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Hermite B-Splines: *n*-Refinability and Mask Factorization

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Abstract: This paper deals with polynomial Hermite splines. In the first part, we provide a simple and fast procedure to compute the refinement mask of the Hermite B-splines of any order and in the case of a general scaling factor. Our procedure is solely derived from the polynomial reproduction properties satisfied by Hermite splines and it does not require the explicit construction or evaluation of the basis functions. The second part of the paper discusses the factorization properties of the Hermite B-spline masks in terms of the augmented Taylor operator, which is shown to be the minimal annihilator for the space of discrete monomial Hermite sequences of a fixed degree. All our results can be of use, in particular, in the context of Hermite subdivision schemes and multi-wavelets.

1. Introduction

Cardinal Hermite interpolation is a classical problem introduced in the seminal papers [1,2]. The idea is to reconstruct a function from samples of it and of its derivatives up to a certain order. It turns out that this kind of interpolation offers more control on the reconstructed data (e.g., tangent and curvature control), making it appealing in many contexts of data processing applications.

Specifically, an *interpolatory Hermite spline* of order r is a piecewise polynomial of degree $2r - 1$ which interpolates *Hermite data*, that is function values and derivatives up to the order $r - 1$.

The basis functions for the space of Hermite splines of order r, with integer knots, correspond to the integer translates of r polynomial functions $\phi_{r;0}, \phi_{r;1}, \ldots, \phi_{r;r-1}$, sometimes named *Hermite B-splines*, supported on $[-1, 1]$, and satisfying the cardinality conditions:

$$\phi_{r;j}^{(i)}(0) = \delta_{ij} \tag{1}$$

where δ is the Kronecker delta.

It is well-known [1,2] that such conditions uniquely determine the basis functions and imply that the Hermite interpolant constructed at integer knots can be written as

$$f = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{r-1} f^{(j)}(k) \phi_{r;j}(-k),$$

for a function $f \in C^{r-1}(\mathbb{R})$.

Hermite B-splines are *refinable* in the sense that there exist $r \times r$ matrices $A^r(k)$, $k \in \{-1, 0, 1\}$, such that the following vector *refinement equation* is satisfied:

$$\phi_r^T = \sum_{k=-1}^{1} \phi_r(2 \cdot -k)^T A^r(k) \tag{2}$$

where we have denoted with ϕ_r the function vector $(\phi_{r;0}, \phi_{r;1}, \ldots, \phi_{r;r-1})^T$.

The refinement property (2) makes Hermite B-splines particularly interesting in the context of vector multi-resolution analysis, multi-wavelets, and Hermite subdivision schemes [3–12].

In this paper, we illustrate the more general refinability property of the Hermite B-spline basis, with respect to any integer scaling (dilation) factor $n \geq 2$. The first goal is to propose a fast procedure for the computation of the mask coefficients associated to their n-refinement equation. Some schemes for the computation of the mask in the binary case have already been proposed in literature. The construction proposed in [13], for example, relies on a recursive procedure for evaluating the explicit expression of the Hermite B-spline vectors of any order. The case of a general dilation factor has been recently studied in [14] and it exploits the refinability properties of the scalar cardinal B-splines with simple knots. Our computation strategy represents a simpler alternative to [13,14]. It is a direct consequence of the polynomial reproduction properties of the Hermite B-splines, which, in turn, are linked to the spectral condition or sum rule property of the associated Hermite subdivision scheme [15–17].

We further discuss the factorization of the matrix mask symbol in terms of proper "annihilators" (compare for example [18]). We give a general result proving that the augmented Taylor operators recently introduced in [19] correspond to the minimal convolution operators annihilating Hermite polynomial sequences up to a fixed degree. They consequently allow for a factorization of the Hermite B-spline mask symbol which highlights the similarity between Hermite B-spline and standard B-splines in the respective contexts of use (multiwavelets and Hermite subdivision on the one side, scalar wavelets and scalar subdivision on the other side).

2. *n*-Refinability of Hermite B-Splines and Subdivision Schemes

Hermite B-splines are n-refinable, with respect to a general dilation factor $n \geq 2$. This follows from the observation that the space of Hermite splines with knots in $\mathbb{Z}/n\mathbb{Z}$ is a subspace of the space with integer knots. Thus there exist finite matrix sequences $(A^{n;r}(k): k = 1 - n, \ldots, n - 1)$, such that the following *n-refinement equation* is satisfied:

$$\phi_r^T = \sum_{k=-n+1}^{n-1} \phi_r(n \cdot -k)^T A^{n;r}(k). \tag{3}$$

From the cardinal interpolation properties of ϕ_r^T, it easily follows that:

1. The central coefficient is given by:

$$A^{n;r}(0) = D = \begin{pmatrix} 1 & & & \\ & \frac{1}{n} & & \\ & & \ddots & \\ & & & \frac{1}{n^{r-1}} \end{pmatrix};$$

2. The matrices $A^{n;r}(k)$, for $k = 1 - n, \ldots, n - 1$, can be explicitly computed by evaluating the elements of the vector ϕ_r and their derivatives up to the order $r - 1$ at k/n, i.e.,

$$A^{n;r}_{i,j}(k) = \phi_{r;j}^{(i)}\left(\frac{k}{n}\right);$$

3. The mask coefficients $A^{n;r}(k)$ satisfy the symmetry and antisymmetry property: $A^{n;r}(k) = S A^{n;r}(-k) S$, with $S = \mathrm{diag}\left(1, -1, 1, \ldots, (-1)^{r-1}\right)$, for $k = -n + 1, \ldots, -1$.

Example 1. *In the case $r = 2$ and general $n \geq 2$, we have*

$$A^{n;2}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}.$$

Furthermore, from the explicit expression of the functions $\phi_{2;0}, \phi_{2;1}$, which can be derived from the cardinality conditions (1), we obtain

$$A^{n;2}(\pm k) = \frac{k - n}{n^3} \begin{pmatrix} (2k + n)(k - n) & \pm k(k - n) \\ \pm 6k & 3k - n \end{pmatrix}, \; k = 1, \ldots, n - 1.$$

In Theorem 1 below we show a strategy to compute the mask $A^{n;r}$, which is based on the polynomial reproduction property of Hermite splines and is simpler than evaluating the functions $\phi_{r;0}, \ldots, \phi_{r;r-1}$ or the strategy presented in [14].

The possibility of expressing Hermite B-splines as n-refinable function vectors allows the construction of corresponding *n-ary Hermite subdivision schemes*. Hermite subdivision schemes [17,20–26] are iterative procedures which, starting from an initial Hermite-type vector sequence $p^0 = (p^0(k): k \in \mathbb{Z})$, generate vector-valued sequences by

$$D^{j+1} p^{j+1} = \mathcal{S}_{A^{n;r}} D^j p^j, \quad j \in \mathbb{N},$$

where $\mathcal{S}_{A^{n;r}}$ is the *n-ary subdivision operator* defined by

$$(\mathcal{S}_{A^{n;r}} p)(i) = \sum_{k \in \mathbb{Z}} A^{n;r}(i - nk) p(k), \quad i \in \mathbb{Z}$$

The advantage of using n-ary in place of binary Hermite B-spline schemes essentially lies in the velocity of the process. Roughly speaking, an n-ary scheme, with $n > 2$, reaches a certain accuracy faster (i.e., in fewer steps) than a binary scheme. Although n-ary scalar subdivision schemes have been the subject of several studies, see for example [27–29] and citations therein, there are still very few results on their Hermite counterparts. The recent paper [30] investigates the ternary Hermite case.

A fast computation strategy for the mask of the Hermite B-splines in the general dilation case as presented in Theorem 1 thus helps the implementation of such schemes, as it allows for an effective iterative interpolation of Hermite data by avoiding the explicit construction of the basis functions and their evaluation at the integers.

3. Spectral Condition and Computation of the Mask

By definition, Hermite B-splines of order r reproduce polynomials up to the degree $2r - 1$ and their derivatives. This means that there exists vector sequences $c^j = (c^j(k): k \in \mathbb{Z})$, such that

$$x^j = \sum_k c^j(k)^T \phi_r(x - k), \quad j = 0, \ldots, 2r - 1. \tag{4}$$

From the refinement equation it is easily proved that the polynomial reproduction condition implies that the infinite block matrix $L = (A^{n;r}(i - nk) : i, k \in \mathbb{Z})$ has eigenvalues $1, \frac{1}{n}, \frac{1}{n^2}, \ldots, \frac{1}{n^{2r-1}}$ with corresponding eigenvectors $c^j, j = 0, \ldots, 2r - 1$.

In fact, (3) and (4) can be written as

$$\Phi(x) = L\,\Phi(nx) \qquad \text{and} \qquad x^j = c^j\,\Phi(x),$$

where $\Phi = (\phi(\cdot - k) : k \in \mathbb{Z}), j = 0, \ldots, 2r - 1$.

Since $x^j = n^{-j}(nx)^j = n^{-j} c^j \Phi(nx)$, we, furthermore, have

$$c^j L = n^{-j} c^j, \quad j = 0, \ldots, 2r - 1. \tag{5}$$

To make notation easier, we denote by v_f the following vector sequence associated to any function $f \in C^{r-1}(\mathbb{R})$:

$$v_f(k) = \begin{bmatrix} f(k) \\ f'(k) \\ \vdots \\ f^{(r-1)}(k) \end{bmatrix}, \quad k \in \mathbb{Z}.$$

Then, from the cardinality properties of ϕ, the coefficient sequences c_k^j are found to be:

$$c^j(k) = v^j(k)$$

where v^j, $j = 0, \ldots, 2r - 1$, are the discrete monomial Hermite sequences:

$$v^j(k) := v_{p_j}(k), \quad p_j(x) = x^j, k \in \mathbb{Z}. \tag{6}$$

The discrete polynomial reproduction condition (5) can also be written in terms of the spectral condition:

$$\sum_k A^{n;r}(i - nk)\, v^j(k) = \frac{1}{n^j}\, v^j(i), \quad i \in \mathbb{Z}, j = 0, \ldots, 2r - 1. \tag{7}$$

This can also be formulated with the help of the subdivision operator $\mathcal{S}_{A^{n;r}}$:

$$\mathcal{S}_{A^{n;r}} v^j = \frac{1}{n^j} v^j, \quad j = 0, \ldots, 2r - 1.$$

An easy computation strategy for the refinement matrix mask of the Hermite B-splines can be obtained by using (7) and support arguments, as shown in the following theorem.

Theorem 1. *For a fixed dilation factor $n \geq 2$ and a given order $r \geq 2$, the mask coefficients $A^{n;r}(k)$, $k = 1 - n, \ldots, n - 1$, associated to the n-refinement equation of the Hermite B-spline, are given by*

$$A^{n;r}(0) = D = \mathrm{diag}\left(1, \frac{1}{n}, \ldots, \frac{1}{n^{r-1}}\right)$$

and

$$A^{n;r}(\pm k) = U(\pm(k - n))\, V\, U(\mp 1)^{-1}, \quad k = 1, \ldots, n - 1,$$

where

$$V = \begin{pmatrix} \frac{1}{n^r} & & & \\ & \frac{1}{n^{r+1}} & & \\ & & \ddots & \\ & & & \frac{1}{n^{2r-1}} \end{pmatrix} = \frac{1}{n^r} D$$

and

$$U(k) = \left(v^r(k), v^{r+1}(k), \ldots, v^{2r-1}(k)\right) \in \mathbb{R}^{r \times r}$$

with the vectors $v^j(k)$, $j = r, \ldots, 2r - 1$ defined as in (6).

Proof. From (5), it follows that the eigenvalues $1, 1/n, \ldots, 1/n^{r-1}$ are associated to the matrix $A^{n;r}(0)$, while the remaining ones $1/n^r, 1/n^{r+2}, \ldots, 1/n^{2r-1}$ are related to the other mask coefficients. In fact one has in particular, for $j = 0, \ldots, 2r - 1$:

$$A^{n;r}(0)\, v^j(0) = \frac{1}{n^j} v^j(0),$$

$$A^{n;r}(i)\, v^j(0) + A^{n;r}(i-n)\, v^j(1) = \frac{1}{n^j} v^j(i), \quad i = 1, \ldots, n-1.$$

We notice that $v^j(0) = 0$, for $j \geq r$, while $v^j(0) = e_j$ for $j = 0, \ldots, r-1$ so that:

- The first equalities give the expected diagonal matrix expression for $A^{n;r}(0)$;
- The remaining equalities just correspond to

$$A^{n;r}(i-n)\, v^j(1) = \frac{1}{n^j} v^j(i), \quad i = 1, \ldots, n-1, \quad j = r, \ldots, 2r-1.$$

The last formula can be written as:

$$A^{n;r}(i-n)\, U(1) = V\, U(i), \quad i = 1, \ldots, n-1$$

from which the result follows for the coefficients with negative indices. The formula for the positive indices coefficients follows from the symmetry and antisymmetry property. □

Example 2. *We apply Theorem 1 for $r = 3$. We have:*

$$U(k) = \begin{pmatrix} k^3 & k^4 & k^5 \\ 3k^2 & 4k^3 & 5k^4 \\ 6k & 12k^2 & 20k^3 \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{27} & & \\ & \frac{1}{81} & \\ & & \frac{1}{243} \end{pmatrix},$$

so, in the case of arity $n = 2, 3$, the positive indexed coefficients are given by:

$$A^{2;3}(1) = U(-1)\, V\, U(-1)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{5}{32} & \frac{1}{64} \\ -\frac{15}{16} & -\frac{7}{32} & -\frac{1}{64} \\ 0 & -\frac{3}{8} & -\frac{1}{16} \end{pmatrix},$$

$$A^{3;3}(1) = U(-2)\, V\, U(-1)^{-1} = \begin{pmatrix} \frac{64}{81} & \frac{16}{81} & \frac{4}{243} \\ -\frac{40}{81} & 0 & \frac{2}{243} \\ -\frac{40}{81} & -\frac{32}{81} & -\frac{10}{243} \end{pmatrix},$$

$$A^{3;3}(2) = U(-1)\, V\, U(-1)^{-1} = \begin{pmatrix} \frac{17}{81} & \frac{2}{27} & \frac{2}{243} \\ -\frac{40}{81} & -\frac{13}{81} & -\frac{4}{243} \\ \frac{40}{81} & \frac{8}{81} & \frac{1}{243} \end{pmatrix}.$$

Note that these are the same masks as obtained in [14] [Example 4.2, 4.3], but the computational effort for our construction is less.

In order to better highlight the implementation simplicity of the procedure, we conclude this section by describing it through the following pseudocode (Algorithm 1), where we have used the explicit expression for the m-th derivative of a monomial of degree j, and the usual convention $\frac{j!}{(j-m)!} = 0$ for $m \geq j$.

Algorithm 1 Mask computation for Hermite B-splines

Require: n, r

1: $D \leftarrow \mathrm{diag}\left(1, \frac{1}{n}, \ldots, \frac{1}{n^{r-1}}\right)$

2: $V \leftarrow 1/n^r D$

3: $S \leftarrow \mathrm{diag}\left(1, -1, \ldots, (-1)^{r-1}\right) D$

4: **for** $j = 0$ **to** $2r - 1$ **do**

5:     compute the column vector $\hat{v}^j = \left[\frac{(-1)^{j-m} j!}{(j-m)!} : m = 0, \ldots, r-1\right]^T$

6: **end for**

7: construct the matrix $\hat{U} = \left[\hat{v}^r, \hat{v}^{r+1}, \ldots, \hat{v}^{2r-1}\right]$

8: $\hat{U} \leftarrow \hat{U}^{-1}$

9: **for** $k = 1$ **to** $n - 1$ **do**

10:     **for** $j = 0$ **to** $2r - 1$ **do**

11:         compute the column vector $v^j = \left[\frac{j!}{(j-m)!}(k-n)^{j-m} : m = 0, \ldots, r-1\right]^T$

12:     **end for**

13:     construct the matrix $U = \left[v^r, v^{r+1}, \ldots, v^{2r-1}\right]$

14:     compute $A(k) = U V \hat{U}$

15:     compute $A(-k) = S A(k) S$

16: **end for**

17: **return** $A(-n+1), \ldots, A(-1), A(0), A(1), \ldots, A(n-1)$

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4. Factorization of the Mask Symbol

Polynomial reproduction properties (or spectral conditions) are strongly connected to the factorizability of the mask *symbol*, given by

$$A^{n;r}(z) = \sum_{k=1-n}^{n-1} A^{n;r}(k) z^k, \quad z \in \mathbb{C} \setminus \{0\},$$

in terms of proper annihilators [16,18]. Such factorizations, in turn, are a major tool for proving convergence and smoothness of Hermite subdivision schemes [16].

For Hermite schemes, operators for factorization purposes have been originally introduced in [15,16], where they are called Taylor operators. Indeed, by adapting the results of [16] from $n = 2$ to general arity $n \geq 2$, there exists a finitely supported mask $B^{n;r}$ such that the Hermite B-spline symbol $A^{r;n}$ satisfies

$$\mathrm{T}^r(z) A^{n;r}(z) = n^{-r+1} B^{n;r}(z) \mathrm{T}^r(z^n), \tag{8}$$

where T^r is the complete Taylor operator of size $(r \times r)$, see [16]. The contractivity of the subdivision operator $\mathcal{S}_{B^{n;r}}$ then implies C^{r-1}-convergence of the scheme $\mathcal{S}_{A^{n;r}}$ [16].

The factorization with respect to T^r holds true whenever the degree of polynomial reproduction of the basis involved is at least $r - 1$. However, since Hermite B-splines of order r have polynomial reproduction degree $2r - 1$, the standard Taylor factorization (8), while still valid, can be "improved".

The fact that the reproduction order is greater than the spline order is termed "polynomial over-reproduction" in [19], and through this over-reproduction, it follows immediately from [19], that $A^{n;r}$ factorizes in the sense of (8) with respect to the augmented Taylor operators $\mathrm{T}^{r;p}$, $p = r - 1, \ldots, 2r - 1$:

$$
T^{r;p} = \begin{pmatrix}
\Delta & -1 & -\frac{1}{2} & \cdots & -\frac{1}{(r-2)!} & -\sum\limits_{k=0}^{p-r+1} G_k^{r-1}\Delta^k \\
& \ddots & \ddots & & \vdots & \vdots \\
& & \ddots & \ddots & \vdots & \vdots \\
& & & \Delta & -1 & -\sum\limits_{k=0}^{p-r+1} G_k^{2}\Delta^k \\
& & & & \Delta & -\sum\limits_{k=0}^{p-r+1} G_k^{1}\Delta^k \\
& & & & & \Delta^{p+2-r}
\end{pmatrix}
\tag{9}
$$

where Δ is the forward difference operator, $\Delta c(i) = c(i+1) - c(i)$, $i \in \mathbb{Z}$, and G_k^j, $k \geq 0, j \geq 1$ are the *coefficients for repeated integration with forward differences* [31]. In general, polynomial over-reproduction allows for factorizations that may lead to high smoothness of the scheme, see [25,32–34].

Similar to (8), through the factorization results of [19,35] we obtain a mask $B^{n;r}$ such that

$$
T^{r;2r-1}(z) A^{n;r}(z) = n^{-2r+1} B^{n;r}(z) T^{r;2r-1}(z^n). \tag{10}
$$

The augmented Taylor operators $T^{r;p}$ generalize the complete Taylor operator T^r. Indeed, we have $T^{r;r-1} = T^r$.

The existence of a factorization as in (8) via certain degree of polynomial reproduction can also be phrased in terms of *minimal annihilators* for the polynomial space, see [18]. Indeed, the complete Taylor operator T^r is unique in the sense that it is a minimal annihilator for the space $\{1, x, \ldots, x^{r-1}\}$.

Following [18], we define a (r, p)-*annihilator operator* as a convolution operator $H^{r;p}$ satisfying

$$
\left(H^{r;p}v^j\right)(k) = \sum_{i\in\mathbb{Z}} H^{r;p}(k-i)v^j(i) = 0, \quad k \in \mathbb{Z}, j = 0, \ldots p, \tag{11}
$$

with v^j as in (6). Here, r denotes the size of the operator and p denotes the maximal degree of polynomials being annihilated. It is shown in [16] that the complete Taylor operator T^r is an $(r, r-1)$-annihilator.

An annihilator $H^{r;p}$ is called *minimal* (with respect to subdivision) if for every subdivision operator $\mathcal{S}_C$ satisfying $\mathcal{S}_C v^j = 0, j = 0, \ldots, p$, there exists a subdivision operator $\mathcal{S}_B$, such that $\mathcal{S}_C = \mathcal{S}_B H^{r;p}$. It is shown in [18] that the complete Taylor operator T^r is indeed a minimal $(r, r-1)$-annihilator.

In the following, we put into evidence that the augmented Taylor operator $T^{r;p}$, in analogy to the complete Taylor operator, is a minimal (r, p)-annihilator. This fact is mentioned in [19], and we provide a formal proof here.

Lemma 1. *The augmented Taylor operator $T^{r;p}$ is a (r, p)-annihilator.*

Proof. We prove this by induction on p. For $p = r - 1$, we know from [18] that the complete Taylor operator $T^{r;r-1}$ is an $(r, r-1)$-annihilator. For the induction step, we assume that $T^{r;p-1}$ is a $(r, p-1)$-annihilator and prove the result for p.

From [19] [Lemma 10] we know $T^{r;p} = \Delta_r \left(I_r - y_{p-r+1}e_r^T\right) T^{r;p-1}$, where

$$
\Delta_r = \begin{pmatrix} I_{r-1} & 0 \\ 0 & \Delta \end{pmatrix}
$$

and $y_j = \left(G_j^{r-1}, \ldots, G_j^1, 0 \right)^T \in \mathbb{R}^r$. Since $\mathrm{T}^{r;p-1}$ annihilates $v^j, j = 0, \ldots, p-1$, we immediately get that $\mathrm{T}^{r;p}$ annihilates $v^j, j = 0, \ldots, p-1$. Therefore, we only need to prove that $\mathrm{T}^{r,p} v^p = 0$.

Ref. [19] [Corollary 16] implies $\mathrm{T}^{r;p-1} v^p = \mathrm{w}_{p-r+1}$, where $\mathrm{w}_j = y_j + e_r = \left(G_j^{r-1}, \ldots, G_j^1, 1 \right)^T$. Therefore,

$$\mathrm{T}^{r;p} v^p = \Delta_r \left(I_r - y_{p-r+1} e_r^T \right) \mathrm{w}_{p-r+1} = \Delta_r \left(\mathrm{w}_{p-r+1} - y_{p-r+1} \right) = \Delta_r e_r = 0.$$

This concludes the induction step. □

Lemma 2. *The augmented Taylor operator $\mathrm{T}^{r;p}$ is a minimal (r, p)-annihilator.*

Proof. We prove this result by induction on p. For $p = r - 1$, the augmented Taylor operator $\mathrm{T}^{r;r-1}$ is just the regular complete Taylor operator of [16] and the minimality result follows from [16,18].

For the induction step, the proof is very similar to the proof of [16] [Proposition 1]. Indeed, suppose that $\mathcal{S}_C v^j = 0$ for $j = 0, \ldots, p$. In particular, $\mathcal{S}_C$ annihilates $v^j, j = 0, \ldots, p - 1$. Therefore, the induction hypothesis implies the existence of a mask B_{p-1}, such that $\mathcal{S}_C = \mathcal{S}_{B_{p-1}} \mathrm{T}^{r;p-1}$. Since $\mathcal{S}_C$ annihilates v^p as well, we have

$$0 = \mathcal{S}_C v^p = \mathcal{S}_{B_{p-1}} \mathrm{T}^{r;p-1} v^p.$$

From [19] [Corollary 16] we know that $\mathrm{T}^{r;p-1} v^p = \mathrm{w}_{p-r+1}$ with $\mathrm{w}_i = \left(G_i^{r-1}, \ldots, G_i^1, 1 \right)^T$. This implies

$$0 = \mathcal{S}_{B_{p-1}} \mathrm{w}_{p-r+1}. \tag{12}$$

Denote by $B_{p-1} = [b^0, \ldots, b^{r-1}]$ the columns of the mask B_{p-1}. Then, (12) implies

$$0 = \sum_{k \in \mathbb{Z}} b^0(i - nk) G_{p-r+1}^{r-1} + \ldots + b^{r-2}(i - nk) G_{p-r+1}^1 + b^{r-1}(i - nk),$$

for all $i \in \mathbb{Z}$. In terms of symbols this means that there exists a vector sequence d, such that

$$d(z)(z^{-n} - 1) = \sum_{\ell=0}^{r-2} b^\ell(z) G_{p-r+1}^{r-1-\ell} + b^{r-1}(z),$$

or equivalently,

$$b^{r-1}(z) = \left[b^0(z), \ldots, b^{r-2}(z), d(z) \right] \begin{pmatrix} -G_{p-r+1}^{r-1} \\ \vdots \\ -G_{p-r+1}^1 \\ z^{-n} - 1 \end{pmatrix}.$$

Define $B_p(z) := \left[b^0(z), \ldots, b^{r-2}(z), d(z) \right]$. With this notation, we have

$$B_{p-1}(z) = B_p(z) \begin{pmatrix} I_{r-1} & -G_{p-r+1}^{(r-1):1} \\ 0 & z^{-n} - 1 \end{pmatrix},$$

with $G_j^{(r-1):1} = \left(G_j^{r-1}, \ldots, G_j^1\right)^T \in \mathbb{R}^{r-1}$. This, together with $\mathcal{S}_C = \mathcal{S}_{B_{p-1}} T^{r;p-1}$, further implies

$$
\begin{aligned}
C(z) &= B_{p-1}(z)\begin{pmatrix} T^{r-1}(z^n) & -\sum_{k=0}^{p-r} G_k^{(r-1):1}(z^{-n}-1)^k \\ 0 & (z^{-n}-1)^{p-r+1} \end{pmatrix} \\
&= B_p(z)\begin{pmatrix} I_{r-1} & -G_{p-r+1}^{(r-1):1} \\ 0 & z^{-n}-1 \end{pmatrix}\begin{pmatrix} T^{r-1}(z^n) & -\sum_{k=0}^{p-r} G_k^{(r-1):1}(z^{-n}-1)^k \\ 0 & (z^{-n}-1)^{p-r+1} \end{pmatrix} \\
&= B_p(z)\begin{pmatrix} T^{r-1}(z^n) & -\sum_{k=0}^{p-r+1} G_k^{(r-1):1}(z^{-n}-1)^k \\ 0 & (z^{-n}-1)^{p-r+2} \end{pmatrix} = B_p(z)T^{r;p}(z^n).
\end{aligned}
$$

This implies $\mathcal{S}_C = \mathcal{S}_{B_p} T^{r;p}$. □

Example 3. *We now use the augmented Taylor operators to factorize the symbols of the Hermite B-spline masks. Recall that, if the spline order is r, then the polynomial reproduction order is $2r-1$.*

The symbols of the augmented Taylor operators in the case $r = 2$, $r = 3$ are, respectively, given by:

$$
T^{2,3} = \begin{pmatrix} -\frac{z-1}{z} & -\frac{5z^2+8z-1}{12z^2} \\ 0 & -\frac{(z-1)^3}{z^3} \end{pmatrix}, \quad
T^{3,5} = \begin{pmatrix} -\frac{z-1}{z} & -1 & -\frac{97z^3+114z^2-39z+8}{360z^3} \\ 0 & -\frac{z-1}{z} & -\frac{9z^3+19z^2-5z+1}{24z^3} \\ 0 & 0 & \frac{(z-1)^4}{z^4}. \end{pmatrix}
$$

From direct computations, it follows that for Hermite B-splines of order $r = 2$, the factors $B^{n;r}(z)$ in case of arity $n = 2$ and $n = 3$ are, respectively, given by:

$$
B^{2;2}(z) = \begin{pmatrix} \frac{3z^2+1}{2z} & -\frac{z}{24} \\ -\frac{6(z-1)^3}{z^2} & \frac{3}{2}z-\frac{1}{2} \end{pmatrix}, \quad
B^{3;2}(z) = \begin{pmatrix} \frac{2z^3+1}{z} & -\frac{1}{12z^2} \\ -\frac{12(z^2-1)(z-1)^2}{z^2} & (2z-1)z \end{pmatrix},
$$

while for Hermite B-splines of order $r = 3$, and arities $n = 2$ and $n = 3$ we have:

$$
B^{2;3}(z) = \begin{pmatrix} -14z+16 & \frac{263z^3-264z^2+39z-8}{30z^2} & \frac{13z}{240}-\frac{1}{90} \\ 30-30z & \frac{37z^3-33z^2+5z-1}{2z^2} & \frac{5z}{48}-\frac{1}{48} \\ 0 & \frac{12(-1+z)^4}{z^3} & \frac{5z^2-4z+1}{2z} \end{pmatrix},
$$

$$
B^{3;3}(z) = \begin{pmatrix} -\frac{110z^4-80z^3-47z+8}{3z^2} & \frac{332z^4-224z^3-71z+8}{15z^2} & \frac{(163z-40)z}{1080} \\ -\frac{5(15z^4-10z^3-6z+1)}{z^2} & \frac{45z^4-28z^3-9z+1}{z^2} & \frac{(21z-5)z}{72} \\ -\frac{120(z^2-2z+1)(-1+z)^3}{z^3} & \frac{24(4z^3-9z^2+6z-1)(-1+z)^2}{z^3} & \frac{14}{3}z^2-\frac{16}{3}z+\frac{5}{3} \end{pmatrix}.
$$

It is worth noticing that, up to a constant factor, the determinant of the generic matrix factor $B^{n;r}(z)$ is the monomial $z^{r(n-1)}$. In other words, the polynomial matrix $\frac{1}{z^{n-1}}B^{n;r}(z)$ is unimodular, so that, from (10),

$$
\det\left(A^{n;r}(z)\right) = n^{1-2r}\det\left(B^{n;r}(z)\right)\det(T^{r,2r-1}(z)^{-1})\det(T^{r,2r-1}(z^n)) = K z^{r(n-1)}\left(\frac{z^n-1}{z-1}\right)^{2r}.
$$

This observation reveals some similarity between the determinant of the symbol of Hermite B-splines and the symbol of the scalar canonical B-splines of degree m, which, in the case of general arity n, possesses $\left(\frac{z^n-1}{z-1}\right)^{m+1}$ as its only polynomial factor [27].

5. Conclusions

We illustrated a simple and fast procedure for the computation of the mask coefficients of Hermite B-spline vectors of any order and for any dilation factor using the polynomial reproduction property satisfied by these splines. Such construction can, in particular, be of use in the context of Hermite subdivision and multi-wavelets. We further showed that the minimal annihilators for the space of monomials up to a degree (possibly larger than the mask's size) are exactly the augmented Taylor operators of [19]. For some examples, the consequent factorization of the Hermite B-spline mask in terms of such annihilators shows a similarity with scalar cardinal B-splines masks in terms of the determinant of the symbol. It is the goal of future research to study this aspect in more detail and to extend the results presented in this paper to the case of Hermite exponential splines, as in [18,36].

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