



A note on second order Riesz transforms in 3-dimensional Lie groups

FABRICE BAUDOUIN AND LI CHEN 

Abstract. We prove explicit L^p bounds for second order Riesz transforms of the sub-Laplacian and of the Laplacian in the Lie groups $\mathbb{H}$, $\mathrm{SU}(2)$, and $\widetilde{\mathrm{SL}}(2)$. Our proof makes use of martingale transform techniques and specific commutation properties between the complex gradient and the sub-Laplacian in those Lie groups.

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1. Introduction. The classical sharp martingale inequalities, going back to the works of Burkholder, have many applications in the study of basic singular integrals on Euclidean spaces. As typical examples, they lead to sharp L^p bounds for Riesz transforms and sharp or dimension-free L^p bounds for second order Riesz transforms. We refer the reader to, for instance, [3, 7, 18] and the overview paper [2]. The purpose of this note is to study explicit L^p bounds for certain subelliptic second order Riesz transforms, in particular Beurling-Ahlfors type operators, on three dimensional Lie groups including the Heisenberg group $\mathbb{H}$, $\mathrm{SU}(2)$, and $\widetilde{\mathrm{SL}}(2)$ (universal cover of $\mathrm{SL}(2)$).

Second order Riesz transforms appear naturally in the study of PDEs (see for instance [17]) and have been extensively studied in the literature. They are mostly interpreted as iterations of Riesz transforms and their adjoints, or second derivatives of the fundamental solution operator for the Laplacian operator. On Euclidean spaces, second order Riesz transforms are well understood as Calderón-Zygmund singular integrals and have bounded L^p norm for

$1 < p < \infty$. A particular interesting example is the classical Beurling-Ahlfors operator on the complex plane defined by

$$Bf(z) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dw.$$

Equivalently,

$$B = R_1^2 - R_2^2 - 2iR_1R_2,$$

where $R_i = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2}$ for $i = 1, 2$ are the Riesz transforms on $\mathbb{R}^2$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian. One can also write $B = \partial^2(-\Delta)^{-1}$, where ∂ is the Cauchy-Riemann operator

$$\partial = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}.$$

The classical Beurling-Ahlfors operator and its generalizations play an important role in quasiconformal mappings. Sharp or dimension-free L^p bounds of second order Riesz transforms have been studied from both deterministic methods and martingale transform techniques using either Poisson or heat extensions, see for instance [5–7, 23]. However, the sharp L^p norm of the Beurling-Ahlfors operator is a long existing open problem.

On Heisenberg groups $\mathbb{H}^n$, second order Riesz transforms are also singular integral operators (see [16, Theorem 3]), which follows from the fundamental solution of the sub-Laplacian L (see for instance [16, 22]). Dimension-free L^p bounds for Riesz transforms have been obtained in [12], see also [21] for the reduced case. However explicit L^p bounds of Riesz transforms and their higher order analogues are not known. We would like to mention that in [4] the authors together with R. Bañuelos gave explicit L^p bounds for generalized Riesz transforms which are commutator of complex gradients and the square root of the non-negative sub-Laplacian. In other settings, dimension-free L^p bounds for second order Riesz transforms have also been studied via deterministic or probabilistic approaches, for instance, on k -forms on complete Riemannian manifolds under curvature assumptions [19, 20], and on discrete groups [1, 14, 15]. In this note, we are interested in extensions of the Beurling-Ahlfors operator and other second order Riesz transforms associated with the sub-Laplacian on the simply connected Lie groups $\mathbb{H}$, $\mathbb{SU}(2)$, and $\widetilde{\mathbb{SL}}(2)$.

Following [9, 11], given $\rho \in \mathbb{R}$, let $\mathfrak{g}(\rho)$ be the three dimensional Lie algebra with basis $\{X, Y, Z\}$ such that

$$[X, Y] = Z, \quad [X, Z] = -\rho Y, \quad [Y, Z] = \rho X.$$

Let $\mathbb{G}(\rho)$ be the three dimensional simply connected Lie group with Lie algebra $\mathfrak{g}(\rho)$. The cases $\rho = 0$, $\rho = 1$, and $\rho = -1$ are corresponding, respectively, to the Heisenberg group $\mathbb{H}$, $\mathbb{SU}(2)$, and $\widetilde{\mathbb{SL}}(2)$ (universal cover of $\mathbb{SL}(2)$). Consider then on $\mathbb{G}(\rho)$ the subelliptic, left-invariant, sub-Laplacian

$$L = X^2 + Y^2.$$

Our main result is the following:

Theorem 1.1. *For any real-valued numbers a, b, c and $\alpha \geq 0$, consider on the Lie group $\mathbb{G}(\rho)$ the operator*

$$S_{a,b,c}^\alpha f = a(-L + \alpha)^{-1} Zf + b(X(-L - \rho + \alpha)^{-1} X - Y(-L - \rho + \alpha)^{-1} Y) f \\ + c(X(-L - \rho + \alpha)^{-1} Y + Y(-L - \rho + \alpha)^{-1} X) f. \quad (1)$$

Then we have for every $p > 1$,

$$\|S_{a,b,c}^\alpha f\|_p \leq \sqrt{2}(|a| + |b| + |c|)(p^* - 1)\|f\|_p,$$

where $p^* = \max\{p, \frac{p}{p-1}\}$.

To prove this result, we write $S_{a,b,c}^\alpha$ in terms of the heat semigroup $P_t = e^{tL}$ and then use martingale techniques similar to [3, Theorem 1.1] together with commutation properties between the complex gradient and the sub-Laplacian which are specific to the 3-dimensional Lie groups under consideration.

This note is organized as follows. In Section 2, we study explicit L^p bounds of the Beurling-Ahlfors type operator $\int_0^\infty P_t W W P_t f dt$ in specific settings including the Heisenberg group $\mathbb{H}$ (more generally $\mathbb{H}^n$), $\mathbb{SU}(2)$, and $\widetilde{\mathbb{SL}}(2)$ and then summarize those findings to give the proof of our main result Theorem 1.1. In Section 3, for the sake of completeness, we discuss the case of second-order Riesz transforms associated with the full-Laplacian.

2. Second order Riesz transforms for the sub-Laplacian.

2.1. On Heisenberg groups. The three dimensional simply connected Heisenberg group is the space $\mathbb{G}(\rho)$ with $\rho = 0$. For the sake of generality, we consider the Heisenberg group $\mathbb{H}^n = \mathbb{R}^{2n+1}$ with arbitrary $n \in \mathbb{N}$. A basis of left-invariant vector fields is

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial z}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z},$$

and hence the sub-Laplacian

$$L = \sum_{j=1}^n (X_j^2 + Y_j^2).$$

Similarly as in $\mathbb{H}$, we have for any $1 \leq j \leq n$,

$$[X_j, Y_j] = Z, \quad [X_j, Z] = [Y_j, Z] = 0.$$

Let $W_j = X_j - iY_j$ be the complex gradient, then

$$W_j L = (L + 2iZ)W_j. \quad (2)$$

In other words, we have $[W_j, L] = 2iZW_j$. Note also that $[W_j, Z] = [L, Z] = 0$.

Using the cylindric coordinates (r, θ, z) as in [8, Section 5.2], the radial sub-Laplacian is then

$$L = \frac{\partial^2}{\partial r^2} + \frac{2n-1}{r} \frac{\partial}{\partial r} + \frac{r^2}{4} \frac{\partial^2}{\partial z^2}.$$

In particular, one can write on $\mathbb{H}$ the left-invariant complex gradient as

$$W = X - iY = e^{-i\theta} \frac{\partial}{\partial r} - \frac{ie^{-i\theta}}{r} \frac{\partial}{\partial r} - \frac{ie^{-i\theta}}{2} r \frac{\partial}{\partial z},$$

and the right-invariant complex gradient as

$$\hat{W} = \hat{X} - i\hat{Y} = e^{-i\theta} \frac{\partial}{\partial r} - \frac{ie^{-i\theta}}{r} \frac{\partial}{\partial r} + \frac{ie^{-i\theta}}{2} r \frac{\partial}{\partial z}.$$

We are concerned with the second order Riesz transforms $W_j(-L)^{-1}W_k$, $1 \leq j, k \leq n$. Denote by $\mathcal{S}(\mathbb{H}^n)$ the Schwartz space of smooth rapidly decreasing functions on the Heisenberg group.

Proposition 2.1. *Let $f \in \mathcal{S}(\mathbb{H}^n)$. Then $W_j(-L)^{-1}W_k f$, $1 \leq j, k \leq n$, are bounded in $L^p(\mathbb{H}^n)$, $p > 1$, with*

$$\|W_j(-L)^{-1}W_k f\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p.$$

Our proof for Proposition 2.1 is through martingale transform techniques. In order to obtain probabilistic representations of $W_j(-L)^{-1}W_k$, $1 \leq j, k \leq n$, we first try to rewrite these operators in form of the Littlewood-Paley type. Formal computation from (2) leads to

$$W_j P_t = e^{-2itZ} P_t W_j$$

and hence $P_t W_j W_k P_t = W_j P_t e^{2itZ} e^{-2itZ} P_t W_k = W_j P_{2t} W_k$. Thus one may have

$$W_j(-L)^{-1}W_k f = 2 \int_0^\infty P_t W_j W_k P_t f dt \quad \forall f \in \mathcal{S}(\mathbb{H}^n). \quad (3)$$

The above computations are only formal since the semigroups associated to $L - 2iZ$ and $L + 2iZ$ are not globally well-defined, see [8, Section 5.2] for more details. However, we can rigorously show the following lemma.

Lemma 2.2. *Let $f \in \mathcal{S}(\mathbb{H}^n)$, then (3) holds.*

Proof. If $f(\mathbf{x}) = f(x, y, z) = e^{i\lambda z} g(x, y)$ for some $\lambda \in \mathbb{R}$ and some function g , we have $Zf = i\lambda f$. It follows from (2) that for any $1 \leq j \leq n$,

$$W_j L f = (L + 2\lambda) W_j f.$$

We deduce then

$$W_j P_t f(0) = e^{2\lambda t} (P_t W_j f)(0). \quad (4)$$

Denote $f_j(x, y, z) = W_j f(x, y, z)$. Observe that $ZW_j f = W_j Zf = i\lambda W_j f$, i.e., $Zf_j = i\lambda f_j$. Therefore

$$W_j W_k P_t f(0) = e^{2\lambda t} W_j (P_t W_k f)(0) = e^{2\lambda t} W_j P_t f_k(0) = e^{4\lambda t} (P_t W_j f_k)(0),$$

and thus applying (4) again,

$$P_t W_j W_k P_t f(0) = e^{4\lambda t} (P_{2t} W_j f_k)(0) = W_j P_{2t} f_k(0) = W_j P_{2t} f_k(0).$$

Now plugging this into the left hand side of (3) leads to

$$2 \int_0^\infty W_j P_{2t} W_k f dt = 2W_j \int_0^\infty P_{2t} dt W_k f = W_j(-L)^{-1}W_k f.$$

For general f , we can conclude by using the Fourier transform of f with respect to the variable z . $\square$

Thanks to the identity (3), we are now able to give a probabilistic proof for the sharp or dimension-free explicit L^p bound of $W_j(-L)^{-1}W_k$, $1 \leq j, k \leq n$, following [3]. We introduce some notations here. Let $(Y_t)_{t \geq 0}$ be the diffusion associated with the sub-Laplacian L starting with a distribution μ and let $(B_t)_{t \geq 0}$ be the Brownian motion on $\mathbb{R}^{2n}$ associated with the Laplace operator $\Delta = \sum_{i=1}^{2n} \frac{\partial^2}{\partial x_i^2}$. Denote $\nabla = (X_1, \dots, X_n, Y_1, \dots, Y_n)$.

Proof of Proposition 2.1. Notice that from (3), one writes

$$W_j(-L)^{-1}W_k f = 2 \int_0^\infty P_t(X_j X_k - Y_j Y_k - i(X_j Y_k + X_k Y_j)) P_t f dt.$$

The rest of the proof is similar to [3, Theorem 1.1] and we write it here for the sake of completeness. Denote

$$S_{jk}^T f := \mathbb{E} \left(\int_0^T A_{jk} \nabla P_{T-t} f(Y_t) \cdot dB_t \mid Y_T = x \right).$$

We claim that

$$W_j(-L)^{-1}W_k f(x) = \lim_{T \rightarrow \infty} S_{jk}^T f,$$

where A_{jk} is a $2n \times 2n$ matrix with entries $a_{jk} = -1$, $a_{j(n+k)} = a_{(n+j)k} = i$, $a_{(n+j)(n+k)} = 1$, and otherwise 0.

For any $T > 0$, it is easy to see that $(P_{T-t} f(Y_t))_{0 \leq t \leq T}$ is a martingale. Then by Itô's formula and the Itô isometry, we have for any $g \in \mathcal{S}(\mathbb{H}^n)$,

$$\begin{aligned} \int_{\mathbb{H}^n} g(x) S_{jk}^T f(x) d\mu(x) &= \mathbb{E} \left(g(Y_T) \int_0^T A_{jk} \nabla P_{T-t} f(Y_t) \cdot dB_t \right) \\ &= 2 \mathbb{E} \left(\int_0^T A_{jk} \nabla P_{T-t} f(Y_t) \cdot \nabla P_{T-t} g(Y_t) dt \right) \\ &= 2 \int_0^T \int_{\mathbb{H}^n} A_{jk} \nabla P_t f(x) \cdot \nabla P_t g(x) d\mu(x) dt \\ &= 2 \int_{\mathbb{H}^n} \int_0^T P_t W_j W_k P_t f(x) dt g(x) d\mu(x). \end{aligned}$$

Thus we deduce that

$$\lim_{T \rightarrow \infty} \int_0^T \int_{\mathbb{H}^n} S_{jk}^T f(x) g(x) d\mu(x) = \int_{\mathbb{H}^n} W_j(-L)^{-1}W_k f(x) g(x) d\mu(x).$$

Observe that the matrix norm of A_{jk} is $\sqrt{2}$, thus by the L^p bound of the martingale transform (see [7]),

$$\|W_j(-L)^{-1}W_k f\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p.$$

□

Remark 2.3. We obtain on $\mathbb{H}$ an explicit bound for the Beurling-Ahlfors operator $B_{\mathbb{H}} = W(-L)^{-1}W$. As singular integral operator, $B_{\mathbb{H}}$ has the form

$$B_{\mathbb{H}}f(0) = \frac{1}{\pi} \int_{\mathbb{H}} (x - iy)^2 (|x + iy|^4 + 16z^2)^{-3/2} f(\mathbf{x}) d\mathbf{x},$$

where $\mathbf{x} = (x, y, z) \in \mathbb{H}$.

Indeed, at the origin one has

$$W(-L)^{-1}Wf(0) = (-L)^{-1}\hat{W}Wf(0) = \int_{\mathbb{H}} G(\mathbf{x})\hat{W}Wf(\mathbf{x})d\mathbf{x} = \int_{\mathbb{H}} W\hat{W}G(\mathbf{x})f(\mathbf{x})d\mathbf{x},$$

where $G(\mathbf{x})$ is the Green function on $\mathbb{H}$ (see for instance [16, Theorem 2]):

$$G(\mathbf{x}) = G(x, y, z) = \frac{1}{4\pi\sqrt{(x^2 + y^2)^2 + 16z^2}}.$$

Direct computation from the cylinder coordinates yields

$$W\hat{W}G(r, \theta, z) = \frac{r^2 e^{-2i\theta}}{\pi(r^4 + 16z^2)^{3/2}}.$$

On the other hand, let $\Pi : \mathbb{H} \rightarrow \mathbb{C}$ be the projection operator, then

$$B_{\mathbb{H}}(f \circ \Pi)(0) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w^2} f(w) dw.$$

This is exactly the classical Beurling-Ahlfors operator.

2.2. On $\mathrm{SU}(2)$. Consider the Lie group $\mathrm{SU}(2)$, which is $\mathbb{G}(\rho)$ with $\rho = 1$. Denote by X, Y, Z the left invariant vector fields on $\mathrm{SU}(2)$ corresponding to the Pauli matrices

$$X = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

for which the following commutation relations hold

$$[X, Y] = Z, \quad [X, Z] = -Y, \quad [Y, Z] = X.$$

We shall be interested in the sub-Laplacian

$$L = X^2 + Y^2.$$

Note that $[L, Z] = 0$. Consider also the complex gradient $W = X - iY$. The Lie algebra structure gives us

$$WL = (L + 2iZ + 1)W, \quad WZ = (Z - i)W. \quad (5)$$

To concretely describe X, Y, Z , and L , one can use the cylindric coordinates introduced in [13]:

$$(r, \theta, z) \rightarrow \exp(r \cos \theta X + r \sin \theta Y) \exp(zZ) = \begin{pmatrix} \cos \frac{r}{2} e^{i\frac{z}{2}} & \sin \frac{r}{2} e^{i(\theta - \frac{z}{2})} \\ -\sin \frac{r}{2} e^{-i(\theta - \frac{z}{2})} & \cos \frac{r}{2} e^{-i\frac{z}{2}} \end{pmatrix}$$

with

$$0 \leq r \leq \pi, \quad \theta \in [0, 2\pi], \quad z \in [-2\pi, 2\pi].$$

Then the vector fields X , Y , and Z read as (see [10])

$$\begin{aligned} X &= \cos(-\theta + z) \frac{\partial}{\partial r} + \sin(-\theta + z) \left(\tan \frac{r}{2} \frac{\partial}{\partial z} + \frac{1}{2} \left(\tan \frac{r}{2} + \frac{1}{\tan \frac{r}{2}} \right) \frac{\partial}{\partial \theta} \right), \\ Y &= -\sin(-\theta + z) \frac{\partial}{\partial r} + \cos(-\theta + z) \left(\tan \frac{r}{2} \frac{\partial}{\partial z} + \frac{1}{2} \left(\tan \frac{r}{2} + \frac{1}{\tan \frac{r}{2}} \right) \frac{\partial}{\partial \theta} \right), \\ Z &= \frac{\partial}{\partial z}. \end{aligned}$$

The left invariant complex gradient $W = X - iY$ becomes

$$W = e^{i(-\theta+z)} \frac{\partial}{\partial r} - ie^{i(-\theta+z)} \left(\tan \frac{r}{2} \frac{\partial}{\partial z} + \frac{1}{2} \left(\tan \frac{r}{2} + \frac{1}{\tan \frac{r}{2}} \right) \frac{\partial}{\partial \theta} \right), \quad (6)$$

and the right invariant complex gradient $\hat{W}$ is as follows

$$\hat{W} = e^{-i\theta} \frac{\partial}{\partial r} + ie^{-i\theta} \left(\tan \frac{r}{2} \frac{\partial}{\partial z} + \frac{1}{2} \left(\tan \frac{r}{2} - \frac{1}{\tan \frac{r}{2}} \right) \frac{\partial}{\partial \theta} \right).$$

Our main result on $\mathbb{SU}(2)$ is the following.

Proposition 2.4. *Let $\alpha \geq 0$. Then the second order Riesz transforms $W(-L - 1 + \alpha)^{-1}W$ is bounded on L^p with*

$$\|W(-L - 1 + \alpha)^{-1}Wf\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p.$$

The proof is similar to that of Proposition 2.1. We observe that formally the relations in (5) lead to

$$WP_t = e^{2itZ+t}P_tW$$

and

$$P_tWWP_t = e^{-2itZ-t}WP_te^{2itZ+t}P_tW = e^{2t}WP_{2t}W.$$

Thus

$$W(-L - 1)^{-1}Wf = 2 \int_0^\infty P_tWWP_t f dt. \quad (7)$$

Indeed, we have

Lemma 2.5. *Let $f \in \mathcal{S}(\mathbb{SU}(2))$, then (7) holds.*

Proof. Similarly as on Heisenberg groups, it suffices to consider $f(r, \theta, z) = e^{i\lambda z}g(r, \theta)$ for some $\lambda \in \mathbb{R}$ and some function g . We have $Zf = i\lambda f$. It follows from (5) that

$$ZWf = W(Z + i)f = i(\lambda + 1)Wf$$

and

$$WLf = (L - 2\lambda - 1)Wf.$$

We deduce then

$$WP_t f = e^{-(2\lambda+1)t} P_t W f. \quad (8)$$

Next applying (5) again, one gets

$$ZWWf = W(Z+i)Wf = WW(Z+2i)f = i(\lambda+2)WWf.$$

Therefore

$$WLWf = (L+2iZ+1)WWf = (L-2\lambda-3)WWf,$$

and

$$WP_t W f = e^{-(2\lambda+3)t} P_t WWf. \quad (9)$$

Now combining (8) and (9), we obtain

$$WWP_t f = e^{-(2\lambda+1)t} W(P_t W f) = e^{-(2\lambda+3)t} e^{-(2\lambda+1)t} P_t WWf = e^{-(4\lambda+4)t} P_t WWf.$$

Using (9) again, then

$$P_t WW P_t f = e^{-(4\lambda+4)t} P_{2t} WWf = e^{-(4\lambda+4)t} e^{2(2\lambda+3)t} W P_{2t} W f = e^{2t} W P_{2t} W f.$$

This leads to

$$\begin{aligned} \int_0^\infty P_t WW P_t dt f &= \int_0^\infty e^{2t} W P_{2t} W f dt = W \int_0^\infty e^{2t} P_{2t} dt W f \\ &= \frac{1}{2} W(-L-1)^{-1} W f. \end{aligned}$$

□

Remark 2.6. The operator $W(-L-1)^{-1}W$ is always well-defined thanks to the spectral decomposition of the sub-Laplacian. Indeed, the eigenvalues of L are

$$-\lambda_{n,k} = -k(k+|n|+1) - \frac{|n|}{2},$$

where $n \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\}$ (see [10] for more details). Denote the corresponding eigenfunctions by $\varphi_{n,k}$. When $k=0$ and $n \geq 0$, one has $W\varphi_{n,k} = 0$. When $k=0$ and $n=-1$, one has $WW\varphi_{-1,0} = 0$ and thus

$$W(-L-1)^{-1}W\varphi_{-1,0} = 0.$$

These indicate that the operator $W(-L-1)^{-1}W$ on eigenfunctions corresponding to eigenvalues $-1/2$ vanishes.

More generally, for any $\alpha \geq 0$, the same proof as above gives

$$W(-L-1+\alpha)^{-1}Wf = 2 \int_0^\infty e^{-2\alpha t} P_t WW P_t dt,$$

where the case $\alpha=1$ is corresponding to the second order Riesz transform $W(-L)^{-1}W$. We can also conclude that $W(-L-1+\alpha)^{-1}W$ is well-defined from spectral theory.

Proof of Proposition 2.4. Due to the identification (7), one has (see Proposition 2.1 and also [3])

$$W(-L - 1 + \alpha)^{-1}Wf(x) = \lim_{T \rightarrow \infty} \mathbb{E} \left(\int_0^T A(T-t) \nabla P_{T-t} f(Y_t) \cdot dB_t \mid Y_T = x \right),$$

where A is a 2×2 matrix with entries $a_{11} = -e^{-2\alpha t}$, $a_{12} = a_{21} = e^{-2\alpha t}i$, and $a_{22} = e^{-2\alpha t}$. Observe that the matrix norm is bounded by $\sqrt{2}$. This leads to the desired estimate. $\square$

Remark 2.7. Similarly as in the Heisenberg case, see Remark 2.3, one may also write $W(-L - 1 + \alpha)^{-1}W$ as an explicit singular integral. Indeed, we consider the operator $W(-L + 1/4)^{-1}W$, then

$$W(-L + 1/4)^{-1}Wf(0) = \int_{\mathbb{SU}(2)} W \hat{W} G(r, \theta, z) f(r, \theta, z) d\mu,$$

where $d\mu = \sin r dr d\theta dz$ is a Haar measure up to a constant such that the Green function $G(r, z)$ of the operator $-L + 1/4$ is given by (see [10, Proposition 3.8])

$$G(r, z) = \frac{1}{32\pi \sqrt{1 - 2 \cos \frac{r}{2} \cos \frac{z}{2} + \cos^2 \frac{r}{2}}}.$$

From direct computation,

$$W \hat{W} G(r, \theta, z) = \frac{\left(\sin \frac{r}{2}\right)^2 e^{i(z-2\theta)}}{128\pi \left(1 - 2 \cos \frac{r}{2} \cos \frac{z}{2} + \cos^2 \frac{r}{2}\right)^{3/2}}.$$

Therefore

$$W(-L + 1/4)^{-1}Wf(0) = \frac{1}{128\pi} \int_{\mathbb{SU}(2)} \frac{\left(\sin \frac{r}{2}\right)^2 e^{i(z-2\theta)}}{\left(1 - 2 \cos \frac{r}{2} \cos \frac{z}{2} + \cos^2 \frac{r}{2}\right)^{3/2}} f(r, \theta, z) d\mu.$$

2.3. On $\widetilde{\mathbb{SL}}(2)$. Consider the basis of the Lie algebra $\mathfrak{sl}(2)$:

$$X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the following relations hold

$$[X, Y] = Z, \quad [X, Z] = Y, \quad [Y, Z] = -X.$$

Consider then the Lie group $\widetilde{\mathbb{SL}}(2)$ which is the simply connected Lie group with Lie algebra $\mathfrak{sl}(2)$. We note that $\widetilde{\mathbb{SL}}(2)$ is the universal cover of $\mathbb{SL}(2)$, the group of 2×2 real matrices of determinant 1. The Lie group $\widetilde{\mathbb{SL}}(2)$ is the group $\mathbb{G}(\rho)$ with $\rho = -1$.

We denote by X, Y, Z the left invariant vector fields on $\widetilde{\mathbb{SL}}(2)$ corresponding to the basis X, Y, Z .

Consider the sub-Laplacian $L = X^2 + Y^2$ and the complex gradient $W = X - iY$. The Lie algebra structure gives us

$$WL = (L + 2iZ - 1)W, \quad WZ = (Z + i)W. \quad (10)$$

Similarly as on $\mathbb{H}$, we can show

Lemma 2.8. *One has for $f \in \mathcal{S}(\widetilde{\mathbb{SL}}(2))$,*

$$W(-L+1)^{-1}Wf = 2 \int_0^\infty P_t W W P_t f dt.$$

Proposition 2.9. *The second order Riesz transform $W(-L+1)^{-1}W$ is bounded on L^p with*

$$\|W(-L+1)^{-1}Wf\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p.$$

The proofs are very similar to the case of $\mathbb{SU}(2)$ so we omit the details here.

2.4. Proof of the main result. In this section, we will prove our main result. First recall that from Bañuelos and Baudoin [3], we have the following general result.

Lemma 2.10. *Consider $\mathbb{G}(\rho)$. Let $A = (a_{ij})_{2 \times 2}$ be a real or complex-valued matrix (may or may not depend on t). Then we have*

$$S_A f = \int_0^\infty P_t (a_{11}X^2 + a_{12}XY + a_{21}YX + a_{22}Y^2) P_t f dt$$

and

$$\|S_A f\|_p \leq \frac{1}{2}(p^* - 1)\|A\|\|f\|_p.$$

In particular, if A is real and orthogonal, then

$$\|S_A f\|_p \leq \frac{1}{2} \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p.$$

On the three dimensional Lie group $\mathbb{G}(\rho)$, it always holds that $[L, Z] = 0$ and $[X, Y] = Z$. Hence taking $a_{12} = 1$, $a_{21} = -1$, and $a_{11} = a_{22} = 0$, we have

$$Z(-L)^{-1} = 2 \int_0^\infty P_t (XY - YX) P_t f dt.$$

More generally, for any $\alpha \geq 0$,

$$Z(-L + \alpha)^{-1} = 2e^{-2\alpha} \int_0^\infty P_t (XY - YX) P_t f dt.$$

As a consequence of Lemma 2.10, we obtain

Corollary 2.11. *Consider a three dimensional model space $\mathbb{G}(\rho)$. Let $1 < p < \infty$ and $\alpha \geq 0$. Then we have*

$$\|Z(-L + \alpha)^{-1}f\|_p \leq \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p.$$

The sub-Laplacian does not commute with X and Y on $\mathbb{G}(\rho)$. However, the complex gradient $W = X - iY$ (and its conjugate) and the sub-Laplacian are well related, see Section 2. Combining Lemmas 2.2, 2.5, and 2.8, one concludes that for $\alpha \geq 0$, there holds

$$W(-L - \rho + \alpha)^{-1}W = 2e^{-2\alpha t} \int_0^\infty P_t W W P_t f dt. \quad (11)$$

Therefore we have from Lemma 2.10 that

Corollary 2.12. *Consider a three dimensional model space $\mathbb{G}(\rho)$. Let $1 < p < \infty$ and $\alpha \geq 0$. Then*

$$\|W(-L - \rho + \alpha)^{-1}Wf\|_p \leq \sqrt{2}(p^* - 1)\|f\|_p.$$

Now we are ready to prove our main result.

Proof of Theorem 1.1. Comparing the real part and the imaginary part for both sides of (11) yields that for $\alpha \geq 0$,

$$\begin{aligned} (X(-L - \rho + \alpha)^{-1}X - Y(-L - \rho + \alpha)^{-1}Y) f &= 2e^{-2\alpha} \int_0^\infty P_t (X^2 - Y^2) P_t f dt, \\ (X(-L - \rho + \alpha)^{-1}Y + Y(-L - \rho + \alpha)^{-1}X) f &= 2e^{-2\alpha} \int_0^\infty P_t (XY + YX) P_t f dt. \end{aligned}$$

Thus by Corollary 2.12, one has

$$\begin{aligned} \|(X(-L - \rho + \alpha)^{-1}X - Y(-L - \rho + \alpha)^{-1}Y) f\|_p &\leq \sqrt{2}(p^* - 1)\|f\|_p, \\ \|(X(-L - \rho + \alpha)^{-1}Y + Y(-L - \rho + \alpha)^{-1}X) f\|_p &\leq \sqrt{2}(p^* - 1)\|f\|_p. \end{aligned} \quad (12)$$

The L^p bound of $S_{a,b,c}^\alpha$ then follows from (12) and Corollary 2.11. $\square$

3. The case of elliptic Laplacians. In this section, we study second order Riesz transforms for the Laplace-Beltrami operator $\mathcal{L} = X^2 + Y^2 + Z^2$ on $\mathbb{G}(\rho)$. In the following, $P_t^\mathcal{L} = e^{t\mathcal{L}}$ denotes the heat semigroup associated with $\mathcal{L}$. Recall that

$$[X, Y] = Z, \quad [X, Z] = -\rho Y, \quad [Y, Z] = \rho X.$$

Hence one computes that

$$W\mathcal{L} = (\mathcal{L} + 2iZ + 2i\rho Z - \rho - \rho^2)W. \quad (13)$$

Following similar arguments as in Section 2, we have

$$W(-\mathcal{L} + \rho(1 + \rho))^{-1}W = 2 \int_0^\infty P_t^\mathcal{L} W W P_t^\mathcal{L} f dt. \quad (14)$$

In addition, since $Z\mathcal{L} = \mathcal{L}Z$, then

$$Z^2(-\mathcal{L})^{-1} = 2 \int_0^\infty Z^2 P_t^\mathcal{L} P_t^\mathcal{L} f dt. \quad (15)$$

Our main result of this section is the following elliptic Laplacian version of Theorem 1.1.

Proposition 3.1. *For any real-valued numbers a, b, c , consider on $\mathbb{G}(\rho)$*

$$S_{a,b,c}^\mathcal{L} f = a(-\mathcal{L})^{-1} Z^2 f + b(X(-\mathcal{L} + \rho(1 + \rho))^{-1} X - Y(-\mathcal{L} + \rho(1 + \rho))^{-1} Y) f \\ + c(X(-\mathcal{L} + \rho(1 + \rho))^{-1} Y + Y(-\mathcal{L} + \rho(1 + \rho))^{-1} X) f.$$

Then we have for every $p > 1$,

$$\|S_{a,b,c}^\mathcal{L} f\|_p \leq \sqrt{2}(|a| + |b| + |c|)(p^* - 1)\|f\|_p.$$

Proof. Using the same strategy as in the proof of Proposition 2.1, the expression (14) gives that

$$W(-\mathcal{L} + \rho(1 + \rho))^{-1} W f(x) = \lim_{T \rightarrow \infty} \mathbb{E} \left(\int_0^T \mathcal{A} \nabla P_{T-t}^\mathcal{L} f(\mathcal{Y}_t) \cdot d\mathcal{B}_t \mid \mathcal{Y}_T = x \right),$$

where $\mathcal{Y}_t$ is the diffusion process associated with the Laplacian $\mathcal{L}$, $\mathcal{B}_t$ is the Brownian motion on $\mathbb{R}^{2n+1}$, and $\mathcal{A}$ is a 3×3 matrix with entries $a_{11} = -1$, $a_{12} = a_{21} = i$, $a_{22} = 1$, and otherwise 0.

Moreover, if the matrix $\mathcal{A}$ above is replaced by the matrix with entry $a_{33} = 1$ and otherwise 0, then the martingale projection gives $Z^2(-\mathcal{L})^{-1}$, thanks to the equation (15). We conclude our proof by applying a similar argument as in Theorem 1.1. $\square$

Finally let us comment on our three examples: $\mathbb{S}\mathbb{U}(2)$, $\mathbb{H}$, and $\widetilde{\mathbb{S}\mathbb{L}}(2)$.

On $\widetilde{\mathbb{S}\mathbb{L}}(2)$. Since $\rho = -1$, the equation (13) becomes $W\mathcal{L} = \mathcal{L}W$ and the constant $\rho(1 + \rho)$ in (14) and Proposition 3.1 is zero. Actually, as a Lie group of compact type, the vector fields X, Y, Z commute with the Laplace-Beltrami operator $\mathcal{L}$ and heat semigroup $P_t^\mathcal{L}$. Then the explicit L^p bound of second order Riesz transforms follow from [3, Theorem 4.1]. That is, for any constant coefficient matrix $A = (a_{ij})_{1 \leq i, j \leq 3}$,

$$\left\| \sum_{i,j=1}^3 a_{ij} \mathcal{R}_i \mathcal{R}_j f \right\|_p \leq (p^* - 1) \|A\| \|f\|_p,$$

where $\mathcal{R}_i$ represents $X(-\mathcal{L})^{-1/2}$, $Y(-\mathcal{L})^{-1/2}$, or $Z(-\mathcal{L})^{-1/2}$.

On $\mathbb{H}$. In this case $\rho = 0$ and we also have $\rho(1 + \rho) = 0$ in Proposition 3.1. Furthermore, we can extend Proposition 3.1 to $\mathbb{H}^n$ by noticing that

$$W_j(-\mathcal{L})^{-1} W_k f = 2 \int_0^\infty P_t^\mathcal{L} (X_j X_k - Y_j Y_k - i(X_j Y_k + X_k Y_j)) P_t^\mathcal{L} f dt.$$

On $\mathrm{SU}(2)$. We have $\rho = 1$ and thus Proposition 3.1 holds with $\rho(1 + \rho) = 2$.

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FABRICE BAUDOIN
Department of Mathematics
University of Connecticut
Storrs CT 06269
USA
e-mail: fabrice.baudoin@uconn.edu

LI CHEN
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803
USA
e-mail: lichen@lsu.edu

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