq \sigma_3 \leq \max\{\tilde{c}_{\mathrm{LLW}}, \sigma_3\}$.

This completes the proof of Proposition 4.5. $\square$

4.3 Proof of Theorem 1.3

Proof of Theorem 1.3. For $i = 1, 2, 3$, let $\overline{c}_i, \underline{c}_i$ be the maximal and minimal spreading speeds defined in (2.1). It follows from definition directly that $\overline{c}_i \geq \underline{c}_i$. By Corollary 3.6, we have $\overline{c}_1 = \underline{c}_1 = \sigma_1$. By Proposition 2.1(ii) and Lemma 3.8, we arrive at $\underline{c}_2 \geq \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\}$, which, together with $\overline{c}_2 \leq \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\}$ in Proposition 4.2, we have $\overline{c}_2 = \underline{c}_2 = \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\}$. Moreover, combining with Propositions 2.1 and 4.5 gives $\overline{c}_3 = \underline{c}_3 = -\max\{\sigma_3, \tilde{c}_{\mathrm{LLW}}\}$. Recalling the c_i as defined in (1.5), we have $\overline{c}_i = \underline{c}_i = c_i$ for all $i = 1, 2, 3$. To complete the proof of Theorem 1.3, it remains to show (1.4).

Observe that the first two items of (1.4) is a direct consequence of Corollary 3.6. Next, we shall show that

$$\liminf_{t\to\infty} \inf_{(c_3+\eta)t<x<(\sigma_1-\eta)t} v(t,x) > 0 \quad \text{for small } \eta > 0. \tag{4.32}$$

Given some small $\eta > 0$, definitions of $\overline{c}_3$ and $\underline{c}_1$ imply the existence of $c_3' \in (c_3, c_3+\eta)$, $\sigma_1' \in (\sigma_1 - \eta, \sigma_1)$ and $T > 0$ such that

$$\inf_{t\geq T} \min\{v(t, c_3' t), v(t, \sigma_1' t)\} > 0.$$

Now, define

$$\delta := \min\left\{\frac{1-b}{2}, \inf_{c_3' T < x < \sigma_1' T} v(T,x), \inf_{t\geq T} \min\{v(t,c_3't), v(t,\sigma_1't)\}\right\} > 0.$$

Observe that $v(t,x)$ and δ form a pair of super- and sub-solutions to the KPP-type equation $\partial_t v = d\partial_{xx} v + rv(1-b-v)$ such that $v(t,x) \geq \delta$ on the parabolic boundary of the domain $\Omega := \{(t,x) : t \geq T,\, c_3' t < x < \sigma_1' t\}$. It follows from the maximum principle that $v \geq \delta$ in Ω. In particular, (4.32) holds.

Similarly, we can show that

$$\liminf_{t\to\infty} \inf_{x<(c_2-\eta)t} u(t,x) > 0 \quad \text{for small } \eta > 0, \tag{4.33}$$

by definition of $\underline{c}_2$ and by (4.19) in Lemma 4.3. Fix small $\eta > 0$. In view of (4.32) and (4.33), the third item of (1.4) holds by applying (a) and (c) in Lemma B.1. Finally, since (4.33) and $\lim\limits_{t\to\infty} \sup\limits_{x<(c_3-\eta)t} v = 0$ (by definition of $\underline{c}_3$), by applying Lemma B.1(b), the fourth item of (1.4) holds true. The proof of Theorem 1.3 is now complete. □

5 The invasion mode due to Tang and Fife

In this section, we assume $\sigma_1 = \sigma_2$ and prove Theorem 1.5.

Proof of Theorem 1.5. For any small $\delta \in (0,1)$, let $(\underline{u}^\delta, \overline{v}^\delta)$ and $(\overline{u}^\delta, \underline{v}^\delta)$ be respectively any solution of

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1-u-av), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t v - d\partial_{xx} v = rv(1+\delta-bu-v), & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1-u-av), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t v - d\partial_{xx} v = rv(1-\delta-bu-v), & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{5.2}$$

with initial data satisfying (H_λ). By comparison, we deduce that

$$(\underline{u}^\delta, \overline{v}^\delta) \preceq (u,v) \preceq (\overline{u}^\delta, \underline{v}^\delta) \ \text{ in } [0,\infty)\times\mathbb{R}. \tag{5.3}$$

Notice that $(\underline{u}^\delta, \overline{v}^\delta)$ is a solution of (5.1) if and only if

$$(\underline{U}^\delta, \overline{V}^\delta) = \left(\underline{u}, \frac{\overline{v}^\delta}{1+\delta}\right) \tag{5.4}$$

is a solution of

$$\begin{cases} \partial_t U - \partial_{xx} U = U(1-U-\overline{a}^\delta V), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t V - d\partial_{xx} V = \overline{r}^\delta V(1-\underline{b}^\delta U - V), & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{5.5}$$

where $\overline{a}^\delta = (1+\delta)a$, $\overline{r}^\delta = (1+\delta)r$ and $\underline{b}^\delta = \frac{b}{1+\delta}$. Observe that $\overline{\sigma}_1^\delta = d(\lambda_v^+ \wedge \sqrt{\frac{\overline{r}^\delta}{d}}) + \frac{\overline{r}^\delta}{\lambda_v^+\wedge\sqrt{\frac{\overline{r}^\delta}{d}}} > \sigma_1 = \sigma_2$ and $0 < \overline{a}^\delta, \underline{b}^\delta < 1$ by choosing δ small enough. By applying Theorem 1.3 to (5.5), we deduce that the rightward and leftward spreading speeds $\overline{c}_1^\delta$ and $\underline{c}_3^\delta$ of $\overline{V}^\delta$ (which is the same as $\overline{v}^\delta$), and the rightward spreading speed $\underline{c}_2^\delta$ of $\underline{U}^\delta$ (same as $\underline{u}^\delta$) exist. Furthermore, they can be characterized by

$$\overline{c}_1^\delta = \overline{\sigma}_1^\delta, \quad \underline{c}_2^\delta = \max\{\underline{c}_{\mathrm{LLW}}^\delta, \underline{\hat{c}}_{\mathrm{nlp}}^\delta\}, \quad \underline{c}_3^\delta = -\max\{\overline{\overline{c}}_{\mathrm{LLW}}^\delta, \overline{\sigma}_3^\delta\}.$$

Precisely, $\underline{c}_{\mathrm{LLW}}^\delta$ (resp. $\overline{\overline{c}}_{\mathrm{LLW}}^\delta$) is the spreading speed for (5.5) as given in Theorem 1.1 (resp. Remark 1.2), $\overline{\sigma}_3^\delta = d(\lambda_v^- \wedge \sqrt{\frac{\overline{r}^\delta}{d}}) + \frac{\overline{r}^\delta(1-\underline{b}^\delta)}{\lambda_v^+ \wedge \sqrt{\frac{\overline{r}^\delta}{d}}}$ and moreover

$$\underline{\hat{c}}_{\mathrm{nlp}}^\delta = \begin{cases} \frac{\overline{\sigma}_1^\delta}{2} - \sqrt{\overline{a}^\delta} + \frac{1-\overline{a}^\delta}{\frac{\overline{\sigma}_1^\delta}{2} - \sqrt{\overline{a}^\delta}}, & \text{if } \overline{\sigma}_1^\delta < 2\lambda_u \text{ and } \overline{\sigma}_1^\delta \le 2(\sqrt{\overline{a}^\delta} + \sqrt{1-\overline{a}^\delta}), \\ \overline{\overline{\lambda}}_{\mathrm{nlp}}^\delta + \frac{1-\overline{a}^\delta}{\overline{\overline{\lambda}}_{\mathrm{nlp}}^\delta}, & \text{if } \overline{\sigma}_1^\delta \ge 2\lambda_u \text{ and } \overline{\overline{\lambda}}_{\mathrm{nlp}}^\delta \le \sqrt{1-\overline{a}^\delta}, \\ 2\sqrt{1-\overline{a}^\delta}, & \text{otherwise}, \end{cases} \tag{5.6}$$

where $\overline{\overline{\lambda}}_{\mathrm{nlp}}^\delta = \frac{1}{2}\left[\overline{\sigma}_1^\delta - \sqrt{(\overline{\sigma}_1^\delta - 2\lambda_u)^2 + 4\overline{a}^\delta}\right]$. Now, by the relation (5.3), we can compare with the spreading speeds $\overline{c}_1$, $\underline{c}_2$ and $\underline{c}_3$ of (u, v):

$$\overline{c}_1 \le \overline{c}_1^\delta, \quad \underline{c}_2 \ge \underline{c}_2^\delta \quad \text{and} \quad \underline{c}_3 \ge \underline{c}_3^\delta. \tag{5.7}$$

It remains to show that, assuming $\sigma_1 = \sigma_2$, we have $\underline{\hat{c}}_{\mathrm{nlp}}^\delta \to \sigma_2$ as $\delta \to 0$. Divide into the two cases:

(i) If $\lambda_u > 1$, then $\sigma_1 = \sigma_2 = 2 < 2\lambda_u$. Since $1 < \sqrt{a} + \sqrt{1-a}$, by choosing $\delta > 0$ small enough, we get $\overline{\sigma}_1^\delta < 2\lambda_u$ and $\overline{\sigma}_1^\delta \le 2(\sqrt{\overline{a}^\delta} + \sqrt{1-\overline{a}^\delta})$, which implies $\underline{\hat{c}}_{\mathrm{nlp}}^\delta = \frac{\overline{\sigma}_1^\delta}{2} - \sqrt{\overline{a}^\delta} + \frac{1-\overline{a}^\delta}{\frac{\overline{\sigma}_1^\delta}{2} - \sqrt{\overline{a}^\delta}} \to 1 - \sqrt{a} + \frac{1-a}{1-\sqrt{a}} = 2 = \sigma_2$ as $\delta \to 0$.

(ii) If $\lambda_u \le 1$, then first claim that

$$\overline{\sigma}_1^\delta \ge \sigma_1 \ge 2\lambda_u, \tag{5.8}$$

which is due to $\overline{\sigma}_1^\delta \ge \sigma_1 = \sigma_2 = \lambda_u + \frac{1}{\lambda_u} \ge 2 \ge 2\lambda_u$.

Next, we claim that

$$\tilde{\lambda}_{\mathrm{nlp}} < \sqrt{1-a}, \tag{5.9}$$

where $\tilde{\lambda}_{\mathrm{nlp}}$ is given in (1.7). To this end, observe that

$$\sigma_1 - 2\sqrt{1-a} < \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a} \tag{5.10}$$

which is a consequence of

$$\begin{aligned}(\sigma_1 - 2\sqrt{1-a})^2 - \left[(\sigma_1 - 2\lambda_u)^2 + 4a\right] &= 4(2 - 2a - \sigma_1\sqrt{1-a}) \\ &\le 4(2 - 2a - 2\sqrt{1-a}) < 0.\end{aligned}$$

From definition of $\tilde{\lambda}_{\mathrm{nlp}} = \frac{1}{2}[\sigma_1 - \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}]$, we deduce (5.9).

By (5.8) and (5.9), we have $\bar{\sigma}_1^\delta \geq 2\lambda_u$ and $\overline{\tilde{\lambda}}_{\rm nlp}^\delta < \sqrt{1-\overline{a}^\delta}$ for δ small, so

$$\hat{\underline{c}}_{\rm nlp}^\delta = \overline{\tilde{\lambda}}_{\rm nlp}^\delta + \frac{1-\overline{a}^\delta}{\overline{\tilde{\lambda}}_{\rm nlp}^\delta} \to \tilde{\lambda}_{\rm nlp} + \frac{1-a}{\tilde{\lambda}_{\rm nlp}} \quad \text{as } \delta \to 0.$$

Since we want $\hat{\underline{c}}_{\rm nlp}^\delta \to \sigma_2$, it remains to show that $\sigma_2 = \tilde{\lambda}_{\rm nlp} + \frac{1-a}{\tilde{\lambda}_{\rm nlp}}$. To this end, recall, from the definition of $\tilde{\lambda}_{\rm nlp}$ (1.7), that

$$\tilde{\lambda}_{\rm nlp} = \frac{\sigma_1 - \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}}{2} = \frac{2(\sigma_1\lambda_u - \lambda_u^2 - a)}{\sigma_1 + \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}}.$$

Using $\sigma_1 = \sigma_2 = \lambda_u + \frac{1}{\lambda_u}$, we deduce

$$\tilde{\lambda}_{\rm nlp} = \frac{\sigma_2 - \sqrt{(\sigma_2 - 2\lambda_u)^2 + 4a}}{2} = \frac{2(1-a)}{\sigma_2 + \sqrt{(\sigma_2 - 2\lambda_u)^2 + 4a}}. \tag{5.11}$$

This implies $\sigma_2 = \tilde{\lambda}_{\rm nlp} + \frac{1-a}{\tilde{\lambda}_{\rm nlp}}$. The proof is now complete.

Hence, by the continuity of $\underline{c}_{\rm LLW}^\delta$ and $\overline{\tilde{c}}_{\rm LLW}^\delta$ in δ (see, e.g. [52, Theorem 4.2 of Ch. 3]), letting $\delta \to 0$ in (5.7) yields

$$\overline{c}_1 \leq \sigma_1, \quad \underline{c}_2 \geq \sigma_2 \quad \text{and} \quad \underline{c}_3 \geq -\max\{\tilde{c}_{\rm LLW}, \sigma_3\}. \tag{5.12}$$

By a quite similar process, we can obtain $(\overline{u}^\delta, \underline{v}^\delta)$ is a solution of (5.2) if and only if

$$(\overline{U}^\delta, \underline{V}^\delta) = \left(\overline{u}^\delta, \frac{\underline{v}^\delta}{1-\delta}\right)$$

is a solution of

$$\begin{cases} \partial_t U - \partial_{xx} U = U(1 - U - \underline{a}^\delta V), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t V - d\partial_{xx} V = \underline{r}^\delta V(1 - \overline{b}^\delta U - V), & \text{in } (0,\infty)\times\mathbb{R}, \end{cases} \tag{5.13}$$

where $\underline{a}^\delta = (1-\delta)a$, $\underline{r}^\delta = (1-\delta)r$ and $\overline{b}^\delta = \frac{b}{1-\delta}$. Observe that $\underline{\sigma}_1^\delta = d(\lambda_v^+ \wedge \sqrt{\frac{r^\delta}{d}}) + \frac{r^\delta}{\lambda_v^+ \wedge \sqrt{\frac{r^\delta}{d}}} < \sigma_1 = \sigma_2$ and $0 < \underline{a}^\delta, \overline{b}^\delta < 1$ for small δ. By exchanging the roles of u and v in (1.1), we may follow the arguments above, and apply Theorem 1.3 once again to deduce that

$$\underline{c}_1 \geq \sigma_1, \quad \overline{c}_2 \leq \sigma_2 \quad \text{and} \quad \overline{c}_3 \leq -\max\{\tilde{c}_{\rm LLW}, \sigma_3\}. \tag{5.14}$$

Theorem 1.5 follows from combining $\underline{c}_i \leq \overline{c}_i$, (5.12), (5.14) and $\sigma_1 = \sigma_2$. $\square$

6 The case $0 < a < 1 < b$ due to Girardin and Lam

The Hamilton-Jacobi approach, which we have so far applied to study the weak competition case ($0 < a, b < 1$), can also be applied to tackle the case ($0 < a < 1 < b$), which was previously studied by Girardin and the third author [26]. This provides an alternative approach which is more transparent than the involved construction of global super- and sub-solutions for the Cauchy problem, as was done in [26]. By arguing similarly as in Theorem 1.3, one can prove the following result.

Theorem 6.1. *Assume $0 < a < 1 < b$ and $\sigma_1 > \sigma_2$. Let (u, v) be the solution of (1.1) such that the initial data satisfies (H_λ). Then there exist $c_1, c_2 \in (0, \infty)$ such that $c_1 > c_2$ and, for each small $\eta > 0$, the following spreading results hold:*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x>(c_1+\eta)t} (|u(t,x)| + |v(t,x)|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{(c_2+\eta)t<x<(c_1-\eta)t} (|u(t,x)| + |v(t,x) - 1|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{x<(c_2-\eta)t} (|u(t,x) - 1| + |v(t,x)|) = 0. \end{cases} \tag{6.1}$$

Precisely, the spreading speeds c_1 and c_2 can be determined as follows:

$$c_1 = \sigma_1, \quad c_2 = \max\{\hat{c}_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\},$$

where σ_1 is defined in (1.3), $\hat{c}_{\mathrm{LLW}}$ denotes the minimal wave speed of (1.1) connecting $(1, 0)$ with $(0, 1)$ and $\hat{c}_{\mathrm{nlp}}$ is given by

$$\hat{c}_{\mathrm{nlp}} = \begin{cases} \frac{\sigma_1}{2} - \sqrt{a} + \frac{1-a}{\frac{\sigma_1}{2} - \sqrt{a}}, & \text{if } \sigma_1 < 2\lambda_u \text{ and } \sigma_1 \le 2(\sqrt{a} + \sqrt{1-a}), \\ \tilde{\lambda}_{\mathrm{nlp}} + \frac{1-a}{\tilde{\lambda}_{\mathrm{nlp}}}, & \text{if } \sigma_1 \ge 2\lambda_u \text{ and } \tilde{\lambda}_{\mathrm{nlp}} \le \sqrt{1-a}, \\ 2\sqrt{1-a}, & \text{otherwise,} \end{cases} \tag{6.2}$$

with

$$\tilde{\lambda}_{\mathrm{nlp}} = \frac{1}{2}\left[\sigma_1 - \sqrt{(\sigma_1 - 2\lambda_u)^2 + 4a}\right]. \tag{6.3}$$

By Theorem 6.1, the spreading speed c_2 is determined by σ_1 (i.e., c_1) and λ_u. In what follows, we explore the relation of c_2 and σ_1 for fixed λ_u. Define the following auxiliary functions:

$$f(\sigma_1) = \frac{\sigma_1}{2} - \sqrt{a} + \frac{1-a}{\frac{\sigma_1}{2} - \sqrt{a}}, \quad g(\sigma_1) = \tilde{\lambda}_{\mathrm{nlp}} + \frac{1-a}{\tilde{\lambda}_{\mathrm{nlp}}},$$

where $\tilde{\lambda}_{\mathrm{nlp}}$ is given by (6.3). It is easily seen that f is decreasing and bijective in $[2\sqrt{1-a}, 2(\sqrt{1-a} + \sqrt{a})]$, while g is decreasing and bijective in

$$\begin{cases} \left[2\sqrt{1-a}, \lambda_u + \sqrt{1-a} + \frac{1-a}{\lambda_u - \sqrt{1-a}}\right] & \text{as } \lambda_u \ge \sqrt{1-a}, \\ \left(\lambda_u + \sqrt{1-a} + \frac{1-a}{\lambda_u - \sqrt{1-a}}, \infty\right) & \text{as } \lambda_u < \sqrt{1-a}. \end{cases}$$

More precisely, it follows that

$$\begin{cases} f^{-1}(c_2) = c_2 - \sqrt{c_2^2 - 4(1-a)} + 2\sqrt{a}, \\ g^{-1}(c_2) = \lambda_u + \frac{c_2 - \sqrt{c_2^2-4(1-a)}}{2} + \frac{a}{\lambda_u - \frac{c_2-\sqrt{c_2^2-4(1-a)}}{2}}. \end{cases}$$

In view of $\tilde{\lambda}_{\rm nlp} \to \lambda_u$ as $\sigma_1 \to \infty$, $g_\infty := g(\infty) = \lambda_u + \frac{1-a}{\lambda_u}$. For fixed λ_u and varied λ_v^+ (or σ_1), by Theorem 6.1 we can rewrite the spreading speed c_2 as follows.

(a) For $g_\infty \leq \hat{c}_{\rm LLW}$, we have the followings:

(a1) If $\lambda_u \geq (\sqrt{a} + \sqrt{1-a})$, then

$$c_2 = \begin{cases} f(\sigma_1), & \text{for } \max\{2\sqrt{dr}, \sigma_2\} \leq \sigma_1 \leq f^{-1}(\hat{c}_{\rm LLW}), \\ \hat{c}_{\rm LLW}, & \text{for } \sigma_1 > f^{-1}(\hat{c}_{\rm LLW}), \end{cases}$$

independent λ_u;

(a2) If $\sqrt{dr} \leq \lambda_u < \sqrt{a} + \sqrt{1-a}$ and $g^{-1}(\hat{c}_{\rm LLW}) > 2\lambda_u$, then

$$c_2 = \begin{cases} f(\sigma_1), & \text{for } \max\{2\sqrt{dr}, \sigma_2\} \leq \sigma_1 < 2\lambda_u, \\ g(\sigma_1), & \text{for } 2\lambda_u \leq \sigma_1 < g^{-1}(\hat{c}_{\rm LLW}), \\ \hat{c}_{\rm LLW}, & \text{for } \sigma_1 \geq g^{-1}(\hat{c}_{\rm LLW}); \end{cases}$$

(a3) If $\lambda_u < \sqrt{dr}$, then

$$c_2 = \begin{cases} g(\sigma_1), & \text{for } \max\{2\sqrt{dr}, \sigma_2\} \leq \sigma_1 < g^{-1}(\hat{c}_{\rm LLW}), \\ \hat{c}_{\rm LLW}, & \text{for } \sigma_1 \geq g^{-1}(\hat{c}_{\rm LLW}); \end{cases}$$

(b) For $g_\infty > \hat{c}_{\rm LLW}$, we have the followings:

(b1) If $\lambda_u \geq (\sqrt{a} + \sqrt{1-a})$, then

$$c_2 = \begin{cases} f(\sigma_1), & \text{for } \max\{2\sqrt{dr}, \sigma_2\} \leq \sigma_1 \leq f^{-1}(\hat{c}_{\rm LLW}), \\ \hat{c}_{\rm LLW}, & \text{for } \sigma_1 > f^{-1}(\hat{c}_{\rm LLW}), \end{cases}$$

independent λ_u;

(b2) If $\sqrt{dr} \leq \lambda_u < \sqrt{a} + \sqrt{1-a}$, then

$$c_2 = \begin{cases} f(\sigma_1), & \text{for } \max\{2\sqrt{dr}, \sigma_2\} \leq \sigma_1 < 2\lambda_u, \\ g(\sigma_1), & \text{for } \sigma_1 \geq 2\lambda_u; \end{cases}$$

(b3) If $\lambda_u < \sqrt{dr}$, then

$$c_2 = g(\sigma_1) \quad \text{for } \sigma_1 \geq \max\{2\sqrt{dr}, \sigma_2\}.$$

For the case (a) $g_\infty \le \hat{c}_{\mathrm{LLW}}$, the relationship between the spreading speeds σ_1 and c_2 given by (a1)-(a3) is illustrated in Figure 2. Therein we may obtain the exact spreading speeds of (1.1), which are determined entirely by λ_u, $\lambda_v^+ \in (0,\infty)$. Traversing all of λ_u, the set of admissible speeds σ_1 and c_2 agrees with [26, Figure 1.1]. Particularly, a direct consequence of Theorem 6.1 is the following proposition, which improves upon [26, Theorem 1.3] by clarifying the role of exponential decay $(\lambda_u,\, \lambda_v^+)$ of the initial data.

Proposition 6.2. *Let $(\overline{c}, \underline{c}) \in (2\sqrt{dr}, \infty) \times (\hat{c}_{\mathrm{LLW}}, \infty)$ such that $\overline{c} > \underline{c}$.*

(a) *If $\underline{c} < f(\overline{c})$, then the pair of spreading speeds $(\overline{c}, \underline{c})$ is not realized by solutions of (1.1) with initial data satisfying* (H_λ).

(b) *If $\underline{c} = f(\overline{c})$, then there exists a unique $\lambda_v^+ = \frac{1}{2d}(\overline{c} - \sqrt{\overline{c}^2 - 4dr})$ such that for $\lambda_u \in [\overline{c}/2, \infty)$, the pair of spreading speeds $(\overline{c}, \underline{c})$ can be realized by solutions of (1.1) with initial data satisfying* (H_λ).

(c) *If $\underline{c} > f(\overline{c})$, then there exists a unique pair (λ_v^+, λ_u) such that the pair of spreading speeds $(\overline{c}, \underline{c})$ can be realized by solutions of (1.1) with initial data satisfying* (H_λ).

Proof. Assertion (a) follows directly from [26, Theorem 1.2]. For assertion (b), $\underline{c} > \hat{c}_{\mathrm{LLW}} \ge 2\sqrt{1-a}$, so that we have $\overline{c} \le 2(\sqrt{a} + \sqrt{1-a})$. Hence it follows directly from (6.2). It remains to show (c).

First, we define $\lambda_v^+ = \frac{\overline{c} - \sqrt{\overline{c}^2 - 4dr}}{2d} \in (0, \sqrt{\frac{r}{d}})$ such that $\overline{c} = \sigma_1 = d\lambda_v^+ + \frac{r}{\lambda_v^+}$. Since σ_1 is strictly monotone in $(0, \sqrt{\frac{r}{d}})$, the choice of such λ_v^+ is unique. Then we shall determine λ_u such that $c_2 = \underline{c} = \tilde{\lambda}_{\mathrm{nlp}} + \frac{1-a}{\tilde{\lambda}_{\mathrm{nlp}}}$.

Since $\underline{c} > f(\overline{c}) \ge 2\sqrt{1-a}$ and $\underline{c} > \hat{c}_{\mathrm{LLW}}$, to satisfy $c_2 = \underline{c}$, by (6.2) we must have $\lambda_u \in (\frac{\overline{c} - \sqrt{\overline{c}^2 - 4a}}{2}, \frac{\overline{c}}{2})$ and

$$\underline{c} = g(\overline{c}) \quad \text{and} \quad \tilde{\lambda}_{\mathrm{nlp}} = \frac{1}{2}\left[\overline{c} - \sqrt{(\overline{c} - 2\lambda_u)^2 + 4a}\right] < \sqrt{1-a}. \tag{6.4}$$

Hence, it suffices to choose the unique $\lambda_u \in (\frac{\overline{c} - \sqrt{\overline{c}^2 - 4a}}{2}, \frac{\overline{c}}{2})$ such that (6.4) holds.

(i) If $\overline{c} \le 2(\sqrt{1-a} + \sqrt{a})$, then observe that when $\lambda_u \in (\frac{\overline{c} - \sqrt{\overline{c}^2 - 4a}}{2}, \frac{\overline{c}}{2})$, $\tilde{\lambda}_{\mathrm{nlp}} \in (0, \overline{c}/2 - \sqrt{a})$ is increasing in λ_u, so that

$$g(\overline{c}) = \tilde{\lambda}_{\mathrm{nlp}} + \frac{1-a}{\tilde{\lambda}_{\mathrm{nlp}}} \in (f(\overline{c}), \overline{c}),$$

is decreasing in λ_u. Noting that $\underline{c} \in (f(\overline{c}), \overline{c})$, we may select the unique $\lambda_u \in (\frac{\overline{c} - \sqrt{\overline{c}^2 - 4a}}{2}, \frac{\overline{c}}{2})$ such that (6.4) holds;

(ii) If $\overline{c} > 2(\sqrt{1-a}+\sqrt{a})$, then to satisfy $\tilde{\lambda}_{\text{nlp}} < \sqrt{1-a}$ in (6.4), it is necessary that $\lambda_u \in (\frac{\overline{c}-\sqrt{\overline{c}^2-4a}}{2}, \frac{\overline{c}-\sqrt{(\overline{c}-2\sqrt{1-a})^2-4a}}{2})$. In this case,

$$\tilde{\lambda}_{\text{nlp}} \in \left(0, \sqrt{1-a}\right) \quad \text{and thus} \quad g(\overline{c}) = \tilde{\lambda}_{\text{nlp}} + \frac{1-a}{\tilde{\lambda}_{\text{nlp}}} \in \left(2\sqrt{1-a}, \overline{c}\right),$$

are also strictly monotone in λ_u, so that there is the unique λ_u such that (6.4) holds.

The proof is now complete. $\square$

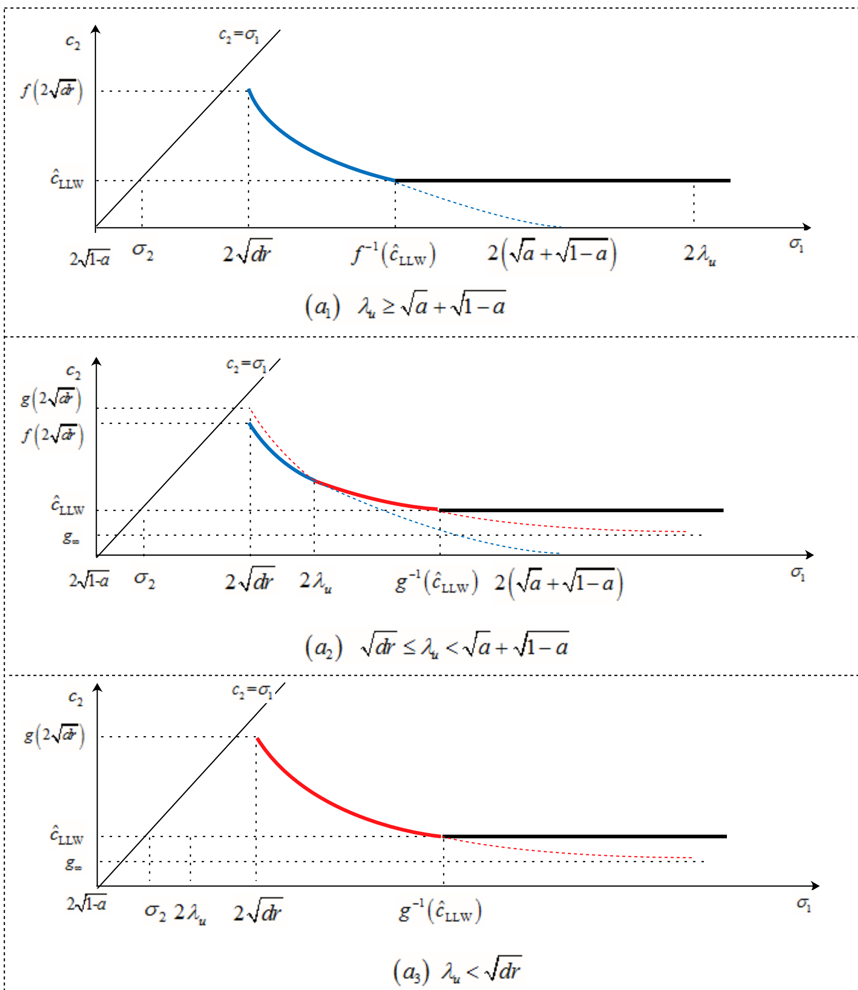

Figure 2: The profile of $c_2(\sigma_1)$ for case (a) $g_\infty \leq \hat{c}_{\text{LLW}}$, which is expressed by the solid line with the blue one representing f and the red one representing g.

7 An extension

In this section, we consider the following competition system with forcing:

$$\begin{cases} \partial_t u - \partial_{xx} u = u(1 - u - av - h(t,x)), & \text{in } (0,\infty)\times\mathbb{R}, \\ \partial_t v - d\partial_{xx} v = rv(1 - bu - v - k(t,x)), & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), & \text{on } \mathbb{R}, \\ v(0,x) = v_0(x), & \text{on } \mathbb{R}, \end{cases} \tag{7.1}$$

where

$$\lim_{t\to\infty} \sup_{x\geq c_0 t} (|h(t,x)| + |k(t,x)|) = 0 \quad \text{ for some } c_0 \in \mathbb{R}. \tag{7.2}$$

We will make an observation in preparation for our forthcoming work on three-species competition systems. Recall the definitions of σ_i $(i = 1, 2, 3)$ from (1.3).

Theorem 7.1. *Let $d, r, b > 0$, $0 < a < 1$ and $\sigma_1 > \sigma_2$. Suppose that $h(x,t), k(x,t)$ are non-negative and satisfy (7.2). Let (u, v) be the solution of (7.1) with the initial data satisfying* (H_λ). *Assume*

$$c_0 < \sigma_2' \qquad \text{where } \sigma_2' = (\lambda_u \wedge \sqrt{1-a}) + \frac{1-a}{\lambda_u \wedge \sqrt{1-a}}.$$

Then,

$$\underline{c}_1 = \overline{c}_1 = \sigma_1, \quad \overline{c}_2 \leq \max\{c_{\mathrm{LLW}}, \hat{c}_{\mathrm{nlp}}\}, \quad \underline{c}_2 \geq \hat{c}_{\mathrm{nlp}},$$

where $\underline{c}_i$, $\overline{c}_i$ $(i = 1, 2)$ are defined in (2.1)*. Furthermore, for each small $\eta > 0$,*

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x>(\sigma_1+\eta)t} (|u(t,x)| + |v(t,x)|) = 0, \\ \lim\limits_{t\to\infty} \sup\limits_{(\overline{c}_2+\eta)t<x<(\sigma_1-\eta)t} (|u(t,x)| + |v(t,x)-1|) = 0, \\ \lim\limits_{t\to\infty} \inf\limits_{(c_0+\eta)t<x<(\underline{c}_2-\eta)t} u(t,x) > 0, \end{cases} \tag{7.3}$$

where σ_1, σ_2 are defined in (1.3) *and c_{LLW}, $\hat{c}_{\mathrm{nlp}}$ are respectively given in Theorem 1.1 and 1.3.*

Proof. The proof can be mimicked after that of Theorem 1.3.
Step 1. The estimates $\overline{c}_1 \leq \sigma_1$ and $\overline{c}_2 \leq \sigma_2$ can be proved by rather similar arguments as in Proposition 2.1, and the details are omitted here.
Step 2. We show that for each small $\eta > 0$,

$$\liminf_{t\to\infty} u(t, (\sigma_2' - \eta)t) > 0 \;\text{ and }\; \liminf_{t\to\infty} v(t, (\sigma_1 - \eta)t) > 0. \tag{7.4}$$

Here, we just show the first one since the proof of the second one is analogous. For the case of $\lambda_u \geq \sqrt{1-a}$, by (7.2) and $c_0 < \sigma_2'$, the system (7.1) is approximately equal to (1.1) in $\{(t,x) : x \geq \frac{c_0+\sigma_2'}{2}t, t \geq T\}$ for sufficient large T, so that we can deduce (7.4) by applying the similar arguments in Steps 4 of the proof of [41, Proposition 2.1]. It remains to consider the case of $\lambda_u < \sqrt{1-a}$.

Fix any $c' \in (\max\{\frac{c_0+\sigma_2'}{2}, 2\sqrt{1-a}\}, \sigma_2')$. It is enough to show that there exist positive constants $\delta, \tilde{\lambda}_1, \tilde{\lambda}_2, T$ such that $\tilde{\lambda}_1 < \tilde{\lambda}_2$ and

$$u(t, x + c't) \geq \frac{\delta}{4} \max\left\{\left[e^{-\tilde{\lambda}_1 x} - e^{-\tilde{\lambda}_2 x}\right], 0\right\} \quad \text{for } t \geq T, x \geq 0. \tag{7.5}$$

This implies $\underline{c}_2 \geq c'$ for each $c' \in (\max\{\frac{c_0+\sigma_2'}{2}, 2\sqrt{1-a}\}, \sigma_2')$, i.e., $\underline{c}_2 \geq \sigma_2'$.

To this end, choose $\delta_1 > 0$ small enough so that

$$\tilde{\lambda}_1 := \frac{1}{2}\left[c' - \sqrt{(c')^2 - 4(1-a-2\delta_1)}\right] > \lambda_u. \tag{7.6}$$

This is possible since $c' < \sigma_2'$ and that $s \mapsto \frac{s-\sqrt{s^2-4(1-a)}}{2}$ is monotone, so that

$$\frac{1}{2}\left[c' - \sqrt{(c')^2 - 4(1-a)}\right] > \frac{1}{2}\left[\sigma_2' - \sqrt{(\sigma_2')^2 - 4(1-a)}\right] = \lambda_u.$$

Next, choose $T > 0$ large so that

$$|h(t,x)| \leq \delta \quad \text{for } \ t \geq T, \ x \geq c't, \tag{7.7}$$

and then choose $\delta \in (0, \delta_1]$ so that

$$u(T, x) \geq \frac{\delta}{4} e^{-\tilde{\lambda}_1(x - c'T)} \quad \text{for } \ x \geq c'T, \tag{7.8}$$

where (7.7) follows from (7.2) by noting $c' > c_0$; and that (7.8) holds due to $u(T,x) \sim e^{-\lambda_u x}$ at ∞ and $\tilde{\lambda}_1 > \lambda_u$ (see, e.g. [52, Corollary 1 of Ch. 1]). By the choice of $\tilde{\lambda}_1 < \tilde{\lambda}_2$, δ, δ_1, T, it follows that

$$\underline{u}(t,x) := \max\left\{\frac{\delta}{4}\left[e^{-\tilde{\lambda}_1(x-c't)} - e^{-\tilde{\lambda}_2(x-c't)}\right], 0\right\}, \tag{7.9}$$

is a sub-solution of the KPP-type equation

$$\partial_t u = \partial_{xx} u + ru(1 - a - h(x,t) - u) \quad \text{ in } \Omega, \tag{7.10}$$

where $\Omega := \{(t,x) : t \geq T, \ x \geq c't\}$.

For $\delta \in (0, \delta_1]$ to be specified later, define

$$\underline{u}(t,x) := \max\left\{\frac{\delta}{4}\left[e^{-\tilde{\lambda}_1(x-c't)} - e^{-\tilde{\lambda}_2(x-c't)}\right], 0\right\}, \tag{7.11}$$

where $\tilde{\lambda}_1$ is given in (7.6) and $\tilde{\lambda}_2 = \frac{1}{2}\left[c' + \sqrt{(c')^2 - 4(1-a-2\delta)}\right]$. We will choose $T > 0$ and $\delta \in (0, \delta_1]$ so that

$$\begin{cases} \partial_t \underline{u} - \partial_{xx}\underline{u} - \underline{u}(1 - a - h(x,t) - \underline{u}) \leq -\underline{u}\,(2\delta - h(x,t) - \underline{u}) \leq 0 & \text{in } \Omega, \\ u(t, c't) \geq 0 = \underline{u}(t, c't) & \text{for } t \geq T, \\ u(T, x) \geq \frac{\delta}{4} e^{-\tilde{\lambda}_1(x-c'T)} \geq \underline{u}(T,x) & \text{for } x \geq c'T, \end{cases}$$

i.e., u and $\underline{u}$ is a pair of super- and sub-solutions of the KPP-type equation

$$\partial_t u = \partial_{xx} u + ru(1 - a - h(x,t) - u) \quad \text{ in } \Omega, \tag{7.12}$$

where $\Omega := \{(t,x) : t \geq T,\ x \geq c' t\}$. Hence, by comparison, (7.5) holds.

To proceed further, as in Section 3, based on the scaling (3.1), we introduce the WKB ansatz w_2^ϵ, which is given by

$$w_2^\epsilon(t,x) = -\epsilon \log u^\epsilon(t,x),$$

satisfying the equation:

$$\begin{cases} \partial_t w_2^\epsilon - \epsilon \partial_{xx} w_2^\epsilon + |\partial_x w_2^\epsilon|^2 + 1 - u^\epsilon - a v^\epsilon - h^\epsilon = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ w_2^\epsilon(0,x) = -\epsilon \log u^\epsilon(0,x), & \text{on } \mathbb{R}. \end{cases}$$

Here $h^\epsilon(t,x) = h(\frac{t}{\epsilon}, \frac{x}{\epsilon})$. By Remark 3.3, we also use the half-relaxed limit method and introduce w_2^* and $w_{2,*}$. By (7.4),

$$\liminf_{\epsilon \to 0} u^\epsilon(t, (\sigma_2' - \eta)t) > 0,$$

and u^ϵ is moreover bounded by 1. We have then, by definitions, that

$$w_2^*(t, (\sigma_2' - \eta)t) = w_{2,*}(t, (\sigma_2' - \eta)t) = 0. \tag{7.13}$$

Step 3. We prove $\underline{c}_1 \geq \sigma_1$.

This follows from (7.4) and definition of $\underline{c}_1$.

Step 4. We prove $\underline{c}_2 \geq \hat{c}_{\text{nlp}}$.

By Step 1 and $h \geq 0$, we have

$$0 \leq \limsup_{\substack{(t',x') \to (t,x) \\ \epsilon \to 0}} v^\epsilon(t',x') \leq \chi_{\{x \leq \sigma_1 t\}}.$$

In view of $\sigma_2' > c_0$, we choose $0 < \eta \ll 1$ such that $\sigma_2' - \eta > c_0$. We then use (7.2) to derive that

$$\lim_{\epsilon \to 0} \sup_{x \geq (\sigma_2' - \eta)t} h^\epsilon(t,x) = 0.$$

Based on (7.13), similar to Lemma 3.4, we can deduce that w_2^* is a viscosity sub-solution of

$$\begin{cases} \min\{\partial_t w + |\partial_x w|^2 + 1 - a\chi_{\{x \leq \sigma_1 t\}}, w\} = 0, & \text{for } x > (\sigma_2' - \eta)t, \\ w(0,x) = \lambda_u x, & \text{for } x \geq 0, \\ w(t, (\sigma_2' - \eta)t) = 0, & \text{for } t \geq 0. \end{cases}$$

We then apply the same arguments developed in Lemmas 3.8 by constructing the same super-solutions, to deduce that $\underline{c}_2 \geq \hat{c}_{\text{nlp}}$.

Step 5. We show $\overline{c}_2 \leq \max\{c_{\text{LLW}}, \hat{c}_{\text{nlp}}\}$ and (7.3).

By (7.2) again, similar to Corollary 3.6, we can get

$$\liminf_{\substack{(t',x')\to(t,x)\\ \epsilon\to 0}} v^{\epsilon}(t',x') \geq \chi_{\{\sigma_2 t<x<\sigma_1 t\}},$$

so that we may use (7.13) to deduce that $w_{2,*}$ is a viscosity super-solution of

$$\begin{cases} \min\{\partial_t w + |\partial_x w|^2 + 1 - a\chi_{\{\sigma_2 t<x<\sigma_1 t\}}, w\} = 0 & \text{for } x > (\sigma_2' - \eta)t, \\ w(0,x) = \lambda_u x, & \text{for } x \geq 0, \\ w(t,(\sigma_2' - \eta)t) = 0, & \text{for } t \geq 0, \end{cases}$$

as in Lemma 4.1. Then we can get $\overline{c}_2 \leq \max\{c_{\text{LLW}}, \hat{c}_{\text{nlp}}\}$ by the same arguments developed in Proposition 4.2. We finally deduce (7.3) by similar arguments as in the proof of Theorem 1.3, which completes the proof. □

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Appendix A Comparison principle

This section is devoted to the proof of a comparison lemma for Hamilton-Jacobi equation for discontinuous super and sub-solutions and for piecewise Lipschitz continuous Hamiltonian. Our proof is inspired by the arguments developed by Ishii [33] and Tourin [51] (see also [2, 11, 25]). Ishii used a crucial observation of [2] to prove the comparison principle for discontinuous super- and sub-soultion of Hamilton-Jacobi equations with nonconvex but continuous Hamiltonian, whereas Tourin gave the uniqueness of continuous solution of Hamilton-Jacobi equations with piecewise Lipschitz continuous Hamiltonian. The uniqueness of viscosity solution for nonlinear first-order partial differential equations was first introduced by Crandall and Lions in [12], then Crandall, Ishii and Lions [13] gave a simpler proof. Ishii [32] study the discontinuous Hamiltonian with time measure and Tourin and Ostrov [46] studied the piecewise Lipschitz continuous, convex Hamiltonian, based on the dynamic programming principle.

Let Ω be a smooth domain in $(0,T]\times\mathbb{R}^N$, which is allowed to be unbounded or even equal to $(0,T]\times\mathbb{R}^N$. We assume without loss that $T = \sup\{t > 0 : (t,x)\in\Omega\}$, and define the parabolic boundary of Ω as

$$\partial_p\Omega = \{(t,x)\in\partial\Omega : t\in[0,T)\}.$$

Consider the following Hamilton-Jacobi equation:

$$\min\{\partial_t w + H(t,x,\partial_x w), w - Lt\} = 0 \quad \text{in } \Omega. \tag{A.1}$$

Let H^* and H_* be, respectively, the upper semicontinuous (usc) and lower semicontinuous (lsc) envelope of H with respect to its first two variables. Precisely,

$$H^*(t,x,p) = \limsup_{(t',x')\to(t,x)} H(t',x',p) \quad \text{and} \quad H_*(t,x,p) = \liminf_{(t',x')\to(t,x)} H(t',x',p).$$

We say that a lower semicontinuous (lsc) function w is a viscosity super-solution of (A.1) if $w - Lt \geq 0$ in Ω, and for all test functions $\varphi \in C^\infty(\Omega)$, if $(t_0,x_0) \in \Omega$ is a strict local minimum point of $w - \varphi$, then

$$\partial_t \varphi(t_0,x_0) + H^*(t_0,x_0,\partial_x\varphi(t_0,x_0)) \geq 0$$

holds; A upper semicontinuous (usc) function w is a viscosity sub-solution of (A.1) if for all test functions $\varphi \in C^\infty(\Omega)$, if $(t_0,x_0) \in \Omega$ is a strict local maximum point of $w - \varphi$ such that $w(t_0,x_0) - Lt_0 > 0$, then

$$\partial_t \varphi(t_0,x_0) + H_*(t_0,x_0,\partial_x\varphi(t_0,x_0)) \leq 0$$

holds. Finally, w is a viscosity solution of (A.1) if and only if w is simultaneously a viscosity super-solution and a viscosity sub-solution of (A.1).

We impose additional assumptions on the domain Ω and the Hamiltonian $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$. Namely, there exists a closed set $\Gamma \subset [0,T] \times \mathbb{R}^N$ and, for each $R > 0$, a continuous function $\omega_R : [0,\infty) \to [0,\infty)$ such that $\omega_R(0) = 0$ and $\omega_R(r) > 0$ for $r > 0$, such that the following holds:

(A1) $H \in C((\Omega \setminus \Gamma) \times \mathbb{R}^N)$;

(A2) For each $(t_0,x_0) \in (\Omega \setminus \Gamma) \cap ((0,T) \times B_R(0))$, there exist a constant $\delta_0 = \delta_0(R) > 0$ such that

$$H(t,y,p) - H(t,x,p) \leq \omega_R\left(|x-y|(1+|p|)\right)$$

for t,x,y,p such that $\|(t,x)-(t_0,x_0)\| + \|(t,y)-(t_0,x_0)\| < \delta_0$ and $p \in \mathbb{R}^N$;

(A3) For each $(t_0,x_0) \in \Omega \cap \Gamma \cap ((0,T) \times B_R(0))$, there exist a constant $\delta_0 = \delta_0(R) > 0$ and a unit vector $(h_0,k_0) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$H^*(s,y,p) - H_*(t,x,p) \leq \omega_R\left((|t-s| + |x-y|)(1+|p|)\right)$$

for all $p \in \mathbb{R}^N$ and s,t,y,x satisfying

$$\begin{cases} \|(t,x)-(t_0,x_0)\| + \|(s,y)-(t_0,x_0)\| < \delta_0, \\ \left\| \frac{(t-s,x-y)}{\|(t-s,x-y)\|} - (h_0,k_0) \right\| < \delta_0; \end{cases}$$

(A4) There exists some $M \geq 0$ such that for each $\lambda \in [0,1)$ and $x_0 \in \mathbb{R}^N$, there exists constants $\bar{\epsilon}(\lambda, x_0) > 0$ and $\bar{C}(\lambda, x_0) > 0$ such that for all $p \in \mathbb{R}^N$, $(t,x) \in \Omega$, if $\epsilon \in [0, \bar{\epsilon}(\lambda, x_0)]$, then

$$H\left(t, x, \lambda p - \frac{\epsilon(x-x_0)}{|x-x_0|^2+1}\right) - M \leq \lambda\left(H(t,x,p) - M\right) + \epsilon \bar{C}(\lambda, x_0).$$

Theorem A.1. *Suppose that H satisfies the hypotheses* (A1)-(A4). *Let $\overline{w}$ and $\underline{w}$ be a pair of super- and sub-solutions of* (A.1) *such that $\overline{w} \geq \underline{w}$ on $\partial_p \Omega$, then*

$$\overline{w} \geq \underline{w} \quad \text{in } \Omega.$$

Remark A.2. *Let*

$$H(t,x,p) = \boldsymbol{H}(p) + R(x/t),$$

where $\boldsymbol{H}$ is convex and coercive in p, and $s \mapsto R(s)$ has bounded variation and satisfies $|R(s)| \leq M$ for some $M \geq 0$. Then it is easy to verify that the hypotheses (A1)-(A4) *hold. In particular, it applies for all our purposes in this paper.*

Our condition (A3) *is a quantitative version of the "local monotonicity condition" that was introduced in [11]. See [11,33] for more examples of Hamiltonians verifying the hypotheses* (A1)-(A4).

Proof. Assume to the contrary that

$$\sup_{\Omega}(\underline{w} - \overline{w}) > 0. \tag{A.2}$$

Step 1. We may assume, without loss of generality, that $M = 0$ in the hypothesis (A4). Indeed, if we make the change of variables $\underline{w}'(t,x) = \underline{w}(t,x) + Mt$ and $\overline{w}'(t,x) = \overline{w}(t,x) + Mt$, then $\underline{w}', \overline{w}'$ are, respectively, a sub-solution and a super-solution of (A.1) with L replaced by $L' = L + M$, and $H(t,x,p)$ replaced by $H'(t,x,p) = H(t,x,p) - M$. This function H' satisfies the hypotheses (A1)-(A4) with $M = 0$. Henceforth in the proof we assume that the hypothesis (A4) holds with $M = 0$.
Step 2. It suffices to show that $\underline{w} \leq \overline{w}$ under the additional assumption that $\underline{w} \leq K$ for some $K > 0$.

Indeed, if $\underline{w}$ is unbounded in Ω, then fix a constant $K > 0$ and take a sequence $\{g_j\}$ of smooth functions satisfying $g_j(r) \nearrow \min\{r, K\}$ and

$$0 \leq g_j'(r) \leq 1, \quad g_j'(r) r \leq r, \quad g_j(r) \leq \min\{r, K\} \qquad \text{for all } r \in \mathbb{R}.$$

Then $\hat{w} := g_j(\underline{w})$ is a viscosity sub-solution of (A.1), since in the region $\{(t,x) :\ \hat{w} - Lt > 0\} \subset \{(t,x) :\ \underline{w} - Lt > 0\}$, we may use the hypothesis (A4) to yield

$$\begin{aligned}\partial_t \hat{w} + H^*(t,x,D\hat{w}) &= g_k'(\underline{w})\partial_t \underline{w} + H^*(t,x,g_k'(\underline{w})D\underline{w}) \\ &\leq g_k'(\underline{w})\left[\partial_t \underline{w} + H^*(t,x,D\underline{w})\right] \leq 0.\end{aligned}$$

By the stability of property of viscosity super and sub-solutions [1, Theorem 6.2], we may let $j \to \infty$ to conclude that $\min\{\underline{w}, K\}$ is a viscosity sub-solution of (A.1) for each $K > 0$. It now remains to prove Theorem A.1 for all bounded above viscosity sub-solutions, since then

$$\min\{\underline{w}, K\} \leq \overline{w} \quad \text{for all } K > 0 \quad \Rightarrow \quad \underline{w} \leq \overline{w}.$$

For $\lambda, \delta \in (0, 1)$, denote

$$W(t,x) = \lambda^2 \underline{w}(t,x) - \overline{w}(t,x) - \delta(\psi(x) + Ct + \frac{1}{T-t}) - \lambda\delta Ct,$$

where $\psi(x) = \frac{1}{2}\log(|x|^2 + 1)$ and $C = \bar{C}(\lambda, 0)$ as in the hypothesis (A4).

Step 3. We choose $\lambda \nearrow 1$, $\delta \in (0, \bar{\epsilon}(\lambda, 0)]$, $R > 0$ and $(t_0, x_0) \in \Omega_R := \Omega \cap [(0,T) \times B_R(0)]$ such that

$$W(t_0, x_0) = \max_{\Omega_R} W(t,x) = \max_{\Omega} W(t,x) > 0. \tag{A.3}$$

From (A.2) and Step 3, we may fix $\lambda \nearrow 1$ and $\delta \searrow 0$ such that

$$\sup_{\Omega} W(t,x) > 0, \quad \text{and} \quad W(t,x) \leq -\frac{\delta}{T-t} \quad \text{on } \partial_p\Omega.$$

Since $\psi(R) \to \infty$ as $R \to \infty$ and $\underline{w} - \overline{w} \leq K$, we arrive at

$$\sup_{(t,x)\in\Omega:\ |x|=R} W(t,x) \to -\infty \quad \text{as } R \to \infty,$$

whence we may fix $R \gg 1$ so that $\max_{\Omega_R} W(t,x) = \max_{\Omega} W(t,x) > 0$ holds. It remains to observe that the maximum (t_0, x_0) in $\overline{\Omega_R}$ is attained in the interior, since $W(t,x) < 0$ when $t = T$ or when $(t,x) \in \partial_p\Omega$.

Step 4. With x_0 as being given in Step 3, fix $\epsilon > 0$ small enough so that

$$\epsilon\bar{C}(\lambda, x_0) \leq \bar{C}(\lambda, 0) \quad \text{and} \quad \delta\epsilon \leq \bar{\epsilon}(\lambda, x_0), \tag{A.4}$$

and define

$$\tilde{W}(t,x) := W(t,x) - \delta\lambda\epsilon\psi(x - x_0) - \frac{1}{2}|t - t_0|^2, \tag{A.5}$$

where $\psi(x) = \frac{1}{2}\log(|x|^2 + 1)$ and $C = \bar{C}(\lambda, 0)$ is as before. Then, (t_0, x_0) is a strict global maximum of $\tilde{W}(t,x)$. Define also

$$\begin{aligned}\Psi_{\alpha,\beta}(t,x,s,y) =& \lambda^2\underline{w}(t,x) - \overline{w}(s,y) - \delta(\psi(x) + Ct + \frac{1}{T-t}) - \lambda\delta(\epsilon\psi(x - x_0) + Ct) \\ &- \frac{\alpha}{2}|x-y|^2 - \frac{\beta}{2}|t-s|^2 - \frac{1}{2}|t-t_0|^2.\end{aligned}$$

Step 5. We claim that there exists $\underline{\alpha} > 0$ such that if $\min\{\alpha, \beta\} \geq \underline{\alpha}$, then

(i) $\Psi_{\alpha,\beta}$ has a local maximum point (t_1, x_1, s_1, y_1) in $\Omega_R \times \Omega_R$;

(ii) $\Psi_{\alpha,\beta}(t_1, x_1, s_1, y_1) \geq \tilde{W}(t_0, x_0) = W(t_0, x_0) > 0$;

(iii) $\beta|t_1-s_1|^2+\alpha|x_1-y_1|^2\to 0,$ as $\min\{\alpha,\beta\}\to\infty;$

(iv) $(t_1,x_1)\to(t_0,x_0)$ and $(s_1,y_1)\to(t_0,x_0)$ as $\min\{\alpha,\beta\}\to\infty,$

where $\Omega_R=\Omega\cap[(0,T)\times B_R(0)]$. Since $\overline{w}\geq 0$ and $\underline{w}\leq K$ by Step 2, we see that $\sup_{\Omega_R\times\Omega_R}\Psi_{\alpha,\beta}\leq K$ independently of α and β, and has a maximum point $(t_1,x_1,s_1,y_1)\in\overline{\Omega}_R\times\overline{\Omega}_R$. Now, by (A.3),

$$\Psi_{\alpha,\beta}(t_1,x_1,s_1,y_1)\geq\max_{\Omega_R}\Psi_{\alpha,\beta}(t,x,t,x)=\tilde{W}(t_0,x_0)=W(t_0,x_0).$$

This proves assertion (ii).

Furthermore, the boundedness also yields $\beta|t_1-s_1|^2+\alpha|x_1-y_1|^2=O(1)$. We claim that $(t_1,x_1)\to(t_0,x_0)$ and $(s_1,y_1)\to(t_0,x_0)$. Indeed, we may pass to a subsequence to get $(\hat{t},\hat{x})$ such that $(t_1,x_1)\to(\hat{t},\hat{x})$ and $(s_1,y_1)\to(\hat{t},\hat{x})$ as $\min\{\alpha,\beta\}\to\infty$. Now, by (ii) we can write

$$\frac{\alpha}{2}|x_1-y_1|^2+\frac{\beta}{2}|t_1-s_1|^2\leq-\tilde{W}(t_0,x_0)+(\tilde{W}(t_1,x_1)+\overline{w}(t_1,x_1))-\overline{w}(s_1,y_1).$$

Letting $\min\{\alpha,\beta\}\to\infty$, then $(t_1,x_1,s_1,y_1)\to(\hat{t},\hat{x},\hat{t},\hat{x})$. Using the fact that $\tilde{W}(t,x)+\overline{w}(t,x)$ (which is essentially $\lambda^2\underline{w}(t,x)$ up to addition of continuous functions) and $-\overline{w}(s,y)$ are both upper semi-continuous in Ω, we may take limsup as $\min\{\alpha,\beta\}\to\infty$ and deduce that

$$0\leq\limsup\left[\frac{\alpha}{2}|x_1-y_1|^2+\frac{\beta}{2}|t_1-s_1|^2\right]\leq-\tilde{W}(t_0,x_0)+\tilde{W}(\hat{t},\hat{x})\leq 0.$$

Since (t_0,x_0) is a strict maximum point of $\tilde{W}$, we must have $(\hat{t},\hat{x})=(t_0,x_0)$. This proves assertions (iii) and (iv).

Finally, $(t_1,x_1,s_1,y_1)\to(t_0,x_0,t_0,x_0)$ and hence must be an interior point of $\Omega_R\times\Omega_R$ when $\min\{\alpha,\beta\}$ is sufficiently large. This proves (i).

Step 6. We show the following inequality:

$$\frac{\delta}{T^2}\leq H^*(s_1,y_1,\alpha(x_1-y_1))-H_*(t_1,x_1,\alpha(x_1-y_1))+|t_1-t_0|. \tag{A.6}$$

Observe that (t_1,x_1) is an interior maximum point of the function $\underline{w}(t,x)-\varphi(t,x)$, where

$$\begin{aligned}\varphi(t,x)=&\frac{1}{\lambda^2}[\overline{w}(s_1,y_1)+\delta(\psi(x)+Ct+\frac{1}{T-t})+\lambda\delta(\epsilon\psi(x-x_0)+Ct)\\&+\frac{\alpha}{2}|x-y_1|^2+\frac{\beta}{2}|t-s_1|^2+\frac{1}{2}|t-t_0|^2].\end{aligned}$$

Also $\underline{w}(t_1,x_1)>0$, which is a consequence of $\overline{w}(s_1,y_1)\geq 0$ and $\Psi_\alpha(t_1,x_1,s_1,y_1)>0$. By definition of $\underline{w}$ being a viscosity sub-solution of (A.1), we have

$$\begin{aligned}&\frac{1}{\lambda^2}\left[\delta(C+\frac{1}{(T-t_1)^2}+\lambda C)+\beta(t_1-s_1)+(t_1-t_0)\right]\\&+H_*\left(t_1,x_1,\frac{1}{\lambda^2}\left(\delta D_x\psi(x_1)+\lambda\delta\epsilon D_x\psi(x_1-x_0)+\alpha(x_1-y_1)\right)\right)\leq 0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\delta(C+\frac{1}{T^2}+\lambda C)&+\beta(t_1-s_1)+(t_1-t_0)\\&+\lambda^2 H_*\left(t_1,x_1,\frac{1}{\lambda}\left(\delta\epsilon D_x\psi(x_1-x_0)+\hat{q}\right)\right)\le 0,\end{aligned} \tag{A.7}$$

where $\hat{q}=\frac{1}{\lambda}\left(\delta D_x\psi(x_1)+\alpha(x_1-y_1)\right)$. In the view of $D\psi(x_1-x_0)=\frac{x_1-x_0}{|x_1-x_0|^2+1}$, we may apply the hypothesis (A4) to get

$$\begin{aligned}&-\delta(C+\frac{1}{T^2}+\lambda C)-\beta(t_1-s_1)-(t_1-t_0)\\&\ge\lambda\left[H_*\left(t_1,x_1,\hat{q}\right)-\delta\epsilon\bar{C}(\lambda,x_0)\right]\\&\ge\lambda H_*\left(t_1,x_1,\frac{1}{\lambda}\left(\delta D_x\psi(x_1)+\alpha(x_1-y_1)\right)\right)-\lambda\delta C,\end{aligned}$$

where we used $\epsilon\bar{C}(\lambda,x_0)\le\bar{C}(\lambda,0)=C$ (due to (A.4)) in the last inequality. Applying the hypothesis (A4) once more, we have

$$\begin{aligned}&-\delta(C+\frac{1}{T^2}+\lambda C)-\beta(t_1-s_1)-(t_1-t_0)\\&\ge\left[H_*\left(t_1,x_1,\alpha(x_1-y_1)\right)-\delta C\right]-\lambda\delta C\\&\ge H_*\left(t_1,x_1,\alpha(x_1-y_1)\right)-\delta C-\lambda\delta C,\end{aligned}$$

and hence

$$\frac{\delta}{T^2}+\beta(t_1-s_1)+(t_1-t_0)+H_*(t_1,x_1,\alpha(x_1-y_1))\le 0. \tag{A.8}$$

In the same way, (s_1,y_1) is a interior minimum point of the function $\overline{w}(s,y)-\psi(s,y)$ with

$$\begin{aligned}\psi(s,y)=&\lambda^2\underline{w}(t_1,x_1)-\delta(\psi(x_1)+Ct_1+\frac{1}{T-t})-\lambda\delta(\psi(x_1-x_0)+Ct_1)\\&-\frac{\alpha}{2}|x_1-y|^2-\frac{\beta}{2}|t_1-s|^2)-\frac{1}{2}|t_1-t_0|^2,\end{aligned}$$

whence

$$\beta(t_1-s_1)+H^*(s_1,y_1,\alpha(x_1-y_1))\ge 0. \tag{A.9}$$

Subtracting (A.8) from (A.9), we obtain (A.6) as claimed.

By Step 5 (iv), we have $(t_1,x_1)\to(t_0,x_0)$ and $(s_1,y_1)\to(t_0,x_0)$ as $\min\{\alpha,\beta\}\to\infty$. On the one hand, if $(t_0,x_0)\notin\Gamma$, then there exists $\alpha_1>0$ such that (t_1,x_1) and (s_1,y_1) enter the $(\delta_0/2)$-neighborhood of (t_0,x_0) whenever $\min\{\alpha,\beta\}\ge\alpha_1$. Now, fix α and let $\beta\to\infty$, then after passing to a sequence, we have

$$t_1,s_1\to\bar{t},\quad x_1\to\bar{x},\quad y_1\to\bar{y}.$$

Furthermore, by Step 5, we have

$$\bar{t} \to t_0, \quad \bar{x} \to x_0, \quad \bar{y} \to x_0, \quad \text{and} \quad \alpha|\bar{x}-\bar{y}|^2 \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{A.10}$$

Hence, we deduce from (A.6) and the hypothesis (A2) that

$$\begin{aligned}\frac{\delta}{T^2} &\le H^*(\bar{t},\bar{y},\alpha(\bar{x}-\bar{y})) - H_*(\bar{t},\bar{x},\alpha(\bar{x}-\bar{y})) + |\bar{t}-t_0| \\ &\le \omega_R\left(\alpha|\bar{x}-\bar{y}|^2 + \frac{1}{\alpha}\right) + o(1),\end{aligned}$$

from which we derive a contradiction for large enough α. This proves Theorem A.1 in case $(t_0, x_0) \in \Omega \setminus \Gamma$.

On the other hand, $(t_0, x_0) \in \Gamma$. Let δ_0 and the unit vector $(h_0, k_0) \in \mathbb{R} \times \mathbb{R}^N$ be given by the hypothesis (A3). Define

$$\begin{aligned}\tilde{\Psi}_{\alpha,\beta}(t,x,s,y) =& \lambda^2 \underline{w}(t,x) - \overline{w}(s-\alpha^{-1/2}h_0, y-\alpha^{-1/2}k_0) \\ &- \delta(\psi(x) + Ct + \frac{1}{T-t}) - \lambda\delta(\epsilon\psi(x-x_0) + Ct) \\ &- \frac{\alpha}{2}|x-y|^2 - \frac{\beta}{2}|t-s|^2 - \frac{1}{2}|t-t_0|^2.\end{aligned} \tag{A.11}$$

By repeating Steps 5 and 6, we can again obtain

$$\frac{\delta}{T^2} \le H^*(\bar{t}-\alpha^{-1/2}h_0, \bar{y}-\alpha^{-1/2}k_0, \alpha(\bar{x}-\bar{y})) - H_*(\bar{t},\bar{x},\alpha(\bar{x}-\bar{y})) + |\bar{t}-t_0| \tag{A.12}$$

for some $\bar{t}, \bar{x}, \bar{y}$ such that for all α large, $(\bar{t},\bar{x}), (\bar{t},\bar{y})$ enter the $(\delta_0/2)$-neighborhood of (t_0, x_0) and (A.10) holds. Furthermore, by verifying that

$$\begin{aligned}\frac{((\bar{t}-\alpha^{-1/2}h_0)-\bar{t}, \bar{y}-\alpha^{-1/2}k_0-\bar{x})}{\|((\bar{t}-\alpha^{-1/2}h_0)-\bar{t}, \bar{y}-\alpha^{-1/2}k_0-\bar{x})\|} &= -\frac{(\alpha^{-1/2}h_0, \bar{x}-\bar{y}+\alpha^{-1/2}k_0)}{\|(\alpha^{-1/2}h_0, \bar{x}-\bar{y}+\alpha^{-1/2}k_0)\|} \\ &= -\frac{(h_0, \sqrt{\alpha}(\bar{x}-\bar{y})+k_0)}{\|(h_0, \sqrt{\alpha}(\bar{x}-\bar{y})+k_0)\|} \\ &\to (h_0, k_0) \quad \text{as } \alpha \to \infty,\end{aligned}$$

we may apply hypothesis (A3) to inequality (A.12) to get

$$\begin{aligned}\frac{\delta}{T^2} &\le \omega_R\left(\left[\alpha^{-1/2}(h_0 + |k_0|) + |\bar{x}-\bar{y}|\right](1+\alpha|\bar{x}-\bar{y}|)\right) + |\bar{t}-t_0| \\ &\le \omega_R\left(|\bar{x}-\bar{y}| + \alpha|\bar{x}-\bar{y}|^2 + C_0(\sqrt{\alpha}|\bar{x}-\bar{y}| + \frac{1}{\sqrt{\alpha}})\right) + |\bar{t}-t_0|.\end{aligned}$$

Letting $\alpha \to \infty$, we can similarly obtain a contradiction. □

A direct consequence of Theorem A.1 is the following uniqueness result.

Corollary A.3. *Assume that the Hamiltonian H satisfies the hypotheses* (A1)-(A4). *Suppose that* (A.1) *has a continuous viscosity solution w. Then, w is the unique viscosity solution of* (A.1) *in the class of all (continuous) viscosity solution.*

Appendix B Two useful lemmas from [41]

In this section, we present two lemmas from [41], which are used in this paper. The first result is used to prove Propositions 2.1 and Corollary 3.6, Lemma 4.3 and Proof of Theorem 1.3.

Lemma B.1 ([41, Lemma 2.3]). *Let $-\infty \leq \underline{c} < \overline{c} \leq \infty$, and let (u, v) be a solution of (1.1) in $\{(t, x) : \underline{c}t \leq x \leq \overline{c}t\}$.*

(a) *If* $\liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} v(t, x) > 0$ *for each* $0 < \eta < (\overline{c} - \underline{c})/2$, *then*

$$\limsup_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} u(t, x) \leq k_1, \quad \liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} v(t, x) \geq k_2,$$

for each $0 < \eta < (\overline{c} - \underline{c})/2$;

(b) *If* $\lim_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} v(t, x) = 0$ *and* $\liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} u(t, x) > 0$ *for each* $0 < \eta < (\overline{c} - \underline{c})/2$, *then*

$$\lim_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} |u(t, x) - 1| = 0, \quad \text{for each } 0 < \eta < (\overline{c} - \underline{c})/2;$$

(c) *If* $\liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} u(t, x) > 0$ *for each* $0 < \eta < (\overline{c} - \underline{c})/2$, *then*

$$\liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} u(t, x) \geq k_1, \quad \limsup_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} v(t, x) \leq k_2$$

for each $0 < \eta < (\overline{c} - \underline{c})/2$;

(d) *If* $\lim_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} u(t, x) = 0$ *and* $\liminf_{t\to\infty} \inf_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} v(t, x) > 0$ *for each* $0 < \eta < (\overline{c} - \underline{c})/2$, *then*

$$\lim_{t\to\infty} \sup_{(\underline{c}+\eta)t<x<(\overline{c}-\eta)t} |v(t, x) - 1| = 0, \quad \text{for each } 0 < \eta < (\overline{c} - \underline{c})/2.$$

Proof. We only prove (c) and the other assertions follow from the similar arguments. Suppose (c) is false, then there exists (t_n, x_n) such that

$$c_n := \frac{x_n}{t_n} \to c \in (\underline{c}, \overline{c}) \text{ and } \lim_{n\to\infty} u(t_n, x_n) < k_1 \text{ or } \lim_{n\to\infty} v(t_n, x_n) > k_2.$$

Define $(u_n, v_n)(t, x) := (u, v)(t_n+t, x_n+x)$. We pass to the limit so that (u_n, v_n) converges in $C_{loc}(\mathbb{R} \times \mathbb{R})$ to an entire solution $(\hat{u}, \hat{v})$ of (1.1). And there exists $\delta > 0$ such that $(\hat{u}, \hat{v})(t, x) \succeq (\delta, 1)$ for $(t, x) \in \mathbb{R}^2$. Let $(\underline{U}, \overline{V})$ be the solutions of the Lotka-Volterra system of ODEs

$$U_t = U(1 - U - aV), \quad V_t = rV(1 - bU - V),$$

with initial data $(U(0), V(0)) = (\delta, 1)$, so that $(\underline{U}, \overline{V})(\infty) = (k_1, k_2)$. By comparison in the time interval $[-T, 0]$, we reveal that for each $T > 0$,

$$(\hat{u}, \hat{v})(t, x) \succeq (\underline{U}, \overline{V})(t + T) \text{ for } (t, x) \in [-T, 0] \times \mathbb{R},$$

so that we in particular have, for every $T > 0$,

$$(\hat{u}, \hat{v})(0, 0) \succeq (\underline{U}, \overline{V})(T).$$

Letting $T \to \infty$, we obtain $(\hat{u}, \hat{v})(0, 0) \succeq (k_1, k_2)$. In particular, we deduce that

$$\lim_{n\to\infty} (u, v)(t_n, x_n) = \lim_{n\to\infty} (u_n, v_n)(0, 0) = (\hat{u}, \hat{v})(0, 0) \succeq (k_1, k_2).$$

This is a contradiction and proves (c). $\square$

The following result is applied to prove Proposition 4.2 and Proposition 4.5.

Lemma B.2 ([41, Lemma 2.4]). *Let $\hat{c} > 0$, $t_0 > 0$, and $(\tilde{u}, \tilde{v})$ be a solution of*

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx} \tilde{u} = \tilde{u}(1 - \tilde{u} - a\tilde{v}), & 0 \le x \le \hat{c}t, t > t_0, \\ \partial_t \tilde{v} - d\partial_{xx} \tilde{v} = r\tilde{v}(1 - b\tilde{u} - \tilde{v}), & 0 \le x \le \hat{c}t, t > t_0, \\ \tilde{u}(t_0, x) = \tilde{u}_0(x), \tilde{v}(t_0, x) = \tilde{v}_0(x), & 0 \le x \le \hat{c}t_0. \end{cases} \tag{B.1}$$

(a) *If $\hat{c} > 2$ and there exists $\hat{\mu} > 0$ such that*

  (i) $\lim_{t\to\infty} (\tilde{u}, \tilde{v})(t, 0) = (k_1, k_2)$ *and* $\lim_{t\to\infty} (\tilde{u}, \tilde{v})(t, \hat{c}t) = (0, 1)$,

  (ii) $\lim_{t\to\infty} e^{\mu t} \tilde{u}(t, \hat{c}t) = 0$ *for each $\mu \in [0, \hat{\mu})$,*

then

$$\lim_{t\to\infty} \sup_{ct < x \le \hat{c}t} \tilde{u}(t, x) = 0 \quad \text{for each } c > c_{\hat{c},\hat{\mu}},$$

where

$$c_{\hat{c},\hat{\mu}} = \begin{cases} c_{\mathrm{LLW}}, & \text{if } \hat{\mu} \ge \lambda_{\mathrm{LLW}}(\hat{c} - c_{\mathrm{LLW}}), \\ \hat{c} - \frac{2\hat{\mu}}{\hat{c} - \sqrt{\hat{c}^2 - 4(\hat{\mu} + 1 - a)}}, & \text{if } 0 < \hat{\mu} < \lambda_{\mathrm{LLW}}(\hat{c} - c_{\mathrm{LLW}}); \end{cases}$$

(b) *If $\hat{c} > 2\sqrt{dr}$ and there exists $\hat{\mu} > 0$ such that*

  (i) $\lim_{t\to\infty} (\tilde{u}, \tilde{v})(t, 0) = (k_1, k_2)$, *and* $\lim_{t\to\infty} (\tilde{u}, \tilde{v})(t, \hat{c}t) = (1, 0)$,

  (ii) $\lim_{t\to\infty} e^{\mu t} \tilde{v}(t, \hat{c}t) = 0$ *for each $\mu \in [0, \hat{\mu})$,*

then

$$\lim_{t\to\infty} \sup_{ct < x \le \hat{c}t} \tilde{v}(t, x) = 0 \quad \text{for each } c > \tilde{c}_{\hat{c},\hat{\mu}},$$

where

$$\tilde{c}_{\hat{c},\hat{\mu}} = \begin{cases} \tilde{c}_{\mathrm{LLW}}, & \text{if } \hat{\mu} \ge \tilde{\lambda}_{\mathrm{LLW}}(\hat{c} - \tilde{c}_{\mathrm{LLW}}), \\ \hat{c} - \frac{2d\hat{\mu}}{\hat{c} - \sqrt{\hat{c}^2 - 4d[\hat{\mu} + r(1-b)]}}, & \text{if } 0 < \hat{\mu} < \tilde{\lambda}_{\mathrm{LLW}}(\hat{c} - \tilde{c}_{\mathrm{LLW}}). \end{cases}$$

Here $c_{\mathrm{LLW}}, \tilde{c}_{\mathrm{LLW}}$ *are given in Theorem 1.1 and Remark 1.2, and*

$$\lambda_{\mathrm{LLW}} = \frac{c_{\mathrm{LLW}} - \sqrt{c_{\mathrm{LLW}}^2 - 4(1-a)}}{2}, \quad \tilde{\lambda}_{\mathrm{LLW}} = \frac{\tilde{c}_{\mathrm{LLW}} - \sqrt{\tilde{c}_{\mathrm{LLW}}^2 - 4dr(1-b)}}{2d}. \tag{B.2}$$

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