Fixed-Time Seeking and Tracking of Time-Varying Extrema

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***Abstract*— Motivated by recent (semi-global practical) fixed-time convergence results in time-invariant model-free optimization problems, in this paper we introduce new tracking bounds and guidelines for the design of extremum seeking controllers in model-free optimization problems with dynamic cost functions. Using semi-global practical input-to-state stability characterizations, we show that the proposed non-smooth ES dynamics are able to significantly reduce the tracking error compared to the traditional smooth algorithms studied in the literature. Moreover, under a suitable tuning of the gains of the algorithm, the nominal average dynamics of the controller are able to achieve *global fixed-time* tracking for a general class of dynamic cost functions. For tuning parameters that do not completely eliminate the tracking error in the nominal average dynamics, but which preserve the continuity of the vector field, we show that "almost complete" error rejection is achieved whenever the gain of the algorithm passes a particular threshold. Numerical results are presented to illustrate the performance of the algorithms.**

I. Introduction

Time-varying, or online, optimization has received significant attention in recent years due to successful applications in areas such as online decision making [1], machine learning [2], and reinforcement learning [3], as well as validations in power systems [4] and transportation systems [5], to name just a few. In this setting, exogenous (sometimes adversarial) time-variation can be induced by dynamic parameters that characterize the cost function or the environment where the system operates. When a model of the time dependency is *unavailable* for the optimization algorithm, feedforward control, so useful in robotic motion planning and set-point tracking, is not feasible anymore. Instead, online time-varying model-free optimization needs, in general, to be formulated as an *approximate tracking* problem, with some residual error accepted due to the unknown variation in the sought optimizer. Since the error is caused by the time-variation in the optimizer, and it is proportional to the size of the optimizer's time derivative, it is natural to treat this derivative as a disturbance and to quantify the residual error using notions such as uniform ultimate boundedness (UUB) [6] or input-to-state stability (ISS) [7], [8]. Examples of applications of ISS tools in optimization problems using continuous-time smooth and hybrid gradient flows have been studied in [9], [10], and [11].

On the other hand, when it comes to *model-free* optimization problems, (averaging-based) *extremum seeking* control (ES) has become a well-established sub-field in the area of adaptive control and self-optimizing control; see [12]–[16]. Yet, in the context of tracking problems with time-varying cost functions, ES has received limited attention. One notable exception is the tracking of optimizers of known functional shapes but unknown coefficients, which was studied in [17, Sec. 5] using the internal model principle. For the traditional gradient descent-based ES, tracking results were studied in [18] using Lie bracket averaging theory and (semi-global practical) UUB tools. A similar smooth ES algorithm for tracking of well-behaved time-varying cost function was also studied in [19].

Contributions: All the existing results in the literature of time-varying optimization via averaging-based ES have relied on controllers whose nominal average systems correspond to smooth (i.e., Lipschitz continuous) gradient flows, typically gradient descent. Thus, such controllers are inherently limited by the tracking properties of their nominal average dynamics, e.g., finite-time or fixed-time perfect tracking is in general not possible. To overcome these limitations, in this paper we introduce a new class of non-smooth ES controllers that are particularly well-suited for the solution of model-free tracking problems. These algorithms are obtained by particular instantiations of the generalized ES introduced in [20] for *time-invariant* cost functions, and, as we will show in this paper, their nominal average dynamics have the ability to achieve fixed-time tracking using discontinuous vector fields, and approximate fixed-time tracking using continuous vector fields. We formalize these properties by borrowing tools from UUB and ISS in singularly perturbed non-smooth dynamics with well-defined average systems [16], [21], [22]. Our theoretical results also provide design guidelines that highlight the effect of the parameters of the ES on the attenuation of the tracking error, characterized by the ISS gain. To the best knowledge of the authors, such results are completely new in the literature of ES.

Organization: The rest of this paper is organized as follows. Section II presents some preliminary definitions and stability notions. Section III characterizes the tracking problem under study in this paper. Section IV presents our main results. Section V presents numerical examples, and finally Section VI ends with some conclusions.

II. Preliminaries

In this paper, we will model our algorithms using the framework of constrained dynamical systems [23], where $x \in \mathbb{R}^n$ is the state of the system evolving according to

$$x \in C, \qquad \dot{x} = F(x), \tag{1}$$

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where $C \subset \mathbb{R}^n$ is a closed set, and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable bounded function that is, in general, continuous for all $x \in \mathbb{R}^n \backslash \{0\}$. For functions F that are discontinuous at the origin, system (1) should be replaced by its Krasovskii regularization [23, Def. 4.13]. A solution of (1) is an absolutely continuous function $x : \text{dom}(x) \to \mathbb{R}^n$ that satisfies: a) $x(0) \in C$; b) $x(t) \in C$, $\forall\ t \in \text{dom}(x)$; and c) $\dot{x}(t) = F(x(t))$ for almost all $t \in \text{dom}(x)$. A solution is said to be complete if $\text{dom}(x) = [0, \infty)$. System (1) is said to render a compact set $\mathcal{A}$ *uniformly globally asymptotically stable* (UGAS) if there exists a (generalized) class $\mathcal{KL}$ function β such that every solution of (1) satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t)$, $\forall\ t \in \text{dom}(x)$. When $\exists\ T^* > 0$ such that $\beta(r, s) = 0$ for all $r > 0$ and $s > T^*$ we say that $\mathcal{A}$ is fixed-time stable.

We also consider ε-perturbed or parameterized dynamical systems of the form $x \in C$, $\dot{x} = F_\varepsilon(x)$, where F_ε is parameterized by a positive constant $\varepsilon > 0$. For these systems, we say that the compact set $\mathcal{A} \subset C$ is Semi-Globally Practically Asymptotically Stable (SGPAS) as $\varepsilon \to 0^+$, if there exists a class $\mathcal{KL}$ function β such that $\forall\ \delta > \nu > 0$, $\exists\ \varepsilon^* > 0$ such that $\forall\ \varepsilon \in (0, \varepsilon^*)$ every solution x with $|x(0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t) + \nu$, $\forall\ t \in \text{dom}(x)$. The notion of SGPAS can be extended to systems that depend on multiple parameters $\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell]^\top$. In this case, and with some abuse of notation, we say that the system renders the set $\mathcal{A}$ SGPAS as $(\varepsilon_\ell, \ldots, \varepsilon_2, \varepsilon_1) \to 0^+$, where the parameters are tuned in order starting from ε_1. We use $\mathbb{S}^1 \subset \mathbb{R}^2$ to denote the unit circle centered at the origin, and $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \mathbb{S}^1$ as the n^{th}-Cartesian product of $\mathbb{S}^1$.

III. Problem Statement

Our goal is to solve the time-varying optimization problem

$$\min_{u \in \mathbb{R}^n}\ \phi(u, \theta), \tag{2}$$

where ϕ is *unknown* but measurable, and θ is a time-varying parameter that evolves according to some *unknown* dynamics of the form:

$$\dot{\theta} = \varepsilon_o \Pi(\theta), \qquad \theta \in \Theta, \tag{3}$$

where $\varepsilon_o > 0$. We make the following qualitative assumptions on (3).

Assumption 1: The set $\Theta \subset \mathbb{R}^p$ is compact, the mapping $\Pi(\cdot)$ is Lipschitz continuous, and the dynamics (3) render forward invariant the set Θ. □

Remark 1: We stress that Assumption 1 is made only to impose enough regularity on the time-variations of the cost function. Knowledge of the explicit forms of the mapping $\Pi(\cdot)$ or the set Θ is not needed in our algorithms. Similarly, the parameter $\varepsilon_o > 0$ is used only to conveniently model how fast the parameter θ changes over time. □

The class of cost functions ϕ considered in this paper are characterized by the following assumption.

Assumption 2: The function ϕ satisfies the following:

1) $\phi(\cdot, \cdot)$ is continuously differentiable.
2) There exists $L > 0$ such that

$$|\nabla_u \phi(u_1, \theta) - \nabla_u \phi(u_2, \theta)| \leq L|u_1 - u_2|, \tag{4}$$

for all $u_1, u_2 \in \mathbb{R}^n$ and all $\theta \in \Theta$.
3) There exists a unique continuously differentiable function $h : \mathbb{R}^p \to \mathbb{R}^n$, such that, for each fixed θ the solution of (2) is given by

$$u^* := h(\theta) = \arg \min_{u \in \mathbb{R}^n} \phi(u, \theta). \tag{5}$$

4) There exists $\kappa > 0$ such that

$$\begin{aligned} \phi(u_2, \theta) \geq \phi(u_1, \theta) + \nabla_u \phi(u_1, \theta)^\top (u_2 - u_1) \\ + \frac{\kappa}{2} |u_2 - u_1|^2, \end{aligned} \tag{6}$$

for all $u_1, u_2 \in \mathbb{R}^n$ and all $\theta \in \Theta$. □

Remark 2: The conditions of Assumption 2 are rather standard in the literature of time-varying optimization [1], [2], particularly when one is interested in establishing exponential (or superlinear) convergence results. Indeed, since θ evolves in a compact set, conditions (4) and (6) will hold for functions ϕ that are (for each $\theta \in \Theta$) globally Lipschitz and strongly convex in u. On the other hand, condition 4) guarantees that problem (2) is well-posed and has a unique solution for each $\theta \in \Theta$. □

Under Assumption 2, the condition number of the cost function ϕ is denoted as $\text{cond}(\phi) := L/\kappa$.

The following lemma, which follows directly by Assumption 1 and item 3) of Assumption 2, provides some useful constants that will emerge in our tracking bounds.

Lemma 1: Suppose that Assumptions 1-2 hold. Then, there exist constants $m_h, m_\Pi > 0$ such that $|\Pi(\theta)| \leq m_\Pi$ and $|\nabla h(\theta)| \leq m_h$ for all $\theta \in \Theta$. □

IV. Tracking Bounds in Extremum Seeking

To track the solutions of problem (2) in a model-free way, i.e., using only measurements of ϕ, we consider a class of generalized gradient-based ES, shown in Figure 1, and characterized by the dynamics

$$\dot{\hat{u}} = -k \left(\frac{\xi}{|\xi|^\alpha} + \frac{\xi}{|\xi|^{-\alpha}} \right), \quad \hat{u} \in \mathbb{R}^n, \tag{7a}$$

$$\dot{\xi} = \frac{1}{\varepsilon_f} \left(-\xi + \phi(u, \theta) M(\mu) \right), \quad \xi \in \mathbb{R}^n, \tag{7b}$$

$$\dot{\mu} = \frac{1}{\varepsilon_p} \mathcal{R}_\kappa \mu, \quad \mu \in \mathbb{T}^n, \tag{7c}$$

where the right-hand side of (7a) is defined to be zero when $\xi = 0$. The parameter α satisfies $\alpha \in [0, 1]$, the input u and the mapping M are defined as

$$u = \hat{u} + \varepsilon_a \mathcal{D}\mu, \quad M(\mu) := 2\varepsilon_a^{-1} \mathcal{D}\mu, \tag{8}$$

and $\mathcal{D}\mu = [\mu_1, \mu_3, \mu_5, \ldots, \mu_{2n-1}]^\top$. As in [24] and [20], the matrix $\mathcal{R}_\kappa$ is block diagonal with skew symmetric blocks given by $\mathcal{R}_i = 2\pi[0, \kappa_i; -\kappa_i, 0] \in \mathbb{R}^{2\times 2}$, $i \in \{1, 2, \ldots, n\}$, where $\kappa = [\kappa_1, \ldots, \kappa_n] \in \mathbb{R}^n$, with $\kappa_i > 0$. The tunable parameters of (7) satisfy the following qualitative properties.

Assumption 3: For each $i \in \{1, 2, \ldots, n\}$ the parameter $\kappa_i > 0$ is a rational number, $\kappa_i \neq \kappa_j$ and $\kappa_i \neq 2\kappa_j$ for all $i \neq j$, and $0 < \varepsilon_o, k < \frac{1}{\varepsilon_f} < \frac{1}{\varepsilon_a} < \frac{1}{\varepsilon_p}$. □

Note that when $\alpha = 0$, the ES dynamics (7) are similar to those studied in [12], [13]. In particular, since the odd components of the solutions of the constrained linear oscillator (7c) are given by $\mu_i(t) = \mu_i(0)\cos\left(\frac{2\pi t}{\varepsilon_p}\kappa_{\frac{i+1}{2}}\right) + \mu_{i+1}(0)\sin\left(\frac{2\pi t}{\varepsilon_p}\kappa_{\frac{i+1}{2}}\right)$, with initial conditions satisfying $\mu_i(0)^2 + \mu_{i+1}(0)^2 = 1$, it follows that if $\mu(0) = [0, 1, 0, 1, \ldots, 0, 1]^\top$ and $\alpha = 0$, then (7) becomes:

$$\dot{\hat{u}} = -2k\xi,$$
$$\dot{\xi} = \frac{1}{\varepsilon_f}\left(-\xi + \phi(\hat{u} + \varepsilon_a \sin(2\pi\kappa t), \theta)\frac{2}{\varepsilon_a}\sin(2\pi\kappa t)\right),$$

which is the standard gradient descent-based extremum seeking (GDES) algorithm studied in [12], [13] and [25] for time-invariant cost functions. On the other hand, as shown in [20], the choices $\alpha \in (0, 1]$ lead to different ES algorithms, termed Fixed-Time ES (FxTES), with substantially different convergence properties.

Note that the interconnection of (3) and (7) leads to a time-invariant system. Therefore, the following definition will be instrumental to characterize the tracking properties of the ES dynamics.

Definition 1: System (7) is said to have *the* (β, γ)*-tracking property with respect to* (3) if $\exists\ \beta \in \mathcal{KL}$ and $\exists\ \gamma \in \mathcal{K}$ such that: $\forall\ k, \varepsilon_o > 0$ and $\forall\ \Delta > \nu > 0$, $\exists\ \varepsilon_f^* > 0$ such that $\forall\ \varepsilon_f \in (0, \varepsilon_f^*)$, $\exists\ \varepsilon_a^* > 0$ such that $\forall\ \varepsilon_a \in (0, \varepsilon_a^*)$, $\exists\ \varepsilon_p^* > 0$ such that $\forall\ \varepsilon_p \in (0, \varepsilon_p^*)$ all solutions of (3) and (7) with initial conditions satisfying:

$$|\hat{u}(0) - u^*(0)| \leq \Delta,\ |\xi(0)| \leq \Delta,\ \mu(0) \in \mathbb{T}^n,\ \theta(0) \in \Theta,$$

also induce the following bound on the input u for all $t \geq 0$:

$$|u(t) - u^*(t)| \leq \beta(|\hat{u}(0) - u^*(0)|, kt) + \gamma\left(\frac{\varepsilon_o}{k}\right) + \nu, \quad (9)$$

and $\limsup_{t\to\infty} |\xi(t)| \in \mathcal{O}(\gamma(\varepsilon_o/k) + \nu + \varepsilon_a)$. □

Definition 1 resembles a semi-global practical ISS bound with respect to the constant "input" ε_0/k [26, Eq. (49)], where we emphasized the transient properties of u, and we divided the residual term of (9) into two terms: the unavoidable "precision error" ν, which can be made arbitrarily small by tuning the parameters $(\varepsilon_p, \varepsilon_a, \varepsilon_f)$; and the additional "tracking error" $\gamma(\varepsilon_o/k)$, which vanishes as $\varepsilon_o \to 0^+$ or $k \to \infty$. Borrowing terminology from the UUB and ISS literature [7], [8], [26], we call the function γ the *ISS gain*, and we note that the ratio ε_o/k quantifies how fast θ changes over time compared to the time-variation of $\hat{u}$ in (7a).

Remark 3: For the closed-loop system with dynamics (3) and (7), the states (θ, μ) are restricted to evolve on the compact sets Θ and $\mathbb{T}^n$, which are forward invariant by assumption, and by design, respectively. Similarly, by the stability and linearity of the low-pass filter in (7b), for a given compact set of initial conditions the trajectories ξ will stay uniformly bounded under bounded inputs u. □

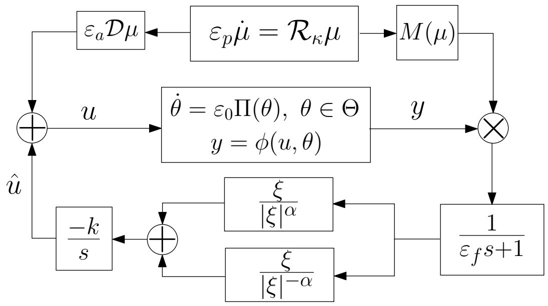

Fig. 1: Scheme of Generalized ES with Time-Varying Costs.

A. Main Result 1: Model-Free Tracking with $\alpha = 0$

Our first result characterizes the pair (β, γ) for the case when $\alpha = 0$ in (7), i.e., for the GDES algorithm. Due to space limitations, in this paper we present only a sketch of the proof.

Theorem 1: Suppose that Assumptions 1-3 hold, and $\alpha = 0$. Then, the ES (7) has *the* (β_0, γ_0)*-tracking property with respect to* (3) with $\beta_0(r, s) := re^{-k\kappa(1-\lambda)s}$, and $\gamma_0(\ell) := \left(\frac{2m_h m_H}{\lambda\kappa}\right)\ell$, where $\lambda \in (0, 1)$. □

Sketch of the Proof: Using a Taylor expansion, leveraging Assumption 3, and computing the nominal average dynamics of system (7) in the slowest time scale, we obtain

$$\dot{\tilde{u}} = -k\nabla_u\phi(\tilde{u} + h(\theta), \theta) - \overbrace{h(\theta)}^{\cdot}, \quad (10)$$

where $\tilde{u} := u - h(\theta)$. Using the following smooth Lyapunov function

$$V(\tilde{u}) := \frac{1}{4}(\tilde{u}^\top\tilde{u})^2, \quad (11)$$

it can be shown that $\dot{V}$ satisfies

$$\dot{V} \leq -2(1-\lambda)k\kappa V, \quad \forall\ |\tilde{u}| > \frac{2m_h m_\Pi}{\lambda\kappa}\cdot\left(\frac{\varepsilon_o}{k}\right),$$

for any $\lambda \in (0, 1)$. By using the Comparison Lemma and standard arguments [6, Ch. 4], it can be shown that system (10) satisfies an ISS-like bound. Using this bound, and similar steps to those in [20], we can apply singular perturbation theory and averaging theory [8] to establish the bound (9). ■

The above result establishes exponential transient via β_0, and a linear gain $\ell \mapsto \gamma_0(\ell)$ that maps the ratio ε_o/k to the first residual term of (9). This result is similar to tracking bounds established in [18] and [19] for ES dynamics based on Lie bracket averaging.

B. Main Result 2: Model-Free Tracking with $\alpha = 1$

The form of γ_o in Theorem 1 indicates a natural limitation of the GDES dynamics. Namely, given a constant gain $k > 0$, the residual tracking error grows linearly as the time-variation of θ increases (i.e., as ε_0 increases in (3)). On the other hand, the next theorem shows that the choice $\alpha = 1$ can effectively annihilate the term $\gamma(\varepsilon_0/k)$ in (9) whenever the gain k passes a particular threshold relative to ε_o.

Theorem 2: Suppose that Assumptions 1-3 hold, and $\alpha = 1$. Then, for all

$$k > 4m_h m_\Pi \text{cond}(\phi)\varepsilon_o,$$

the ES (7) has *the* (β_1, γ_1)*-tracking property with respect to* (3) with $\gamma_1 := 0$ and

$$\beta_1(r,s) := c_1 \tan\left(\max\left\{0, -c_2 s + \arctan\left(c_3 r^{\frac{2}{\alpha}}\right)\right\}\right)^{\frac{\alpha}{2}}, \tag{12}$$

where $c_1, c_2, c_3 > 0$. □

Sketch of the Proof: As in Theorem 1, we compute the nominal average dynamics of the ES algorithm. In this case, we obtain:

$$\dot{\tilde{u}} = -k\left(\frac{\nabla_u \phi(\tilde{u}+h(\theta),\theta)}{|\nabla_u \phi(\tilde{u}+h(\theta),\theta)|^{\alpha}} + \frac{\nabla_u \phi(\tilde{u}+h(\theta),\theta)}{|\nabla_u \phi(\tilde{u}+h(\theta),\theta)|^{-\alpha}}\right) - \dot{\overbrace{h(\theta)}}. \tag{13}$$

Using again the Lyapunov function (11), leveraging the co-coercivity and monotonicity properties of $\nabla\phi$, and setting $\alpha = 1$, we can obtain

$$\dot{V} \le -\frac{k}{L}c_1 V^{\frac{3}{4}} - \frac{2k}{L}c_2 V^{\frac{5}{4}} + \left(\varepsilon_o c_3 - \frac{k}{L}c_1\right)V^{\frac{3}{4}},$$

where

$$c_1 = \frac{\kappa}{2^{\frac{1}{2}}}, \quad c_2 = \frac{\kappa^3}{2^{\frac{3}{2}}}, \; c_3 = 2^{\frac{3}{2}} m_h m_\Pi. \tag{14}$$

Therefore, if k satisfies

$$k > \frac{\varepsilon_0 c_3 L}{c_1} = 4\varepsilon_0 m_h m_\Pi \left(\frac{L}{\kappa}\right), \tag{15}$$

then $\dot{V} \le -\frac{k}{L}\left(c_1 V^{\frac{3}{4}} + c_2 V^{\frac{5}{4}}\right)$, which establishes uniform global fixed-time stability of the origin $\tilde{u} = 0$ via [27, Lem. 2], with fixed-time convergence bound $T^* = \frac{4\pi L}{k\kappa^2}$. The class $\mathcal{KL}$ function β_1 follows by [20, Lem. 3]. From here, we follow similar steps as those in [20]. ■

In Theorem 2, the function β_1 satisfies $\beta_1(r,s) = 0$ for all $r > 0$ and $s > T_F^* = \frac{4\pi L}{k\kappa^2}$ [27, Lemma 2], where the constants (κ, L) come from Assumption 2. In other words, β_1 has the fixed-time convergence property, and the bound (9) establishes a semi-global practical fixed-time tracking property for the ES algorithm. To our knowledge, this is the first bound of this form established for ES in time-varying optimization problems.

Remark 4: The result of Theorem 2 shows that the choice $\alpha = 1$ can effectively eliminate the effect of $\dot{\theta}$ in the residual term of the bound (9). However, while this property is appealing, it comes at the price of obtaining an ES algorithm with a discontinuous right-hand side, which might induce undesirable chattering near the optimal trajectory u^*. Indeed, a closer look to the proof of Theorem 2, reveals that -on average- the ES algorithm shares similarities to sliding mode controllers. □

Next, to avoid the chattering phenomenon induced by the choice $\alpha = 1$, we consider a selection of α that preserves the continuity of the vector field, and provides a balance between the results of Theorems 1 and 2.

C. Main Result 3: Model-Free Tracking with $\alpha \in (0,1)$

When $\alpha \in (0,1)$, the tracking properties of system (7) are characterized by the following theorem, which provides desirable structures for the pair (β, γ).

Theorem 3: Suppose that Assumptions 1-3 hold, and $\alpha \in (0,1)$. Then, the ESC (7) has *the* $(\beta_{01}, \gamma_{01})$*-tracking property with respect to* (3) with the same β_{01} (12), and

$$\gamma_{01}\left(\frac{\varepsilon_o}{k}\right) := \rho^{-1}\left(\frac{\varepsilon_o}{k}\cdot w\right), \tag{16}$$

where $w > 0$ is a positive constant, and $\rho^{-1}(\cdot)$ is the inverse of the function $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ defined as

$$\rho(s) = \epsilon_1 s^{1-\alpha} + \epsilon_2 s^{1+\alpha}, \tag{17}$$

with $\epsilon_1 = (1/4)^{\frac{1-\alpha}{4}}$ and $\epsilon_2 = (1/4)^{\frac{1+\alpha}{4}}$. □

Sketch of the Proof: As in the proof of Theorems 1 and 2, we use the Lyapunov function (11) to study the behavior of the nominal average dynamics of the ES algorithm, which are given by (13). When $\alpha \in (0,1)$, the time-derivative of V now satisfies

$$\dot{V} \le -\frac{k}{L}\tilde{c}_1 V^{1-\frac{\alpha}{4}} - \frac{k}{L}\tilde{c}_2 V^{1+\frac{\alpha}{4}} + \sigma(\tilde{u}),$$

where $\sigma(\tilde{u}) := -\frac{k\underline{c}}{L}\left(V^{1-\frac{\alpha}{4}} + V^{1+\frac{\alpha}{4}}\right) + \varepsilon_o c_3 V^{\frac{3}{4}}$. When $\tilde{u} \neq 0$, the term $\sigma(\tilde{u})$ is non-positive whenever

$$\varepsilon_o c_3 \le \frac{k\underline{c}}{L}\left(\left(\frac{1}{4}\right)^{\frac{1-\alpha}{4}}|\tilde{u}|^{1-\alpha} + \left(\frac{1}{4}\right)^{\frac{1+\alpha}{4}}|\tilde{u}|^{1+\alpha}\right).$$

Defining $\rho(|\tilde{u}|) := \left(\frac{1}{4^{\frac{1-\alpha}{4}}}|\tilde{u}|^{1-\alpha} + \frac{1}{4^{\frac{1+\alpha}{4}}}|\tilde{u}|^{1+\alpha}\right)$, which satisfies $\rho \in \mathcal{K}_\infty$ because $\alpha \in (0,1)$, we obtain:

$$\dot{V} \le -\frac{k}{L}\tilde{c}_1 V^{1-\frac{\alpha}{4}} - \frac{k}{L}\tilde{c}_2 V^{1+\frac{\alpha}{4}}, \; \forall\, |\tilde{u}| \ge \rho^{-1}\left(\frac{\varepsilon_o}{k}\cdot\frac{c_3 L}{\underline{c}}\right)$$

which establishes uniform global fixed-time ISS of the origin $\tilde{u} = 0$ with ε_0/k as "input", via [28, Thm. 4]. From here, we can follow similar steps as in [20], using averaging tools for nonsmooth systems with ISS bounds [8]. ■

To study the structure of the gain γ_{01} given by (16), we show in Figure 2 (solid line) different plots of the mapping $\ell \mapsto \rho^{-1}(\ell)$ for three different values of $\alpha \in (0,1)$. First, we can start by considering only the first term in (17), which leads to the gain

$$\gamma_L\left(\frac{\varepsilon_o}{k}\right) = \sqrt{2}\left(\frac{\varepsilon_o}{k}\cdot w\right)^{\frac{1}{1-\alpha}}.$$

The function $\gamma_L(\cdot)$ is shown in Figure 2, in red color for different values of α. It can be seen that $\gamma_L(s) \approx 0$ whenever $s \ll 1$ and $\alpha \approx 1$. Thus, for any $\varepsilon_o > 0$, if k is such that

$$\frac{\varepsilon_o}{k}\cdot w < 1, \tag{18}$$

then, "almost" perfect tracking is achieved.

On the other hand, if we consider only the second term in (17) the resulting gain is

$$\gamma_H\left(\frac{\varepsilon}{k}\right) = \sqrt{2}\left(\frac{\varepsilon_o}{k}\cdot w\right)^{\frac{1}{1+\alpha}},$$

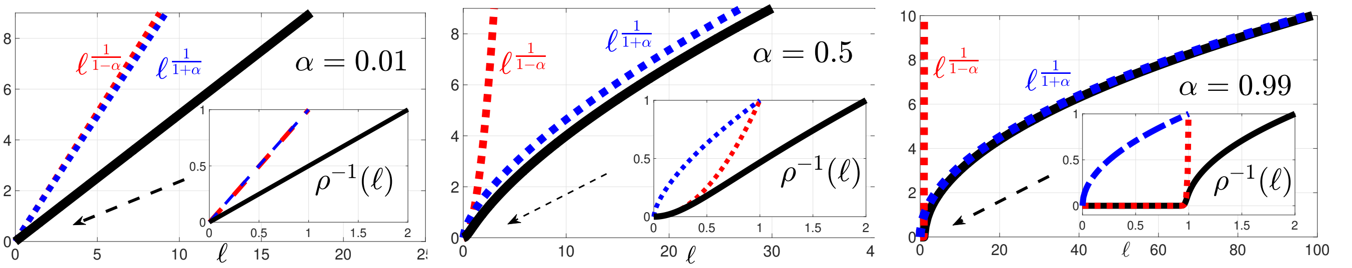

Fig. 2: ISS gain ρ^{-1} (solid black line) when $\alpha = 0.01, 0.5, 0.99$. Dotted red lines and blues lines show the gains obtained by considering only each of the individual components of (17).

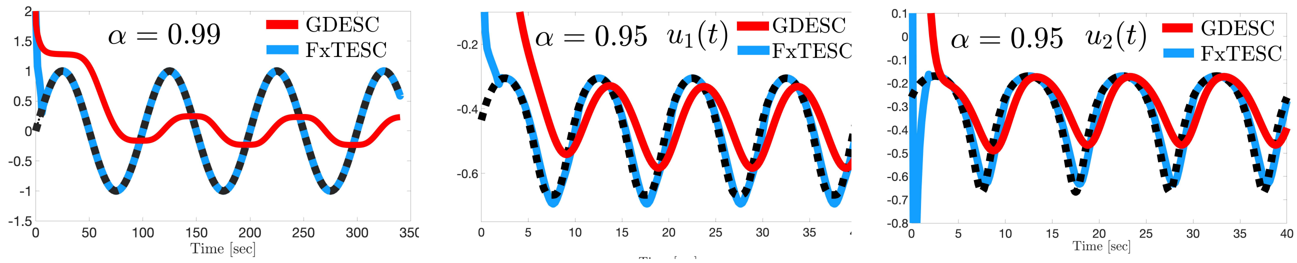

Fig. 3: Left: Comparison between GDES and FxTES with $\alpha = 0.99$ in a scalar tracking problem. Center and Right: Evolution of inputs u_1 and u_2 in a multivariable tracking problem. In this case, the FxTESC implements $\alpha = 0.95$. The black dashed line corresponds to the optimal trajectory.

which is shown in Figure 2, in blue color, for different values of α. As $\alpha \to 1$ the function $\gamma_H(s)$ converges to a square root function. Therefore, it provides a suitable attenuation (better than linear) whenever $s \gg 1$.

The actual function ρ^{-1}, shown in black color, incorporates the advantages of both γ_L and γ_H. As shown in the inset of the right plot of Figure 2, for $\alpha = 0.99$ and small ratios ε_o/k, the gain approximately eliminates the tracking error. This qualitative structure provides a guideline for the choice of α in the non-smooth ESC algorithm in time-varying optimization problems.

D. *Further technical considerations*

The complete proofs of Theorems 1-3, omitted due to space limitations, are based on averaging theory for non-smooth systems, and uniform ultimate bounds similar to those obtained in [18], [19], [22], [29], in the context of ISS for systems with multiple time scales. However, we do not directly use standard ISS results since in our case the time-variation of the cost, embedded on θ, is not driven by an arbitrary input but rather by the exosystem (3). This allows us to impose enough regularity on θ and its time derivatives to avoid situations where fast variations of the cost interfere with the computation of the average system; see [22, pp. 233]. Indeed, our Definition 1 is somehow similar to the notion of *semi-global practical derivative* ISS [22, Def. 10].

V. Numerical Examples

To illustrate the results of Theorems 1-3, we present two numerical examples.

A. *Scalar Fixed-Time Model-Free Tracking*

Consider a parameterized cost function given by $\phi(\theta, u) = (u-\theta)^2$, where the time-varying parameter θ satisfies $\theta(t) = \sin(\varepsilon_o t)$ for all $t \geq 0$. Thus, we can take $\Theta = [-1, 1]$. This sinusoidal variation can be generated by a time-invariant oscillator of the form (7c) with suitable initial conditions. The gradient of the cost function satisfies

$$\begin{aligned}|\nabla_u \phi(u_1, \theta) - \nabla_u \phi(u_2, \theta)| &= |2(u_1 - \theta) - 2(u_2 - \theta)| \\ &\leq 2|u_1 - u_2|, \quad \forall\, \theta \in \Theta,\end{aligned}$$

and also $|2(u_1-\theta)|^2 = 4|u_1-\theta|^2 = 2\phi(\theta, u)$, for all $\theta \in \Theta$. Thus, inequalities (4) and (6) hold with $L = 2$ and $\kappa = 1$. Moreover, since the unique minimizer of ϕ is given by $h(\theta) = \theta$, Assumption 2 is satisfied. We implemented the generalized ES (7) with parameters $k = 1$, $\varepsilon_p = 5 \times 10^{-4}$, $\varepsilon_a = 1 \times 10^{-2}$, and $\varepsilon_f = 1 \times 10^{-1}$. The trajectories generated by the algorithm with $\alpha \in \{0, 0.99\}$ are shown in Figure 3. For $\alpha = 0$, the ES algorithm corresponds to the GDES, whose trajectories are shown with red color. On the other hand, the trajectories associated with $\alpha = 0.9$ are shown in blue color. The dashed black line corresponds to the optimal trajectory. Since the value of k guarantees that the argument of γ_{10} in (16) is less than one, the tracking error is essentially removed, which is consistent with the behavior observed in Figure 3.

B. *Multivariable Fixed-Time Model-Free Tracking*

To test the tracking performance of the ES (7) in a multivariable optimization problem, we now consider the cost function $\phi(\theta, u) = \frac{1}{2}u^\top Q(\theta)u + b^\top u$, where $b = [1, 1]$

and Q is given by:

$$Q(\theta) := \begin{bmatrix} 2+\theta_1 & 0.5 \\ 0.5 & 3+\theta_2 \end{bmatrix}, \tag{19}$$

where the parameters θ_1 and θ_2 are again generated by oscillators of the form (7c), such that $\theta_1(t) = \sin(2\varepsilon_o t)$, and $\theta_2(t) = 2\sin(5\varepsilon_o t)$, for all $t \geq 0$. Thus, in this case $\theta = [\theta_1, \theta_2]^\top \in \Theta := [-1,1] \times [-2,2]$. For each $t \geq 0$, the minimizer of ϕ is well-defined, and given by $u^*(t) = -Q(\theta(t))^{-1}b$. Indeed, the time-varying matrix Q is uniformly positive definite, and Assumption 1 holds with $\kappa = 0.5$ and $L = 5.11$. The ES (7) is implemented with parameters $k = 0.25$, $\varepsilon_p = 1 \times 10^{-2}$, $\varepsilon_a = 1 \times 10^{-2}$, and $\varepsilon_f = 1 \times 10^{-1}$. The center and right plots of Figure 3 show the trajectories of the ES with $\alpha = 0.99$ (FxTES) and $\alpha = 0$ (GDES). Given that inequality (18) is satisfied, the FxTES greatly reduces the residual term γ_{01} in (16), leading to almost perfect tracking.

VI. Conclusions

We introduced several new results in the context of time-varying optimization and tracking using extremum seeking control. In particular, we proposed a class of non-smooth ES algorithms able to significantly reduce the tracking error under appropriate tuning of the parameters of the algorithm. The tracking error reduction was characterized by class $\mathcal{K}$ functions that map the "speed" of the time-varying optimizer to the size of the residual ball where the trajectories of the ES algorithm converge. A qualitative study of these functions (or gains) can provide design guidelines for the ES controllers in problems with dynamic cost functions.

References

[1] E. Hazan, *Introduction to Online Convex Optimization*. MIT Press, 2020.

[2] E. Hazan, A. Agarwal, and S. Kale, "Logarithmic regret algorithms for online convex optimization," *Machine Learning*, vol. 69, pp. 169–192, 2007.

[3] D. Bertsekas, *Reinforcement Learning and Optimal Control*. Athena Scientific, 2019.

[4] M. Colombino, E. Dall'Anese, and A. Bernstein, "Online optimization as a feedback controller: Stabiliy tracking," *IEEE Transactions on Control of Network Systems, to appear.*, vol. 7, pp. 422–432, 2020.

[5] P. Jaillet and M. R. Wagner, *The Vehicle Routing Problem: Latest Advances and New Challenge*. Springer, 2008, ch. Online Vehicle Routing Problems: A Survey, pp. 221–237.

[6] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall, 2002.

[7] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, pp. 435–443, 1989.

[8] D. Nešíc and A. R. Teel, "Input-to-state stability for nonlinear time-varying systems via averaging," *Mathematics of Controls, Signals and Systems*, vol. 14, pp. 257–280, 2001.

[9] A. Cherukuri, E. Mallada, S. Low, and J. Cortés, "The role of convexity in saddle-point dynamics: Lyapunov function and robustness," *IEEE Transactions on Automatic Control.*, vol. 63, no. 8, pp. 2449–2464, 2018.

[10] G. Bianchin, J. Cortes, J. I. Poveda, and E. DallAnese, "Time-varying optimization of LTI systems via projected primal-dual gradient flows," *IEEE Transaction on Control of Network Systems, appeared online.*, 2021.

[11] G. Bianchin, J. I. Poveda, and E. DallAnese, "Online optimization of switched LTI systems using continuous-time and hybrid accelerated gradient flows," *arXiv preprint arXiv:2008.03903*, 2020.

[12] K. Ariyur and M. Krstić, *Real-Time Optimization by Extremum-Seeking Control*. Hoboken, NJ: Wiley, 2003.

[13] Y. Tan, D. Nešić, and I. Mareels, "On non-local stability properties of extremum seeking controllers," *Automatica*, vol. 42, no. 6, pp. 889–903, 2006.

[14] H. Durr, M. S. Stankovic, C. Ebenbauer, and K. H. Johansson, "Lie bracket approximation of extremum seeking systems," *Automatica*, vol. 49, no. 6, pp. 1538–1552, 2013.

[15] D. Nešić, A. Mohammadi, and C. Manzie, "A framework for extremum seeking control of systems with parameter uncertainties," *IEEE Trans. Autom. Control*, vol. 58, pp. 435–448, 2013.

[16] J. I. Poveda and N. Li, "Robust hybrid zero-order optimization algorithms with acceleration via averaging in time," *Automatica*, vol. 123, p. 109361, 2021.

[17] M. Krstić, "Performance improvement and limitations in extremum seeking control," *Systems and Control Letters*, vol. 39, pp. 313–326, 2000.

[18] A. Scheinker and M. Krstić, "Extremum seeking-based tracking for unknown systems with unknown control directions," *51st IEEE Conference on Decision and Control*, pp. 6065–6070, 2012.

[19] V. Grushkovskaya, H-B., Durr, C. Ebenbauer, and A. Zuyev, "Extremum seeking for time-varying functions using lie bracket approximations," *IFAC-Papers Online*, vol. 50, pp. 5522–5528, 2017.

[20] J. I. Poveda and M. Krstić, "Non-smooth extremum seeking control with user-prescribed fixed-time convergence," *IEEE Transactions on Automatic and Control, DOI 10.1109/TAC.2021.3063700*, pp. 1–8, 2021.

[21] W. Wang, A. R. Teel, and D. Nešić, "Analysis for a class of singularly perturbed hybrid systems via averaging," *Automatica*, pp. 1057–1068, 2012.

[22] W. Wang, D. Nesic, and A. R. Teel, "Input-to-state stability for a class of hybrid dynamical systems via averaging," *Mathematics of Control, Signals, and Systems volume*, vol. 23, pp. 223–256, 2012.

[23] R. Goebel, R. Sanfelice, and A. R. Teel, *Hybrid Dynamical System*. Princeton, NJ: Princeton University Press, 2012.

[24] J. I. Poveda and A. R. Teel, "A framework for a class of hybrid extremum seeking controllers with dynamic inclusions," *Automatica*, vol. 76, pp. 113–126, 2017.

[25] D. Nešić, Y. Tan, W. Moase, and C. Manzie, "A unifying approach to extremum seeking: Adaptive schemes based on estimation of derivatives," *in Proc. of IEEE Conference on Decision and Control*, 2010.

[26] A. R. Teel, L. Moreau, and D. Nesic, "A unified framework for input-to-state stability in systems with two time scales," *IEEE Transactions on Automatic Control*, pp. 1526–1544, 2003.

[27] S. Parsegov, A. Polyakov, and P. Shcherbakov, "Nonlinear fixed-time control protocol for uniform allocation of agents on a segment," *51st IEEE Conference on Decision and Control*, pp. 7732–7737, 2012.

[28] F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti, "Finite-time and fixed-time input-to-state stability: Explicit and implicit approaches," *Systems & Control Letters*, vol. 144, no. 104775, 2020.

[29] P. D. Christofides and A. R. Teel, "Singular perturbation and input-to-state stability," *IEEE Trans. Autom. Control.*, vol. 41, no. 11, pp. 1645–1650, 1996.