Journal of Statistical Physics (2021) 184:22
https://doi.org/10.1007/s10955-021-02803-4

# Thouless–Anderson–Palmer Equations for the Ghatak–Sherrington Mean Field Spin Glass Model

**Antonio Auffinger**[1] · **Cathy Xi Chen**[1]

Received: 6 April 2021 / Accepted: 13 July 2021 / Published online: 13 August 2021
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**Abstract**
We derive the Thouless–Anderson–Palmer (TAP) equations for the Ghatak and Sherrington model (J Phys C 10(16):3149–3156, 1977). Our derivation, based on the cavity method, holds at high temperature and at all values of the crystal field. It confirms the prediction of Yokota (J Phys Condens Matter 4(10):2615–2622, 1992).

**Keywords** TAP Equations · Ghatak–Sherrington · Spin glasses

**Mathematics Subject Classification** Primary 60F10 · 82D30

## 1 Introduction and Main Results

The Hamiltonian of the Ghatak and Sherrington (GS) spin-glass model is defined as the random function

$$H_N(\boldsymbol{\sigma}) = \frac{\beta}{\sqrt{N}} \sum_{1\le i<j\le N} g_{ij}\sigma_i\sigma_j + D\sum_{i=1}^{N}\sigma_i^2 + h\sum_{i=1}^{N}\sigma_i, \tag{1}$$

where $S \ge 1$ is a fixed integer, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N) \in \Sigma_N = \{0, \pm 1, \dots, \pm S\}^N$. The parameters $\beta \ge 0$, $D \in \mathbb{R}$, $h \in \mathbb{R}$ represent the inverse temperature, crystal field and external field respectively, and $g_{ij}$ are i.i.d. standard Gaussian random variables for $1 \le i < j \le N$. This model was introduced by Ghatak and Sherrington [11] as a generalization of the classical Sherrington–Kirkpatrick (SK) model [17]. It is supposed to model an induced spin glass and an anisotropic extension of the SK model [11].

As in the SK model, the study of thermodynamic quantities of the GS model has required significant efforts by many physicists and mathematicians. In particular, it has been predicted

Communicated by Federico Ricci-Tersenghi.

✉ Antonio Auffinger
tuca@northwestern.edu

Cathy Xi Chen
xchen56@math.northwestern.edu

1 Department of Mathematics, Northwestern University, Evanston, USA

the existence of multiple phase transitions as the temperature decreases to zero, including a second replica symmetric phase at low temperature, a phenomena indicative of inverse freezing [14]. This is in sharp contrast with the SK model (and the $p$-spin). We refer the reader to [9–13,15,21] and the references therein for a brief history and importance of the GS model in the physics community. In the mathematics literature, the most notable progress was Panchenko's result establishing the Parisi formula for the GS model [16], which holds for the SK model defined on the compact product measure as well.

In this paper, we study the behavior of the thermal average of the magnetization

$$m = (m_1, \dots, m_N) = (\langle \sigma_1 \rangle, \dots, \langle \sigma_N \rangle)$$

and its second moment

$$p = (p_1, \dots, p_N) = (\langle \sigma_1^2 \rangle, \dots, \langle \sigma_N^2 \rangle),$$

where for a function $f$ on $\Sigma_N$, we denote $\langle f \rangle$ the average under the Gibbs measure $G_N$, defined as

$$G_N(\{\boldsymbol{\sigma}\}) = \frac{\exp(H_N(\boldsymbol{\sigma}))}{Z_N},$$

with

$$Z_N = \sum_{\boldsymbol{\sigma}} \exp(H_N(\boldsymbol{\sigma})).$$

It has been predicted (in the case $S = 1, h = 0$ [15,21]) that these pairs of random variables satisfy at high temperature (in a sense that will be made precise later) a system of coupled self consistent equations given by

$$m_i \approx \frac{2\sinh(\beta\xi_i)}{\exp(\beta\Delta_i) + 2\cosh(\beta\xi_i)} \quad p_i \approx \frac{2\cosh(\beta\xi_i)}{\exp(\beta\Delta_i) + 2\cosh(\beta\xi_i)} \tag{2}$$

with

$$\xi_i = \frac{1}{\sqrt{N}} \sum_j g_{ij} m_j - \frac{\beta}{N} m_i \sum_j g_{ij}^2 \left(p_j - m_j^2\right),$$

and

$$\Delta_i = -D - \frac{\beta}{2N} \sum_j g_{ij}^2 \left(p_j - m_j^2\right).$$

Equations (2) are the analogue of the well-studied TAP equations

$$m_i \approx \tanh\left(h + \sum_{k \neq i} g_{ik} m_k - \beta^2 (1-q) m_i\right) \tag{3}$$

in the SK model [20].

In the mathematics community, there have been several approaches to rigorously understand the TAP equations. First, in the SK model, Talagrand [19] and Chatterjee [6] established (3) at high temperature. At low temperature, a version of (3) where one decomposes the Gibbs measure into "pure states" was established by Auffinger–Jagannath [4]. A very fruitful approach to TAP was introduced by Bolthausen through an iteration scheme that shares some connections to message passing algorithms [5]. Bolthausen's iteration was recently shown to indeed approximate the magnetizations by Chen–Tang [8]. A dynamical method

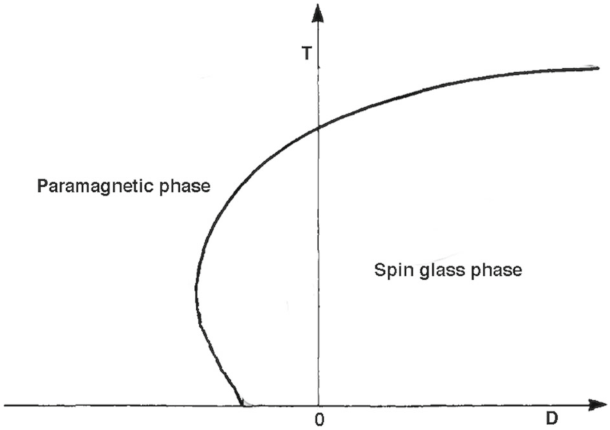


**Fig. 1** The predicted phase diagram for the GS model when $S = 1$ and $h = 0$ as described in Mottishaw–Sherrington [15]. Here, $T = \beta^{-1}$. The TAP equations (2) should be valid in the paramagnetic phase. Note that for certain values of the crystal field $D$ the values of $\beta$ in the paramagnetic phase are the union of two disjoint intervals, a behavior different than the SK model

to derive (3) was also very recently proposed by Adhikari–Brennecke–von Soosten–Yau [1]. The TAP equations were also viewed as critical point solutions of the TAP functional and studied in [2,3,7,18].

Different than the SK case or the mixed $p$-spin, the TAP equations for the GS model depend on two set of parameters. This creates a few roadblocks to understand its validity. For instance, in the physics community, there is still a debate of what should be the correct analogue of the de Almeida–Thouless line and for which set of parameters $(\beta, D, h)$ one should expect (2) to be true. The $(\beta, D)$ phase diagram in the case $S = 1, h = 0$ is thoroughly discussed by physicists in the papers of Lage and de Almeida [13], Mottishaw and Sherrignton [15] and da Costa et al. [10]. In this case, it is predicted that for certain values of the crystal field $D$ the values of $\beta$ for which (2) holds is the union of two disjoint intervals, a behavior not present in the SK model. See Fig. 1 for a phase diagram.

The main goal of this paper is to derive a rigorous interpretation of (2) at high temperature for all values of the crystal field. We do not expect that our bounds on $\beta$ are optimal. As illustrated by the discussion above, deriving the exact conditions when (2) holds is an interesting and challenging direction to pursue.

We will now state our results. Let $X$ be a standard Gaussian random variable. Given $(\beta, D, h)$ as above consider the system of equations in $\mathbb{R}^2$ given by

$$p = \mathbb{E}\left[\frac{\sum_{\gamma=1}^{S}\gamma^2\cdot 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}\right], \tag{4}$$

$$q = \mathbb{E}\left[\frac{\sum_{\gamma=1}^{S}\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}\right]^2. \tag{5}$$

This system is the analogue of the fixed-point equation $q = \mathbb{E}\tanh^2(\beta\sqrt{q}X + h)$ that appears in the SK model. Our first result shows that for $\beta$ small, this system of equations has a unique solution.

**Proposition 1** *There exists a* $\tilde{\beta} > 0$ *such that for all* $0 \le \beta < \tilde{\beta}$, $h \ge 0$, *and* $D \in \mathbb{R}$, *the system of Eqs.* (4) *and* (5) *has a unique solution.*

Assume that $\beta < \tilde{\beta}$ and the pair $(p, q)$ is the unique solution of (4) and (5). Our main result describes the validity of the TAP equations in the $L^2$ sense as follows.

**Theorem 1** *There exists some* $K, \hat{\hat{\beta}} > 0$, *such that for all* $0 \le \beta < \hat{\hat{\beta}}$, $h \ge 0$ *and* $D \in \mathbb{R}$, *we have for all* $N \ge 1$ *and* $k \ge 1$

$$\mathbb{E}\left[\langle\sigma_N\rangle - \frac{\sum_{\gamma=1}^{S}\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\beta\xi_N+h\right)\right]\exp\left(\gamma^2\Delta\right)}{1+\sum_{\gamma=1}^{S}2\mathrm{ch}\left[\gamma\left(\beta\xi_N+h\right)\right]\exp\left(\gamma^2\Delta\right)}\right]^{2k} \le \frac{K}{N^k}, \tag{6}$$

$$\mathbb{E}\left[\left\langle\sigma_N^2\right\rangle - \frac{\sum_{\gamma=1}^{S}\gamma^2\cdot 2\mathrm{ch}\left[\gamma\left(\beta\xi_N+h\right)\right]\exp\left(\gamma^2\Delta\right)}{1+\sum_{\gamma=1}^{S}2\mathrm{ch}\left[\gamma\left(\beta\xi_N+h\right)\right]\exp\left(\gamma^2\Delta\right)}\right]^{2k} \le \frac{K}{N^k}, \tag{7}$$

*where*

$$\xi_N = \sum_{i\le N-1} g_{iN}\langle\sigma_i\rangle - \beta(p-q)\langle\sigma_N\rangle, \quad \text{and} \quad \Delta = D + \frac{1}{2}\beta^2(p-q).$$

The proof of Theorem 1 follows the cavity approach as in Sects. 1.6 and 1.7 of Talagrand's book [19]. The main difference between the SK model and the GS model is that we now need to control the self-overlap, and relate it to the solution $(p, q)$. This requires new estimates and a careful analysis of the fixed point equation. The rest of the paper is organized as follows. In the next section, we show concentration of the overlap and self-overlap, the main tool to prove Theorem 1. In Sect. 3, we provide the proof of Theorem 1. The proof of Proposition 1 is left to the last section.

## 2 Concentration of Overlaps

We denote the overlap between configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$, and the self-overlap of $\boldsymbol{\sigma}$ respectively by

$$R_{1,2} = \frac{1}{N}\sum_{i\le N}\sigma_i^1\sigma_i^2 \quad \text{and} \quad R_{1,1} = \frac{1}{N}\sum_{i\le N}(\sigma_i)^2.$$

In this section, we use the cavity method to show concentration of overlaps $R_{1,2}$ and $R_{1,1}$. We assume from now on that $\beta < \tilde{\beta}$ and $(p, q)$ is the solution given in Proposition 1.

**Proposition 2** *There exists a* $\hat{\beta} > 0$, *such that for all* $\beta < \hat{\beta}$, *we have*:

$$\mathbb{E}\left\langle\left(R_{1,2}-q\right)^2\right\rangle \le \frac{16S^4}{N},$$
$$\mathbb{E}\left\langle\left(R_{1,1}-p\right)^2\right\rangle \le \frac{16S^4}{N}.$$

We start with some notations and preliminary results needed to prove Proposition 2. For $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$, $\boldsymbol{\rho} = (\sigma_1, \ldots, \sigma_{N-1})$, we write

$$\begin{aligned} H_N(\boldsymbol{\sigma}) &= \frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N} g_{ij}\sigma_i\sigma_j + D\sum_{i=1}^{N}\sigma_i^2 + h\sum_{i=1}^{N}\sigma_i \\ &= H_{N-1}(\boldsymbol{\rho}) + \sigma_N \cdot \frac{\beta}{\sqrt{N}} \sum_{i<N} g_{iN}\sigma_i + D\sigma_N^2 + h\sigma_N, \end{aligned}$$

where with a slight abuse of notation,

$$H_{N-1}(\boldsymbol{\rho}) = \frac{\beta}{\sqrt{N}} \sum_{1 \le i < j \le N-1} g_{ij}\sigma_i\sigma_j + D\sum_{i \le N-1}\sigma_i^2 + h\sum_{i \le N-1}\sigma_i. \tag{8}$$

With the notation above, we have the following identity. Its proof is identical to the proof of Proposition 1.6.1 in [19].

**Proposition 3** *Given a function $f$ on $\Sigma_N$, it holds that*

$$\langle f(\boldsymbol{\sigma})\rangle = \frac{\left\langle \mathrm{Av}\left(f(\boldsymbol{\sigma})\exp\left(\sigma_N \cdot \frac{\beta}{\sqrt{N}}\sum_{i<N} g_{iN}\sigma_i + D\sigma_N^2 + h\sigma_N\right)\right)\right\rangle_-}{\left\langle \mathrm{Av}\left(\exp\left(\sigma_N \cdot \frac{\beta}{\sqrt{N}}\sum_{i<N} g_{iN}\sigma_i + D\sigma_N^2 + h\sigma_N\right)\right)\right\rangle_-},$$

*where* Av *means average over $\sigma_N = 0, \pm 1, \ldots, \pm S$, and $\langle\cdot\rangle_-$ is the average under the Gibbs measure with respect to the Hamiltonian $H_{N-1}$.*

Now consider the interpolated Hamiltonian:

$$\begin{aligned} H_t(\boldsymbol{\sigma}) = H_{N-1}(\boldsymbol{\rho}) &+ \sigma_N\left[\sqrt{t}\frac{\beta}{\sqrt{N}}\sum_{i<N} g_{iN}\sigma_i + \sqrt{1-t}\,\beta z\sqrt{q}\right] \\ &+ (1-t)\cdot\frac{\beta^2}{2}(p-q)\sigma_N^2 + D\sigma_N^2 + h\sigma_N, \end{aligned}$$

where $z$ is a standard Gaussian random variable independent of $g_{ij}$. We denote the overlap of the first $N-1$ coordinates by

$$R^-_{l,l'} = \frac{1}{N}\sum_{i<N}\sigma_i^l\sigma_i^{l'}.$$

To simplify the notation, let $\epsilon_l = \sigma_N^l$, and write

$$R_{l,l'} = R^-_{l,l'} + \frac{\epsilon_l\epsilon_{l'}}{N}. \tag{9}$$

**Lemma 1** *We have*:

$$\mathbb{E}\left\langle\sigma_N^2\right\rangle_0 = p \quad \textit{and} \quad \mathbb{E}\left\langle\sigma_N\right\rangle_0^2 = q,$$

*where $p$ and $q$ satisfy the Eqs. (4) and (5), and $\langle\cdot\rangle_0$ is the average under the Gibbs measure with respect to the interpolated Hamiltonian $H_t(\boldsymbol{\sigma})$ at $t = 0$.*

***Proof*** Note that

$$H_0(\boldsymbol{\sigma}) = H_{N-1}(\boldsymbol{\rho}) + \sigma_N \cdot \beta z\sqrt{q} + \frac{\beta^2}{2}(p-q)\sigma_N^2 + D\sigma_N^2 + h\sigma_N.$$

Applying Proposition 3 with $H_0$, we get:

$$\begin{aligned}\mathbb{E}\langle\sigma_N^2\rangle_0 &= \mathbb{E}\frac{\left\langle \mathrm{Av}\left(\sigma_N^2\exp\left(\sigma_N\cdot\beta z\sqrt{q}+\frac{\beta^2}{2}(p-q)\sigma_N^2+D\sigma_N^2+h\sigma_N\right)\right)\right\rangle_-}{\left\langle \mathrm{Av}\left(\exp\left(\sigma_N\cdot\beta z\sqrt{q}+\frac{\beta^2}{2}(p-q)\sigma_N^2+D\sigma_N^2+h\sigma_N\right)\right)\right\rangle_-}\\ &= \mathbb{E}\left[\frac{\sum_{\gamma=1}^S\gamma^2\cdot 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}\right]=p.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\sigma_N\rangle_0 &= \frac{\left\langle \mathrm{Av}\left(\sigma_N\exp\left(\sigma_N\cdot\beta z\sqrt{q}+\frac{\beta^2}{2}(p-q)\sigma_N^2+D\sigma_N^2+h\sigma_N\right)\right)\right\rangle_-}{\left\langle \mathrm{Av}\left(\exp\left(\sigma_N\cdot\beta z\sqrt{q}+\frac{\beta^2}{2}(p-q)\sigma_N^2+D\sigma_N^2+h\sigma_N\right)\right)\right\rangle_-}\\ &= \left[\frac{\sum_{\gamma=1}^S\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}\right],\\ \mathbb{E}\langle\sigma_N\rangle_0^2 &= \left[\frac{\sum_{\gamma=1}^S\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}\right]^2=q.\end{aligned}$$

□

Let

$$u_{\boldsymbol{\sigma}}=\frac{\beta}{\sqrt{N}}\sum_{i<N}g_{iN}\sigma_i\sigma_N,\quad v_{\boldsymbol{\sigma}}=\beta z\sqrt{q}\sigma_N,\quad y_{\boldsymbol{\sigma}}=\frac{\beta^2}{2}(p-q)\sigma_N^2,\quad\text{and}$$
$$\omega_{\boldsymbol{\sigma}}=\exp\left(H_{N-1}(\boldsymbol{\rho})+D\sigma_N^2+h\sigma_N\right).$$

Then the interpolated Hamiltonian can be written as

$$H_t(\boldsymbol{\sigma})=\sqrt{t}u_{\boldsymbol{\sigma}}+\sqrt{1-t}v_{\boldsymbol{\sigma}}+(1-t)y_{\boldsymbol{\sigma}}+\log(\omega_{\boldsymbol{\sigma}}).$$

Let $\langle\cdot\rangle_t$ be an average for the corresponding Gibbs measure. We write

$$\nu_t(f)=\mathbb{E}\langle f\rangle_t,\quad\text{and}\quad \nu_t'(f)=\frac{d}{dt}(\nu_t(f)).$$

Then $\mathbb{E}\left\langle\left(R_{1,2}-q\right)^2\right\rangle$ and $\mathbb{E}\left\langle\left(R_{1,1}-p\right)^2\right\rangle$ in Proposition 2 are equal to $\nu_1\left(\left(R_{1,2}-q\right)^2\right)$ and $\nu_1\left(\left(R_{1,1}-p\right)^2\right)$. Set

$$U(\boldsymbol{\sigma}^l,\boldsymbol{\sigma}^{l'})=\frac{1}{2}\left(\mathbb{E}u_{\boldsymbol{\sigma}^l}u_{\boldsymbol{\sigma}^{l'}}-\mathbb{E}v_{\boldsymbol{\sigma}^l}v_{\boldsymbol{\sigma}^{l'}}\right),\quad\text{and}\quad V(\boldsymbol{\sigma}^l)=-y_{\boldsymbol{\sigma}^l}.$$

Then we have

$$\mathbb{E}u_{\boldsymbol{\sigma}^l}u_{\boldsymbol{\sigma}^{l'}}=\epsilon_l\epsilon_{l'}\cdot\beta^2R_{l,l'}^-,\quad \mathbb{E}v_{\boldsymbol{\sigma}^l}v_{\boldsymbol{\sigma}^{l'}}=\epsilon_l\epsilon_{l'}\cdot\beta^2 q,$$

and

$$U(\boldsymbol{\sigma}^l, \boldsymbol{\sigma}^{l'}) = \epsilon_l \epsilon_{l'} \cdot \frac{\beta^2}{2}(R^-_{l,l'} - q), \tag{10}$$

$$V(\boldsymbol{\sigma}^l) = -\frac{\beta^2}{2}(p-q)\epsilon_l^2. \tag{11}$$

**Lemma 2** *If $f$ is a function on $(\Sigma_N)^n$, then*

$$\begin{aligned}\nu_t'\left(f\left(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n\right)\right) &= \sum_{1\le l,l'\le n} \nu_t\left(U\left(\boldsymbol{\sigma}^l,\boldsymbol{\sigma}^{l'}\right)f\right) - 2n\sum_{l\le n}\nu_t\left(U\left(\boldsymbol{\sigma}^l,\boldsymbol{\sigma}^{n+1}\right)f\right)\\ &\quad - n\nu_t\left(U\left(\boldsymbol{\sigma}^{n+1},\boldsymbol{\sigma}^{n+1}\right)f\right) + n(n+1)\nu_t\left(U\left(\boldsymbol{\sigma}^{n+1},\boldsymbol{\sigma}^{n+2}\right)f\right)\\ &\quad + \sum_{l\le n}\nu_t\left(V\left(\boldsymbol{\sigma}^l\right)f\right) - n\nu_t\left(V\left(\boldsymbol{\sigma}^{n+1}\right)f\right).\end{aligned} \tag{12}$$

***Proof*** The proof is similar to Lemma 1.4.2 in [19] with the correct derivatives. The first two lines on the right side of (12) are the same as those in the SK model, while the last line arises due to $y_{\boldsymbol{\sigma}}$. □

If we combine Lemma 2 with Eqs. (10) and (11), we get the following.

**Lemma 3** *Let $f$ be a function on $(\Sigma_N)^n$, then for $0 < t < 1$, we have*

$$\begin{aligned}\nu_t'(f) = \beta^2\Big[&\sum_{1\le l<l'\le n}\nu_t\left(\epsilon_l\epsilon_{l'}\left(R^-_{l,l'}-q\right)f\right) - n\sum_{l\le n}\nu_t\left(\epsilon_l\epsilon_{n+1}\left(R^-_{l,n+1}-q\right)f\right)\\ &+\frac{n(n+1)}{2}\nu_t\left(\epsilon_{n+1}\epsilon_{n+2}\left(R^-_{n+1,n+2}-q\right)f\right) + \frac{1}{2}\sum_{l\le n}\nu_t\left(\epsilon_l^2\left(R^-_{l,l}-p\right)f\right)\\ &-\frac{n}{2}\nu_t\left(\epsilon_{n+1}^2\left(R^-_{n+1,n+1}-p\right)f\right),\end{aligned} \tag{13}$$

*and also*

$$\nu_t'(f) = A_1 + A_2 - B, \tag{14}$$

*where*

$$\begin{aligned}A_1 := \beta^2\Big[&\sum_{1\le l<l'\le n}\nu_t\left(\epsilon_l\epsilon_{l'}\left(R_{l,l'}-q\right)f\right) - n\sum_{l\le n}\nu_t\left(\epsilon_l\epsilon_{n+1}\left(R_{l,n+1}-q\right)f\right)\\ &+\frac{n(n+1)}{2}\nu_t\left(\epsilon_{n+1}\epsilon_{n+2}\left(R_{n+1,n+2}-q\right)f\right)\Big],\end{aligned} \tag{15}$$

$$A_2 := \beta^2\Big[\frac{1}{2}\sum_{l\le n}\nu_t\left(\epsilon_l^2\left(R_{l,l}-p\right)f\right) - \frac{n}{2}\nu_t\left(\epsilon_{n+1}^2\left(R_{n+1,n+1}-p\right)f\right)\Big], \tag{16}$$

$$\begin{aligned}B := \frac{\beta^2}{N}\Big[&\sum_{1\le l<l'\le n}\nu_t\left(\epsilon_l^2\epsilon_{l'}^2 f\right) - n\sum_{l\le n}\nu_t\left(\epsilon_l^2\epsilon_{n+1}^2 f\right) + \frac{n(n+1)}{2}\nu_t\left(\epsilon_{n+1}^2\epsilon_{n+2}^2 f\right)\\ &+\frac{1}{2}\sum_{l\le n}\nu_t\left(\epsilon_l^4 f\right) - \frac{n}{2}\nu_t\left(\epsilon_{n+1}^4 f\right)\Big].\end{aligned} \tag{17}$$

***Proof*** The result follows by replacing Eqs. (10) and (11) in (12) and by some straightforward algebra. □

**Lemma 4** *For a function* $f \geq 0$ *on* $(\Sigma_N)^n$, *we have*

$$\nu_t(f) \leq \exp\left(6n^2\beta^2 S^4\right)\nu_1(f).$$

***Proof*** Note that $R^-_{l,l'} \leq S^2$ and $p, q \in [0, S^2]$. So $|R^-_{l,l'} - q| \leq 2S^2$ for $l \neq l'$, $|R^-_{l,l} - p| \leq 2S^2$, and $(p-q)^2 \leq S^2$. Thus, by (13), we have

$$\begin{aligned}|\nu_t'(f)| &\leq \left(\frac{n(n-1)}{2} + n^2 + \frac{n(n+1)}{2} + \frac{n}{2} + \frac{n}{2}\right)\beta^2 \cdot 2S^4\nu_t(f)\\ &= (2n^2+n)\beta^2 \cdot 2S^4\nu_t(f) \leq 6n^2\beta^2S^4\nu_t(f).\end{aligned}$$

Thus,

$$\left|\frac{\nu_t'(f)}{\nu_t(f)}\right| \leq 6n^2\beta^2S^4.$$

Let

$$g(1-t) := \log\left[\nu_t(f)\right] \quad \text{for} \quad t \in [0, 1].$$

Then

$$\left|g'(1-t)\right| = \left|\frac{\nu_t'(f)}{\nu_t(f)}\right| \leq 6n^2\beta^2S^4.$$

Therefore,

$$g(1-t) = g(0) + \int_0^{1-t} g'(s)ds \leq g(0) + \int_0^{1-t} |g'(s)|ds \leq g(0) + (1-t)6n^2\beta^2S^4,$$

i.e.

$$\log[\nu_t(f)] \leq \log[\nu_1(f)] + (1-t)6n^2\beta^2S^4.$$

Hence,

$$\nu_t(f) \leq \exp\left[(1-t)6n^2\beta^2S^4\right]\nu_1(f) \leq \exp\left(6n^2\beta^2S^4\right)\nu_1(f), \text{ as desired.}$$

□

Now combining Lemmas 3 and 4, for a function $f$ on $(\Sigma_N)^n$, and $0 < t < 1$, we have

$$\begin{aligned}|\nu_t'(f)| &\leq \left(\frac{n(n-1)}{2} + n^2 + \frac{n(n+1)}{2} + \frac{n}{2} + \frac{n}{2}\right)\beta^2 \cdot 2S^4\nu_t|f|)\\ &\leq 6n^2\beta^2S^4\exp\left(6n^2\beta^2S^4\right)\nu_1(|f|),\end{aligned} \tag{18}$$

which will be used several times in the following proof of Proposition 2.

***Proof of Proposition 2*** We will show concentration of $R_{1,2}$ first. Recall that $\epsilon_l = \sigma_N^l$. Using symmetry among sites, we can write

$$\nu_1\left(\left(R_{1,2}-q\right)^2\right) = \frac{1}{N}\sum_{i\leq N}\nu_1\left[\left(\sigma_i^1\sigma_i^2 - q\right)\left(R_{1,2}-q\right)\right] = \nu_1(f), \tag{19}$$

where

$$f := (\epsilon_1\epsilon_2 - q)\left(R_{1,2}-q\right).$$

By (9),

$$f = (\epsilon_1\epsilon_2 - q)\left(\frac{\epsilon_1\epsilon_2}{N} + R^-_{1,2} - q\right) = \frac{1}{N}\left[(\epsilon_1\epsilon_2)^2 - \epsilon_1\epsilon_2 q\right] + (\epsilon_1\epsilon_2 - q)\left(R^-_{1,2} - q\right).$$

Lemma 1 implies

$$\nu_0\left[(\epsilon_1\epsilon_2 - q)\left(R^-_{1,2} - q\right)\right] = \nu_0\left(\epsilon_1\epsilon_2 - q\right)\nu_0\left(R^-_{1,2} - q\right) = 0,$$

and hence

$$\nu_0(f) = \frac{1}{N}\nu_0\left[(\epsilon_1\epsilon_2)^2 - \epsilon_1\epsilon_2 q\right] \le \frac{2S^4}{N}. \tag{20}$$

Using $|\epsilon_1\epsilon_2 - q| \le 2S^2$, we have

$$|f| = \left|(\epsilon_1\epsilon_2 - q)\left(R_{1,2} - q\right)\right| \le 2S^2\left|R_{1,2} - q\right|.$$

Now apply inequality (18) with $n = 2$ to get

$$|\nu_1(f) - \nu_0(f)| \le 24\beta^2 S^4 \exp\left(24\beta^2 S^4\right)\nu_1(|f|). \tag{21}$$

Therefore, combining (21) with (19) and (20), we obtain

$$\nu_1\left(\left|R_{1,2} - q\right|^2\right) \le \frac{2S^4}{N} + 24\beta^2 S^4 \exp\left(24\beta^2 S^4\right)\nu_1\left(\left|R_{1,2} - q\right|^2\right).$$

Choose $\beta_0$ such that

$$24\beta_0^2 S^4 \exp\left(24\beta_0^2 S^4\right) \le \frac{7}{8},$$

then we have

$$\nu_1\left(\left|R_{1,2} - q\right|^2\right) \le \frac{2S^4}{N} + \frac{7}{8}\nu_1\left(\left|R_{1,2} - q\right|^2\right),$$

and hence

$$\nu_1\left(\left|R_{1,2} - q\right|^2\right) \le \frac{16S^4}{N}. \tag{22}$$

We use a similar method to show concentration of $R_{1,1}$. We can write

$$\nu_1\left(\left(R_{1,1} - p\right)^2\right) = \frac{1}{N}\sum_{i\le N}\nu_1\left[\left((\sigma_i^1)^2 - p\right)\left(R_{1,1} - p\right)\right] = \nu_1(\kappa), \tag{23}$$

where

$$\kappa := \left(\epsilon_1^2 - p\right)\left(R_{1,1} - p\right).$$

It follows that

$$\kappa = \left(\epsilon_1^2 - p\right)\left(\frac{\epsilon_1^2}{N} + R^-_{1,1} - p\right) = \frac{1}{N}\left[\left(\epsilon_1^2\right)^2 - \epsilon_1^2 p\right] + \left(\epsilon_1^2 - p\right)\left(R^-_{1,1} - p\right),$$

and by Lemma 1

$$\nu_0\left[\left(\epsilon_1^2 - p\right)\left(R^-_{1,1} - p\right)\right] = \nu_0\left(\epsilon_1^2 - p\right)\nu_0\left(R^-_{1,1} - p\right) = 0.$$

Hence,

$$\nu_0(\kappa) = \frac{1}{N}\nu_0\left[\left(\epsilon_1^2\right)^2 - \epsilon_1^2 p\right] \leq \frac{1}{N}\left[\nu_0\left(S^4\right) - \nu_0\left(\epsilon_1^2\right)p\right] \leq \frac{1}{N}(S^4 - p^2) \leq \frac{S^4}{N}. \quad (24)$$

By definition of $\kappa$, $\nu_1\left(\left(R_{1,1} - p\right)^2\right) = \nu_1(\kappa)$. Also note that $R_{1,1}, \epsilon_1^2, p \in [0, S^2]$. We have

$$\left|\epsilon_1^2 - p\right| \leq S^2 \quad \text{and} \quad |R_{1,1} - p| \leq S^2,$$

thus,

$$|\kappa| = \left|\left(\epsilon_1^2 - p\right)\left(R_{1,1} - p\right)\right| \leq S^4 |R_{1,1} - p|.$$

Applying inequality (18) with $n = 1$, we get

$$|\nu_1(\kappa) - \nu_0(\kappa)| \leq 6\beta^2 S^4 \exp\left(6\beta^2 S^4\right)\nu_1(|\kappa|). \quad (25)$$

Therefore, combining (25) with (23) and (24), we obtain

$$\nu_1\left(\left|R_{1,1} - p\right|^2\right) \leq \frac{S^4}{N} + 6\beta^2 S^4 \exp\left(6\beta^2 S^4\right)\nu_1\left(\left|R_{1,1} - p\right|^2\right).$$

Choosing $\beta_1$ such that

$$6\beta_1^2 S^4 \exp\left(6\beta_1^2 S^4\right) \leq \frac{15}{16},$$

we have

$$\nu_1\left(\left|R_{1,1} - p\right|^2\right) \leq \frac{S^4}{N} + \frac{15}{16}\nu_1\left(\left|R_{1,1} - p\right|^2\right),$$

i.e.

$$\nu_1\left(\left|R_{1,1} - p\right|^2\right) \leq \frac{16S^4}{N}.$$

Now take $\hat{\beta} = \min(\beta_0, \beta_1)$. Then for all $\beta < \hat{\beta}$, we have:

$$\nu_1\left(\left|R_{1,2} - q\right|^2\right) \leq \frac{16S^4}{N} \quad \text{and} \quad \nu_1\left(\left|R_{1,1} - p\right|^2\right) \leq \frac{16S^4}{N}.$$

□

Based on Proposition 2, we use an inductive argument to control the higher moments of the overlaps $R_{1,2}$ and $R_{1,1}$.

**Proposition 4** *There exist* $\hat{\hat{\beta}} > 0$, *and some constant* $C > 0$ *such that for all* $\beta < \hat{\hat{\beta}}$ *and any* $k \geq 1$, *we have*:

$$\nu_1\left(\left(R_{1,2} - q\right)^{2k}\right) \leq \left(\frac{Ck}{N}\right)^k, \quad (26)$$

$$\nu_1\left(\left(R_{1,1} - p\right)^{2k}\right) \leq \left(\frac{Ck}{N}\right)^k, \quad (27)$$

*where* $C$ *does not depend on* $N$ *or* $k$.

Before proving the above proposition, we show the following lemma first.

**Lemma 5** *Given a function $f$ on $(\Sigma_N)^n$, and $\tau_1, \tau_2 > 0$ with $\frac{1}{\tau_1} + \frac{1}{\tau_2} = 1$, we have*

$$|\nu_1(f) - \nu_0(f)| \le \exp\left[6n^2\beta^2 S^4\right]\left(2n^2\beta^2 S^2 \left[\nu_1(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}} \left[\nu_1(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}} + n^2\beta^2 S^4 \left(2 + \frac{3}{N}\right)\nu_1(|f|)\right), \tag{28}$$

*and*

$$|\nu_1(f) - \nu_0(f)| \le \exp\left[6n^2\beta^2 S^4\right]\left(2n^2\beta^2 S^2 \left[\nu_1(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}} \left[\nu_1(|R_{1,1} - p|^{\tau_2})\right]^{\frac{1}{\tau_2}} + n^2\beta^2 S^4 \left(4 + \frac{3}{N}\right)\nu_1(|f|)\right). \tag{29}$$

***Proof*** We show (28) first. Note that

$$|\nu_1(f) - \nu_0(f)| = \left|\int_0^1 \nu_t'(f)dt\right| \le \sup_{0<t<1} \left|\nu_t'(f)\right|.$$

Also know that $|\epsilon_l\epsilon_{l'}| \le S^2$ and $(p-q)^2 \le S^2$. Now apply Hölder's inequality, we will have for $\frac{1}{\tau_1} + \frac{1}{\tau_2} = 1$, and $l \ne l'$,

$$\left|\nu_t\left(\epsilon_l\epsilon_{l'}\left(R_{l,l'} - q\right)f\right)\right| \le S^2\nu_t\left(|f||R_{l,l'} - q|\right) \le S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}}.$$

Also we have

$$\nu_t\left(\epsilon_l^2\epsilon_{l'}^2 f\right) \le S^4\nu_t(|f|).$$

According to Eqs. (15) to (17),

$$\begin{aligned}
|A_1| &\le \left(\frac{n(n-1)}{2} + n^2 + \frac{n(n+1)}{2}\right)\beta^2 S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}} \\
&= 2n^2\beta^2 S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}}, \\
|A_2| &\le \beta^2\left[\frac{n}{2}\cdot 2S^4\nu_t(|f|) + \frac{n}{2}\cdot 2S^4\nu_t(|f|)\right] = 2\beta^2 n S^4\nu_t(|f|), \\
|B| &\le \frac{\beta^2}{N}\left(\frac{n(n-1)}{2} + n^2 + \frac{n(n+1)}{2} + \frac{n}{2} + \frac{n}{2}\right)S^4\nu_t(|f|) \\
&= \frac{\beta^2}{N}\left(2n^2 + n\right)S^4\nu_t(|f|) \le \frac{3\beta^2 n^2 S^4}{N}\nu_t(|f|).
\end{aligned}$$

Hence based on Eq. (14), we have

$$\begin{aligned}
|\nu_t'(f)| &\le |A_1| + |A_2| + |B| \\
&\le 2n^2\beta^2 S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}} + n^2\beta^2 S^4\left(2 + \frac{3}{N}\right)\nu_t(|f|).
\end{aligned}$$

Now by Lemma 4, we get

$$|\nu_t'(f)| \le \exp\left[6n^2\beta^2 S^4\right]\left(2n^2\beta^2 S^2\left[\nu_1(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_1(|R_{1,2} - q|^{\tau_2})\right]^{\frac{1}{\tau_2}} + n^2\beta^2 S^4\left(2 + \frac{3}{N}\right)\nu_1(|f|)\right).$$

Therefore,

$$|\nu_1(f)-\nu_0(f)| \le \sup_{0\le t\le 1}\left|\nu_t'(f)\right| \le \exp\left[6n^2\beta^2S^4\right]\Bigg(2n^2\beta^2S^2\left[\nu_1(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}$$
$$\left[\nu_1(|R_{1,2}-q|^{\tau_2})\right]^{\frac{1}{\tau_2}} + n^2\beta^2S^4\left(2+\frac{3}{N}\right)\nu_1(|f|)\Bigg).$$

Now we show the inequality (29). Note that we have for $\frac{1}{\tau_1}+\frac{1}{\tau_2}=1$, and $l=l'$,

$$\left|\nu_t\left(\epsilon_l\epsilon_{l'}\left(R_{l,l}-p\right)f\right)\right| \le S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,1}-p|^{\tau_2})\right]^{\frac{1}{\tau_2}}.$$

Considering different upper bounds for $|A_1|$ and $|A_2|$, we obtain

$$|A_1| \le \left(\frac{n(n-1)}{2}+n^2+\frac{n(n+1)}{2}\right)\beta^2S^2\cdot 2S^2\nu_t\left(|f|\right) = 4n^2\beta^2S^4\nu_t\left(|f|\right),$$
$$|A_2| \le \left(\frac{n}{2}+\frac{n}{2}\right)\beta^2S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,1}-p|^{\tau_2})\right]^{\frac{1}{\tau_2}}$$
$$\le n^2\beta^2S^2\left[\nu_t(|f|^{\tau_1})\right]^{\frac{1}{\tau_1}}\left[\nu_t(|R_{1,1}-p|^{\tau_2})\right]^{\frac{1}{\tau_2}}.$$

Following the same method as above, we show (29) as desired. □

We now prove Proposition 4.

***Proof of Proposition 4*** We will show (26) first. For $1\le s\le N$, let

$$A_s = \frac{1}{N}\sum_{s\le i\le N}(\sigma_i^1\sigma_i^2-q).$$

Then, $A_1 = R_{1,2}-q$. By Proposition 2, we have for $k=1$, $\nu_1(A_1^2)\le\frac{C}{N}$. Note that constant $C$ may vary from step to step. We will prove by induction over $k$ that we have

$$\forall s\le N,\quad \nu_1\left(A_s^{2k}\right)\le\left(\frac{Ck}{N}\right)^k.$$

Note that when $s=N$,

$$|A_N| = \left|\frac{\epsilon_1\epsilon_2-q}{N}\right| \le \frac{2S^2}{N},$$

hence, for some constant $C$, we have

$$\nu_1\left(A_s^{2k}\right)\le\left(\frac{2S^2}{N}\right)^{2k}\le\left(\frac{Ck}{N}\right)^k.$$

For the rest of the proof for (26), we can assume $s<N$. Notice that symmetry among sites implies

$$\nu_1\left(A_s^{2k+2}\right) = \frac{1}{N}\sum_{s\le i\le N}\nu_1\left(\left(\sigma_i^1\sigma_i^2-q\right)A_s^{2k+1}\right) = \frac{N-s+1}{N}\nu_1(f)\le|\nu_1(f)|, \tag{30}$$

where

$$f = (\epsilon_1\epsilon_2-q)\,A_s^{2k+1}.$$

We will use Lemma 5 to control $|\nu_1(f)|$. First, we evaluate $\nu_0(f)$. Let

$$\tilde{A} = \frac{1}{N}\sum_{s\le i\le N-1}\left(\sigma_i^1\sigma_i^2 - q\right).$$

By Lemma 1, we obtain

$$\nu_0\left((\epsilon_1\epsilon_2 - q)\,\tilde{A}_s^{2k+1}\right) = \nu_0(\epsilon_1\epsilon_2 - q)\nu_0\left(\tilde{A}_s^{2k+1}\right) = 0.$$

Note that $|\epsilon_1\epsilon_2 - q| \le 2S^2$. Thus,

$$|\nu_0(f)| = \left|\nu_0\left((\epsilon_1\epsilon_2 - q)\,A_s^{2k+1}\right) - \nu_0\left((\epsilon_1\epsilon_2 - q)\,\tilde{A}_s^{2k+1}\right)\right| \le 2S^2\nu_0\left(\left|A_s^{2k+1} - \tilde{A}_s^{2k+1}\right|\right). \tag{31}$$

We use the inequality

$$|x^{2k+1} - y^{2k+1}| \le (2k+1)|x-y|(x^{2k} + y^{2k})$$

for $x = A_s$ and $y = \tilde{A}$. Since

$$|x - y| = \left|\frac{\epsilon_1\epsilon_2 - q}{N}\right| \le \frac{2S^2}{N},$$

applying Lemma 4 for $n = 2$, we deduce from (31) that

$$\begin{aligned}|\nu_0(f)| &\le \frac{4S^4(2k+1)}{N}\left(\nu_0\left(A_s^{2k}\right) + \nu_0\left(\tilde{A}^{2k}\right)\right)\\ &\le \frac{4S^4(2k+1)}{N}\exp(24\beta^2S^4)\left(\nu_1\left(A_s^{2k}\right) + \nu_1\left(\tilde{A}^{2k}\right)\right).\end{aligned}$$

Because $s < N$, we observe that $\tilde{A}$ and $A_{s+1}$ are equal in distribution under $\nu_1$. Therefore, the induction hypothesis yields

$$|\nu_0(f)| \le \frac{8S^4(2k+1)}{N}\exp(24\beta^2S^4)\left(\frac{Ck}{N}\right)^k \le \left(\frac{C(k+1)}{N}\right)^{k+1}.$$

Note that

$$f = (\epsilon_1\epsilon_2 - q)\,A_s^{2k+1} \le 2S^2A_s^{2k+1}.$$

We apply Eq. (28) in Lemma 5 with $n = 2$, $\tau_1 = \frac{2k+2}{2k+1}$ and $\tau_2 = 2k+2$, combining with (30), to get

$$\begin{aligned}\nu_1\left(A_s^{2k+2}\right) \le |\nu_1(f)| \le |\nu_0(f)| + \exp\left[24\beta^2S^4\right]\\ \left(16\beta^2S^4\left[\nu_1\left(A_s^{2k+2}\right)\right]^{\frac{1}{\tau_1}}\left[\nu_1\left(|R_{1,2} - q|^{2k+2}\right)\right]^{\frac{1}{\tau_2}}\right.\\ \left.+\,4\beta^2S^4\left(2 + \frac{3}{N}\right)\cdot\frac{N}{N-s+1}\nu_1\left(A_s^{2k+2}\right)\right).\end{aligned}$$

Note that $xy \leq x^{\tau_1} + y^{\tau_2}$. Then the above inequality becomes

$$\nu_1\left(A_s^{2k+2}\right) \leq \left(\frac{C(k+1)}{N}\right)^{k+1} + \exp\left[24\beta^2 S^4\right]$$
$$\left(\left[16\beta^2 S^4 + 4\beta^2 S^4\left(2+\frac{3}{N}\right)\cdot\frac{N}{N-s+1}\right]\nu_1\left(A_s^{2k+2}\right)\right.$$
$$\left.+\,\nu_1\left(|R_{1,2}-q|^{2k+2}\right)\right). \tag{32}$$

Since when $n=1$, we have $A_1 = R_{1,2} - q$. Then the previous inequality implies that for $\beta$ small enough and some $C$ big enough, we have

$$\nu_1\left(\left(R_{1,2}-q\right)^{2k+2}\right) \leq \left(\frac{C(k+1)}{N}\right)^{k+1}. \tag{33}$$

Using (33) in (32) yields that for the other values of $s$ as well, we have

$$\nu_1\left(A_s^{2k+2}\right) \leq \left(\frac{C(k+1)}{N}\right)^{k+1}.$$

Now for $1 \leq s \leq N$, let

$$B_s = \frac{1}{N}\sum_{s\leq i\leq N}\left(\left(\sigma_i^1\right)^2 - p\right),$$

then, $B_1 = R_{1,1} - p$. Following the same method as above, we can show (27) as desired. □

## 3 TAP Equations for the Ghatak–Sherrington Model

In this section, we prove Theorem 1. Set $\beta_-$ to be

$$\frac{\beta_-}{\sqrt{N-1}} = \frac{\beta}{\sqrt{N}}.$$

Note that for some positive constant $K$,

$$|\beta - \beta_-| \leq \frac{K}{N}.$$

Let $p_- = p_-(N-1)$ and $q_- = q_-(N-1)$ be so that

$$p_- = \mathbb{E}\left[\frac{\sum_{\gamma=1}^S \gamma^2\cdot 2\mathrm{ch}\left[\gamma\left(\sqrt{q_-}\beta_- X + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\sqrt{q_-}\beta_- X + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}\right],$$

$$q_- = \mathbb{E}\left[\frac{\sum_{\gamma=1}^S \gamma\cdot 2\mathrm{sh}\left[\gamma\left(\sqrt{q_-}\beta_- X + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\sqrt{q_-}\beta_- X + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}\right]^2.$$

Note that $(p_-, q_-)$ exists and is unique for small $\beta$ due to Proposition 1. We have the following lemma.

**Lemma 6** *There exist* $K, \hat{\hat{\beta}} > 0$ *such that for all* $\beta < \hat{\hat{\beta}}$, $D, h \in \mathbb{R}$, *we have*

$$|p - p_-| \le \frac{K}{N}, \tag{34}$$

$$|q - q_-| \le \frac{K}{N}. \tag{35}$$

***Proof*** We will show (34) first, and (35) follows similarly. Let

$$\phi(X) = \frac{\sum_{\gamma=1}^{S} \gamma^2 \cdot 2\text{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\text{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)},$$

and

$$\psi(X) = \frac{\sum_{\gamma=1}^{S} \gamma \cdot 2\text{sh}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\text{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}.$$

Define

$$G(\beta, p, q) = \mathbb{E}\left[\phi(X)\right],$$

and

$$F(\beta, p, q) = \mathbb{E}\left[\psi(X)\right]^2.$$

Also define $p(\beta), q(\beta)$ by

$$p(\beta) = G(\beta, p(\beta), q(\beta)), \quad \text{and} \quad q(\beta) = F(\beta, p(\beta), q(\beta)).$$

Note that $p(\beta)$ and $q(\beta)$ are well-defined for small $\beta$ due to Proposition 1. We have

$$q'(\beta) = \frac{\frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial p} \cdot p'(\beta)}{1 - \frac{\partial F}{\partial q}}, \quad \text{and} \quad p'(\beta) = \frac{\frac{\partial G}{\partial \beta} + \frac{\partial G}{\partial q} \cdot q'(\beta)}{1 - \frac{\partial G}{\partial p}}.$$

Plugging the equation of $q'(\beta)$ into $p'(\beta)$, we obtain

$$p'(\beta) = \frac{\frac{\partial G}{\partial \beta}\left(1 - \frac{\partial F}{\partial q}\right) + \frac{\partial G}{\partial q} \cdot \frac{\partial F}{\partial \beta}}{\left(1 - \frac{\partial G}{\partial p}\right)\left(1 - \frac{\partial F}{\partial q}\right) - \frac{\partial G}{\partial q} \cdot \frac{\partial F}{\partial p}}.$$

Now we calculate the partial derivatives of $G(\beta, p, q)$ and $F(\beta, p, q)$. To simplify the notation, we introduce the following functions. Define

$$f(X) = \frac{1}{1 + \sum_{\gamma=1}^{S} 2\text{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}, \tag{36}$$

$$\kappa_{ch}(X) = 2\text{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right), \tag{37}$$

$$\kappa_{sh}(X) = 2\text{sh}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right). \tag{38}$$

Then

$$\phi(X)=\sum_{\gamma=1}^{S}\gamma^2\cdot\kappa_{ch}(X)f(X),\quad\text{and}\quad\psi(X)=\sum_{\gamma=1}^{S}\gamma\cdot\kappa_{sh}(X)f(X).$$

Define

$$\theta(X)=\sum_{\gamma=1}^{S}\gamma^4\cdot\kappa_{ch}(X)f(X),\quad\text{and}\quad\eta(X)=\sum_{\gamma=1}^{S}\gamma^3\cdot\kappa_{sh}(X)f(X).$$

Since $\beta<\hat{\hat{\beta}}$, and $D,h\in\mathbb{R}$, it's clear that $\phi(X)$, $\psi(X)$, $f(X)$, $\theta(X)$, and $\eta(X)$ are bounded functions. By a direct differentiation and noting that $|\sinh(x)|\leq\cosh(x)$, it is easy to see that the functions $\phi'(X)$, $\psi'(X)$, $f'(X)$, $\theta'(X)$, and $\eta'(X)$ are also bounded. For the explicit bounds, please check inequalities from (50) to (53). In addition, using Gaussian integration by parts yields that

$$\begin{aligned}
\frac{\partial G}{\partial\beta}&=\mathbb{E}\left[\sqrt{q}\left(\eta'(X)-\psi'(X)\phi(X)-\psi(X)\phi'(X)\right)+\beta(p-q)\left(\theta(X)-\phi^2(X)\right)\right],\\
\frac{\partial G}{\partial p}&=\mathbb{E}\left[\frac{\beta^2}{2}\left(\theta(X)-\phi^2(X)\right)\right],\\
\frac{\partial G}{\partial q}&=\mathbb{E}\left[\frac{\beta}{2\sqrt{q}}\left(\eta'(X)-\psi'(X)\phi(X)-\psi(X)\phi'(X)\right)+\frac{\beta^2}{2}\left(\phi^2(X)-\theta(X)\right)\right],\\
\frac{\partial F}{\partial\beta}&=2\mathbb{E}\left[\sqrt{q}\left(\psi'(X)\phi(X)+\psi(X)\phi'(X)-3\psi^2(X)\psi'(X)\right)\right.\\
&\quad\left.+\beta(p-q)\psi(X)\left(\eta(X)-\psi(X)\phi(X)\right)\right],\\
\frac{\partial F}{\partial p}&=\mathbb{E}\left[\beta^2\psi(X)\left(\eta(X)-\psi(X)\phi(X)\right)\right],\\
\frac{\partial F}{\partial q}&=\mathbb{E}\left[\frac{\beta}{\sqrt{q}}\left(\psi'(X)\phi(X)+\psi(X)\phi'(X)-3\psi^2(X)\psi'(X)\right)\right.\\
&\quad\left.+\beta^2\left(\psi^2(X)\phi(X)-\psi(X)\eta(X)\right)\right].
\end{aligned}$$

It follows that all the partial derivatives of $G(\beta,p,q)$ and $F(\beta,p,q)$ are bounded, hence so is $p'(\beta)$. Note that for some positive number $K$,

$$|\beta-\beta_-|\leq\frac{K}{N}.$$

By the mean value theorem,

$$|p-p_-|\leq\frac{K}{N}.$$

Similarly, $|q-q_-|\leq\frac{K}{N}$ is satisfied. □

Before the proof of Theorem 1, we state a result which is the analogue of Theorem 1.7.11 in [19]. Consider independent standard Gaussian random variables $y_i$ and $\xi$, which are independent of the randomness of $\langle\cdot\rangle$, and denote $\mathbb{E}_\xi$ the expectation with respect to the random variable $\xi$ only.

**Theorem 2** ([19, Theorem 1.7.11]) *Assume $\beta < \hat{\hat{\beta}}$, and $D, h \in \mathbb{R}$. Let $U$ be an infinitely differentiable function on $\mathbb{R}$ with derivatives given by $U^{(l)}$. Assume for all $l$ and $b$, the $l$th derivative of $U$ satisfies*

$$\mathbb{E}|U^{(l)}(z)|^b < \infty$$

*where $z$ is a Gaussian random variable. Then, using the notation $\dot{\sigma}_i = \sigma_i - \langle \sigma_i \rangle$, we have for $k = 1, 2$*

$$\mathbb{E}\left(\left\langle U\left(\frac{1}{\sqrt{N}}\sum_{i\leq N} y_i \dot{\sigma}_i\right)\right\rangle - \mathbb{E}_\xi U\left(\xi\sqrt{p-q}\right)\right)^{2k} \leq \frac{K}{N^k},$$

*where $p$ and $q$ satisfy the Eqs. (4) and (5) respectively, and the constant $K$ depends on $U$, $\beta$, but not on $N$.*

***Proof*** The proof of Theorem 2 is similar to Talagrand's proof of the Theorem 1.7.11 [19] except for the following differences. Let $\dot{S}_l = \frac{1}{\sqrt{N}}\sum_{i\leq N} y_i \dot{\sigma}_i^l$ and $\mathbb{E}_0$ denote the expectation with respect to $y_i$ and $\xi_l$ only. Set

$$T_{l,l} = \mathbb{E}_0(\dot{S}_l^2) - \mathbb{E}_0(\xi_l\sqrt{p-q})^2 = \frac{1}{N}\sum_{i\leq N}(\dot{\sigma}_i^l)^2 - (p-q),$$

and for $l \neq l'$, let

$$T_{l,l'} = \mathbb{E}_0(\dot{S}_l \dot{S}_{l'}) - \mathbb{E}_0\left[\xi_l \xi_{l'}(p-q)\right] = \frac{1}{N}\sum_{i\leq N}\left(\dot{\sigma}_i^l\right)\left(\dot{\sigma}_i^{l'}\right).$$

We claim that there exists a positive number $K$ such that

$$\forall r, \quad \mathbb{E}\left\langle T_{l,l'}^{2r}\right\rangle \leq \frac{K}{N^r}.$$

We explain the case $l = l'$ first. Since

$$(\dot{\sigma}_i^l)^2 = \left(\sigma_i^l - \langle\sigma_i\rangle\right)^2 = (\sigma_i^l)^2 - 2\sigma_i^l\left\langle\sigma_i^l\right\rangle + \langle\sigma_i\rangle^2,$$

it follows that

$$\begin{aligned} T_{l,l} &= \frac{1}{N}\sum_{i\leq N}\left(\dot{\sigma}_i^l\right)^2 - (p-q) = \frac{1}{N}\sum_{i\leq N}\left(\left(\sigma_i^l\right)^2 - 2\sigma_i^l\left\langle\sigma_i^l\right\rangle + \langle\sigma_i\rangle^2\right) - (p-q) \\ &= \frac{1}{N}\sum_{i\leq N}\left[\left(\sigma_i^l\right)^2 - p\right] + \frac{1}{N}\sum_{i\leq N}\left[\left(\langle\sigma_i\rangle^2 - q\right) + 2\left(q - \sigma_i^l\left\langle\sigma_i^l\right\rangle\right)\right]. \end{aligned}$$

We control the first and second term of this sum separately. By Proposition 4,

$$\mathbb{E}\left\langle\left[\frac{1}{N}\sum_{i\leq N}\left(\sigma_i^l\right)^{2r} - p\right]^2\right\rangle = \mathbb{E}\left\langle(R_{1,1} - p)^{2r}\right\rangle \leq \left(\frac{Cr}{N}\right)^r. \tag{39}$$

For the second term of the sum, we use the fact that for any $A$ and $B$, we have the inequality $(A+B)^{2r} \leq 2^{2r}(A^{2r}+B^{2r})$. Applying Jensen's inequality and Proposition 2, we have

$$\begin{aligned}
&\mathbb{E}\left\langle \left[ \frac{1}{N} \sum_{i \leq N} \left( \langle \sigma_i \rangle^2 - q \right) + 2\left( q - \sigma_i^l \left\langle \sigma_i^l \right\rangle \right) \right]^{2r} \right\rangle \\
&\leq 2^{2r} \left( \mathbb{E}\left\langle \left[ \frac{1}{N} \sum_{i \leq N} \left( \langle \sigma_i \rangle^2 - q \right) \right]^{2r} \right\rangle + \mathbb{E}\left\langle \left[ \frac{1}{N} \sum_{i \leq N} 2\left( q - \sigma_i^l \left\langle \sigma_i^l \right\rangle \right) \right]^{2r} \right\rangle \right) \\
&\leq 2\left( \mathbb{E}\left\langle (R_{1,2} - q)^{2r} \right\rangle + 4\mathbb{E}\left\langle (q - R_{1.2})^{2r} \right\rangle \right) \leq \left( \frac{Cr}{N} \right)^r . \qquad (40)
\end{aligned}$$

Therefore, combining (39) and (40), we obtain that for some positive number $K$

$$\begin{aligned}
\mathbb{E}\langle T_{l,l}^{2r} \rangle &\leq 2 \left( \mathbb{E}\left\langle \left[ \frac{1}{N} \sum_{i \leq N} \left( \sigma_i^l \right)^2 - p \right]^{2r} \right\rangle + \mathbb{E}\left\langle \left[ \frac{1}{N} \sum_{i \leq N} \left( \langle \sigma_i \rangle^2 - q \right) + 2\left( q - \sigma_i^l \left\langle \sigma_i^l \right\rangle \right) \right]^{2r} \right\rangle \right) \\
&\leq \frac{K}{N^r}.
\end{aligned}$$

This is similar for the case $l \neq l'$. Hence, for all $l$ and $l'$, and some positive number $K$, we have

$$\forall r, \quad \mathbb{E}\left\langle T_{l,l'}^{2r} \right\rangle \leq \frac{K}{N^r}.$$

□

Now we state two corollaries of the Theorem 2, which are equivalent of Talagrand's Corollary 1.7.13 and 1.7.15 [19].

**Corollary 1** *There exists a* $K, \hat{\hat{\beta}} > 0$ *such that for all* $\beta < \hat{\hat{\beta}}$, $D, h \in \mathbb{R}$, *and* $\epsilon \in \Sigma$ *we have*

$$\mathbb{E}\left( \left\langle \exp \frac{\epsilon\beta}{\sqrt{N}} \sum_{i \leq N} y_i \sigma_i \right\rangle - \exp\left[ \frac{\epsilon^2 \beta^2}{2}(p-q) \right] \exp \frac{\epsilon\beta}{\sqrt{N}} \sum_{i \leq N} y_i \langle \sigma_i \rangle \right)^{2k} \leq \frac{K}{N^k}, \qquad (41)$$

*and*

$$\begin{aligned}
&\mathbb{E}\left( \left\langle \frac{1}{\sqrt{N}} \sum_{i \leq N} y_i \dot{\sigma}_i \exp \frac{\epsilon\beta}{\sqrt{N}} \sum_{i \leq N} y_i \sigma_i \right\rangle - \epsilon\beta(p-q) \exp\left[ \frac{\epsilon^2 \beta^2}{2}(p-q) \right] \exp \frac{\epsilon\beta}{\sqrt{N}} \sum_{i \leq N} y_i \langle \sigma_i \rangle \right)^{2k} \\
&\leq \frac{K}{N^k} \qquad (42)
\end{aligned}$$

*where K does not depend on N.*

***Proof*** The proof is identical to the proof of Corollary 1.7.13 in [19]. □

For the rest of the section, we will use the following lemma.

**Lemma 7** *If* $\left|\frac{A'}{B'}\right| \leq B$ *and* $B \geq 1$, *we have*

$$\left|\frac{A'}{B'} - \frac{A}{B}\right| \leq |A - A'| + |B - B'|.$$

**Corollary 2** *Let*

$$\mathcal{E} = \exp\left(\frac{\epsilon\beta}{\sqrt{N}}\sum_{i \leq N} y_i\sigma_i + \epsilon^2 D + \epsilon h\right). \tag{43}$$

*Recall that* Av *denotes average over* $\epsilon \in \Sigma$. *There exists a constant* $K > 0$ *and* $\hat{\hat{\beta}} > 0$ *such that for all* $\beta < \hat{\hat{\beta}}$, *and* $D, h \in \mathbb{R}$, *we have*

$$\mathbb{E}\left(\frac{\langle \mathrm{Av}\epsilon\mathcal{E}\rangle}{\langle \mathrm{Av}\mathcal{E}\rangle} - \frac{\sum_{\gamma=1}^{S}\gamma \cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}\right)^{2k}$$
$$\leq \frac{K}{N^k}, \tag{44}$$

$$\mathbb{E}\left(\frac{\langle \mathrm{Av}\epsilon^2\mathcal{E}\rangle}{\langle \mathrm{Av}\mathcal{E}\rangle} - \frac{\sum_{\gamma=1}^{S}\gamma^2 \cdot 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}\right)^{2k}$$
$$\leq \frac{K}{N^k}, \tag{45}$$

$$\mathbb{E}\left(\frac{1}{\sqrt{N}}\sum_{i\leq N} y_i\frac{\langle\sigma_i \mathrm{Av}\mathcal{E}\rangle}{\langle \mathrm{Av}\mathcal{E}\rangle} - \beta(p-q)\frac{\langle \mathrm{Av}\epsilon\mathcal{E}\rangle}{\langle \mathrm{Av}\mathcal{E}\rangle} - \frac{1}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle\right)^{2k}$$
$$\leq \frac{K}{N^k}. \tag{46}$$

***Proof*** Define

$$A(\epsilon) = \left\langle \exp\frac{\epsilon\beta}{\sqrt{N}}\sum_{i\leq N} y_i\sigma_i\right\rangle - \exp\left[\frac{\epsilon^2\beta^2}{2}(p-q)\right]\exp\frac{\epsilon\beta}{\sqrt{N}}\sum_{i\leq N} y_i\langle\sigma_i\rangle.$$

Note that $A(0) = 0$. Deducing from (41), for $\gamma = 1, \ldots, S$, we have

$$\mathbb{E}\left(\gamma A(\gamma)\exp(\gamma^2 D + \gamma h) - \gamma A(-\gamma)\exp(\gamma^2 D - \gamma h)\right)^{2k} \leq \frac{K}{N^k},$$

and

$$\mathbb{E}\left(A(\gamma)\exp(\gamma^2 D + \gamma h) + A(-\gamma)\exp(\gamma^2 D - \gamma h) + A(0)\right)^{2k} \leq \frac{K}{N^k}.$$

Hence,

$$\mathbb{E}\left(\sum_{\gamma=1}^{S}\left[\gamma A(\gamma)\exp(\gamma^2 D + \gamma h) - \gamma A(-\gamma)\exp(\gamma^2 D - \gamma h)\right]\right)^{2k} \leq \frac{K}{N^k},$$

i.e.

$$\mathbb{E}\left(\langle \mathrm{Av}\epsilon\mathcal{E}\rangle - \sum_{\gamma=1}^{S}\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)\right)^{2k} \le \frac{K}{N^k},$$

$$\mathbb{E}\left(\left\langle \mathrm{Av}\epsilon^2\mathcal{E}\right\rangle - \sum_{\gamma=1}^{S}\gamma^2\cdot 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)\right)^{2k} \le \frac{K}{N^k},$$

and

$$\mathbb{E}\left(\langle \mathrm{Av}\mathcal{E}\rangle - \left[1+\sum_{\gamma=1}^{S}2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)\right]\right)^{2k} \le \frac{K}{N^k}. \tag{47}$$

Equations (44) and (45) follow from Lemma 7. Using the same method, we get from (42)

$$\mathbb{E}\left(\left\langle \frac{1}{\sqrt{N}}\sum_{i\le N}y_i\dot{\sigma}_i\mathrm{Av}\mathcal{E}\right\rangle - \beta(p-q)\sum_{\gamma=1}^{S}\gamma\cdot\exp\left[\frac{\gamma^2\beta^2}{2}(p-q)\right]2\mathrm{sh}\right.$$
$$\left.\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle + h\right)\right]\right)^{2k} \le \frac{K}{N^k}. \tag{48}$$

Combining (48) with (47) and using Lemma 7, we obtain

$$\mathbb{E}\left(\frac{\left\langle\frac{1}{\sqrt{N}}\sum_{i\le N}y_i\dot{\sigma}_i\mathrm{Av}\mathcal{E}\right\rangle}{\langle\mathrm{Av}\mathcal{E}\rangle} - \beta(p-q)\right.$$
$$\left.\frac{\sum_{\gamma=1}^{S}\gamma\cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^{S}2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle+h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}\right)^{2k} \le \frac{K}{N^k}. \tag{49}$$

Note that

$$\frac{\left\langle\frac{1}{\sqrt{N}}\sum_{i\le N}y_i\dot{\sigma}_i\mathrm{Av}\mathcal{E}\right\rangle}{\langle\mathrm{Av}\mathcal{E}\rangle} = \frac{1}{\sqrt{N}}\sum_{i\le N}y_i\frac{\langle\sigma_i\mathrm{Av}\mathcal{E}\rangle}{\langle\mathrm{Av}\mathcal{E}\rangle} - \frac{1}{\sqrt{N}}\sum_{i\le N}y_i\langle\sigma_i\rangle.$$

Combining (49) with (44) proves (46). □

Finally, we turn to the proof of Theorem 1.

***Proof of Theorem 1*** First, we show (6). Recall that the Hamiltonian (8) is the Hamiltonian of an $(N-1)$-spin system with parameter

$$\beta_- = \beta\sqrt{1-\frac{1}{N}} \le \beta,$$

$$\mathcal{E} = \exp\left(\frac{\epsilon\beta_-}{\sqrt{N-1}}\sum_{i\le N-1}g_{iN}\sigma_i + \epsilon^2 D + \epsilon h\right) = \exp\left(\frac{\epsilon\beta}{\sqrt{N}}\sum_{i\le N-1}g_{iN}\sigma_i + \epsilon^2 D + \epsilon h\right).$$

By Proposition 3, we have

$$\langle \sigma_N \rangle = \frac{\langle \mathrm{Av}\epsilon\mathcal{E}\rangle_-}{\langle \mathrm{Av}\mathcal{E}\rangle_-}.$$

Next, applying (44) to the $(N-1)$-spin system and the sequence $y_i = g_{iN}$, we obtain

$$\mathbb{E}\left(\langle\sigma_N\rangle - \frac{\sum_{\gamma=1}^S \gamma \cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle_- + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle_- + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}\right)^{2k} \le \frac{K}{N^k}.$$

Now to show (6), it suffices to show that

$$\mathbb{E}\Bigg(\frac{\sum_{\gamma=1}^S \gamma \cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta^2}{2}(p-q)\right]\right)} - \frac{\sum_{\gamma=1}^S \gamma \cdot 2\mathrm{sh}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle_- + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}{1+\sum_{\gamma=1}^S 2\mathrm{ch}\left[\gamma\left(\frac{\beta}{\sqrt{N}}\sum_{i\le N} g_{iN}\langle\sigma_i\rangle_- + h\right)\right]\exp\left(\gamma^2\left[D+\frac{\beta_-^2}{2}(p_- - q_-)\right]\right)}\Bigg)^{2k} \le \frac{K}{N^k}.$$

Let

$$f(x,y) = \frac{\sum_{\gamma=1}^S \gamma \cdot 2\mathrm{sh}(\gamma y)e^{\gamma^2 x}}{1+\sum_{\gamma=1}^S 2\mathrm{ch}(\gamma y)e^{\gamma^2 x}},$$

$$x_1 = D + \frac{\beta^2}{2}(p-q), \quad y_1 = \frac{\beta}{\sqrt{N}}\sum_{i\le N-1} g_{iN}\langle\sigma_i\rangle - \beta^2(p-q)\langle\sigma_N\rangle + h,$$

and

$$x_2 = D + \frac{\beta_-^2}{2}(p_- - q_-), \quad y_2 = \frac{\beta}{\sqrt{N}}\sum_{i\le N-1} g_{iN}\langle\sigma_i\rangle_- + h.$$

We claim

$$\mathbb{E}\left[f(x_1,y_1) - f(x_2,y_2)\right]^{2k} \le \frac{K}{N^k}.$$

By taking the partial derivatives of $f(x,y)$, it is straight-forward to show that $f(x,y)$ is a Lipschitz function with respect to both $x$ and $y$. There exists a positive number $L$ such that

$$|f(x_1,y) - f(x_2,y)| \le L|x_1 - x_2|,$$

and

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|.$$

Thus using the fact that for any $A$ and $B$, $(A+B)^{2k} \leq 2^{2k}(A^{2k}+B^{2k})$, we obtain

$$\mathbb{E}\,[f(x_1, y_1) - f(x_2, y_2)]^{2k} \leq 2^{2k} L^{2k} \left[\mathbb{E}\,(x_1 - x_2)^{2k} + \mathbb{E}\,(y_1 - y_2)^{2k}\right].$$

By Lemmas 5 and 6, it follows that

$$\begin{aligned} |x_1 - x_2| &= \left|\frac{\beta^2}{2}(p-q) - \frac{\beta_-^2}{2}(p_- - q_-)\right| \\ &\leq \left|\frac{p-q}{2}\left(\beta^2 - \beta_-^2\right)\right| + \left|\frac{\beta_-^2}{2}\,(p - p_-)\right| + \left|\frac{\beta_-^2}{2}\,(q - q_-)\right| \leq \frac{K}{N}, \end{aligned}$$

i.e.

$$\mathbb{E}\,(x_1 - x_2)^{2k} \leq \frac{K}{N^k}.$$

Now applying (46) to the $(N-1)$-spin system, we get

$$\mathbb{E}\left(\frac{1}{\sqrt{N-1}}\sum_{i\leq N-1} g_{iN}\,\langle\sigma_i\rangle - \beta_-(p_- - q_-)\,\langle\sigma_i\rangle - \frac{1}{\sqrt{N-1}}\sum_{i\leq N-1} g_{iN}\,\langle\sigma_i\rangle_-\right)^{2k} \leq \frac{K}{N^k}.$$

If we multiply both sides by $\beta_-^{2k}$, using $\left|\beta^2 - \beta_-^2\right| \leq \frac{K}{N}$ and Lemma 6 again, we have

$$\mathbb{E}\left(\frac{\beta}{\sqrt{N}}\sum_{i\leq N-1} g_{iN}\,\langle\sigma_i\rangle - \beta^2(p-q)\,\langle\sigma_i\rangle - \frac{\beta}{\sqrt{N}}\sum_{i\leq N-1} g_{iN}\,\langle\sigma_i\rangle_-\right)^{2k} \leq \frac{K}{N^k},$$

i.e.

$$\mathbb{E}\,(y_1 - y_2)^{2k} \leq \frac{K}{N^k}.$$

Therefore, we have

$$\mathbb{E}\,[f(x_1, y_1) - f(x_2, y_2)]^{k} \leq 2^{2k} L^{2k} \left[\mathbb{E}\,(x_1 - x_2)^{2k} + \mathbb{E}\,(y_1 - y_2)^{2k}\right] \leq \frac{K}{N^k}.$$

Similarly, we can show (7) using the same method. □

## 4 Proof of Proposition 1

In this proof, we use the same notation as in the proof of Lemma 6.

***Proof*** Recall that

$$\phi(X) = \frac{\sum_{\gamma=1}^{S} \gamma^2 \cdot 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)},$$

and

$$\psi(X) = \frac{\sum_{\gamma=1}^{S} \gamma \cdot 2\mathrm{sh}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}{1 + \sum_{\gamma=1}^{S} 2\mathrm{ch}\left[\gamma\left(\sqrt{q}\beta X + h\right)\right] \exp\left(\gamma^2\left[D + \frac{\beta^2}{2}(p-q)\right]\right)}.$$

We define functions $G(\beta, p, q) = \mathbb{E}[\phi(X)]$ and $F(\beta, p, q) = \mathbb{E}[\psi(X)]^2$, and hence the Eqs. (4) and (5) become

$$p = G(\beta, p, q), \quad \text{and} \quad q = F(\beta, p, q).$$

Define a self-mapping $T : [0, S^2] \times [0, S^2] \to [0, S^2] \times [0, S^2]$ by

$$T(p, q) := (G(\beta, p, q), F(\beta, p, q)).$$

By the contraction mapping theorem, it suffices to show that there exists a $\tilde{\beta} > 0$ such that for all $0 \le \beta < \tilde{\beta}$, $h \ge 0$, and $D \in \mathbb{R}$, $T$ is a contraction.

We have

$$\phi(X) = \sum_{\gamma=1}^{S} \gamma^2 \cdot \kappa_{ch}(X) f(X) \le S^2, \quad \psi(X) = \sum_{\gamma=1}^{S} \gamma \cdot \kappa_{sh}(X) f(X) \le S, \tag{50}$$

$$\theta(X) = \sum_{\gamma=1}^{S} \gamma^4 \cdot \kappa_{ch}(X) f(X) \le S^4, \quad \text{and} \quad \eta(X) = \sum_{\gamma=1}^{S} \gamma^3 \cdot \kappa_{sh}(X) f(X) \le S^3, \tag{51}$$

where $f$, $\kappa_{sh}$ and $\kappa_{ch}$ are given by (36), (37), and (38). Calculating the derivatives of the above functions, we obtain

$$\phi'(X) = \sqrt{q}\beta\,(\eta(X) - \phi(X)\psi(X)) \le 2\sqrt{q}\beta S^3, \quad \psi'(X) = \sqrt{q}\beta\left(\phi(X) - \psi^2(X)\right) \le 2\sqrt{q}\beta S^2, \tag{52}$$

and

$$\eta'(X) = \sqrt{q}\beta\,(\theta(X) - \eta(X)\psi(X)) \le 2\sqrt{q}\beta S^4. \tag{53}$$

Therefore, we have the following:

$$\frac{\partial G}{\partial p} = \mathbb{E}\left[\frac{\beta^2}{2}\left(\theta(X) - \phi^2(X)\right)\right] \le S^4\beta^2 := L_1,$$

$$\frac{\partial G}{\partial q} = \mathbb{E}\left[\frac{\beta}{2\sqrt{q}}\left(\eta'(X) - \psi'(X)\phi(X) - \psi(X)\phi'(X)\right) + \frac{\beta^2}{2}\left(\phi^2(X) - \theta(X)\right)\right] \le 4S^4\beta^2 := L_2,$$

$$\frac{\partial F}{\partial p} = \mathbb{E}\left[\beta^2\psi(X)\,(\eta(X) - \psi(X)\phi(X))\right] \le 2S^4\beta^2 := L_3,$$

$$\frac{\partial F}{\partial q} = \mathbb{E}\left[\frac{\beta}{\sqrt{q}}\left(\psi'(X)\phi(X) + \psi(X)\phi'(X) - 3\psi^2(X)\psi'(X)\right) + \beta^2\left(\psi^2(X)\phi(X) - \psi(X)\eta(X)\right)\right] \le 12S^4\beta^2 := L_4.$$

By Cauchy's inequality, we have

$$|G(p_1, q_1) - G(p_2, q_2)| \le L_1|p_1 - p_2| + L_2|q_1 - q_2| \le \sqrt{L_1^2 + L_2^2}\sqrt{|p_1 - p_2|^2 + |q_1 - q_2|^2},$$

and similarly,

$$|F(p_1, q_1) - F(p_2, q_2)| \le L_3|p_1 - p_2| + L_4|q_1 - q_2| \le \sqrt{L_3^2 + L_4^2}\sqrt{|p_1 - p_2|^2 + |q_1 - q_2|^2}.$$

Hence,

$$
\begin{aligned}
&|T(p_1, q_1) - T(p_2, q_2)| \\
&= \sqrt{[G(p_1, q_1) - G(p_2, q_2)]^2 + [F(p_1, q_1) - F(p_2, q_2)]^2} \\
&\leq \sqrt{\sum_{i=1}^{4} L_i^2} \cdot \sqrt{|p_1 - p_2|^2 + |q_1 - q_2|^2} = \sqrt{165} S^4 \beta^2 \sqrt{|p_1 - p_2|^2 + |q_1 - q_2|^2}.
\end{aligned}
$$

To make this map a contraction map, we need $\sqrt{165} S^4 \beta^2 < 1$. Let $\tilde{\beta} = \frac{1}{\sqrt[4]{165} S^2}$. Thus, for all $0 \leq \beta < \tilde{\beta}$, $h \geq 0$, and $D \in \mathbb{R}$, $T$ is a contraction mapping.

□

**Acknowledgements** Both authors would like to thank Wei-Kuo Chen for several suggestions on a previous version of this work, including a simplification of the proof of Proposition 1. They thank two anonymous referees for many inputs that significantly improved the presentation of the paper. They also would like to thank Si Tang for early discussions and help with computer simulations. This research partially supported by NSF Grant CAREER DMS-1653552, Simons Foundation/SFARI (597491-RWC), and NSF Grant 1764421.

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