Fixed-Time Nash Equilibrium Seeking in Time-Varying Networks

Jorge I. Poveda, Miroslav Krstić, and Tamer Başar

***Abstract*— In this paper we introduce first-order and zeroth-order Nash equilibrium seeking dynamics with fixed-time and practical fixed-time convergence certificates for non-cooperative games having finitely many players. The first-order algorithms achieve exact convergence to the Nash equilibrium of the game in a finite time that can be additionally upper bounded by a constant that is independent of the initial conditions of the actions of the players. Moreover, these fixed-time bounds can be prescribed a priori by the system designer under an appropriate tuning of the parameters of the algorithms. When players have access only to measurements of their cost functions, we consider a class of distributed multi-time scale zeroth-order model-free adaptive dynamics that achieve semi-global practical fixed-time stability, qualitatively preserving the fixed-time bounds of the first-order dynamics as the time scale separation increases. Moreover, by leveraging the property of fixed-time input-to-state stability, further results are obtained for mixed games where some of the players implement different seeking dynamics. Fast and slow switching communication graphs are also incorporated using tools from hybrid systems. We consider potential games as well as general non-potential strongly monotone games. Numerical examples illustrate our results.**

***Index Terms*— Learning in Games, Nash equilibria, Non-cooperative games, Extremum Seeking.**

I. INTRODUCTION

SINCE the concept of Nash equilibrium (NE) was generally introduced in [2], different iterative Nash equilibrium seeking (NES) algorithms have been extensively studied in the literature of economics, computer science, and engineering; see for instance [3]–[6] and references therein.

In the control's literature, Nash seeking algorithms are generally studied from a dynamical systems perspective, either by using discrete-time models [5]–[8], or continuous-time models; see [9]–[12]. Some works have also used hybrid dynamics to model game-theoretic settings that involve continuous-time and discrete-time dynamics [13], [14]. Irrespective of the nature of the algorithms, a characteristic feature in this domain is the satisfaction of suitable *stability*, *convergence* and *robustness* properties that are critical for practical applications that use feedback mechanisms. Nevertheless, the design of *high-performance* NES dynamics that not only possess suitable stability and robustness properties, but that also converge rapidly to the NE of the game, with a graceful transient performance, has proven to be a persistent challenge. As a matter of fact, most of the algorithms in the literature have established either asymptotic convergence results [6], [8], [10], [15]–[17], or exponential convergence under strong monotonicity properties, e.g., [18], [7], [12]. However, even exponential convergence may not be suitable for high-performance engineering applications where *certificates* in terms of convergence times are required. Indeed, standard asymptotic, exponential, and even finite-time stability results encode an intrinsic dependence of the convergence time on the distance between the NE of the game and the initial conditions of the actions of the players. For learning dynamics with exponential convergence rates, the convergence time to the NE grows logarithmically with the distance between the NE and the initial conditions of the algorithm. In the finite-time case, the convergence time usually grows in a linear manner. These properties preclude the existence of *convergence time certificates* in settings where the actions of the players are not constrained *a priori* to compact sets. Moreover, even if the system designer can restrict the actions of the players to compact sets, standard asymptotic bounds usually rely on *worst-case* convergence times established from the largest compact set of possible initial conditions, which have limited practical use in engineering applications.

Contributions: In this paper, we introduce new classes of model-based and model-free NES dynamics with *fixed-time* convergence properties. These Fixed-time Nash Equilibrium Seeking (FXNES) dynamics guarantee that the actions of the players converge to the NE of the game before a fixed time that is independent of their initial conditions, and which can be prescribed *a priori* by the system designer. Moreover, the NES dynamics can be implemented in games where players have access only to real-time *measurements* of their *costs*. Specifically, the contributions of this paper are:

1) We present a new class of FXNES dynamics with uniform global fixed-time stability (UGFXS) properties. To our best knowledge, such type of stability property, introduced in [19], has not been established before in the context of Nash equilibrium seeking. The FXNES dynamics make use of local Oracles that provide real-time measurements of the gradients of the cost functions. Depending on the level of information available

J. I. Poveda is with the Department of Electrical, Computer and Energy Engineering, University of Colorado, Boulder, CO, USA. E-mail: jorge.poveda@colorado.edu.

M. Krstić is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA, USA. E-mail: krstic@ucsd.edu.

Tamer Başar is with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, IL. E-mail: basar1@uiuc.edu.

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to the players, the dynamics can be implemented either as fully decoupled update rules, or as coupled dynamics that make use of the complete pseudo-gradient of the game. We characterize their convergence properties in pseudo-gradient dominated potential games, as well as in strongly monotone games that are not necessarily potential games.

2) Motivated by realistic competitive settings where players implement the fastest algorithm at their disposal, we study the stability and convergence properties of games with *mixed dynamics*. In this setting, a subset of the players implement the FXNES dynamics, while other players implement standard gradient play dynamics [9], [15], [18], and/or rescaled gradient play update rules. We establish uniform global asymptotic and exponential stability properties for potential games, and also fixed-time input-to-state (ISS) stability properties for the FXNES dynamics with respect to the reaction curves of the players. In the presence of stubborn players, the fixed-time ISS result reduces to exact fixed-time convergence to the reaction curve of the players that implement the FXNES dynamics.

3) In Section V we present the main results of the paper: We introduce novel *model-free adaptive* versions of all the FXNES dynamics. The adaptive dynamics are based on novel tools recently developed for non-smooth and hybrid extremum seeking systems [20]–[22], which extend the traditional results in smooth extremum seeking control [23]–[25] that have been instrumental for the design of model-free NES dynamics with asymptotic and exponential convergence rates; see [26], [15], [18], [17]. In the model-free setting, the coupled NES dynamics implement a distributed consensus mechanism operating on a faster time scale, which permits a distributed implementation of the algorithm. The resulting closed-loop system is then analyzed using tools from singular perturbation theory for non-smooth and hybrid systems in order to show that the actions of the players converge to (a neighborhood) of the NE via the same generalized class $\mathcal{KL}_{\mathcal{T}}$ function obtained in the Oracle-based scenario, provided the parameters of the algorithm are appropriately tuned to induce enough time scale separation in the system. To the best of our knowledge, these results are the first in the literature on model-free fixed-time NES.

4) An additional novelty of this paper is the introduction of time-varying graphs in the model-free dynamics. Whereas switching graphs are now fairly standard in the multi-agent control literature (see e.g., [27]), they have not been systematically used in the context of averaging-based adaptive NES due to the challenges that emerge when the singularly perturbed system becomes a switched system with slow or arbitrarily fast switching graphs. We overcome these issues by using tools from set-valued hybrid systems theory [28].

Additional Contributions with Respect to [1]: Earlier, partial results of this paper appeared in the Proceedings of the IEEE CDC 2020 [1]. The results of [1] involved only dynamics with *homogeneous* exponents and gains, which led to significant simplifications in the proofs; further, [1] only considered dynamics with coupled normalization terms. Additionally, [1] only studied homogeneous dynamics and static graphs. On the other hand, the results of this paper substantially extend those of [1] by introducing new *decoupled* NES dynamics, novel convergence results for non-potential games and algorithms with *heterogeneous* gains and exponents (requiring different proof techniques), stability results for model-based and model-free *mixed dynamics*, results for slow and fast *switching graphs*, as well as new numerical illustrative examples for static and dynamic players. This paper also presents the complete stability analysis of the algorithms.

The paper is organized as follows. In Section II, we introduce some preliminaries. Section III defines the NES problem. In Section IV, we present the FXNES dynamics, and we also provide stability and convergence guarantees. Section V presents the model-free algorithms along with their respective analyses, and Section VI ends with the conclusions.

II. PRELIMINARIES

Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $z \in \mathbb{R}^n$, we use $|z|_{\mathcal{A}} := \min_{s\in\mathcal{A}} \|z - s\|_2$. When $\mathcal{A} = \{a\}$, we simply use $|z - a|$. We use $\mathbb{S}^1 := \{z \in \mathbb{R}^2 : z_1^2 + z_2^2 = 1\}$ to denote the unit circle in $\mathbb{R}^2$, and $r\mathbb{B}$ to denote a closed ball in the Euclidean space, of radius $r > 0$, and centered at the origin. The set of positive rational numbers is denoted by $\mathbb{Q}_{>0}$. We use $I_n \in \mathbb{R}^{n\times n}$ to denote the identity matrix, and $\mathcal{C}^1$ to denote the class of continuously differentiable functions. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A function $\tilde{\beta}$ is of class $\mathcal{KL}_{\mathcal{T}}$ if $\tilde{\beta} \in \mathcal{KL}$, and additionally, there exists a continuous function $T : \mathbb{R} \to \mathbb{R}_{\geq 0}$, called the settling time function, such that $\tilde{\beta}(r, s) = 0$ for all $s > T(r)$ and all $r > 0$. When there exists $T^* > 0$ such that $T(r) \leq T^*$ for all $r > 0$, we say that $\tilde{\beta}$ has the *fixed-time convergence property*. A directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is characterized by the set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$, and the set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For a given node $i \in \mathcal{V}$, we denote its set of neighbors as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A graph is said to be balanced if $\sum_{j=1}^{p} a_{ij} = \sum_{j=1}^{p} a_{ji}$, where $[a_{ij}]$ is the entry (i, j) of the adjacency matrix of the graph. Given smooth functions $J_i : \mathbb{R}^n \to \mathbb{R}$, for $i \in \mathcal{V}$, the *pseudogradient* $G := [G_1, \ldots, G_N]^\top$ is defined as (see [9, Eq. (3.9)]):

$$G(x) := \left[\nabla_1 J_1(x)^\top, \nabla_2 J_2(x)^\top, \ldots, \nabla_N J_N(x)^\top\right]^\top, \quad (1)$$

where each element $\nabla_i J_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ is defined as $\nabla_i J_i(x) := [\frac{\partial J_i(x)}{\partial x_{i,1}}, \frac{\partial J_i(x)}{\partial x_{i,2}}, \ldots, \frac{\partial J_i(x)}{\partial x_{i,m_i}}]^\top$, for all $i \in \mathcal{V}$. In this paper, we will model our algorithms as constrained set-valued dynamical systems [28], with state $x \in \mathbb{R}^n$, and dynamics

$$x \in C, \qquad \dot{x} \in F(x), \quad (2)$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping that is outer-semicontinuous, locally bounded, and convex-valued, and where $C \subset \mathbb{R}^n$ is a closed set. Solutions to system (2) are absolutely continuous functions $x : \text{dom}(x) \to \mathbb{R}^n$ that satisfy: a) $x(0) \in C$; b) $x(t) \in C$ for all $t \in \text{dom}(x)$; and c) $\dot{x}(t) \in F(x(t))$ for almost all $t \in \text{dom}(x)$. A solution is said to be complete if $\text{dom}(x) = [0, \infty)$. When $C = \mathbb{R}^n$ and F is a single-valued Lipschitz continuous function, system (2) becomes a standard ODE with unique solutions. To study system (2) we will use the following stability notions.

Definition 1: A compact set $\mathcal{A} \subset C$ is said to be *uniformly globally asymptotically stable* (UGAS) for system (2) if there exists a class $\mathcal{KL}$ function β such that every solution x satisfies the bound $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t)$, $\forall\ t \in \text{dom}(x)$. When $\beta(r,s) = c_1 r \exp(-c_2 s)$ for some $c_1, c_2 > 0$, the set $\mathcal{A}$ is said to be *uniformly globally exponentially stable* (UGES). If $\beta \in \mathcal{KL}_{\mathcal{T}}$, the set $\mathcal{A}$ is said to be *uniformly globally finite-time stable (UGFTS)*. If, additionally, the settling time function of β is uniformly bounded by some $T^* > 0$, then the set $\mathcal{A}$ is said to be *uniformly globally fixed-time stable* (UGFXS). □

We will also consider ε-parameterized systems of the form

$$x \in C, \quad \dot{x} \in F_\varepsilon(x), \tag{3}$$

where $\varepsilon > 0$ is a tunable parameter. For these systems, we will study semi-global practical stability properties.

Definition 2: A compact set $\mathcal{A} \subset C$ is said to be β-*Semi-Globally Practically Asymptotically Stable* (SGP-AS) as $\varepsilon \to 0^+$, if there exists $\beta \in \mathcal{KL}$ such that for each pair $\Delta > \nu > 0$ there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ every solution of (3) with $|x(0)|_{\mathcal{A}} \leq \Delta$ satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t) + \nu$, $\forall\ t \in \text{dom}(x)$. When β has an exponential form, we say that $\mathcal{A}$ is *semi-globally practically exponentially stable* (SGP-ES). If $\beta \in \mathcal{KL}_{\mathcal{T}}$, the set $\mathcal{A}$ is said to be *semi-globally practically finite-time stable* (SGP-FTS). If, additionally, β has the fixed-time convergence property, we say that $\mathcal{A}$ is *semi-globally practically fixed-time stable* (SGP-FXS). □

The notions of SGP-AS (-ES, -FTX, -FXS) can be extended to systems that depend on multiple parameters $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell]^\top$. In this case, we say that system (3) renders the set $\mathcal{A}$ β-SGP-AS as $(\varepsilon_\ell, \ldots, \varepsilon_2, \varepsilon_1) \to 0^+$, where the parameters are tuned in order starting from ε_1.

Remark 1: Semi-global practical asymptotic stability is always a "finite-time" convergence property. Namely, for all complete solutions, SGP-AS immediately guarantees that the condition $|x(t)|_{\mathcal{A}} \leq 2\nu$ holds after some finite $\overline{T}_{\nu,\Delta}$. However, the *structure* of β will determine the transient behavior of the solutions x, and also how $\overline{T}_{\nu,\Delta}$ depends on $x(0)$. □

We will also use the following auxiliary lemma, corresponding to [29, Lemma 1] and [30, Lemma 2].

Lemma 1: Consider the dynamics (2) with $C = \mathbb{R}^n$ and F being singled-valued with a unique equilibrium point at the origin, and suppose $\exists\ \rho_1, \rho_2 > 0$, $\beta_1 \in (0,1)$, $\beta_2 > 1$, and a positive definite, radially unbounded and smooth Lyapunov function V satisfying $D^*V(x) \leq -\rho_1 V(x)^{\beta_1} - \rho_2 V(x)^{\beta_2}$, for all $x \in \mathbb{R}^n$, where D^*V is the Dini derivative of V [29]. Then, the origin is UGFXS for (2) with $T^* = \frac{1}{\rho_1(1-\beta_1)} + \frac{1}{\rho_2(\beta_2 - 1)}$. If $\beta_1 = 1 - \frac{1}{2\alpha}$, $\beta_2 = 1 + \frac{1}{2\alpha}$, and $\alpha > 1$, then the origin is UGFXS for (2) with $T^* = \frac{\alpha\pi}{\sqrt{\rho_1\rho_2}}$. □

III. PROBLEM STATEMENT AND MOTIVATION

In a wide variety of applications, a group of agents compete to minimize their own individual cost functions by controlling their respective actions. When the cost functions of the agents also depend on each other's actions, the strategic interactions between the agents can be modeled as a non-cooperative game. In particular, a non-cooperative game is described by three elements. The first element is the set of players, which we will conveniently denote as $\mathcal{V} = \{1, 2, \ldots, N\}$. The second element is the joint action set, denoted as $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \ldots \times \mathcal{S}_N$, which represents all the possible actions that the players can take. Finally, the third element is the vector of cost functions $J = [J_1, J_2, \ldots, J_N]^\top$, where the real-valued $\mathcal{C}^1$ function $J_i : \mathcal{S} \to \mathbb{R}$ represents the cost function of the i^{th} player. In this paper, we study games in which the number of alternatives available to each player is a continuum. More precisely, the action of each player is represented by a vector x_i in the set $\mathcal{S}_i = \mathbb{R}^{m_i}$, where m_i is an integer. We define $n := \sum_{i=1}^N m_i$, such that $\mathcal{S} = \mathbb{R}^n$, and we use $x_{-i} \in \mathbb{R}^{n-m_i}$ to denote the vector of actions of the players other than player i. We also use $J_i(x_i, x_{-i})$ to explicitly denote the dependence of J_i on the actions x, and we consider the following smoothness assumption which is standard in the literature [7], [12]:

Assumption 1: There exists $L > 0$ such that $|G(x) - G(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$. □

Our main goal is to design an update rule for the actions of the players to converge to a point $x^* \in \mathbb{R}^n$ that satisfies

$$J_i(x_i^*, x_{-i}^*) = \inf_{x_i \in \mathbb{R}} J_i(x_i, x_{-i}^*), \quad \forall\ i \in \{1, 2, \ldots, N\}. \tag{4}$$

Such actions describe a *Nash equilibrium* (NE) [2], which is a desirable state where every player of the game has no individual incentive to deviate to another action. While a plethora of Nash equilibrium seeking (NES) algorithms exist in the literature, see e.g., [5] and references therein, in this paper we are interested in *continuous-time* dynamic strategies that can be implemented by the players in order to exactly converge to x^* in finite time, and regardless of the initial choices for the actions, i.e., in a fixed time. Any continuous-time algorithm with this property cannot be modeled by a Lipschitz continuous autonomous ordinary differential equation due to the lack of uniqueness of solutions backward in time. Yet, by achieving fixed-time convergence, these algorithms can achieve dramatic improvements in terms of transient behavior for high-performance applications. As an illustrative example, for a duopoly game with costs[1] $J_i(x) = 0.5(x_i^2 - x_i x_j)$, $i \in \{1, 2\}$, $j \neq i$, Figure 1 compares the reachable sets in the time domain $[0, 80]$ of one of the fixed-time Nash equilibrium seeking (FXNES) dynamics introduced in this paper (c.f. (8)), compared to the well-known gradient-play (GP) dynamics [9]:

$$\dot{x}_i = -k_i \nabla_i J_i(x), \quad k_i > 0, \quad \forall\ i \in \mathcal{V}. \tag{5}$$

As seen in Figure 1, the trajectories of the FXNES dynamics converge exactly to the NE of the game before a fixed time $T_1^* > 0$. Indeed, we will show that the FXNES dynamics induce uniform global fixed-time convergence to x^* via a generalized $\mathcal{KL}_{\mathcal{T}}$ function β_1 of the form

$$|\tilde{x}(t)| \leq c_1 \tan\left(\max\left\{0, -c_2 t + \arctan\left(c_3 |\tilde{x}(0)|^{c_4}\right)\right\}\right)^{c_5}, \tag{6}$$

where $\tilde{x} := x - x^*$, $c_i > 0$ for $i \in \{1, 2, \ldots, 5\}$. This bound will guarantee that $|\tilde{x}(t)| = 0$ before the time

$$T_1^* = \frac{\pi N^{\frac{a}{4}}}{2a\underline{k}\kappa}, \tag{7}$$

[1] Note that this 2-player game admits the unique NE ($x_1^* = x_2^* = 0$).

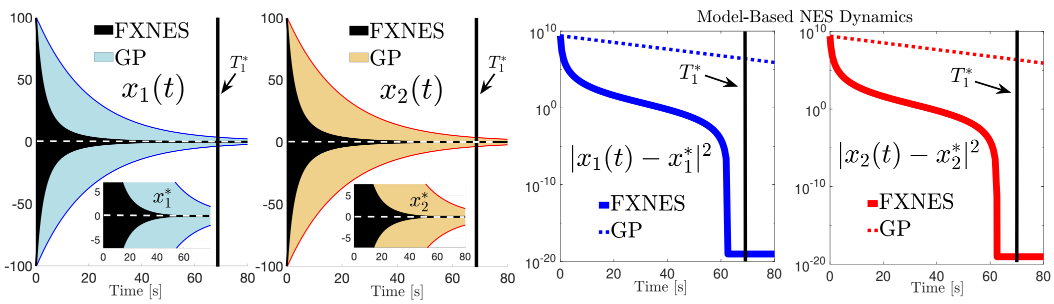

Fig. 1: A duopoly game with Oracles. Left: Comparison between reachable sets of the GP dynamics (5) and the decoupled FXNES dynamics (8) with $a = b = 0.5$ for all $t \in [0, 80]$. Right: One trajectory of each algorithm, initialized at $x_i(0) = 5 \times 10^4$, illustrating the exponential and the fixed-time convergence property, respectively. Simulations used $\kappa = 0.5$, $k_i = 0.1$, $a_i = 0.5 = b$ for all $i \in \{1, 2\}$.

where $\underline{k}$ is the smallest gain used by the dynamics of the players, $\kappa > 0$ is a constant that characterizes the strong monotonicity properties of the game, and $a \in (0,1)$ is a tunable parameter of the controller. Since the bound (7) is independent of the initial conditions of the dynamics, it provides a *convergence certificate* for the algorithm.

In settings where players have access to measurements of J_i instead of the gradients ∇J_i, dynamics of the form (5) or (8) cannot be directly implemented. In this case, each player needs to use zeroth-order or *model-free* adaptive algorithms, and we are interested in algorithms that additionally retain the qualitative properties of the FXNES dynamics. Figure 2 shows the results of implementing one of the model-free FXNES dynamics studied in this paper, in the same duopoly example considered in Figure 1, and compared to the standard model-free NES algorithm of [18] using the same tunable parameters. On the right plot, it can be seen that the adaptive dynamics approximately recover the fixed-time convergence of the model-based dynamics. In particular, as we will show in Section V, for arbitrarily large compact sets of initial conditions, and for arbitrarily small values of $\nu > 0$, the parameters of the model-free dynamics can be tuned to guarantee that the actions of the players satisfy the bound $|\tilde{x}(t)| \leq \beta_1(|\tilde{x}(0)|, t) + \nu$, $\forall\, t \geq 0$, with β_1 given by (6). This type of bound can induce a drastic improvement in the transient performance of the algorithms.

IV. FIXED TIME NES WITH GRADIENT ORACLES

We start by considering a class of *model-based* FXNES dynamics suitable for games where players have access to local Oracles that provide real-time *measurements* of the values of the gradients $\nabla_i J_i$. The results of this section will be instrumental for the model-free games considered in the next section, where players do not have access to the mathematical forms of J_i or ∇J_i, thus ruling out strategies that require explicit evaluations of these functions.

A. Decoupled FXNES Dynamics

To achieve fixed-time NES, we first consider the following dynamics implemented by each player i of the game:

$$\dot{x}_i = -\nabla_i J_i(x) \left(\frac{k_i}{|\nabla_i J_i(x)|^a} + \frac{k_i}{|\nabla_i J_i(x)|^{-b}} \right), \quad (8)$$

where $k_i > 0$ is a tunable gain, $a \in (0,1)$ and $b > 0$ are tunable parameters that are homogeneous for all players, and where the right-hand side of (8) is defined to be zero whenever $\nabla_i J_i(x) = 0$. Note that the dynamics (8) are fully decoupled and continuous, but not Lipschitz continuous. To characterize the stability and convergence properties, we first focus on a class of games that we term *pseudogradient-dominated potential games*, which satisfy the following.

Assumption 2: There exists a $\mathcal{C}^2$ radially unbounded function $P : \mathbb{R}^N \to \mathbb{R}$, called the potential function, such that:

(a) The gradient of P satisfies $\nabla_i P(x) = \nabla_i J_i(x)$, for all $i \in \mathcal{V}$, and for all $x \in \mathbb{R}^n$.
(b) The NE exists, satisfies $x^* = \arg\min_{x\in\mathbb{R}^n} P(x)$, and it is unique.
(c) There exists $\kappa > 0$ such that $P(x) - P(x^*) \leq \frac{1}{2\kappa}|G(x)|^2$, for all $x \in \mathbb{R}^N$. □

Potential games are ubiquitous in the literature [5]. They arise in applications such as congestion control, resource allocation problems, and oligopoly games with quadratic costs [5], [18]. Pseudo-gradient dominated potential games are standard potential games that also satisfy the inequality of item (c). Note that any potential game with a strongly convex potential function satisfies this inequality. In this particular case, existence and uniqueness of x^* follows from [9, Thm. 2]. However, in general, the inequality in (c) does not imply convexity of P, but rather the property of *invexity*. Invex games have only recently been explored in the literature [31], [32].

Proposition 1: Suppose that Assumptions 1-2 hold. Then, the NES dynamics (8) render UGFXS the NE $x^* \in \mathbb{R}^n$, with a settling time function bounded by:

$$T_1^* = \frac{1}{\underline{k}\kappa} \left(\frac{2^{\frac{3a}{4}} \kappa^{\frac{a}{2}}}{a} + \frac{N^{\frac{b}{2}}}{2^{\frac{3b}{4}} \kappa^{\frac{b}{2}} b} \right), \quad (9)$$

where $\underline{k} = \min_i\ k_i$. □

Proof: Consider the Lyapunov function

$$V(x) = \frac{1}{2}(P(x) - P(x^*))^2, \quad (10)$$

which, under Assumption 2, is positive definite with respect to x^*, and also radially unbounded due to items (b)-(c) of Assumption 2, which imply a quadratic growth on V [33, Thm. 2]. Note that if $x(t_0)$ is such that $\nabla_i J_i(x(t_0)) = 0$, for all $i \in \mathcal{V}$, then $\dot{x} = 0$, and, by Assumption 2 we must have $x(t) = x^*$ for all $t \geq 0$. Moreover, the time derivative of V

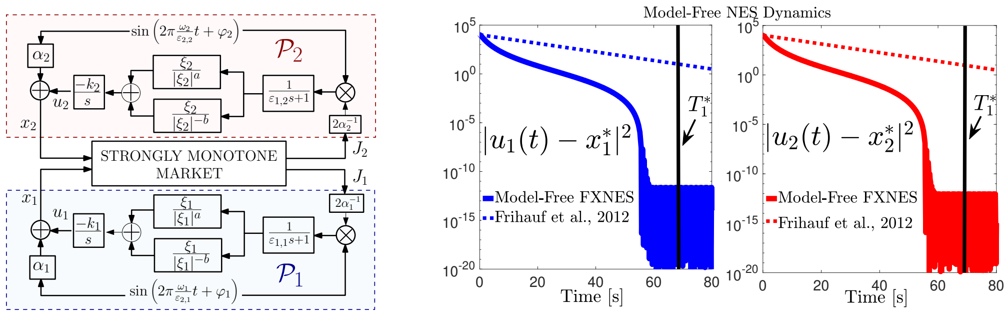

Fig. 2: A duopoly game with no gradient Oracles. Left: Scheme of model-free adaptive FXNES dynamics. Right: Comparison between trajectories generated by the model-free dynamics of [18] and the model-free adaptive FXNES dynamics, using the same tunable parameters.

satisfies $\dot V(x) = \sqrt{2}V(x)^{\frac{1}{2}}\nabla P(x)^\top \dot x$, and using property (a) of Assumption 2, we obtain:

$$\begin{aligned}\dot V(x) &\le -V(x)^{\frac{1}{2}}\sum_{i=1}^{N}|\nabla_i J_i(x)|^2\left(\frac{\underline{k}\sqrt{2}}{|\nabla_i J_i(x)|^a}+\frac{\underline{k}\sqrt{2}}{|\nabla_i J_i(x)|^{-b}}\right)\\ &= -\underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\sum_{i=1}^{N}\left(\xi_i^{\frac{2-a}{2}}+\xi_i^{\frac{2+b}{2}}\right),\quad \xi_i := |\nabla_i J_i(x)|^2,\\ &= -\underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\sum_{i=1}^{N}\xi_i^{1-\frac{a}{2}} - \underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\sum_{i=1}^{N}\xi_i^{1+\frac{b}{2}},\end{aligned}$$

where $\underline{k} := \min_i k_i$. Note that if x is such that $x \neq x^*$ but $\nabla_i J_i(x) = 0$ for some $i \in \mathcal{V}$, the above inequalities still hold with $\xi_i = 0$ for such i. Using Lemma 7 in the Appendix:

$$\begin{aligned}\dot V(x) &\le -\underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\left(\left(\sum_{i=1}^{N}\xi_i\right)^{1-\frac{a}{2}} + N^{-\frac{b}{2}}\left(\sum_{i=1}^{N}\xi_i\right)^{1+\frac{b}{2}}\right)\\ &= -\underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\left(|G(x)|^{2-a}+N^{-\frac{b}{2}}|G(x)|^{2+b}\right). \qquad (11)\end{aligned}$$

and using Lemma 9 in the Appendix, we can further upper bound the time derivative of V in (11) as follows

$$\begin{aligned}\dot V(x) &\le -4\underline{k}\kappa\left(2^{-\frac{3a}{4}}(\kappa)^{-\frac{a}{2}}V(x)^{\frac{4-a}{4}} + N^{-\frac{b}{2}}2^{\frac{3b}{4}}(\kappa)^{\frac{b}{2}}V(x)^{\frac{4+b}{4}}\right).\\ &\le -4\underline{k}\kappa\left(c_1 V(x)^{1-\frac{a}{4}} + c_2 V(x)^{1+\frac{b}{4}}\right). \qquad (12)\end{aligned}$$

By Lemma 1, the last inequality implies UGFXS, with bound for the settling time function given by $T_1^* = \frac{1}{4\underline{k}\kappa}\left(\frac{4}{c_1 a}+\frac{4}{c_2 b}\right)$. The result follows by direct substitution of c_1 and c_2. ■

Unlike existing results in the literature of Nash seeking, Proposition 1 not only establishes global asymptotic or exponential stability but also fixed-time convergence to the NE. Moreover, equation (9) provides an explicit bound for the convergence time. Note that equation (9) depends only on the exponents (a, b), the constant κ, the number of players N, and the minimum gain k_i. Thus, if the constants (κ, N, a, b) are known, any time $T_1^* > 0$ can be prescribed a priori by the system designer via the assignment of the gains k_i. Note that unlike existing fixed-time optimization algorithms [34], the normalizing term in (8) is the partial derivative of the cost of each player with respect to its own action. Thus, system (8) can be seen as a fixed-time pseudo-gradient flow.

Remark 2: When $a = b$ in (8), a sharper bound can be obtained for T_1^* by using the second part of Lemma 1. In this case, Proposition 1 holds with $T_1^* = \frac{\pi}{2a\underline{k}\kappa\sqrt{c_1c_2}}$, where the constants c_1, c_2 are given in (12). By direct substitution of c_1 and c_2 we obtain precisely the bound (7). The $\mathcal{KL}_{\mathcal{T}}$ bound (6) follows by applying [20, Lemma 3]. □

B. Heterogeneous FXNES Dynamics

The decoupled dynamics (8) implement homogeneous exponents (a, b). To relax this condition, we now consider the following coupled FXNES dynamics with *heterogeneous* exponents $a_i \in (0, 1)$, $b_i > 0$, implemented by each player i:

$$\dot x_i = -\nabla_i J_i(x)\left(\frac{k_i}{|G(x)|^{a_i}}+\frac{k_i}{|G(x)|^{-b_i}}\right), \qquad (13)$$

where G is the pseudogradient of the game, and the right-hand side of (13) is defined to be zero when $G(x) = 0$.

The following result establishes that the learning dynamics (13) induce global fixed-time stability in any game satisfying Assumptions 1-2, with a convergence bound independent of the number of players.

Proposition 2: Suppose that Assumptions 1-2 hold. Then, the learning dynamics (13) render UGFXS the NE $x^* \in \mathbb{R}^n$, with a settling time function bounded by:

$$T_2^* = \frac{1}{\underline{k}\kappa\min\{1/\bar\alpha_a, \underline{\gamma}_b\}}\left(\frac{1}{\min_i a_i}+\frac{1}{\min_i b_i}\right), \qquad (14)$$

where $\underline{k} = \min_i k_i$, $\bar\alpha_a = \max_i\left(\frac{2^{\frac{3}{4}}L}{\sqrt{\kappa}}\right)^{a_i}$, and $\underline{\gamma}_b = \min_i(2^{\frac{3}{4}}\sqrt{\kappa})^{b_i}$. □

Proof: To prove Proposition 2, we first establish the following auxiliary Lemma.

Lemma 2: Suppose that Assumptions 1 and 2 hold, and let V be given by (10). Then, the time derivative of V along the trajectories of (13) satisfies

$$\dot V(x) \le \begin{cases} -\tilde\rho V(x)^{1-\frac{\underline{a}}{4}}, & \text{if } V(x)\le 1,\\ -\tilde\rho V(x)^{1+\frac{\underline{b}}{4}}, & \text{if } V(x)\ge 1,\end{cases} \qquad (15)$$

where $\tilde\rho := 4\kappa\underline{k}\min\{1/\bar\alpha_a, \underline{\gamma}_b\}$, with $\underline{a} = \min_i a_i$, and $\underline{b} = \min_i b_i$.

Proof of Lemma 2: For each player $i \in \mathcal{V}$, we define the function $\psi_i : \mathbb{R}^n\backslash\{x^*\} \to \mathbb{R}_{>0}$ given by

$$\psi_i(x) := \frac{1}{|G(x)|^{a_i}}+\frac{1}{|G(x)|^{-b_i}}. \qquad (16)$$

The time derivative of V can be written as $\dot V(x) = -\sqrt{2}V(x)^{\frac{1}{2}}\sum_{i=1}^{N}k_i\psi_i(x)\nabla_i J_i(x)^\top\nabla_i J_i(x)$, which satisfies

$\dot{V}(x) \leq -\underline{k}\sqrt{2}V(x)^{\frac{1}{2}} \sum_{i=1}^{N} |\nabla_i J_i(x)|^2 \psi_i(x)$, for all $x \neq x^*$, where $\underline{k} := \min_i k_i$. Using Lemmas 8 and 9 in the Appendix we can further upper bound $\dot{V}$ as follows:

$$\begin{aligned}\dot{V} &\leq -V(x)^{\frac{1}{2}} \sum_{i=1}^{N} |\nabla_i J_i(x)|^2 \left(\frac{\underline{k}\sqrt{2}}{\alpha_{a_i} V(x)^{\frac{a_i}{4}}} + \frac{\underline{k}\sqrt{2}\gamma_{b_i}}{V(x)^{-\frac{b_i}{4}}} \right) \\ &\leq -V(x)^{\frac{1}{2}} \sum_{i=1}^{N} |\nabla_i J_i(x)|^2 \left(\frac{\underline{k}\sqrt{2}}{\bar{\alpha}_a V(x)^{\frac{a_i}{4}}} + \frac{\underline{k}\sqrt{2}\underline{\gamma}_b}{V(x)^{-\frac{b_i}{4}}} \right),\end{aligned}$$

where $\bar{\alpha}_a := \max_i \ \alpha_{a_i}$ and $\underline{\gamma}_b := \min_i \ \gamma_{b_i}$. We now consider two possible cases when $x \neq x^*$:

1) If $V(x) \leq 1$, by Lemma 5 in the Appendix we obtain:

$$\begin{aligned}\dot{V} &\leq -V(x)^{\frac{1}{2}} \sum_{i=1}^{N} |\nabla_i J_i(x)|^2 \left(\frac{\underline{k}\sqrt{2}}{\bar{\alpha}_a V(x)^{\frac{\underline{a}}{4}}} + \frac{\underline{k}\sqrt{2}\underline{\gamma}_b}{V(x)^{-\frac{\bar{b}}{4}}} \right) \\ &= -V(x)^{\frac{1}{2}} \left(\frac{\underline{k}\sqrt{2}}{\bar{\alpha}_a V(x)^{\frac{\underline{a}}{4}}} + \frac{\underline{k}\sqrt{2}\underline{\gamma}_b}{V(x)^{-\frac{\bar{b}}{4}}} \right) \sum_{i=1}^{N} |\nabla_i J_i(x)|^2 \\ &= -\underline{k}\sqrt{2} \left(\frac{V(x)^{\frac{1}{2}}}{\bar{\alpha}_a V(x)^{\frac{\underline{a}}{4}}} + \frac{V(x)^{\frac{1}{2}}\underline{\gamma}_b}{V(x)^{-\frac{\bar{b}}{4}}} \right) |G(x)|^2, \\ &\leq -\underline{k}\gamma_2\sqrt{2} \left(\frac{V(x)^{1-\frac{\underline{a}}{4}}}{\bar{\alpha}_a} + \underline{\gamma}_b V(x)^{1+\frac{\bar{b}}{4}} \right),\end{aligned}$$

which implies $\dot{V}(x) \leq -\rho_s V(x)^{1-\frac{\underline{a}}{4}}$ with $\rho_s := \frac{4\underline{k}\kappa}{\bar{\alpha}_a}$.

2) If $V(x) \geq 1$, by Lemma 6 in the Appendix, and the same steps as above, we obtain

$$\begin{aligned}\dot{V} &\leq -V(x)^{\frac{1}{2}} \left(\frac{\underline{k}\sqrt{2}}{\bar{\alpha}_a V(x)^{\frac{\bar{a}}{4}}} + \frac{\underline{k}\sqrt{2}\underline{\gamma}_b}{V(x)^{-\frac{\underline{b}}{4}}} \right) \sum_{i=1}^{N} |\nabla_i J_i(x)|^2, \\ &\leq -\underline{k}\gamma_2\sqrt{2} \left(\frac{V(x)^{1-\frac{\bar{a}}{4}}}{\bar{\alpha}_a} + \underline{\gamma}_b V(x)^{1+\frac{\underline{b}}{4}} \right),\end{aligned}$$

which implies $\dot{V} \leq -\rho_\ell V(x)^{1+\frac{\underline{b}}{4}}$ with $\rho_\ell := 4\underline{k}\kappa\underline{\gamma}_b$.

The above inequalities of items 1) and 2) imply (15). □

Finally, using Lemma 2, it now follows that if $V(x(0)) > 1$, then $V(x(t)) \leq 1$ for all $t \geq T'$, with $T' = \frac{4}{\tilde{\rho}\underline{b}}$. Similarly, for any $x(t_0)$ such that $V(x(t_0)) \leq 1$, we have that $V(x(t)) = 0$ for all $t \geq t_0 + T''$, with $T'' = \frac{4}{\tilde{\rho}\underline{a}}$. Thus, global fixed-time convergence occurs for all $t \geq T_2^* := T' + T''$. If $x(t_0)$ is such that $G(x(t_0)) = 0$, then by definition we have $\dot{x} = 0$, but under Assumption 2 and the fact that $\dot{V} < 0$ for all $x \neq x^*$, we must also have $x(t) = x^*$ for all $t \geq t_0$. ■

Remark 3: The fixed-time characterization given by equation (14) shows the dependence of the bound T_2^* on the minimum values of the exponents (a_i, b_i) among all players of the game, and also on the minimum gain k_i, and the parameter κ. Note that, unlike T_1^* in (9), the bound (14) does not depend on the number of players in the game. □

Remark 4: The learning dynamics (13) implement a normalizing term that is a function of the overall pseudo-gradient G. Therefore, these dynamics are not suitable for fully decentralized implementations. Nevertheless, as we will show in Section V, the term $|G(x)|$ can be computed by each player in a distributed way via multi-time scale techniques, approximately preserving the convergence bounds as the time scale separation increases. □

An important class of games in the context of fast NES algorithms are the so-called *strongly monotone games* [7], [12], [35], characterized by the following assumption.

Assumption 3: There exists $\kappa > 0$ such that the mapping $x \mapsto G(x)$ satisfies $(x_1 - x_2)^\top (G(x_1) - G(x_2)) \geq \kappa |x_1 - x_2|^2$, for all $x_1, x_2 \in \mathbb{R}^n$. □

For general (not necessarily potential) strongly monotone games, the existence of a unique NE is guaranteed [9, Thm. 2]. However, the analysis of fast NES dynamics is more challenging. Indeed, even for settings where *exponential* convergence is desired, standard convergence results generally require additional joint strong-convexity/Lipschitz conditions on the cost functions of the players. In our case, the following proposition establishes that the learning dynamics (13) render UGFXS the NE of any strongly monotone game when the exponents of the normalizing term are homogeneous. However, we impose neither homogeneity conditions on the gains k_i, nor the existence of a potential function.

Proposition 3: Suppose that Assumptions 1 and 3 hold. Let the players implement the NES dynamics (13) with $a_i = a \in (0,1)$ and $b_i = b > 0$, for all $i \in \mathcal{V}$. Then, the NE x^* is UGFXS, with a settling time function bounded by

$$T_3^* = \frac{1}{\underline{k}\kappa} \left(\frac{(2\bar{k}L^2)^{\frac{a}{2}}}{a} + \frac{1}{(2\underline{k}\kappa^2)^{\frac{b}{2}} b} \right). \tag{17}$$

where $\bar{k} = \max_i \ k_i$, and $\underline{k} = \min_i \ k_i$. □

Proof: Let $K := \text{diag}(k)$, where $k = [k_1 \mathbf{1}_{m_1}^\top, k_2 \mathbf{1}_{m_2}^\top, \ldots, k_N \mathbf{1}_{m_N}^\top]^\top$. We consider the quadratic Lyapunov function

$$V(x) = \frac{1}{2}(x - x^*)^\top K^{-1}(x - x^*), \tag{18}$$

which satisfies $\frac{1}{2\bar{k}}|x - x^*|^2 \leq V(x) \leq \frac{1}{2\underline{k}}|x - x^*|^2$ due to the fact that $k_i > 0$ for all $i \in \mathcal{V}$. Define the function $\psi : \mathbb{R}^n \backslash \{x^*\} \to \mathbb{R} > 0$ as $\psi(x) := \left(\frac{1}{|G(x)|^a} + \frac{1}{|G(x)|^{-b}} \right)$. It follows that for all $x \neq x^*$ the time-derivative of V along the trajectories of the dynamics (13) satisfies

$$\begin{aligned}\dot{V} &= -(x - x^*)^\top K^{-1} K G(x)\psi(x) = -(x - x^*)^\top G(x)\psi(x) \\ &\leq -\kappa|x - x^*|^2 \psi(x) \leq -2\underline{k}\kappa V(x)\psi(x),\end{aligned} \tag{19}$$

and using Lemma 10 in the Appendix we obtain:

$$\begin{aligned}\dot{V}(x) &\leq -2\underline{k}\kappa V(x) \left(\frac{1}{\gamma_1^{\frac{a}{2}} V(x)^{\frac{a}{2}}} + \frac{\gamma_2^{\frac{b}{2}}}{V(x)^{-\frac{b}{2}}} \right) \\ &\leq -2\underline{k}\kappa \left(\frac{1}{\gamma_1^{\frac{a}{2}}} V(x)^{1-\frac{a}{2}} + \gamma_2^{\frac{b}{2}} V(x)^{1+\frac{b}{2}} \right),\end{aligned}$$

for all $x \in \mathbb{R}^n$. The result follows by Lemma 1. ■

Remark 5: For strongly monotone games (non necessarily potential), the condition $a = b$ and the alternative Lyapunov function $V_2(x) = 0.5V(x)^2$, with V given by (18), can lead to the sharper convergence bound $T_3^{**} = \frac{\pi}{2\underline{k}\kappa a} \left(\frac{L^2 \bar{k}}{\kappa^2 \underline{k}} \right)^{\frac{a}{4}}$, which shows the role of the *condition number* (L/κ) of G, as well as the ratio $(\bar{k}/\underline{k})$. When the strongly monotone game is also a potential game, and the gains are also homogeneous, then the

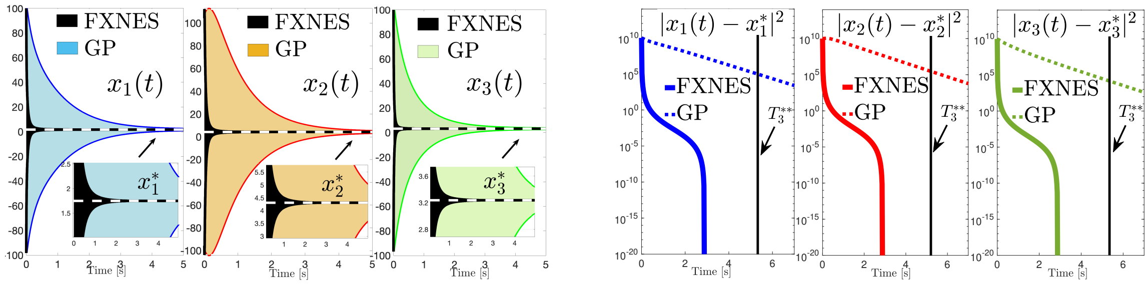

Fig. 3: Left: Comparison of reachable sets of the GP dynamics (5) and the FXNES dynamics (13). Right: Time history of one solution generated by the dynamics (5) (dashed line) and (13) (solid line), with identical initialization and gains.

result of Proposition 3 recovers the results of [34] established for standard optimization problems. □

Example 1: To illustrate the result of Proposition 3 in non-potential games, consider a game with three players, having cost functions of the form

$$J_i(x)=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}D_{jk}^{i}x_jx_k+\sum_{j=1}^{N}d_j^ix_j+c^i, \qquad (20)$$

where $D_{j,k}^i, d_j^i, c^i \in \mathbb{R}$ are parameters associated with the cost of the i^{th} player, for all $j,k \in \mathcal{V}=\{1,2,3\}$. Let $D^1=[27,-12,6;-12,-7.11,-5.33;6,-5.33,-16]$, $D^2=[-13.5,-24,-6;-24,32,-21.33;-6,-21.33,-24]$, $D^3=[-9,12,3;12,3.55,-5.33;3,-5.33,24]$, $d^1=[-15,-6.66,-30]$, $d^2=[-22.5,-26.66,-50]$, $d^3=[-30,-13.33,-60]$, and $c_i=0$ for all i. This game is not a potential game, and therefore it does not satisfy Assumption 2. However, the game satisfies Assumptions 1 and 3 with $L=55$, $\kappa=8.70$, and $x^*=[1.74,4.30,3.23]$. We implement the FXNES dynamics (13) with $a=b=0.5$, and also the gradient play dynamics (5). In both cases we use the same gains $k_i=0.1$ for all $i\in\mathcal{V}$. The left plot of Figure 3 compares the reachable sets of both algorithms in the time interval $[0,5]$ for initial conditions $u(0)$ in the set $[-100,100]\times[-100,100]\times[-100,100]$. The right plot shows *one* solution of both dynamics with identical initialization, as well as the estimate T_3^{**} from Remark 5. □

C. Mixed Gradient Dynamics

In the previous subsections, we established different fixed-time convergence results for non-cooperative games when *all* players agree to implement the learning dynamics (8) or (13). We now take a departure from this setting by studying a more realistic and challenging scenario, where players having different NES dynamics at their disposal aim to learn the NE x^*. In particular, we consider a mixed-game characterized by a partition of the set of players $\mathcal{V}$, given by $\mathcal{V}=\mathbf{F}\cup\mathbf{R}\cup\mathbf{G}$, where $\mathbf{F}$ denotes the players who implement the FXNES dynamics (8) (who we call F-players), $\mathbf{G}$ denotes the players who implement the gradient play dynamics (5) (who we call G-players), which are a particular case of (8) with $a_i=b_i=0$; and $\mathbf{R}$ corresponds to the R-players, who implement the rescaled gradient dynamics $\dot{x}_i=-k_i\nabla_iJ_i(x)|\nabla_iJ_i(x)|^{-a_i}$, with $a_i\in(0,1)$, which correspond to (8) with $b_i=\infty$. These dynamics are also common in the literature of NES and optimization, see [18], [9].

Proposition 4: Suppose that Assumptions 1-2 hold and consider the mixed-game $\mathcal{V}=\mathbf{F}\cup\mathbf{R}\cup\mathbf{G}$, where $a_i\in(0,1)$ and $b_i=a_i$ in (8). Then, the NE x^* is UGAS, and if $\mathbf{R}=\{\emptyset\}$, the actions of the players satisfy the exponential bound $|\tilde{x}(t)|\leq\frac{L}{\kappa}|\tilde{x}(0)|e^{-\underline{k}\kappa t}$, for all $t\geq 0$, where $\tilde{x}=x-x^*$, and $\underline{k}=\min_i\ k_i$. □

Proof: Without loss of generality, let $\mathbf{F}:=\{1,2,\ldots,N_F\}$, $\mathbf{R}=\{N_F+1,N_F+2,\ldots,N_F+N_R\}$, and $\mathbf{G}:=\{N_F+N_R+1,N_F+N_R+2,\ldots,N\}$ be the set of players who implement the FXNES dynamics, the rescaled gradient dynamics, and the gradient dynamics, respectively. Define the functions $\psi_i(x):=\left(\frac{1}{|\nabla_iJ_i(x)|^{a_i}}+\frac{1}{|\nabla_iJ_i(x)|^{-a_i}}\right)$. Using the Lyapunov function (10), we can compute the time derivative of V, which satisfies

$$\dot{V}(x)\leq-\underline{k}V(x)^{\frac{1}{2}}\sqrt{2}\Bigg[\sum_{i=1}^{N_F}|\nabla_iJ_i(x)|^2\psi_i(x)+\sum_{i=N_F+1}^{N_F+N_R}|\nabla_iJ_i(x)|^{2-a_i}+\sum_{i=N_F+N_R+1}^{N}|\nabla_iJ_i(x)|^2\Bigg],$$

which implies $\dot{V}(x)\leq-\underline{k}\sqrt{2}V(x)^{\frac{1}{2}}\eta(x)<0$, for all $\forall\ x\neq x^*$, because $\eta(x)>0$ for all $x\neq x^*$. This establishes UGAS of x^*. When $\mathbf{R}=\{\emptyset\}$, and since $|\nabla_iJ_i(x)|^2\psi_i(x)\geq|\nabla_iJ_i(x)|^2$ due to Lemma 4, for all $x\in\mathbb{R}^n$, and all $i\in\mathcal{V}$, we have that $\dot{V}$ can be bounded as

$$\begin{aligned}\dot{V}(x)&\leq-\underline{k}V(x)^{\frac{1}{2}}\sqrt{2}\left(\sum_{i=n+1}^{N}|\nabla_iJ_i(x)|^2+\sum_{i=1}^{n}|\nabla_iJ_i(x)|^2\right),\\&\leq-\underline{k}V(x)^{\frac{1}{2}}\sqrt{2}|G(x)|^2\leq-4\underline{k}\kappa V(x),\end{aligned}$$

where the last inequality used the fact that $|G(x)|^2\geq\sqrt{8\kappa^2V(x)}$. By a standard Comparison Lemma, and due to Assumptions 1-2, we obtain $\frac{\kappa^2}{8}|x(t)-x^*|^4\leq V(x(t))\leq V(x(0))e^{-4\underline{k}\kappa t}$ and $V(x(0))e^{-4\underline{k}\kappa t}\leq\frac{1}{8\kappa^2}L^4|x(0)-x^*|^4e^{-4\underline{k}\kappa t}$, which leads to the desired bound. ■

The result of Proposition 4 implies that, for each compact set of initial conditions, all actions of the players are uniformly bounded, and converge to x^* in the limit. We now leverage this uniform boundedness property to assert a stronger result for the F-players. In particular, when $\mathbf{R}=\{\emptyset\}$, we will establish that the players who implement the FXNES dynamics converge in a fixed-time -with respect to their initial conditions-to a neighborhood of their reaction curves ℓ_i, which are the F-players' best response strategies given the actions of the G-players, see [16, Def. 4.3]. We also quantify the F-player's convergence certificates, thus providing the means for influencing success in the transient phase of mixed games.

To achieve this goal, and to avoid set-valued reaction curves, we focus our attention on quadratic games with cost functions of the form (20), and scalar actions x_i. To compute the reaction curves of the F-players, we use $x_g \in \mathbb{R}^{|\mathbf{G}|}$ to denote the vector of actions of the G-players, and $x_f \in \mathbb{R}^{|\mathbf{F}|}$ to denote the vector of actions of the F-players. By writing the overall vector of actions of the game as $x := [x_f^\top, x_g^\top]^\top$, the re-organized pseudogradient of the game takes the form

$$\tilde{G}(x) = \begin{bmatrix} D_a & D_b \\ D_c & D_d \end{bmatrix} \begin{bmatrix} x_f \\ x_g \end{bmatrix} + \begin{bmatrix} d_f \\ d_g \end{bmatrix}, \tag{21}$$

where $D_a \in \mathbb{R}^{|\mathbf{F}|\times|\mathbf{F}|}$, $D_d \in \mathbb{R}^{|\mathbf{F}|\times|\mathbf{G}|}$, $D_d \in \mathbb{R}^{|\mathbf{G}|\times|\mathbf{F}|}$. Under Assumption 3, the NE of the game is given by the pair (x_f^*, x_g^*) that satisfies [16, Prop. 4.6]:

$$\begin{bmatrix} x_f^* \\ x_g^* \end{bmatrix} = \begin{bmatrix} D_a & D_b \\ D_c & D_d \end{bmatrix}^{-1} \begin{bmatrix} d_f \\ d_g \end{bmatrix}.$$

Moreover, under Assumption 1 we have that $D_b = D_c^\top$, $D_a > 0$, and the inverse of the Schur complement, $S^{-1} := (D_d - D_c D_a^{-1} D_b)^{-1}$, also exists [36, pp. 123-124]. Using (21), the best response function $x_f^r : \mathbb{R}^{|\mathbf{G}|} \to \mathbb{R}^{|\mathbf{F}|}$ for the F-players can be computed as:

$$x_f^r(x_g) = -D_a^{-1}(D_b x_g + d_f). \tag{22}$$

Using (22), the next proposition establishes a global fixed-time input-to-state stability result [37] that bounds the error with respect to the reaction curve of the F-players, i.e., $\tilde{x}_f = x_f - x_f^r$, taking as input the error $\tilde{x}_g := x_g - x_g^*$ of the G-players, where $x^* = [x_f^*, x_g^*]$ is the NE of the game.

Proposition 5: Suppose that Assumptions 1-3 hold. Then, there exists $\beta \in \mathcal{KL}_\mathcal{T}$ such that

$$|\tilde{x}_f(t)| \leq \beta(|\tilde{x}_f(0)|, t) + \gamma_5 \|\tilde{x}_g\|_{[0,\infty]}, \tag{23}$$

for all $t \geq 0$, and the settling time function of β is bounded by $T_5^* > 0$, with

$$T_5^* := \frac{2}{\underline{k}_f \lambda_{\min}(D_a)} \left(\frac{(2\lambda_{\min}(D_a))^{\frac{a}{2}}}{a} + \frac{N^{\frac{b}{2}}}{(2\lambda_{\min}(D_a))^{\frac{b}{2}} b} \right),$$
$$\gamma_5 := 2\frac{\lambda_{\max}(D_a)}{\lambda_{\min}(D_a)^3} \cdot \frac{\bar{k}_g}{\underline{k}_f} \cdot |D_b||S|,$$

where $\bar{k}_g$ is the largest gain used by the G-Players, and $\underline{k}_f$ is the smallest gain used by the F-Players. □

Remark 6: Since the asymptotic gain γ_5 is proportional to $\bar{k}_g|D_b||S|$, it follows that whenever the G-players are stubborn (i.e., $\bar{k}_g = 0$) or there is no coupling between G-players and F-players (i.e., $D_b = 0$), the F-players render UGFXS the point $x_f^r(x_g(0))$ in their best response curve. □

Before presenting the proof of Proposition 5, we consider a numerical example that illustrates the fixed-time ISS behavior that arises in mixed games with F-players.

Example 2 (Market with Mixed Dynamics): Consider the duopoly market example of [18, Sec. II], where two firms that produce the same good compete for profit by controlling their prices x_1 and x_2. Since in (4) we consider minimization, the negative of the profits of the companies $i \in \{1, 2\}$ have the form $J_i = -s_i(x_i - m_i)$, where s_i is the number of sales, and m_i is the marginal cost. The sales of the firms have the forms $s_1 = S_d - s_2$ and $s_2 = \frac{1}{p}(x_1 - x_2)$, where S_d is the total consumer demand, which is assumed to be constant, and $p > 0$ is the preference of the consumer for the product of firm 1. Substitution into J_i leads to the quadratic costs given by $J_1 = (x_1^2 - x_1 x_2 - (m_1 + S_d p)x_1 + m_1 x_2 + pS_d m_1)/p$ and $J_2 = (x_2^2 - x_1 x_2 + m_2 x_1 - m_2 x_2)/p$. For the purpose of simulation, we use the same parameters considered in [18], i.e., $S_d = 100$, $p = 0.2$, $m_1 = m_2 = 30$. With these parameters, the duopoly becomes a *strongly monotone market* that also satisfies Assumption 1. To attain Nash seeking, both firms implement the fastest algorithm at their disposal: firm 1 implements the GP dynamics (5), while firm 2 implements the FXNES dynamics (8) with $a = b = 0.5$. The gain k_2 of the second firm is tuned to guarantee fixed-time ISS to the reaction curve x_2^r in at most 1 second. The results are presented in Figure 4. The reaction curves are shown with dotted lines. The inset shows that the convergence is approximately linear for both firms after the price of the second firm has quickly converged to the reaction curve x_2^r. Figure 4 shows that $|x_2 - x_2^r|$ converges to the worst-case theoretical ultimate bound $\gamma_5|\tilde{x}_g(0)|$ in approximately 1 s. □

Proof of Proposition 5: Using (21), the dynamics of the F-players can be written as $\dot{x}_f = -K_f \Psi(x)(D_a x_f + D_b x_g + d_f)$, where K_f is a diagonal matrix with diagonal components given by the gains $k_{f,i} > 0$, and $\Psi(x) \in \mathbb{R}^{|\mathcal{F}|\times|\mathcal{F}|}$ is a diagonal matrix with diagonal components given by $\psi_i(x) := |\nabla_i J_i(x)|^{-a} + |\nabla_i J_i(x)|^b$, which in this case satisfy:

$$\psi_i(x) = |D_{a,i} x_f + D_{b,i} x_g + d_{f,i}|^{-a} + |D_{a,i} x_f + D_{b,i} x_g + d_{f,i}|^b.$$

The dynamics of the G-players can be written as $\dot{x}_g = -K_g(D_c x_f + D_d x_g + d_g)$. To study the convergence of x_f to x_f^r, we define the error coordinates $\tilde{x}_f := x_f - x_f^r$ and $\tilde{x}_g = x_g - x_g^*$. Then, the dynamics of the G-players are

$$\begin{aligned} \dot{x}_g &= -K_g\big(D_c(\tilde{x}_f - D_a^{-1}(D_b(\tilde{x}_g + x_g^*) + d_f)) \\ &\quad + D_d(\tilde{x}_g + x_g^*) + d_g\big), \\ &= -K_g D_c \tilde{x}_f + K_g(D_c D_a^{-1} D_b - D_d)\tilde{x}_g - K_g d_g \\ &\quad + K_g D_c D_a^{-1} d_f + K_g(D_c D_a^{-1} D_b - D_d)x_g^*. \end{aligned}$$

The NE satisfies $x_g^* = -S^{-1}(d_g - D_c D_a^{-1} d_f)$, thus the dynamics of the $\mathcal{G}$-players can be written as $\dot{x}_g = -K_g D_c \tilde{x}_f - K_g S \tilde{x}_g$, and ψ_i can be written in the $\tilde{x}_f$-coordinates as

$$\psi_i(\tilde{x}_f) = \frac{1}{|D_{a,i}\tilde{x}_f|^a} + \frac{1}{|D_{a,i}\tilde{x}_f|^b}. \tag{24}$$

Similarly, the error dynamics of the F-players are given by $\dot{\tilde{x}}_f = \dot{x}_f - \dot{x}_f^r$, which can be written as:

$$\begin{aligned} \dot{\tilde{x}}_f &= -K_f \Psi(x)\left(D_a x_f + D_b x_g + d_f\right) - D_a^{-1}[D_b(K_g D_c \tilde{x}_f \\ &\quad + K_g S \tilde{x}_g)], \\ &= -K_f \Psi(x)(D_a(\tilde{x}_f + x_f^r) + D_b x_g + d_f) \\ &\quad - D_a^{-1}[D_b(K_g D_c \tilde{x}_f + K_g S \tilde{x}_g)], \end{aligned}$$

and using the definition of x_f^r:

$$\begin{aligned}\dot{\tilde{x}}_f &= -K_f\Psi(x)\left(D_a(\tilde{x}_f - D_a^{-1}[D_b x_g + d_f]) + D_b x_g + d_f\right)\\ &\quad - D_a^{-1}[D_b(K_g D_c\tilde{x}_f + K_g S\tilde{x}_g)],\\ &= -K_f\Psi(x)D_a\tilde{x}_f - D_a^{-1}M\tilde{x}_f - D_a^{-1}B\tilde{x}_g, \end{aligned}\tag{25}$$

where $M := D_bK_gD_c = D_bK_gD_b^\top$ and $B := D_bK_gS$. We now consider the quadratic Lyapunov function $V(\tilde{x}_f) = \frac{1}{2}\tilde{x}_f^\top D_a\tilde{x}_f$, which is positive definite and has time derivative

$$\dot{V}(\tilde{x}_f) = -\tilde{x}_f^\top D_a^\top K_f\Psi(\tilde{x}_f)D_a\tilde{x}_f - \tilde{x}_f^\top M\tilde{x}_f - \tilde{x}_f^\top B\tilde{x}_g.$$

Expanding the first term, and using the diagonal structure of $K_f\Psi(\tilde{x}_f)$, we obtain

$$\begin{aligned}\dot{V} \leq &-\sum_{i=1}^{|\mathbf{F}|} k_{f,i}\left(\frac{(D_{a,i}\tilde{x}_f)^\top D_{a,i}\tilde{x}_f}{|D_{a,i}\tilde{x}_f|^a} + \frac{(D_{a,i}\tilde{x}_f)^\top D_{a,i}\tilde{x}_f}{|D_{a,i}\tilde{x}_f|^{-b}}\right)\\ &- \tilde{x}_f^\top M\tilde{x}_f + \bar{k}_g|D_b||S||\tilde{x}_g||\tilde{x}_f|,\end{aligned}$$

where $\bar{k}_g := \max\ k_{g_i}$. Since $M \succeq 0$, we have that

$$\begin{aligned}\dot{V}(\tilde{x}_f) \leq &-\underline{k}_f\sum_{i=1}^{|\mathbf{F}|}\left(|D_{a,i}\tilde{x}_f|^{2-a} + |D_{a,i}\tilde{x}_f|^{2+b}\right)\\ &+ \bar{k}_g|D_b||S||\tilde{x}_g||\tilde{x}_f|.\end{aligned}$$

where $\underline{k}_f := \min\ k_{f_i}$. Using the fact that $|D_{a,i}\tilde{x}_f|^{2-a} + |D_{a,i}\tilde{x}_f|^{2+b} \geq |D_{a,i}\tilde{x}_f|^2$ for all $\tilde{x}_f$ and $i \in \mathbf{F}$, we can write:

$$\begin{aligned}\dot{V}(\tilde{x}_f) \leq &-\frac{\underline{k}_f}{2}\sum_{i=1}^{|\mathbf{F}|}\left(|D_{a,i}\tilde{x}_f|^{2-a} + |D_{a,i}\tilde{x}_f|^{2+b}\right)\\ &-\frac{\underline{k}_f}{2}\sum_{i=1}^{|\mathbf{F}|}|D_{a,i}\tilde{x}_f|^2 + \bar{k}_g|D_b||S||\tilde{x}_g||\tilde{x}_f|.\end{aligned}$$

Let $\tilde{a} := 1 - 0.5a$ and $\tilde{b} = 1 + 0.5b$. Using Lemma 7 in the Appendix, we can further upper bound $\dot{V}$ as follows:

$$\begin{aligned}\dot{V}(\tilde{x}_f) \leq &-\frac{\underline{k}_f}{2}\left(\sum_{i=1}^{|\mathcal{F}|}|D_{a,i}\tilde{x}_f|^2\right)^{\tilde{a}} - \frac{\underline{k}_f}{N^{\frac{b}{2}}}\left(\sum_{i=1}^{|\mathbf{F}|}|D_{a,i}\tilde{x}_f|^2\right)^{\tilde{b}}\\ &-\frac{\underline{k}_f}{2}\sum_{i=1}^{|\mathbf{F}|}|D_{a,i}\tilde{x}_f|^2 + \bar{k}_g|D_b||S||\tilde{x}_f||\tilde{x}_g|,\\ \leq &-\frac{\underline{k}_f}{2}\left(|D_a\tilde{x}_f|^2\right)^{\tilde{a}} - \underline{k}_f N^{-\frac{b}{2}}\left(|D_a\tilde{x}_f|^2\right)^{\tilde{b}}\\ &-\frac{\underline{k}_f}{2}|D_a\tilde{x}_f|^2 + \bar{k}_g|D_b||S||\tilde{x}_f||\tilde{x}_g|.\end{aligned}$$

Using the inequality $|D_a\tilde{x}_f|^2 \geq 2\lambda_{\min}(D_a)V(\tilde{x}_f)$, we obtain

$$\begin{aligned}\dot{V}(\tilde{x}_f) \leq &-\frac{\underline{k}_f}{2}(2\lambda_{\min}(D_a)V(\tilde{x}_f))^{\tilde{a}} - \frac{\underline{k}_f}{2N^{\frac{b}{2}}}(2\lambda_{\min}(D_a)V(\tilde{x}_f))^{\tilde{b}}\\ &-\frac{\underline{k}_f}{2}\lambda_{\min}(D_a)^2|\tilde{x}_f|^2 + \bar{k}_g|D_b||S||\tilde{x}_f||\tilde{x}_g|,\end{aligned}$$

which implies that $\dot{V}$ can be upper bounded as: $\dot{V}(\tilde{x}_f) \leq -c_1V(\tilde{x}_f)^{\tilde{a}} - c_2V(\tilde{x}_f)^{\tilde{b}}$, for all $|\tilde{x}_f| \geq \frac{2\bar{k}_g|D_b||S|}{\underline{k}_f\lambda_{\min}(D_a)^2}|\tilde{x}_g|$. By [37, Thm. 4], this implies *uniform global fixed-time input-to-state stability* for system (25) with input $|\tilde{x}_g|$. ■

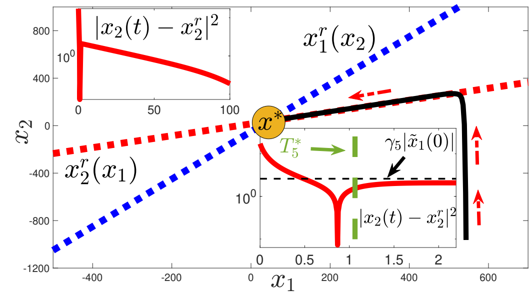

Fig. 4: Actions of players in Example 2. The centered inset shows in logarithmic scale the evolution in time of the distance $|x_2 - x_2^r|$. The black dashed line corresponds to the theoretical worst case ISS-bound, and the green dashed line indicates the time T_5^*.

Remark 7: In quadratic games with stubborn players (i.e, $k_{g,i} = 0$), it is well-known that system (5) guarantees exponential convergence to a neighborhood of the reaction curves of the active players [18, Theorem 2]. Proposition 5 generalizes this result for mixed games with F-players. □

The FXNES dynamics considered in this section make use of Oracles that provide real-time measurements or evaluations of the partial gradients $\nabla_i J_i$. Examples of applications where gradients (or signals proportional to the gradients) are available (or can be constructed) from measurements include multi-zone lighting control systems [38, pp. 79-92], where the zones can be modeled as players having access to measurements of each zone's illuminance, and formation-based games in mobile robots, where the distance between robots is usually proportional to the gradient of a potential function [39]. On the other hand, individual Oracles that perform gradient *evaluations* of $\nabla_i J_i(\cdot)$ might require access to the actions of the other players, complicating a distributed implementation. To prevent this, in the next section we introduce a class of *adaptive distributed model-free* FXNES dynamics for games where players have access only to *measurements* of their cost or payoff functions, e.g., the profit of firms in an market [18, Sec. II], the power generated by wind turbines in a wind farm [40], etc. These model-free dynamics will be distributed and will leverage the results established in Propositions 1-5.

V. FIXED-TIME NES WITHOUT ORACLES

We now consider *model-free* adaptive FXNES dynamics suitable for non-cooperative games with no gradient Oracles. To implement the adaptive dynamics, players need to have access only to real-time measurements of their costs functions J_i. Thus, the algorithms can be seen as *payoff-based* dynamics [5] or *zeroth-order* algorithms [15], [18], [22], which have not been studied before in the context of fixed-time NES.

To implement the model-free dynamics, each player i now updates its own action using the following equation:

$$x_i(t) = u_i(t) + \alpha_i\hat{\mu}_i(t),\ \ \hat{\mu}_i(t) := \sin\left(2\pi\frac{\omega_i}{\varepsilon_{2,i}}t + \varphi_i\right),\tag{26}$$

for all $t \geq 0$, where $\varphi_i \in (-\pi,\pi)$, $\alpha_i := \alpha\rho_{0,i} \in (0,1)$, $\omega_i > 0$, and $\varepsilon_{2,i} := \varepsilon_2\rho_{2,i} > 0$, $\varepsilon_2 > 0$, $\alpha > 0$, $\rho_{0,i}, \rho_{2,i} > 0$, and where u_i is an auxiliary state that we call the *nominal* action. To simplify our presentation and computations, we will focus on scalar actions $x_i \in \mathbb{R}$, for all $i \in \mathcal{V}$. However, all our results can also be applied to vector-valued actions by using (26) with multiple sinusoids with different frequencies $\omega_{i,j}$ assigned to the components $x_{i,j}$ of x_i, where $j \in \{1, 2, \dots, m_i\}$.

To update their nominal actions u_i, each player needs to sense its own cost function to generate the following signal:

$$F_i(x,t) := \frac{2}{\alpha_i} J_i(x)\hat{\mu}_i(t). \tag{27}$$

Note that F_i encodes global information of the game via the measured signal $J_i(x)$. Using F_i, we will construct fully decoupled and also distributed model-free FXNES dynamics. We will work with the following assumption, which is standard in the literature of averaging-based equilibrium seeking [15], [18].

Assumption 4: Let $\tilde{\omega}_i := \omega_i/\rho_{2,i}$. Then, $\tilde{\omega}_i \in \mathbb{Q}_{>0}$, $\tilde{\omega}_i \neq \tilde{\omega}_j$, $\tilde{\omega}_i \neq 2\tilde{\omega}_j$, and $\tilde{\omega}_i \neq 3\tilde{\omega}_j$ for all $i \neq j$. □

A. Decoupled Model-Free FXNES

The first model-free dynamics that we consider are fully decoupled, and given by the differential equations:

$$\dot{u}_i = -\zeta_i \left(\frac{k_i}{|\zeta_i|^a} + \frac{k_i}{|\zeta_i|^{-b}} \right), \; \dot{\zeta}_i = -\frac{1}{\varepsilon_{1,i}}\Big(\zeta_i - F_i(x,t)\Big), \tag{28}$$

where $a \in (0,1)$, $b > 0$, $\zeta_i \in \mathbb{R}$ is an individual auxiliary state used by each player, $\varepsilon_{1,i} := \varepsilon_1\rho_{1,i}$, $\rho_{1,i} > 0$, $\varepsilon_1 > 0$, F_i is given by (27), and the right-hand side of $\dot{u}_i$ is defined to be zero whenever $\zeta_i = 0$. These dynamics are continuous but not Lipschitz continuous. Figure 2 shows a scheme where two players implement the model-free FXNES dynamics (26)-(28) in a strongly monotone market (c.f. Example 1). As shown in the right plot of the figure, the nominal action u_i converges to the NE of the market in approximately the same amount of time as in the model-based FXNES dynamics. Note that, due to the presence of the probing signal in (26), the actual prices x_i will now converge to an $\mathcal{O}(\alpha)$-neighborhood of NE of the game.

The following Theorem is the first main result of this paper. The proof, presented in Section V-C, exploits the fixed-time convergence bounds established in Section IV.

Theorem 1: Suppose that Assumptions 1, 2 and 4 hold, and let T_1^* be given by (9). Then, $\forall\ \Delta > \nu > 0$, $\exists\ \varepsilon_1^* > 0$, s.t. $\forall\ \varepsilon_1 \in (0, \varepsilon_1^*)$ $\exists\ \alpha^* > 0$, s.t. $\forall\ \alpha \in (0,\alpha^*)$ $\exists\ \varepsilon_2^* > 0$, s.t. $\forall\ \varepsilon_2 \in (0,\varepsilon_2^*)$ all actions of the players with $|u(t_0) - x^*| \leq \Delta$, $|\zeta(t_0)| \leq \Delta$, and dynamics (26)-(28), satisfy:

$$|x(t) - x^*| \leq \beta(|u(t_0) - x^*|, t - t_0) + \nu, \quad \forall\, t \geq t_0 \geq 0, \tag{29}$$

where $\beta \in \mathcal{KL}_{\mathcal{T}}$, and $\beta(r,s) = 0, \quad \forall\ s > T_1^*$ and $\forall\ r \geq 0$. □

The result of Theorem 1 establishes a semi-global practical fixed-time convergence bound for the actions of the players using the model-free FXNES dynamics (28), with sufficiently small parameters $(\varepsilon_2, \alpha, \varepsilon_1)$. In this bound, the function β has the fixed-time convergence property and the residual error ν can be made arbitrarily small.

As in Section IV-C, for quadratic games with cost functions of the form (20), we can also consider *model-free* mixed dynamics, where the F-players implement the dynamics (28), and the G-players use $a = b = 0$, which recovers the model-free NES algorithms of [15], [18].

Corollary 1: Consider the mixed game under the definitions and assumptions of Proposition 5. Then, if Assumption 4 holds, $\forall\ \Delta > \nu > 0$, $\exists\ \varepsilon_1^* > 0$, s.t. $\forall\ \varepsilon_1 \in (0,\varepsilon_1^*)$, $\exists\ \alpha^* > 0$, s.t. $\forall\ \alpha \in (0,\alpha^*)$, $\exists\ \varepsilon_2^* > 0$, s.t. $\forall\ \varepsilon_2 \in (0,\varepsilon_2^*)$ all actions of the players with $|u(t_0) - x^*| \leq \Delta$, $|\zeta(t_0)| \leq \Delta$, and dynamics (26)-(28), satisfy:

$$|\tilde{x}_f(t)| \leq \beta(|\tilde{u}_f(t_0)|, t - t_0) + \gamma_5\|\tilde{x}_g\|_{[0,\infty]} + \nu, \tag{30}$$

for all $t \geq t_0 \geq 0$, where $\beta \in \mathcal{KL}_{\mathcal{T}}$ and $\beta(r,s) = 0$ for all $s > T_5^*$ and all $r \geq 0$. □

The results of Theorem 1 and Corollary 1 differ from previous results in the literature, e.g., [15], [18], which established only (semi-global practical) asymptotic or exponential results using standard averaging tools for Lipschitz continuous differential equations, which are not applicable in our setting. As shown in Figure 2, the fixed-time convergence property of β induces a dramatic improvement in the transient performance of the controller. When a lower bound for κ is known a priori, a minimum universal gain $\underline{k}$ can be selected to prescribe the convergence time. Note, however, that the parameters $(\varepsilon_{2,i}, \alpha_i, \varepsilon_{1,i})$ are always dependent on the set of initial conditions of the actions of the players (i.e., Δ), as well as the desired residual error (i.e., ν). In general, this dependence is unavoidable in averaging-based dynamics.

B. Distributed Model-Free FXNES with Dynamic Graphs

We now consider *model-free* FXNES dynamics with *heterogeneous* exponents for games for which there does not necessarily exist a potential function. In this case, players are allowed to share information with neighbors who are characterized by a time-varying communication graph $\mathcal{G}$. Specifically, each player i is endowed with an auxiliary vector state $\xi_i := [\xi_{i1}, \xi_{i2}, \dots, \xi_{i,N}]^\top \in \mathbb{R}^N$, which has as many components as there are players in the game. In this case, the nominal action u_i is updated as

$$\dot{u}_i = -\xi_{ii}\left(\frac{k_i}{\left(\xi_i^\top \xi_i\right)^{\frac{a_i}{2}}} + \frac{k_i}{\left(\xi_i^\top \xi_i\right)^{-\frac{b_i}{2}}} \right), \tag{31a}$$

$$\dot{\xi}_{ij} = \frac{1}{\varepsilon_1}\Big(\sum_{k\in\mathcal{N}_i(t)} \Big(\xi_{kj} - \xi_{ij}\Big) + \upsilon_{ij}\left(F_i(u,t) - \xi_{ij}\right)\Big), \tag{31b}$$

where F_i is defined as in (27), $\upsilon_{ij} = 1$ for $i = j$, $\upsilon_{ij} = 0$ for all $i \neq j$, and $\dot{u}_i := 0$ whenever $\xi_i^\top \xi_i = 0$. We make the following assumption on the time-varying graph $t \mapsto \mathcal{G}(t)$.

Assumption 5: The time-varying communication graph $t \mapsto \mathcal{G}(t)$ is strongly connected for all $t \geq 0$, and any two consecutive switching times (t_i, t_{i+1}) of the graph satisfy $t_{i+1} - t_i \geq \eta_1$, for some $\eta_1 > 0$. □

The dynamic communication scenario considered in Assumption 5 departs from the traditional time-invariant setting considered in the literature of averaging-based NES dynamics, e.g., [15], [18], [17], [26]. In particular, under Assumption 5, the model-free learning dynamics become a switching system.

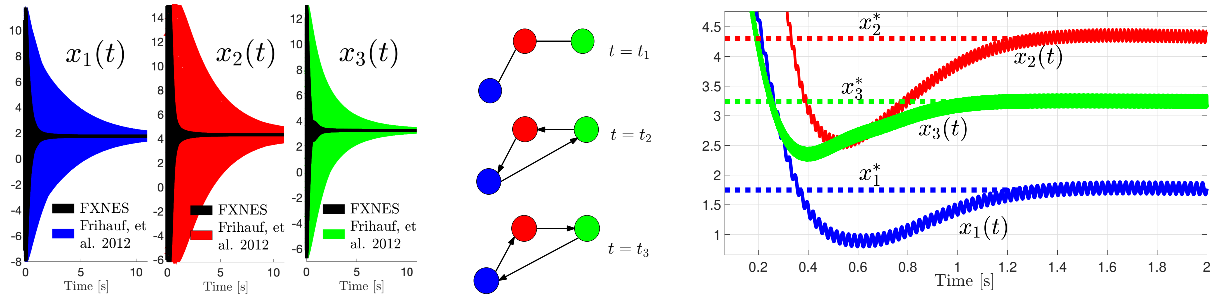

Fig. 5: Trajectories of the actions of the players in Example 3. The left plot compares the reachable sets of the model-free FXNES dynamics with the reachable sets generated by the dynamics of [18]. The graph switches between three configurations, shown in the centered plot.

The following Theorem is the second main result of this paper. The proof, presented in Section V-C, exploits the fixed-time convergence bounds established in Proposition 2 (for potential games) and Proposition 3 (for general strongly monotone games), as well as recent averaging results for set-valued systems [41], [21], [22].

Theorem 2: Suppose that Assumptions 1, 4, and 5 hold. Let $\mathcal{G}(t)$ be balanced for all $t \geq 0$. Then, if either Assumption 2 or Assumption 3 also hold, it follows that $\forall\ \Delta > \nu > 0$, and $\forall\ \eta_1 > 0$, $\exists\ \varepsilon_1^* > 0$, s.t. $\forall\ \varepsilon_1 \in (0, \varepsilon_1^*)$, $\exists\ \alpha^* > 0$, s.t. $\forall\ \alpha \in (0, \alpha^*)$, $\exists\ \varepsilon_2^* > 0$, s.t. $\forall\ \varepsilon_2 \in (0, \varepsilon_2^*)$ all actions of the players with $|u(t_0) - x^*| \leq \Delta$, $|\xi(t_0)| \leq \Delta$, and dynamics (26) and (31) satisfy:

$$|x(t)-x^*| \leq \beta(|u(t_0)-x^*|, t-t_0)+\nu, \quad \forall\ t \geq t_0 \geq 0, \quad (32)$$

where $\beta \in \mathcal{KL}_{\mathcal{T}}$, $\beta(r,s) = 0$, $\forall\ s > T_S^*$ and $\forall\ r \geq 0$, and: T_S^* is given by (14) for games that satisfy Assumption 2; T_S^* is given by (17) for games that satisfy Assumption 3. □

In essence, the result of Theorem 2 establishes a (semi-global practical) fixed-time convergence result to the NE x^*, for any balanced graph that is strongly connected and switching arbitrarily fast.

Next, we present a Corollary that covers the case when the communication graph is not necessarily balanced. Here, the (semi-global practical) fixed-time convergence result holds when η_1 is sufficiently large, i.e., under "slow" switching.

Corollary 2: Suppose that Assumptions 1, 4, and 5 hold. Then, if either Assumption 2 or Assumption 3 also hold, it follows that $\forall\ \Delta > \nu > 0$, $\exists\ \varepsilon_1^* > 0$ s.t. $\forall\ \varepsilon_1 \in (0, \varepsilon_1^*)$, $\exists\ \alpha^* > 0$ s.t. $\forall\ \alpha \in (0, \alpha^*)$, $\exists\ \varepsilon_2^* > 0$ s.t. $\forall\ \varepsilon_2 \in (0, \varepsilon_2^*)$ s.t. $\exists\ \eta_1^* > 0$ s.t. $\forall\ \eta_1 > \eta_1^*$ all actions of the players with $|u(t_0) - x^*| \leq \Delta$, $|\xi(t_0)| \leq \Delta$, and dynamics (26) and (31) satisfy the bound (32). □

The results of Theorem 2 and Corollary 2 permit the incorporation of time-varying graphs (slow and arbitrarily fast) into the fast dynamics of the averaging-based NES algorithms. For standard multi-agent model-free optimization problems, related techniques with *asymptotic* convergence bounds were presented in [14] and [42].

Remark 8: While Assumption 5 considers graphs that are strongly connected for all times, it is possible to consider graphs that are sporadically disconnected. Indeed, this case can be modeled as a switching system with stable modes (i.e., with strongly connected graphs) and unstable modes (i.e., with disconnected graphs). Using this framework, the stability properties of the model-free dynamics can be studied using the tools from [21, Sec. 5] imposing a minimum dwell-time and a time activation constraint in the unstable modes.

We conclude this section with a numerical example that illustrates the performance of the model-free FXNES dynamics (26)-(31) in settings where the cost functions of the players are now generated by *dynamical systems* that have well-defined input-to-output steady state cost functions.

Example 3: Consider a non-cooperative game with three players, where each player i is represented by a scalar *dynamical system* with state θ_i and output y_i, with dynamics

$$\varepsilon_0 \dot{\theta}_i = -\mathcal{A}_i \theta_i + \mathcal{B}_i x_i, \qquad y_i = -0.5\theta^\top D^i \theta - \theta^\top d^i, \quad (33)$$

with $\mathcal{A} = [2, 1.5, 1.5]$, $\mathcal{B} = [3, 2, 3]$, $1 \gg \varepsilon_0 > 0$, and (D^i, d^i) are taken from [43, Section VI]. Since each dynamical system (33) has an exponentially stable equilibrium given by $\theta_1^* = 3/2u_1$, $\theta_2^* = 4/3u_2$, $\theta_3^* = 2u_3$, the steady state input-to-output maps obtained by substituting θ^* for θ in the outputs y_i are also quadratic functions J_i as a function of u. Indeed, they are precisely the same cost functions considered in Example 1. However, in this case, each player implements the dynamics (31) in feedback loop with the dynamics (33) using in equation (27) the signal y_i instead of J_i. The graph switches every 0.1 seconds. Figure 5 shows the emerging behavior for values of $\varepsilon_0 = 4 \times 10^{-6}$ and the same control parameters considered in Example 1. The left plot compares the reachable set of the dynamics in the time interval $[0, 10]$, versus the model-free dynamics of [18]. The right plot shows the time history of the actions of the players. Note that, whereas the stability results of Theorems 1-2 did not cover this scenario, a bound of the form (32) can be readily established via singular perturbation theory in the appropriate time scale [20]. □

C. Proof of Theorems 1-2

We carry out the proofs of Theorems 1-2 simultaneously. We organize the proof in six main steps.

Step 1: Setting up the model. We start by writing the dynamics (28) and (31) as a time-invariant system of the form (2). To do this, we can use trigonometric identities to write the probing sinusoid signal $\hat{\mu}_i$ of each player as $\hat{\mu}_i(t) = \cos(\varphi_i)\sin(2\pi\frac{\omega_i}{\varepsilon_{2,i}}t) + \sin(\varphi_i)\cos(2\pi\frac{\omega_i}{\varepsilon_{2,i}}t)$. Since for any $\varphi_i \in$

$(-\pi, \pi)$, the vector $[\cos(\varphi_i), \sin(\varphi_i)]^\top$ describes the Cartesian coordinates of a point on the unit circle, we can generate each dither signal $\hat{\mu}_i$ by using a linear dynamic oscillator with state $\bar{\mu}_i := [\hat{\mu}_i, \tilde{\mu}_i]^\top \in \mathbb{R}^2$, restricted to evolve in $\mathbb{S}^1$, given by

$$\begin{bmatrix} \dot{\hat{\mu}}_i \\ \dot{\tilde{\mu}}_i \end{bmatrix} = \frac{2\pi}{\varepsilon_{2,i}} \mathcal{R}_{\omega_i} \begin{bmatrix} \hat{\mu}_i \\ \tilde{\mu}_i \end{bmatrix}, \quad \mathcal{R}_{\omega_i} := \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}, \quad (34)$$

for all $i \in \mathcal{V}$. Indeed, since $\bar{\mu}_i$ is restricted to $\mathbb{S}^1$, it follows that $\hat{\mu}_i(0)^2 + \tilde{\mu}_i(0)^2 = 1$, and each trajectory $t \mapsto \hat{\mu}_i(t)$ generated by (34) coincides with the dither signal of (26) with $\varphi_i = \arctan(\hat{\mu}_i(0)/\tilde{\mu}_i(0))$. After defining (34), we proceed to define the vector of actions of the game $u := [u_1, u_2, \ldots, u_N]^\top$, the matrix $\mathbf{K} := \text{diag}([k_1, k_2, \ldots, k_N])$, the vectors $\mu := [\hat{\mu}_1, \tilde{\mu}_1, \hat{\mu}_2, \tilde{\mu}_2, \ldots, \hat{\mu}_N, \tilde{\mu}_N]^\top$, $\zeta := [\zeta_1, \zeta_2, \ldots, \zeta_N]^\top$, $\mathbf{J}(x, \hat{\mu}) := [J_1(x)\hat{\mu}_1, J_2(x)\hat{\mu}_2, \ldots, J_N(x)\hat{\mu}_N]^\top$ and $\Psi(\zeta) := [\psi(\zeta_1), \psi(\zeta_2), \ldots, \psi(\zeta_N)]^\top$, the matrix $A := \text{diag}([\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N]^\top)$, and $E(x, \hat{\mu}) := 2(A^{-1}\mathbf{J}(x, \hat{\mu}))$, where the real valued mapping $z \mapsto \psi(z)$ is defined as $\psi(z) := (z^\top z)^{-a} + (z^\top z)^b$, for all $z \in \mathbb{R}^N$. Next, we define a block diagonal matrix $\mathcal{R}_{\tilde{\omega}} \in \mathbb{R}^{2N \times 2N}$, which is parametrized by a vector of gains $\tilde{\omega} = [\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_N]^\top$ (c.f. Assumption 4), with the i^{th} diagonal block of $\mathcal{R}_{\tilde{\omega}}$ defined as (34). Using this construction, the overall dynamics can be written as

$$(u, \zeta, \mu) \in \mathbb{R}^N \times M\mathbb{B} \times \mathbb{T}^N, \begin{cases} \dot{u} = -\mathbf{K}\text{diag}(\zeta)\Psi(\zeta), \\ \varepsilon_1 \dot{\zeta} = -\Lambda\Big(\zeta - E(x, \hat{\mu})\Big), \\ \varepsilon_2 \dot{\mu} = 2\pi \mathcal{R}_{\tilde{\omega}} \mu, \end{cases} \quad (35)$$

where $\mathbb{T}^N := \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, and Λ is a diagonal matrix with entries given by $1/\rho_{1,i}$, for $i \in \mathcal{V}$. The constant $M > 0$ is introduced only for the purpose of analysis to restrict the state ζ to evolve in a compact set. This constant can be selected arbitrarily large to capture every solution of interest, and in the next steps we will establish a lower bound on M to guarantee that every solution of (35) with $|u(0) - u^*| \leq \Delta$ and $|\zeta(0)| \leq \Delta$ is complete under appropriate tuning of the parameters.

Similarly, in order to write the overall dynamics (31) in vectorial form, we first define the following additional vectors and matrices: $\xi_D := [\xi_{11}, \xi_{22}, \ldots, \xi_{NN}]^\top$, $\xi_i := [\xi_{i1}, \xi_{i2}, \ldots, \xi_{iN}]^\top$, $\xi := [\xi_1^\top, \xi_2^\top, \ldots, \xi_N^\top]^\top$, $\tilde{\Psi}(\xi) := [\tilde{\psi}_1(\xi_1), \tilde{\psi}_2(\xi_2), \ldots, \tilde{\psi}_N(\xi_N)]^\top$, $\mathbf{b}_i := [\upsilon_{i1}, \upsilon_{i2}, \ldots, \upsilon_{iN}]^\top$, $\mathbf{b} := [\mathbf{b}_1^\top, \mathbf{b}_2^\top, \ldots, \mathbf{b}_N^\top]^\top$, $\tilde{E} := E(x, \hat{\mu}) \otimes \mathbf{1_N}$, where for each player i the real valued mapping $z \mapsto \tilde{\psi}_i(z)$ is defined as $\tilde{\psi}_i(z) := (z^\top z)^{-a_i} + (z^\top z)^{b_i}$, for all $z \in \mathbb{R}^N$. We also define the matrices $\mathbf{B} := \text{diag}(\mathbf{b})$ and $\mathbf{L}_\sigma = L_\sigma \otimes I_{N \times N}$, where L_σ is the Laplacian matrix of the graph $\mathcal{G}$, and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{G}$ is a switching signal mapping to the finite collection of possible digraphs $\mathcal{G}$. By using these definitions, the overall dynamics can be written in compact form as (2), with states $(u, \xi, \mu) \in \mathbb{R}^N \times M\mathbb{B} \times \mathbb{T}^N$, and dynamics:

$$\dot{u} = -\mathbf{K}\text{diag}(\xi_D)\tilde{\Psi}(\xi), \quad (36a)$$

$$\varepsilon_1 \dot{\xi} = -(\mathbf{L}_\sigma + \mathbf{B})\xi + \mathbf{B}\tilde{E}, \quad (36b)$$

$$\varepsilon_2 \dot{\mu} = 2\pi \mathcal{R}_{\tilde{\omega}} \mu. \quad (36c)$$

To analyze the dynamics (36) under arbitrarily fast switching of the graph, let us consider the set-valued mapping $\mathcal{T} : \mathbb{R}^{NN} \rightrightarrows \mathbb{R}^{NN}$, defined at each point ξ as $\mathcal{T}(\xi) := \overline{\text{con}} \bigcup_{\sigma \in \mathcal{G}} (\mathbf{L}_\sigma + \mathbf{B})\xi$. Using $\mathcal{T}$, we write the switching system (36) as a dynamical system of the form (2) with states $(u, \xi, \mu) \in \mathbb{R}^N \times M\mathbb{B} \times \mathbb{T}^N$, and dynamics

$$\dot{u} = -\mathbf{K}\text{diag}(\xi_D)\tilde{\Psi}(\xi), \quad (37a)$$

$$\varepsilon_1 \dot{\xi} \in -\mathcal{T}(\xi) + \mathbf{B}E(x, \hat{\mu}), \quad (37b)$$

$$\varepsilon_2 \dot{\mu} = 2\pi \mathcal{R}_{\tilde{\omega}} \mu. \quad (37c)$$

In particular, by the Relaxation Theorem, the set of solutions of (37) is dense in the set of solutions of (36), and every solution of (36) is also a solution of (37), [28, Cor. 4.24].

Step 2: First Application of Averaging Theory: We now proceed to analyze the dynamics (35) and (37) via averaging theory for non-smooth systems, [21], [22], [41], [44]. To do this, we use the following lemma from [22, Appendix A].

Lemma 3: Suppose that Assumption 4 holds. Then, there exists $\theta > 0$ such that every solution μ of the linear oscillators of (34) with $\varepsilon_2 = 1$ and $\bar{\mu}(0) \in \mathbb{T}^n$ satisfies:

$$\frac{1}{\ell\theta} \int_0^{\ell\theta} \hat{\mu}(t)\hat{\mu}(t)^\top dt = \frac{1}{2} I_n, \quad \int_0^{\ell\theta} \hat{\mu}(t) dt = 0, \quad (38)$$

for all $\ell \in \mathbb{Z}_{\geq 1}$, where $\hat{\mu} := [\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_N]^\top$. □

By using the properties (38), we can proceed to average the dynamics of (u, ζ) in (35), and (u, ξ) in (36), along the trajectories $t \mapsto \hat{\mu}(t)$. To compute the average, let $\bar{\alpha} := \max_i \alpha_i$, and consider a Taylor expansion of each cost function $J_i(u + A\hat{\mu})$ around u, given by $J_i(u + A\hat{\mu}) = J_i(u) + \sum_{k=1}^N \alpha_k \hat{\mu}_k \frac{\partial J_i(u)}{\partial u_k} + \mathcal{O}_i(\bar{\alpha}^2)$, where $\mathcal{O}_i(\bar{\alpha}^2)$ represents higher order terms that are bounded on compact sets, and which can be made arbitrarily small by decreasing $\bar{\alpha}$. Using properties (38), it then follows that $\frac{1}{\ell\theta} \int_0^{\ell\theta} \hat{\mu}_i(s) J_i(u + A\hat{\mu}(s)) ds = \frac{\alpha_i}{2} \frac{\partial J_i(u)}{\partial u_i} + \mathcal{O}_i(\bar{\alpha}^2)$, for each $i \in \mathcal{V}$. Therefore, we have $\frac{1}{\ell\theta} \int_0^{\ell\theta} E\left(u + \hat{\mu}(s), \hat{\mu}(s)\right) ds = G(u) + \mathcal{O}_i(\bar{\alpha})$, where G is the pseudo-gradient of the game. We then obtain the following average dynamics of system (35) with states $(u^A, \zeta^A) \in \mathbb{R}^N \times M\mathbb{B}$:

$$\dot{u} = -\mathbf{K}\text{diag}(\zeta)\Psi(\zeta), \ \varepsilon_1 \dot{\zeta} = -\Lambda\Big(\zeta - G(\hat{u})\Big) + \mathcal{O}(\bar{\alpha}), \quad (39)$$

as well as the average dynamics of system (37), with states $(u^A, \xi^A) \in \mathbb{R}^N \times M\mathbb{B}$ and dynamics

$$\dot{u} = -\mathbf{K}\text{diag}(\xi_D)\tilde{\Psi}(\xi) \quad (40a)$$

$$\varepsilon_1 \dot{\xi} \in -\mathcal{T}(\xi) + \mathbf{B}\left(G(u) \otimes \mathbf{1_N}\right) + \mathcal{O}(\bar{\alpha}). \quad (40b)$$

We analyze systems (39) and (40) as $\mathcal{O}(\bar{\alpha})$-perturbed versions of *nominal average* systems with dynamics

$$\dot{u} = -\mathbf{K}\text{diag}(\zeta)\Psi(\zeta), \quad \varepsilon_1 \dot{\zeta} = -\Lambda\Big(\zeta - G(\hat{u})\Big), \quad (41)$$

and

$$\dot{u} = -\mathbf{K}\text{diag}(\xi_D)\tilde{\Psi}(\xi), \quad \varepsilon_1 \dot{\xi} \in -\mathcal{T}(\xi) + \mathbf{B}\left(G(u) \otimes \mathbf{1_N}\right), \quad (42)$$

respectively.

Step 3: Second Application of Averaging Theory: For ε_1 sufficiently small, the nominal average dynamics (41) and (42) are also in standard singular perturbation form; see [44],

[41], [21], [22]. To find the *boundary layer* dynamics, let $\tau = t/\varepsilon_1$, and consider system (41) in the τ-time scale with state $(u^A, \zeta^A) \in \mathbb{R}^N \times M\mathbb{B}$, and dynamics given by

$$\frac{\partial u}{\partial \tau}^A = -\varepsilon_1 \mathbf{K}\text{diag}(\zeta^A)\Psi(\zeta^A), \ \frac{\partial \zeta^A}{\partial \tau} = -\Lambda\Big(\zeta^A - G(u^A)\Big). \tag{43}$$

Similarly, (42) with state $(u^A, \xi^A) \in \mathbb{R}^N \times M\mathbb{B}$ becomes:

$$\frac{\partial u}{d\tau}^A = -\varepsilon_1 \mathbf{K}\text{diag}(\xi^A_D)\tilde{\Psi}(\xi^A), \tag{44}$$

$$\frac{d\xi^A}{d\tau} \in -\mathcal{T}(\xi^A) + \mathbf{B}\left(G(u^A) \otimes \mathbf{1_N}\right). \tag{45}$$

Setting $\varepsilon_1 = 0$, we obtain the boundary layer dynamics

$$\frac{\partial \zeta_{bl}}{\partial \tau} = -\Lambda\Big(\zeta_{bl} - G(u_{bl})\Big), \quad \text{and} \tag{46a}$$

$$\frac{d\xi_{bl}}{d\tau} \in -\mathcal{T}(\xi_{bl}) + \mathbf{B}\left(G(u_{bl}) \otimes \mathbf{1_N}\right), \tag{46b}$$

respectively, with u_{bl} acting as a constant. Since Λ is diagonal and positive definite, the linear boundary layer dynamics (46a) render the equilibrium $\zeta^*_{bl} = G(u_{bl})$ globally exponentially stable. Similarly, since $\upsilon_{ii} = 1$ for all $i \in \mathcal{V}$, for each $\sigma \in \mathcal{G}$ the matrix $-(L_\sigma \otimes I_{N\times N} + B)$ is Hurwitz [27, Lemma 1.6]. Moreover, by Assumption 5 and [45, Lemma 5], if the graph $\mathcal{G}$ is also balanced, the matrix $\mathcal{M}_\sigma := (L_\sigma \otimes I_{N\times N} + B) + (L_\sigma \otimes I_{N\times N} + B)^\top$ is positive definite for each $\sigma \in \mathcal{G}$. Since the set of possible graphs satisfying Assumption 5 with N players is compact, there exists $\lambda^* > 0$ such that $\min\{\text{eig}(\mathcal{M}_\sigma)\} > \lambda^*$ for all $\sigma \in \mathcal{G}$. Using this property, we can study the stability properties of system (46b) via the change of coordinates $\tilde{\xi}_{bl} = \xi_{bl} - \mathbf{1_N} \otimes G(u_{bl})$, which leads to:

$$\frac{d\tilde{\xi}_{bl}}{d\tau} \in -\mathcal{T}(\tilde{\xi}_{bl} + \mathbf{1_N} \otimes G(u_{bl})) + \mathbf{B}\left(G(u_{bl}) \otimes \mathbf{1_N}\right). \tag{47}$$

Recall that the mapping $\mathcal{T}(\cdot)$ describes a convex combination of linear systems with matrices $\mathbf{L}_\sigma + \mathbf{B}$, where $\sigma \in \mathcal{G}$. If we substitute $\mathcal{T}(\cdot)$ by one of these systems, (46b) becomes

$$\begin{aligned} -(\mathbf{L}_\sigma + \mathbf{B})(\tilde{\xi}_{bl} + \mathbf{1_N} \otimes G(u_{bl})) + \mathbf{B}\left(G(u_{bl}) \otimes \mathbf{1_N}\right) \\ = -\left(\mathbf{L}_\sigma + \mathbf{B}\right)\tilde{\xi}_{bl}, \end{aligned} \tag{48}$$

where we used the definition of $\mathbf{L}_\sigma$ and the facts that $(L_\sigma \otimes I_{N\times N})(\mathbf{1_N} \otimes G(u_{bl})) = 0$, and $\mathbf{B}(\mathbf{1_N} \otimes G(u_{bl})) = \mathbf{B}\left(G(u_{bl}) \otimes \mathbf{1_N}\right)$. Let $V(\tilde{\xi}_{bl}) = 0.5\tilde{\xi}_{bl}^\top \tilde{\xi}_{bl}$. Since for each $\sigma \in \mathcal{G}$ the matrix $\mathbf{L}_\sigma + \mathbf{B}$ is Hurwitz, it follows that $\nabla V(\tilde{\xi}_{bl})^\top(-\left(\mathbf{L}_\sigma + \mathbf{B}\right)\tilde{\xi}_{bl}) \leq -\lambda^*|\tilde{\xi}_{bl}|^2$. Using the fact that a common Lyapunov function for a finite collection of continuous vector fields is also a Lyapunov function for their convex combination [46, Cor. 2.3], and the quadratic form of $V(\tilde{\xi}_{bl})$, we obtain that system (46b) renders the origin exponentially stable, and every solution converges to $\xi^*_{bl} = \mathbf{1}_N \otimes G(u_{bl})$, which satisfies $\xi^{*\top}_{bl,i}\xi^*_{bl,i} = G(u_{bl})^\top G(u_{bl})$, for all $i \in \mathcal{V}$.

Therefore, it follows that the singularly perturbed systems (41) and (42) have well-defined *reduced systems*, which correspond to the right hand side of $\dot{u}$ in (41) and (42), with ζ and ξ substituted by ζ^*_{bl} and ξ^*_{bl}, respectively. Thus, their respective reduced systems, with state $z \in \mathbb{R}^N$, are:

$$\dot{z} = \mathbf{K}\text{diag}(G(z))\Upsilon(z), \quad \text{and} \quad \dot{z} = \mathbf{K}\text{diag}(G(z))\tilde{\Upsilon}(z), \tag{49}$$

respectively, where Υ is given by $\Upsilon(z) := [\psi(G_1(z)), \psi(G_2(z)), \ldots, \psi(G_N(z))]^\top \in \mathbb{R}^N$, and $\tilde{\Upsilon}(z) := [\tilde{\psi}_1(G(z)), \tilde{\psi}_2(G(z)), \ldots, \tilde{\psi}_N(G(z))]^\top \in \mathbb{R}^N$. Note that systems (49) are precisely the FXNES dynamics (8) and (13), whose stability properties were already established in Propositions 1-5.

Step 4: β-SGPAS for Second Singularly Perturbed System: Using the result of Step 3, we proceed to apply averaging results for non-smooth systems in order to establish suitable stability properties for the singularly perturbed dynamics (41) and (42). In particular, by [21, Lemma 6], systems (41) and (42) render the compact set $\mathcal{A} := \{x^*\} \times M\mathbb{B}$ β-SGPAS as $\varepsilon_1 \to 0^+$, with the same $\mathcal{KL}$ bound β obtained in Propositions 1-5, i.e., (41) and (42) are actually β-SGPFXS. In particular, $\forall\ \delta > \nu > 0$, $\exists\ \varepsilon^*_1 > 0$ such that $\forall\ \varepsilon_1 \in (0, \varepsilon^*_1)$ every solution of systems (41) and (42) with $u(0) \in \{x^*\} + \delta\mathbb{B}$ satisfies the following bound:

$$|\eta^a(t)|_{\mathcal{A}} \leq \beta(|\eta^a(0)|_{\mathcal{A}}, t) + \nu, \tag{50}$$

for all $t \in \text{dom}(\eta^A)$, where $\eta^A := (u^A, \xi^A)$ for (41), and $\eta^A := (u^A, \zeta^A)$ for (42), and where $\beta \in \mathcal{KL}_{\mathcal{T}}$ comes from Propositions 1-5. Since the right-hand side of (42) is outer-semicontinuous, locally bounded, and convex-valued [28, Ch. 5], we obtain via [28, Thm. 7.21] and [22, Prop. 6] that the $\mathcal{O}(\bar{\alpha})$-perturbed systems (39) and (40) render β-SGPAS as $(\alpha, \varepsilon_1) \to 0^+$ the same compact set $\mathcal{A}$, with the same $\beta \in \mathcal{KL}_{\mathcal{T}}$, i.e., $\forall\ \Delta > \nu > 0\ \exists\ \varepsilon^*_1 > 0$, such that $\forall\ \varepsilon_1 \in (0, \varepsilon^*_1)$ $\exists\ \alpha^* > 0$ such that $\forall\ \alpha \in (0, \alpha^*)$ every solution of (39) and (40) with $|x^A(0) - x^*| \leq \Delta$ satisfies a bound of the form (50).

Step 5: β-SGPAS for First Singularly Perturbed System: Having established stability properties for the average dynamics (39) and (40), we can now use averaging theory to directly establish stability properties for the singularly perturbed system (37). Indeed, given that the oscillator dynamics in (35) and (37) evolve in (and render UGAS) the set $\mathbb{T}^n$, by averaging results for non-smooth systems [41, Thm. 2] we obtain that systems (35) and (37) render the compact set $\mathcal{A} \times \mathbb{T}^N$ β-SGPAS as $(\varepsilon_2, a, \varepsilon_1) \to 0^+$ with the same $\mathcal{KL}_{\mathcal{T}}$ bound β of Step 4. This establishes that for each feasible tuple of parameters (k_i, α_i, b_i) of player i, and $\forall\ \Delta > \nu > 0\ \exists\ \varepsilon^*_1 > 0$ such that $\forall\ \varepsilon_1 \in (0, \varepsilon^*_1)$ $\exists\ \alpha^* \in (0, \nu/2)$, such that $\forall\ \alpha \in (0, \alpha^*)$ $\exists\ \varepsilon^*_2 > 0$ such that $\forall\ \varepsilon_2 \in (0, \varepsilon^*_2)$ each solution satisfies

$$|\mathbf{y}(t)|_{\mathcal{A}\times\mathbb{T}^N} \leq \beta(|\mathbf{y}(0)|_{\mathcal{A}\times\mathbb{T}^N}, t) + \frac{\nu}{2}, \tag{51}$$

for all $t \in \text{dom}(\mathbf{y})$, where $\mathbf{y} := (\hat{u}, \zeta, \mu)$ for system (35), and $\mathbf{y} := (\hat{u}, \xi, \mu)$ for system (37). Given that by definition of solutions we have that $|\zeta, \mu|_{M\mathbb{B}\times\mathbb{T}^N} = 0$ and $|\xi, \mu|_{M\mathbb{B}\times\mathbb{T}^N} = 0$, and therefore $|\mathbf{y}|_{\mathcal{A}\times\mathbb{T}^N} = |u - x^*|$, it then follows that the vector of actions of the players satisfies the bound $|u(t) - x^*| \leq \beta(|u(0) - x^*|, t) + \frac{\nu}{2}$, for all $t \in \text{dom}(\mathbf{y})$. Using $\bar{\alpha} = \max_{i\in\mathcal{V}} \alpha_i$, the previous $\mathcal{KL}_{\mathcal{T}}$ bound, $\bar{\alpha} \in (0, \min\{\nu/2, \alpha^*\})$, the triangle inequality, and the fact that $x = u + A\hat{\mu}$, we obtain the convergence bounds (29) and (32).

Step 6: Existence of Complete Solutions: Finally, we exploit the linear structure of the dynamics of the states ζ and ξ to show that, provided M is sufficiently large, the restriction of the states ζ and ξ to the compact set $M\mathbb{B}$ is inconsequential to establish the existence of complete solutions for the model-free FXNES dynamics from initial conditions satisfying $|\zeta(0)| \leq \Delta$ and $|\xi(0)| \leq \Delta$. Without loss of generality, we assume that $\nu < 1$. Due to the bound (51), the fact that for any $M > 0$ the set $M\mathbb{B}$ is compact, and the forward invariance of the compact set $\mathbb{T}^n$, for sufficiently small parameters $(\varepsilon_2, a, \varepsilon_1)$ (see Step 5) the model-free FXNES dynamics have no finite escape times from any given arbitrarily large compact set of initial conditions. Thus, any maximal solution of the restricted dynamics (35) and (37) with a bounded time domain must stop due to ζ or ξ leaving the set $M\mathbb{B}$. To show that this cannot occur when M is sufficiently large, let $\Delta > \nu$ be given. Define the set $K_0 := \{u \in \mathbb{R}^n : |u - x^*| \leq \beta(\Delta, 0) + 1\}$. This set is compact due to the continuity of β. Therefore, there exists $m_1 > 0$ such that $K_0 \subset \{x^*\} + m_1\mathbb{B}$. By continuity of the pseudogradient $G(\cdot)$, there exits $m_2 > 0$ such that $|G(u)| \leq m_2$ for all u such that $|u - x^*| \leq m_1$. Let $g(u) := (G(u) \otimes \mathbf{1_N})$. By exponential stability and linearity of the ξ-dynamics in (42), for each bounded signal $t \mapsto g(u(t))$, the trajectories of ξ in (42) satisfy the bound

$$|\xi(t)| \leq c \exp\left(-\frac{\lambda}{\varepsilon_1}t\right)|\xi(0)| + c\|\mathbf{B}\| \sup_{0\leq\tau\leq t}|g(u(\tau))|, \quad (52)$$

for some $c > 0$ and $\lambda > 0$. Thus, by setting $M := c(\Delta + \|\mathbf{B}\|Nm_2) + 1$, and using the result of Step 4, there exists $\varepsilon_1^* > 0$ such that for all $\varepsilon_1 \in (0, \varepsilon_1^*)$ every solution of system (42) with $|\hat{u}^A(0) - x^*| \leq \Delta$ and $|\xi^A(0)| \leq \Delta$ is complete and satisfies inequality (50) with $\nu = \tilde{\nu}/(2Lc\|\mathbf{B}\|)$, and $\Delta > \nu > \tilde{\nu} > 0$. Moreover, since $\beta \in \mathcal{KL}_{\mathcal{T}}$, there exists a $T^* > 0$ such that $|\hat{u}^A(t) - x^*| \leq 2\nu$ for all $t \geq T^*$. By Lipschitz continuity of the pseudogradient, this implies that $|G(u^A(t))| \leq \tilde{\nu}/(c\|\mathbf{B}\|)$ for all $t \geq T^*$. Therefore, by using (52), the trajectories ξ satisfy $|\xi(t)| \leq \tilde{\nu}$, for all $t \geq T^*$. By [28, Cor. 7.7], it follows that there exists an Ω-limit set $\Omega_{\varepsilon_1,\Delta}$ satisfying $\Omega_{\varepsilon_1} \subset (x^* + \nu\mathbb{B}) \times \tilde{\nu}\mathbb{B}$, that is asymptotically stable for system (42) with basin of attraction containing the compact set $(u^* + \Delta\mathbb{B}) \times \Delta\mathbb{B}$. Now, since system (40) is an $\mathcal{O}(\bar{\alpha})$-perturbed version of (42), by closeness of solutions between nominal and perturbed well-posed systems [28, Prop. 6.34], for any $\varepsilon_1 \in (0, \varepsilon_1^*)$ and any $\epsilon > 0$ there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*)$ the trajectories of (40) will remain ϵ-close to the trajectories of (42) on arbitrarily large compact time domains, and will also satisfy a bound of the form (50). Moreover, by closeness of solutions between the trajectories of the average dynamics (40) and the (u, ξ)-components of the original dynamics (37), for any $\varepsilon_1 \in (0, \varepsilon_1^*)$ and $\alpha \in (0, \alpha_{\varepsilon_1}^*)$, and $\epsilon > 0$, there exits $\varepsilon_2^* > 0$ such that for all $\varepsilon_2 \in (0, \varepsilon_2^*)$ the trajectories of the FxTNES dynamics and the average dynamics (40) will also be ϵ-close on arbitrarily large compact time domains [41, Thm. 1], and will also satisfy a bound of the form (51). Thus, using $\epsilon \in (0, 1/2)$ and the definition of M, there exist parameters $(\varepsilon_2, \alpha, \varepsilon_1)$ such that every trajectory ξ of system (37) with $|\xi(0)| \leq \Delta$ and $|u(0) - x^*| \leq \Delta$ will necessarily remain in the interior of $M\mathbb{B}$, which contradicts the assumption that ξ and ζ leave the set $M\mathbb{B}$. An essentially identical argument holds for ζ in (35). This establishes the result for Theorems 1 and 2. Corollary 1 follows exactly the same steps, with the only difference that instead of using averaging results for systems having a UGAS compact set, we use averaging results for non-smooth systems having a well-defined average system with ISS properties [47, Thm. 2]. ■

D. Proof of Corollary 2

To prove Corollary 2, we first assign a bijection between the set of possible graphs $\mathcal{G}$ satisfying Assumption 5 and the set $Q := \{1, 2, 3, \ldots, \bar{q}\}$, where $\bar{q} \in \mathbb{Z}_{>0}$. In this way, the learning dynamics (36) with switching graphs can be modeled as a switching system with states $(u, \xi, \mu, q) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{T}^N \times Q$, where each value of q represents a different configuration of the graph. We proceed to induce an average dwell-time constraint on how frequently the signal q switches. To do this, we interconnect the dynamics (36) with the following hybrid automaton that we use to generate the switching signal q:

$$(q, \tau) \in Q \times [0, N_1], \quad \begin{cases} \dot{q} = 0 \\ \dot{\tau} \in [0, \eta_1] \end{cases}, \quad (53a)$$

$$(q, \tau) \in Q \times [1, N_1], \quad \begin{cases} q^+ \in Q \backslash \{q\} \\ \tau^+ = \tau - 1 \end{cases}, \quad (53b)$$

where $N_1 > 1$ is called the chatter bound, and $1/\eta_1$ is called the dwell-time. Every trajectory q generated by system (53) satisfies an average-dwell time constraint with dwell-time $1/\eta_1$, and every signal satisfying an average dwell-time constraint with dwell-time $1/\eta_1$ can be generated by the automaton (53) under an appropriate choice of initial conditions [28, Ch. 2]. Therefore, the closed-loop system is now a hybrid dynamical system with continuous-time dynamics evolving on the set $\mathbb{R}^N \times M\mathbb{B} \times \mathbb{T}^N \times Q \times [0, N_1]$, given by

$$\dot{u} = -\mathbf{K}\text{diag}(\xi_D)\tilde{\Psi}(\xi), \quad \varepsilon_2\dot{\mu} = 2\pi R_\kappa \mu \quad (54a)$$

$$\varepsilon_1\dot{\xi} = -(\mathbf{L}_\sigma + \mathbf{B})\xi + \mathbf{B}\tilde{E}, \quad \dot{q} = 0, \quad \dot{\tau} \in [0, \eta_1], \quad (54b)$$

and discrete-time dynamics evolving on the set $\mathbb{R}^N \times M\mathbb{B} \times \mathbb{T}^N \times Q \times [1, N_1]$, and given by

$$u^+ = u, \ \xi^+ = \xi, \ \mu^+ = \mu, \ q^+ \in Q\backslash\{q\}, \ \tau^+ = \tau - 1 \quad (55)$$

Note that when $\eta_1 = 0$, the state τ is constant, and the system behaves as having a constant logic mode q for all time, i.e., with a static graph. In this case, since each graph is strongly connected and $\upsilon_{ii} = 1$, for each fixed q the singled-valued boundary layer dynamics (46b) render the equilibria exponentially stable. Thus, all the steps from the proof of Theorem 2 can be applied to conclude the bound (51) for each fixed graph, i.e, when $\eta_1 = 0$, the set $\{u^*\} \times M\mathbb{B} \times \mathbb{T}^N \times Q \times [0, N_1]$ is SGPAS as $(\varepsilon_2, \alpha, \varepsilon_1) \to 0^+$ with $\beta \in \mathcal{KL}_{\mathcal{T}}$. By [28, Cor. 7.28] it follows that the model-free FXNES dynamics render the set $\{u^*\} \times M\mathbb{B} \times \mathbb{T}^N \times Q \times [0, N_1]$ SGPAS as $(\eta_1, \varepsilon_2, \alpha, \varepsilon_1) \to 0^+$, that is, SGPAS with $\beta \in \mathcal{KL}_{\mathcal{T}}$ is preserved under sufficiently slow switching (i.e., sufficiently small η_1).

VI. CONCLUSIONS

We introduced a novel class of Nash equilibrium seeking dynamics for non-cooperative games. The novelty of the dynamics lies in their ability to achieve Nash seeking in a fixed time, which is independent of the initial conditions of the actions of the players. In the model-free scenario, the dynamics achieve semi-global practical fixed-time stability, and the actions of the players converge to a neighborhood of the NE via a class $\mathcal{KL}_{\mathcal{T}}$ function with a settling time function having a uniform bound. Future research directions will consider fixed-time seeking and tracking results for generalized time-varying NES problems over graphs, as well as the effect of saturation mechanisms in fixed-time seeking dynamics.

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APPENDIX

We present some auxiliary lemmas:

Lemma 4: $\xi^a + \frac{1}{\xi^a} \geq 2$ for all $a > 0$ and $\xi > 0$. □

Lemma 5: Let $v : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $a_i, b_i > 0$ for all $i \in \mathcal{V}$. Then, for all $x \in \mathbb{R}^n$ such that $v(x) \leq 1$, the following holds:

$$\left(\frac{1}{v(x)^{\bar{a}}} + \frac{1}{v(x)^{-\underline{b}}}\right) \geq \left(\frac{1}{v(x)^{\tilde{a}_i}} + \frac{1}{v(x)^{-\tilde{b}_i}}\right), \tag{56a}$$

$$\left(\frac{1}{v(x)^{\tilde{a}_i}} + \frac{1}{v(x)^{-\tilde{b}_i}}\right) \geq \left(\frac{1}{v(x)^{\underline{a}}} + \frac{1}{v(x)^{-\bar{b}}}\right), \tag{56b}$$

with $\underline{a} := \min_i \tilde{a}_i$, $\bar{b} := \max_i \tilde{b}_i$, $\bar{a} := \max_i \tilde{a}_i$, $\underline{b} := \min_i \tilde{b}_i$. □

Proof: For any $v(x) \in (0,1]$, the function $f(s) = v(x)^s$ is monotonically *decreasing* in $s \geq 0$. Therefore, since $\bar{a} \geq \tilde{a}_i \geq \underline{a} > 0$ for all i, we have $v(x)^{\underline{a}} \geq v(x)^{\tilde{a}_i} \geq v(x)^{\bar{a}}$, which is equivalent to $\frac{1}{v(x)^{\bar{a}}} \geq \frac{1}{v(x)^{\tilde{a}_i}} \geq \frac{1}{v(x)^{\underline{a}}}$. Similarly, since $\bar{b} \geq \tilde{b}_i \geq \underline{b} > 0$ for all i, we obtain $v(x)^{\underline{b}} \geq v(x)^{\tilde{b}_i} \geq v(x)^{\bar{b}}$, which is equivalent to $\frac{1}{v(x)^{-\underline{b}}} \geq \frac{1}{v(x)^{-\tilde{b}_i}} \geq \frac{1}{v(x)^{-\bar{b}}}$. ■

Lemma 6: Let $v : \mathbb{R}^n \to \mathbb{R}_{>0}$ and $a_i, b_i > 0$ for all $i \in \mathcal{V}$. Then, for all $x \in \mathbb{R}^n$ such that $v(x) \geq 1$, the following holds:

$$\left(\frac{1}{v(x)^{\underline{a}}} + \frac{1}{v(x)^{-\bar{b}}}\right) \geq \left(\frac{1}{v(x)^{\tilde{a}_i}} + \frac{1}{v(x)^{-\tilde{b}_i}}\right), \tag{57a}$$

$$\left(\frac{1}{v(x)^{\tilde{a}_i}} + \frac{1}{v(x)^{-\tilde{b}_i}}\right) \geq \left(\frac{1}{v(x)^{\bar{a}}} + \frac{1}{v(x)^{-\underline{b}}}\right), \tag{57b}$$

with $\underline{a} := \min_i \tilde{a}_i$, $\bar{b} := \max_i \tilde{b}_i$, $\bar{a} := \max_i \tilde{a}_i$, $\underline{b} = \min_i \tilde{b}_i$. □

Proof: For any $v(x) > 1$, the function $f(s) = v(x)^s$ is monotonically *increasing* in $s \geq 0$. Therefore, since $\bar{a} \geq \tilde{a}_i > \underline{a} > 0$ for all i, we have $v(x)^{\bar{a}} \geq v(x)^{\tilde{a}_i} \geq v(x)^{\underline{a}}$, which is equivalent to $\frac{1}{v(x)^{\underline{a}}} \geq \frac{1}{v(x)^{\tilde{a}_i}} \geq \frac{1}{v(x)^{\bar{a}}}$. Similarly, since $\bar{b} \geq \tilde{b}_i \geq \underline{b} > 0$ for all i, we obtain $v(x)^{\bar{b}} \geq v(x)^{\tilde{b}_i} \geq v(x)^{\underline{b}}$, which is equivalent to $\frac{1}{v(x)^{-\bar{b}}} \geq \frac{1}{v(x)^{-\tilde{b}_i}} \geq \frac{1}{v(x)^{-\underline{b}}}$. ■

Lemma 7: [48, Lem. 3.1 & 3.2] Let $\xi_i \geq 0$ for all $i \in \mathcal{V}$. If $p \in (0,1]$, then $\sum_{i=1}^N \xi_i^p \geq \left(\sum_{i=1}^N \xi_i\right)^p$. If $p \geq 1$, then $\sum_{i=1}^N \xi_i^p \geq N^{1-p}\left(\sum_{i=1}^N \xi_i\right)^p$. □

Lemma 8: Suppose that Assumptions 1-2 hold, and let V be given by (10). Then, $\forall\, p \in (0,1)$ and $\forall\, x \in \mathbb{R}^n$ the following inequality holds: $|G(x)|^p \leq \alpha_p V(x)^{\frac{p}{4}}$, with $\alpha_p := \left(\frac{8L^4}{\kappa^2}\right)^{\frac{p}{4}}$.

Proof: Item (c) of Assumption 2 implies that V also satisfies [33, Thm. 2]:

$$\kappa^2|x - x^*|^4 \leq 8V(x), \quad \forall\, x \in \mathbb{R}^N. \tag{58}$$

Using (58), the Lipschitz property of G, and the fact that $G(x^*) = 0$, we obtain that $|G(x)|^4 \leq L^4|x - x^*|^4$, which implies $\frac{\kappa^2}{8L^4}|G(x)|^4 \leq V(x)$. Since the function $f(z) = z^p$ is monotone increasing for $z > 0$ and $p > 0$, the above inequality implies that $|G(x)|^p \leq \left(\frac{8L^4}{\kappa^2}V(x)\right)^{\frac{p}{4}}$. □

Lemma 9: Suppose that Assumption 2 holds, and let V be given by (10). Then, for all $p > 0$ and for all $x \in \mathbb{R}^n$: $\gamma_p V(x)^{\frac{p}{4}} \leq |G(x)|^p$, where $\gamma_p := 2^{\frac{3p}{4}}\kappa^{\frac{p}{2}}$.

Proof: Directly from item (c) of Assumption 2, and the definition of a potential game, we obtain: $V(x) \leq \frac{1}{8\kappa^2}|\nabla P(x)|^4 = \frac{1}{8\kappa^2}|G(x)|^4$, which implies the bound because $p > 0$. □

Lemma 10: Suppose that Assumptions 1 and 3 hold. Let V be given by (18). Then, for all $p > 0$, the following inequalities hold: $\frac{1}{\gamma_2^{\frac{p}{2}} V(x)^{\frac{p}{2}}} \geq \frac{1}{|G(x)|^p} \geq \frac{1}{\gamma_1^{\frac{p}{2}} V(x)^{\frac{p}{2}}}$, and $\frac{1}{\gamma_1^{-\frac{p}{2}} V(x)^{-\frac{p}{2}}} \geq \frac{1}{|G(x)|^{-p}} \geq \frac{1}{\gamma_2^{-\frac{p}{2}} V(x)^{-\frac{p}{2}}}$ for all $x \neq x^*$, where $\gamma_1 = 2L^2\bar{k}$ and $\gamma_2 = 2\kappa^2\underline{k}$.

Proof: The strong monotonicity property of G implies the bound $|G(x)|^2 \geq \kappa^2|x - x^*|^2$ for all $x \in \mathbb{R}^n$. Using this bound, as well as (18), we obtain: $|G(x)|^p \geq \left(2\kappa^2\underline{k}\right)^{\frac{p}{2}} V(x)^{\frac{p}{2}} = \gamma_2^{\frac{p}{2}} V(x)^{\frac{p}{2}}$. Similarly, using the Lipschitz property on G, we have $|G(x)|^2 \leq L^2|x - x^*|^2$, which implies $|G(x)|^p \leq (2L^2\bar{k})^{\frac{p}{2}} V(x)^{\frac{p}{2}} = \gamma_1^{\frac{p}{2}} V(x)^{\frac{p}{2}}$. The above inequalities establish the result. □

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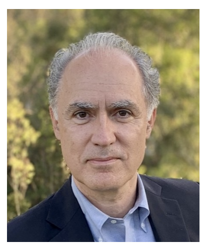
Jorge I. Poveda is an Assistant Professor in the ECEE Department at the University of Colorado, Boulder. He received his M.Sc. and Ph.D. degrees in Electrical and Computer Engineering from UC Santa Barbara, in 2016 and 2018, respectively. He was a Postdoctoral Fellow at Harvard University in 2018. He has received the CCDC Outstanding Scholar Fellowship and Best Ph.D. Dissertation awards at UCSB, the NSF CRII and CAREER awards, the AFOSR Young Investigator award, and the campus-wide RIO Faculty Fellowship at CU Boulder.



Miroslav Krstić is a Distinguished Professor of Mechanical and Aerospace Engineering, holds the Alspach endowed chair, and is the founding director of the Cymer Center for Control Systems and Dynamics at UC San Diego. He also serves as Senior Associate Vice Chancellor for Research at UCSD. As a graduate student, Krstic won the UC Santa Barbara Best Ph.D. Dissertation award and student best paper awards at CDC and ACC. Krstic has been elected Fellow of seven scientific societies IEEE, IFAC, ASME, SIAM, AAAS, IET (UK), and AIAA, and as a foreign member of the Serbian Academy of Sciences and Arts and of the Academy of Engineering of Serbia. He has received the Bellman Award, the SIAM Reid Prize, ASME Oldenburger Medal, Nyquist Lecture Prize, Paynter Outstanding Investigator Award, Ragazzini Education Award, IFAC TC Nonlinear Contr. Systems Award, Chestnut textbook prize, Contr. Syst. Society Distinguished Member Award, the PECASE, NSF CAREER, and ONR Young Investigator awards, the Axelby and Schuck paper prizes, and the first UCSD Research Award given to an engineer.

Tamer Başar has been with the University of Illinois Urbana-Champaign (UIUC) since 1981, where he is currently Swanlund Endowed Chair Emeritus; Center for Advanced Study Professor Emeritus of Electrical and Computer Engineering; Research Professor at the Coordinated Science Laboratory; and Research Professor at the Information Trust Institute. He has served at UIUC as the Director of the Center for Advanced Study during 2014-2020. He received B.S.E.E. from Robert College, Istanbul, and M.S., M.Phil, and Ph.D. from Yale University. He has received several awards and recognitions over the years, including the highest awards of IEEE CSS, IFAC, AACC, and ISDG, the IEEE Control Systems Award, and a number of international honorary doctorates and professorships. He is a member of the US National Academy of Engineering, and Fellow of IEEE, IFAC and SIAM, and has served as presidents of IEEE CSS, ISDG, and AACC. He has over 1000 publications in systems, control, communications, and dynamic games, including books on game theory, robust control, network security and stochastic networked control.