On properties of the spherical mixed vector p-spin model

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Abstract

This paper studies properties of the mixed spherical vector p-spin model. At zero temperature, we establish and investigate a Parisi type formula for the ground state energy. At finite temperature, we provide some properties of minimizers of the Crisanti–Sommers formula recently obtained in Ko (2018). In particular, we extend some of the one-dimensional Parisi measure results of Auffinger and Chen (2015) to the vector case.
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1. Introduction and main results

1.1. Introduction

Spherical spin glass models are one of the main sources of ideas and techniques in the theoretical study of disordered complex systems. These models are simple enough to produce explicit computations while retaining many of the intriguing phenomena of high-dimensional random systems. Their energy landscape provides a metaphor to explain several phenomena in other areas of science, including biology, chemistry, data science, and economy.

One of these explicit computations was the limiting free energy discovered by Crisanti and Sommers in [9] for the spherical p-spin model with one dimensional spins. This formula is

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the analogue of the classical Parisi formula for the Sherrington–Kirkpatrick model [18] and it was rigorously proven for even-p-spin models by Talagrand in [19] and extended to general mixed p-spin models by Chen in [7]. These variational formulas and their minimizers have deep importance to describe and classify the energy landscape of such systems. We refer the readers to [1,2,4,6,10,11,14] and the references therein for results in this direction.

The rigorous study of spherical spin models with vector spins started with the work of Panchenko and Talagrand [17] with the first non-trivial bounds for the free energy. Recently, Ko [12] provided a proof of the limiting free energy for these models and its Crisanti–Sommers analogue [13]. These results came after important contribution of Panchenko in the study of vector p-spins on the hypercube [15–17]. As far as we know, there is no rigorous study on the role and properties of the minimizers of such models. The goal of this paper is to provide further properties of the model. Our main results are the following:

1. We provide a characterization of the minimizer of the Crisanti–Sommers formula, the order parameter of the model, in Theorem 2.
2. We extend the results of [12] to zero temperature, proving an exact formula for the ground state energy. These results are stated in Sections 5 and 6.

Let us now describe the spherical model with vector spins and state some of its fundamental results. Fix $m \geq 1$ and for $N \geq 1$, let S_N be the sphere in $\mathbb{R}^N$ of radius $\sqrt{N}$. We denote a configuration of the vector spin by

$$\vec{\sigma} = (\vec{\sigma}_1, \ldots, \vec{\sigma}_N) \in S_N^m \text{ where } S_N^m = \{\vec{\sigma} \in (\mathbb{R}^N)^m | \vec{\sigma}(j) \in S_N \text{ for } j = 1, \ldots, m\}.$$

Here the jth coordinate of $\vec{\sigma}$ is denoted by $\vec{\sigma}(j)$ and the vector entries of $\vec{\sigma}$ are denoted by

$$\vec{\sigma}_i = (\vec{\sigma}_i(1), \ldots, \vec{\sigma}_i(m)) \in \mathbb{R}^m, \quad 1 \leq i \leq N.$$

For $p \geq 2$, we denote the p-spin Hamiltonian of the jth copy by

$$H_{N,p}(\vec{\sigma}(j)) = \frac{1}{N^{\frac{p-1}{2}}} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1, \ldots, i_p} \vec{\sigma}_{i_1}(j) \cdots \vec{\sigma}_{i_p}(j),$$

where $g_{i_1, \ldots, i_p}$ are i.i.d. standard Gaussians for all $p \geq 2$ and indices $(i_1, \ldots, i_p)$. The corresponding mixed p-spin Hamiltonian for the jth copy at inverse temperatures $(\vec{\beta}_p)_{p \geq 2}$, where $\vec{\beta}_p = (\vec{\beta}_p(j))_{1 \leq j \leq m}$, can be expressed as

$$H_N^j(\vec{\sigma}) = \sum_{p \geq 2} \vec{\beta}_p(j) H_{N,p}(\vec{\sigma}(j)). \tag{1}$$

Here we only consider mixed even p-spin models, *i.e.* $\beta_p(j) = 0$ for all $1 \leq j \leq m$ and odd $p \geq 3$. Moreover, we assume that the inverse temperature of each jth copy satisfy

$$\sum_{p \geq 2} 2^p \vec{\beta}_p^2(j) < \infty,$$

so that (1) is well-defined.

We define the Hamiltonian of m copies mixed p-spin models of spherical spin glasses by

$$H_N(\vec{\sigma}) = \sum_{j=1}^m H_N^j(\vec{\sigma}).$$

If, for any $1 \leq k, \ell \leq m$, we introduce the function

$$\xi_{k,\ell}(x) = \sum_{p \geq 2} \beta_p(k)\beta_p(\ell)x^p, \tag{2}$$

then it is not difficult to check that, for two arbitrarily spin configurations $\vec{\sigma}^1$ and $\vec{\sigma}^2$,

$$\mathbb{E}\left[H_N^k(\vec{\sigma}^1(k))H_N^\ell(\vec{\sigma}^2(\ell))\right] = N\xi_{k,\ell}(R_{1,2}^{k,\ell}) \text{ for all } 1 \le k, \ell \le m,$$

where

$$R_{1,2}^{k,\ell} = \frac{1}{N} \sum_{j=1}^N \vec{\sigma}_j^1(k)\vec{\sigma}_j^2(\ell)$$

is the overlap between the corresponding coordinates of the vector configurations $\vec{\sigma}^1$ and $\vec{\sigma}^2$. The overlap matrix between configurations $\vec{\sigma}^\ell$ and $\vec{\sigma}^{\ell'}$ is expressed as

$$R_{\ell,\ell'} = R(\vec{\sigma}^\ell, \vec{\sigma}^{\ell'}) = (R_{\ell,\ell'}^{kk'})_{1 \le k,k' \le m} = \frac{1}{N} \sum_{i=1}^N \vec{\sigma}_i^\ell \otimes \vec{\sigma}_i^{\ell'}$$

where $\otimes$ is the outer product on vectors in $\mathbb{R}^m$.

Let $\mathbb{M}$ be the space of $m \times m$ semi-symmetric positive definite matrices with entries in $[-1, 1]$ and diagonals entries equal to 1. We now describe the Crisanti–Sommers formula for the free energy of a system of vector spins with constrained self-overlap $Q \in \mathbb{M}$. For any positive semidefinite matrix $A = (A_{i,j})_{1 \le i,j \le m}$, let

$$\xi(A) := \sum_{p \ge 2} (\vec{\beta}_p \otimes \vec{\beta}_p) \odot A^{\circ p} = (\xi_{i,j}(A_{i,j}))_{1 \le i,j \le m}$$

where $\xi_{i,j}$ are defined in (2) and $A^{\circ p}$ denotes the pth Hadamard power of the matrix A (element-wise multiplication). Clearly, we have the matrices

$$\xi'(A) = \sum_{p \ge 2} p(\vec{\beta}_p \otimes \vec{\beta}_p) \odot A^{\circ(p-1)} \text{ and } \xi''(A) = \sum_{p \ge 2} p(p-1)(\vec{\beta}_p \otimes \vec{\beta}_p) \odot A^{\circ(p-2)}.$$

Given any $\epsilon > 0$ and $Q \in \mathbb{M}$, denote the set of spins with constrained self overlaps by

$$\mathcal{Q}_N^\epsilon = \left\{ \vec{\sigma} \in S_N^m \,\middle|\, \|R(\vec{\sigma}, \vec{\sigma}) - Q\|_\infty \le \epsilon \right\},$$

where $\|A\|_\infty = \sup_{1 \le i,j \le m} |A_{ij}|$, for any $m \times m$ matrix A. For an external field $\vec{h} \in \mathbb{R}^m$ and any $\beta > 0$, we define the free energy as

$$F_N^{\epsilon,Q}(\beta) = \frac{1}{N} \mathbb{E} \log \int_{\mathcal{Q}_N^\epsilon} \exp \beta \left(H_N(\vec{\sigma}) + \sum_{j=1}^m \vec{h}(j) \sum_{i=1}^N \vec{\sigma}_i(j) \right) d\lambda_N^n(\vec{\sigma}),$$

where the reference measure $\lambda_N^m = \lambda_N^{\otimes m}$ is the product of Haar measures λ_N on S_N with normalization $\lambda_N(S_N) = 1$. Here the parameter β is the so-called inverse temperature. Moreover, denote $\xi_\beta = \beta^2 \xi$ and $\vec{h}_\beta = \beta \vec{h}$.

For a measurable function $f : S_N^m \times S_N^m \to \mathbb{R}$ we also set

$$\langle f \rangle_{\epsilon,Q} = \frac{\int_{(\mathcal{Q}_N^\epsilon)^2} f(\vec{\sigma}^1, \vec{\sigma}^2) \exp(H_N(\vec{\sigma}^1) + H_N(\vec{\sigma}^2)) d\lambda_N^m(\vec{\sigma}^1) d\lambda_N^m(\vec{\sigma}^2)}{\int_{(\mathcal{Q}_N^\epsilon)^2} \exp(H_N(\vec{\sigma}^1) + H_N(\vec{\sigma}^2)) d\lambda_N^m(\vec{\sigma}^1) d\lambda_N^m(\vec{\sigma}^2)}. \tag{3}$$

For a matrix-valued function $A(\vec{\sigma}^1, \vec{\sigma}^2) = (a_{ij}(\vec{\sigma}^1, \vec{\sigma}^2))_{1 \le i,j \le m}$, we let $\langle A(\vec{\sigma}^1, \vec{\sigma}^2) \rangle_{\epsilon,Q}$ denote the matrix $(\langle a_{ij}(\vec{\sigma}^1, \vec{\sigma}^2) \rangle_{\epsilon,Q})_{1 \le i,j \le n}$. Observe that for any f continuous, the map

$$Q \mapsto \langle f \rangle_{\epsilon,Q} \tag{4}$$

is continuous on $(\mathbb{M}, \|\cdot\|_\infty)$.

We will now recall the formula for the free energy obtained in [13]. Denote a right-continuous non-decreasing function by

$$x(t) : [0, m] \to [0, 1] \text{ such that } x(0) = 0 \text{ and } x(m) = 1 \tag{5}$$

and a 1-Lipschitz monotone matrix path in the space of $m \times m$ positive semidefinite matrices by

$$\Psi(t) : [0, m] \to \mathbb{S}_+^m \text{ such that } \text{Trace}(\Psi(t)) = t \text{ and } \Psi(0) = 0 \text{ and } \Psi(m) = Q. \tag{6}$$

where $\mathbb{S}_+^m$ is the space of $m \times m$ positive semidefinite matrices.

Set

$$t_x := x^{-1}(1) = \inf\{t \in [0, m] | x(t) = 1\}$$

and

$$s_x := x^{-1}(0) = \sup\{t \in [0, m] | x(t) = 0\}.$$

Assuming $t_x < m$, then $\forall t_x < \hat{T} < m$, we can define the quantity

$$
\mathscr{C}_{\beta,Q}(x, \Phi) = \frac{1}{2} \left[\int_0^m x(t) \langle \xi_\beta'(\Phi(t)) + \vec{h}_\beta \vec{h}_\beta^T, \Phi'(t) \rangle dt + \log |\Phi(m) - \Phi(\hat{T})| \right.
$$
$$
\left. + \int_0^{\hat{T}} \langle \hat{\Phi}(t)^{-1}, \Phi'(t) \rangle dt \right] \tag{7}
$$

where $\hat{\Phi}(t) : [0, m] \to \mathbb{R}^{m \times m}$ is a decreasing matrix path given by

$$\hat{\Phi}(t) = \int_t^m x(s) \Phi'(s) ds.$$

Furthermore, for any $\Lambda \in \mathbb{S}_+^m$ satisfying $\Lambda > \int_0^m x(s) \xi_\beta''(\Phi(s)) \odot \Phi'(s) ds$, we have a continuous form of the Parisi formula as follows:

$$
\mathscr{P}_{\beta,Q}(x, \Lambda, \Phi) = \frac{1}{2} \left[\int_0^m \langle \xi_\beta''(\Phi(q)) \odot \Phi'(q), (\Lambda - D_{\beta,Q}^x(q))^{-1} \rangle dq \right.
$$
$$
+ \langle \vec{h}_\beta \vec{h}_\beta^T, (\Lambda - D_{\beta,Q}^x(0))^{-1} \rangle
$$
$$
\left. - \int_0^m x(q) \langle \xi_\beta''(\Phi(q)) \odot \Phi(q), \Phi'(q) \rangle dq + \langle \Lambda, Q \rangle - m - \log |\Lambda| \right] \tag{8}
$$

where $D_{\beta,Q}^x(q) := \int_q^m x(s) \xi_\beta''(\Phi(s)) \odot \Phi'(s) ds$.

Denote $\mathscr{M}_0$ the collection of all 1-Lipschitz monotone matrix paths that satisfy (6) and denote by $\mathscr{N}_0$ the collection of all nonnegative nondecreasing and right-continuous functions on $[0, m)$ and $\mathscr{N}$ the collection of all nonnegative nondecreasing and right-continuous functions on $[0, m)$ satisfying the assumption that there exists $0 < c < m$ such that $\alpha(t)$ is constant for all $t \in [c, m]$.

We will also need a discrete version of (7) and (8) that we describe now. Consider a discrete monotone matrix path encoded by an increasing sequence of real numbers and a monotone sequence of $n \times n$ symmetric positive semidefinite matrices,

$$0 = x_0 \leq x_1 \leq \ldots \leq x_{r-2} \leq x_{r-1} \leq 1,$$
$$0 = Q_0 \leq Q_1 \leq \ldots \leq Q_{r-2} \leq Q_{r-1} \leq Q_r = Q,$$

where $r \geq 1$. We denote $\underline{x} = (x_k)_{k=0}^{r-1}$ and $\underline{Q} = (Q_k)_{k=1}^{r}$. The discrete Crisanti–Sommers formula is given by

$$\mathscr{C}_r(\underline{x}, \underline{Q}) = \frac{1}{2}[\langle \vec{h}\vec{h}^T, D_1 \rangle + \frac{1}{x_{r-1}} \log |Q - Q_{r-1}| - \Sigma_{1 \leq k \leq r-2} \frac{1}{x_k} \log \frac{|D_{k+1}|}{|D_k|}$$

$$+ \langle Q_1, D_1^{-1} \rangle + \Sigma_{1 \leq k \leq r-1} x_k \cdot \mathrm{Sum}(\xi(Q_{k+1}) - \xi(Q_k))], \tag{9}$$

where $D_p = \Sigma_{p \leq k \leq r-1} x_k(Q_{k+1} - Q_k)$ for $1 \leq p \leq r-1$, and

$$\mathrm{Sum}(A) = \sum_{1 \leq i, j \leq m} A_{ij}.$$

and the discrete Parisi formula is given by

$$\mathscr{C}_r(\Lambda, \underline{x}, \underline{Q}) = \frac{1}{2}[\langle \vec{h}\vec{h}^T, \Lambda^{-1} \rangle + \langle \Lambda, Q \rangle - m - \log |\Lambda|$$

$$+ \sum_{1 \leq k \leq r-1} \log \frac{|\Lambda_{k+1}|}{|\Lambda_k|} + \langle \xi'(Q_1), \Lambda_1^{-1} \rangle$$

$$- \sum_{1 \leq k \leq r-1} x_k \cdot \mathrm{Sum}(\theta(Q_{k+1}) - \theta(Q_k))] \tag{10}$$

where $\Lambda_r = \Lambda$, $\Lambda_p = \Lambda - \sum_{p \leq k \leq r-1} x_k(\xi'(Q_{k+1}) - \xi'(Q_k))$ for $1 \leq p \leq r-1$.

The limit of the free energy with self overlaps constrained to Q can be expressed as

Theorem 1 (*[13], Theorems 1–3, Proposition 1*). *The limit of the free energy with self overlaps constrained to Q is*

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} F_N^{\epsilon, Q}(\beta) = \inf_{r, \underline{x}, \underline{Q}} \mathscr{C}_r(x, \underline{Q}) = \inf_{x, \Phi \in \mathscr{N}_0 \times \mathscr{M}_0} \mathscr{C}_{\beta, Q}(x, \Phi)$$

$$= \inf_{x, \Phi \in \mathscr{N}_0 \times \mathscr{M}_0} \mathscr{P}_{\beta, Q}(x, \Lambda, \Phi). \tag{11}$$

The last two infimums are over $x(t)$ and $\Phi(t)$ defined in (5) and (6) such that $|Q - \Phi(t_x)| > 0$ and they are both attained.

1.2. Main results

Our first main result explains the role of the minimizers of the Crisanti–Sommers formula in the case $n \geq 2$.

Theorem 2. *Assume that the pair (x, Φ) is a minimizer of the Crisanti–Sommers formula (11) and write $\mu_P([0, q]) = x(q)$. For any $F = (F_{i,j})_{1 \leq i, j \leq m} : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ continuous and bounded,*

$$\lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E} \langle F(R_{1,2}) \rangle_{\epsilon, Q} = \int_0^m F \circ \Phi(t) d\mu_P(t).$$

Remark 1 (*Uniqueness of the Parisi Pair (x, Φ)*). By choosing $F(X) = (\mathrm{trace}(X))_{1 \leq i, j \leq m}$ and using the fact that $\mathrm{Trace}(\Phi(t)) = t$, one can see that the minimizing measure μ_P is unique. Similarly, for any $t \in \mathrm{supp} \mu_P$, the value of $\Phi(t)$ is also unique. However, for any $t \notin \mathrm{supp} \mu_P$, we can modify $\Phi(t)$ arbitrarily as $\Phi(t)$ will not change the corresponding value of $\mathscr{C}(x, \Phi)$.

Our second main result is a proof of a Parisi type formula for the ground state energy GSE

$$GSE := \lim_{N\to\infty} \max_{\vec{\sigma}\in(S_N)^m} \frac{H_N(\vec{\sigma})}{N},$$

and the constrained GSE, defined as

$$GSE(Q) = \lim_{\epsilon\to 0}\lim_{N\to\infty} \max_{R(\vec{\sigma},\vec{\sigma})\in\mathcal{Q}_N^\epsilon} \frac{H_N(\vec{\sigma})}{N}.$$

Set

$$\mathcal{K}(Q) := \left\{ (L,\alpha,\Phi) \in \mathbb{S}_+^m \times \mathcal{N} \times \mathcal{M} : L \right.$$
$$\left. > \int_0^m \alpha(s)\Phi'(s)ds \text{ and } \Phi \text{ constrained on } Q \right\}.$$

For any $(L,\alpha,\Phi) \in \mathcal{K}(Q)$, define

$$\mathcal{C}(L,\alpha,\Phi) = \frac{1}{2}\left[\langle \xi'(Q) + \vec{h}\vec{h}^T, L\rangle + \int_0^m \langle (L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}, \Phi'(t)\rangle dt \right.$$
$$\left. - \int_0^m \langle \xi''(\Phi(t))\odot\Phi'(t), \int_0^t \alpha(s)\Phi'(s)ds\rangle dt \right].$$

We can now state the second main result of this paper. Parisi type formulas for the ground state energy were obtained in the past for other models, see [5,8].

Theorem 3 (*Parisi's Formula for the Ground State Energy.*). *For any vector mixed p-spin model and any constraint Q we have*

$$GSE(Q) = \inf_{(L,\alpha,\Phi)\in\mathcal{K}(Q)} \mathcal{C}(L,\alpha,\Phi).$$

Moreover, there exists a matrix Q_∞ such that

$$GSE = \inf_{(L,\alpha,\Phi)\in\mathcal{K}(Q_\infty)} \mathcal{C}(L,\alpha,\Phi), \tag{12}$$

and the minimizers (L_0,α_0,Φ_0) of (12) *satisfy*

$$\alpha_0 := \lim_{\beta\to\infty} \beta x_{\beta,Q_\infty} \text{ vaguely on } [0,m),$$

$$\Phi_0 := \lim_{\beta\to\infty} \Phi_{\beta,Q_\infty} \text{ uniformly and } \Phi_0' := \lim_{\beta\to\infty} \Phi_{\beta,Q_\infty}' \text{ uniformly,}$$

$$L_0 := \lim_{\beta\to\infty} \int_0^m \beta x_{\beta,Q_\infty}(s)\Phi_{\beta,Q_\infty}'(s)ds.$$

Remark 2. The matrix Q_∞ is "explicit" and it is derived as the limit as $\beta \to \infty$ of minimizers Q_β of (11) as $\beta \to \infty$. This is explained in Section 5.1.

Remark 3. The vague convergence of $(\beta\Phi_{\beta,Q_\infty})_{(\beta>0)}$ on $[0,m)$ means that $\lim_{\beta\to\infty} \beta x_{\beta,Q_\infty}(s) = \alpha_0(s)$ at all points of continuity of α_0 on $[0,m)$.

2. Examples

Before turning into the proofs of the theorems we provide two examples where explicit computations are amenable.

2.1. A full replica symmetry breaking (FRSB) example

Let $n = 2$ and consider the constraint

$$Q = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

For $\beta > \sqrt{\frac{2}{\text{tr}(Q^2)}}$, define $\vec{\beta} := (\beta, \beta)$. For any matrix A, define $ch A : (ch(a_{i,j}))_{1 \le i,j \le 2}$ and $sh A : (sh(a_{i,j}))_{1 \le i,j \le 2}$. Moreover, define $E = \vec{e} \otimes \vec{e}$, where $\vec{e} = (1, 1)$. We now set the model with $\vec{h} = 0$ and

$$\xi(A) = (\vec{\beta} \otimes \vec{\beta}) \odot (ch A - E).$$

Define a matrix path as $\Phi(q) = \frac{q}{2} Q$ and $\phi(q) = \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle^{-\frac{1}{2}}$.

$$\phi''(q) = \frac{1}{4} \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle^{-\frac{5}{2}}$$
$$[3 \langle \xi'''(\Phi(q)), \Phi'(q)^{\circ 3} \rangle^2 - 2 \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle \langle \xi''''(\Phi(q)), \Phi'(q)^{\circ 4} \rangle].$$

Since the copies are at the same temperature, the conclusion that $\phi''(q) < 0$ can be deduced by the fact that $th^2(1.1) < \frac{2}{3}$ and then $3 sh^2 q < 2 ch^2 q$, for $q < 1.1$. Hence $\phi(q)$ is concave on $[0, 2]$.

Since $\phi(0) < \sqrt{2}$ and $\phi(2) > 0$, there exists a unique q_0 such that $\phi(q_0) = \frac{1}{\sqrt{2}}(2 - q_0)$. Define a distribution function as follows:

$$x(q) = \begin{cases} -\sqrt{2}\phi'(q) & 0 \le q \le q_0, \\ 1 & q \ge q_0. \end{cases} \tag{13}$$

Thus we claim that (x, Φ) is a minimizer of $\mathscr{C}$ for the model ξ and the proof is deferred to Section 7.

2.2. A replica symmetric example

Consider the multi-dimensional SK model, *i.e.* for $A \in \mathbb{S}_+^m, \xi(A) = (\vec{\beta}_2 \otimes \vec{\beta}_2) \circ A^{\circ 2}$. Formula (12) can be explicitly solved and we find that the multi-dimensional SK model is replica symmetric at zero temperature.

Proposition 4. *The multi-dimensional SK model is replica symmetric at zero temperature, that is, the minimizer (L_0, α_0, Φ_0) is given by*

$$L_0 = Q^{\frac{1}{2}}(Q^{\frac{1}{2}}(\xi'(Q) + \vec{h}\vec{h}^T)Q^{\frac{1}{2}})^{-\frac{1}{2}}Q^{\frac{1}{2}}, \quad \alpha_0 = 0 \quad \text{and} \quad \Phi_0 = \frac{t}{m}Q \tag{14}$$

and the corresponding GSE is equal to

$$Sum\left((Q^{\frac{1}{2}}(\xi'(Q) + \vec{h}\vec{h}^T)Q^{\frac{1}{2}})^{\frac{1}{2}}\right).$$

The proof of Proposition 4 follows from Proposition 22 stated in Section 7.

3. Further properties of the Crisanti–Sommers formula at positive temperature

In this section, we go back to the Crisanti–Sommers formula at positive temperature and list a sequence of properties of their minimizers. The proof of these propositions will be given in Section 6.

The first two facts concern the role of 0 in the support of μ_P.

Proposition 5. *If the external field vanishes, i.e., $\vec{h} = 0$ and for $p \geq 4$, there exists $\vec{\beta}_p$ such that $\vec{\beta}_p \neq 0$, then $0 \in supp\ \mu_P$.*

The next result shows that 0 can only be isolated point of the Parisi measure if the model does not have an SK component.

Proposition 6. *If $2\langle \vec{\beta}_2 \otimes \vec{\beta}_2, \Phi'(0)^{\circ 2} \rangle \neq \langle \hat{\Phi}(0)^{-1}\Phi'(0), \hat{\Phi}(0)^{-1}\Phi'(0) \rangle$ and $0 \in supp\ \mu_P$, then there exists $\hat{q} > 0$ such that $\mu_P([0, \hat{q}]) = \mu_P(\{0\})$.*

We say two points x, y are consecutive isolated points of the support of a measure μ if x, y are isolated points in supp μ with $x < y$ and $\mu((x, y)) = 0$. The next proposition is a weak 1-RSB criterion for the minimizer.

Proposition 7. *If $t \mapsto \langle \xi''(\Phi(t)), \Phi'(t)^{\circ 2} \rangle^{-\frac{1}{2}}$ is convex in an interval I, then supp μ_P contains at most 2 consecutive isolated points. In particular, if $\xi(A) = (\vec{\beta}_p \otimes \vec{\beta}_p) \odot A^{\circ p}$, then supp μ_P contains at most 2 consecutive isolated points.*

The next proposition is the high-dimensional analogue of Theorem 3 from [3] and shows that accumulation points in the support must be continuity points of the measure μ_P.

Proposition 8. *Suppose that there exist an increasing sequence $(u_l^-)_{l \geq 1}$ and a decreasing sequence $(u_l^+)_{l \geq 1}$ of supp μ_P such that $\lim_{l \to \infty} u_l^- = u_0 = \lim_{l \to \infty} u_l^+$. Moreover, $\Phi'(u_0) > 0$. Then μ_P is continuous at u_0.*

The next criterion goes in the opposite direction and provides an example where a jump discontinuity must be present.

Proposition 9. *For any $u_0 \in supp\ \mu_P$, if $\langle \xi''(Q), \Phi'(u_0)^{\circ 2} \rangle < \langle Q^{-1}\Phi'(u_0)Q^{-1}, \Phi'(u_0) \rangle$, then the Parisi measure μ_P has a jump discontinuity at u_0. In particular, if $Sum(\xi''(Q) \odot Q^{\circ 2}) \leq m$, μ_P has a jump discontinuity at t_x.*

As in the one dimensional case, when an interval is in the support, μ_P has an explicit formula.

Proposition 10. *If $(a, b) \subseteq supp\ \mu_P$ with $0 \leq a < b < m$, then*

$$\mu_P([0, u]) = \frac{\langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3} \rangle}{2 Trace\left((\Phi'(u)^{\frac{1}{2}} (\xi''(\Phi(u)) \odot \Phi'(u))\Phi'(u)^{\frac{1}{2}})^{\frac{3}{2}} \right)}$$

for all $u \in (a, b)$.

Last, a property concerning the multidimensional SK model.

Proposition 11. *If $\Phi'(t) > 0$ for any $t \in [0, m]$, then the SK model is at most 1 replica symmetric breaking (1RSB).*

4. Proof of Theorem 2

In this section we provide a proof of Theorem 2.

Proof of Theorem 2. For each $1 \leq i \leq m$, denote the ith unit vector in $\mathbb{R}^m$ by $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Given a vector $\vec{\alpha}$, let $\vec{\alpha} \oplus_i \vec{\alpha}$ as $\vec{\alpha} \otimes \vec{e}_i + \vec{e}_i \otimes \vec{\alpha}$. In this proof we drop from our notation the dependencies on β and Q.

For any $p \geq 2$, we consider arbitrarily $\vec{c}_p$ that also satisfies the requirement of the inverse temperature, As the Crisanti–Sommers functional is differentiable at each $\vec{\beta}_p(i)$, where $p \geq 2$, $1 \leq i \leq n$, we consider $\vec{\beta}_p + t\vec{c}_p$ as inverse temperature and compute the first derivative of the Crisanti–Sommers functional at $t = 0$.

$$
\begin{aligned}
\frac{d\mathscr{C}}{dt}\Big|_{t=0} &= \frac{1}{2} p \int_0^m x(t) \langle \vec{\beta}_p \otimes \vec{c}_p + \vec{c}_p \otimes \vec{\beta}_p, \Phi(t)^{\circ(p-1)} \odot \Phi'(t) \rangle dt \\
&= \frac{p}{2} \int_0^m \int_0^t \langle \vec{\beta}_p \otimes \vec{c}_p + \vec{c}_p \otimes \vec{\beta}_p, \Phi_{ij}(t)^{p-1} \Phi_{ij}'(t) \rangle d\mu_P(s) dt \\
&= \frac{p}{2} \int_0^m \int_s^m \langle \vec{\beta}_p \otimes \vec{c}_p + \vec{c}_p \otimes \vec{\beta}_p, \Phi_{ij}(t)^{p-1} \Phi_{ij}'(t) \rangle dt \, d\mu_P(s) \\
&= \frac{1}{2} \langle \vec{\beta}_p \otimes \vec{c}_p + \vec{c}_p \otimes \vec{\beta}_p, Q^{\circ p} - \int_0^m \Phi(s)^{\circ p} d\mu_P(s) \rangle \\
&= \langle \vec{\beta}_p \otimes \vec{c}_p, Q^{\circ p} - \int_0^m \Phi(s)^{\circ p} d\mu_P(s) \rangle. \tag{15}
\end{aligned}
$$

On the hand, integration by parts implies

$$
\mathbb{E}\left[\frac{\partial H_N(\vec{\sigma})}{\partial t} H_N(\vec{\sigma}) \right] = N \langle \vec{c}_p \otimes \vec{\beta}_p, R_{1,2} \rangle
$$

and

$$
\mathbb{E}\left\langle \frac{\partial H_N(\vec{\sigma})}{\partial t} \right\rangle_{\epsilon, Q} = N \langle \vec{c}_p \otimes \vec{\beta}_p, Q^{\circ p} - \mathbb{E}\langle R_{1,2}^{\circ p} \rangle_{\epsilon, Q} \rangle. \tag{16}
$$

Last, we note that the differential of $F_N^\epsilon(Q)$ at $\beta_p(i)$ is given by $\frac{1}{N} \mathbb{E}\langle \frac{\partial H_N(\vec{\sigma})}{\partial t} \rangle_{\epsilon, Q}$, so by (16)

$$
\frac{d}{dt} F_N^\epsilon(Q) = \langle \vec{c}_p \otimes \vec{\beta}_p, Q^{\circ p} - \mathbb{E}\langle R_{1,2}^{\circ p} \rangle_{\epsilon, Q} \rangle. \tag{17}
$$

Moreover, by Hölder's inequality, F_N^ϵ is convex at $\beta_p(i)$, and this combined with the fact that $\mathscr{C}$ is both convex and differentiable at $\beta_p(i)$, we get

$$
\lim_{N \to \infty} \lim_{\epsilon \to 0} \frac{\partial F_N^\epsilon}{\partial t} = \frac{\partial \mathscr{C}}{\partial t}.
$$

A combination of (11), (15), and (17) yields

$$
\lim_{N \to \infty} \lim_{\epsilon \to 0} \langle \vec{c}_p \otimes \vec{\beta}_p, Q^{\circ p} - \mathbb{E}\langle R_{1,2}^{\circ p} \rangle_{\epsilon, Q} \rangle = \langle \vec{\beta}_p \otimes \vec{c}_p, Q^{\circ p} - \int_0^m \Phi(s)^{\circ p} d\mu_P(s) \rangle
$$

which is equivalent to say that, for each $1 \leq i \leq m$,

$$
\langle \vec{\beta}_p \otimes \vec{c}_p, \lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle R_{1,2}^{\circ p} \rangle_{\epsilon, Q} - \int_0^m \Phi(s)^{\circ p} d\mu_P(s) \rangle = 0.
$$

As c_p is chosen arbitrarily, we get the following relation,

$$
\lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle R_{1,2}^{\circ p} \rangle_{\epsilon, Q} = \int_0^m \Phi(t)^{\circ p} d\mu_P(t).
$$

Since the even polynomials are dense on $C[0, 1]$, we get the desired conclusion. $\quad\square$

5. Proof of Theorem 3

This section covers the proof of Theorem 3. A little of preparation is needed and the proof requires two initial steps that we split in the following two subsections. The first subsection is devoted to construct the matrix Q_∞, the second subsection provides a characterization for points in the support of the minimizer μ_β and useful bounds for terms in the Crisanti–Sommers functional, the proof of Theorem 3 is presented in the third and last subsection.

5.1. Construction of the matrix Q_∞

We first modify the setting of Theorem 1 to achieve convergence of the minimizer for the ground state energy. Let $\mathcal{M}$ be the collection of all continuously differentiable Lipschitz monotone matrix path with Lipschitz derivatives that satisfies (6) without satisfying the assumption that $\mathrm{Trace}(\Phi(t)) = t$, but satisfying the assumption that for any $t \in \mathrm{supp}\ \mu$, $\mathrm{Trace}(\Phi(t)) = t$.

We first claim that any matrix path minimizer of $\mathscr{C}$ lies in $\mathcal{M}$. Moreover, we claim that any discrete path corresponds to a matrix path in $\mathcal{M}$, hence

$$\inf_{r,\underline{x},\underline{Q}} \mathscr{C}_r(\underline{x}, \underline{Q}) \geq \inf_{x,\Phi\in\mathcal{N}\times\mathcal{M}} \mathscr{C}_{\beta,Q}(x, \Phi). \tag{18}$$

We leave the proof of this claim to next section. Moreover, as

$$\inf_{r,\underline{x},\underline{Q}} \mathscr{C}_r(\underline{x}, \underline{Q}) \leq \inf_{x,\Phi\in\mathcal{N}_0\times\mathcal{M}_0} \mathscr{C}_{\beta,Q}(x, \Phi) \text{ and } \inf_{x,\Phi\in\mathcal{N}\times\mathcal{M}} \mathscr{C}_{\beta,Q}(x, \Phi)$$

$$\geq \inf_{x,\Phi\in\mathcal{N}_0\times\mathcal{M}_0} \mathscr{C}_{\beta,Q}(x, \Phi),$$

we obtain that $\inf_{r,\underline{x},\underline{Q}} \mathscr{C}_r(\underline{x}, \underline{Q}) \leq \inf_{x,\Phi\in\mathcal{N}\times\mathcal{M}} \mathscr{C}_{\beta,Q}(x, \Phi)$ and therefore

$$\inf_{r,\underline{x},\underline{Q}} \mathscr{C}_r(\underline{x}, \underline{Q}) = \inf_{x,\Phi\in\mathcal{N}\times\mathcal{M}} \mathscr{C}_{\beta,Q}(x, \Phi).$$

Since any path Φ in $\mathcal{M}$ is continuously differentiable and Φ' is Lipschitz and uniformly bounded, then by Arzela–Ascoli theorem, for any $\beta \geq 0$, the minimizer $\Phi_{\beta,Q}$ is also a continuously differentiable function with Lipschitz and uniformly bounded derivative. Similarly, for any $\beta > 0$, we get a subsequence $\{\beta_n\}_{n\geq 0}$ such that $\{\Phi_{\beta_n,Q}\}$ converges to a continuously differentiable path Φ_0. We will use this property from now on.

Fix $\beta > 0$. Let $(x_{\beta,Q}, \Phi_{\beta,Q})$ be an optimizer of (11) in $\mathcal{N}_0 \times \mathcal{M}_0$. The following lemma shows that there exists $Q_\beta \in \mathcal{M}$ such that $\mathscr{C}_{\beta,Q_\beta}(x_{\beta,Q_\beta}, \Phi_{\beta,Q_\beta}) = \sup_{Q\in\mathcal{M}} \mathscr{C}_{\beta,Q}(x_{\beta,Q}, \Phi_{\beta,Q})$. For $Q \in \mathcal{M}$, let

$$\mathscr{C}(Q) = \inf_{x,\Phi} \mathscr{C}_{\beta,Q}(x, \Phi) = \mathscr{C}_{\beta,Q}(x_{\beta,Q}, \Phi_{\beta,Q}).$$

Lemma 12. *For any $\beta > 0$, the map $Q \mapsto \mathscr{C}(Q)$ is continuous. Furthermore, there exists $Q_\beta \in \mathcal{M}$ such that*

$$\mathscr{C}(Q_\beta) = \sup_{Q\in\mathcal{M}} \mathscr{C}(Q).$$

Proof. We start by proving continuity of $\mathscr{C}$. It suffices to show that for any sequence $\{Q_n\} \in \mathcal{M}$ converging to $Q \in \mathcal{M}$, $\{\mathscr{C}(x_n, \Phi_n)\}$ converges to $\mathscr{C}(x, \Phi)$, where (x_n, Φ_n) and (x, Φ) are minimizers of (11) with constraints Q_n and Q, respectively.

Let $F = (F_{i,j})_{1 \le i,j \le m} : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ be a continuous and bounded function. Since $Q_n \to Q$ as $n \to \infty$, we obtain from (4),

$$\lim_{n \to \infty} \lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle F(R_{1,2}) \rangle_{\epsilon, Q_n} = \lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle F(R_{1,2}) \rangle_{\epsilon, Q}.$$

On the other hand, Theorem 2 implies

$$\lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle F(R_{1,2}) \rangle_{\epsilon, Q} = \int_0^m F \circ \Phi(t) d\mu(t)$$

and

$$\lim_{N \to \infty} \lim_{\epsilon \to 0} \mathbb{E}\langle F(R_{1,2}) \rangle_{\epsilon, Q_n} = \int_0^m F \circ \Phi_n(t) d\mu_n(t).$$

Combining the above displays we obtain

$$\lim_{n \to \infty} \int_0^m F \circ \Phi_n(t) d\mu_n(t) = \int_0^m F \circ \Phi(t) d\mu(t). \tag{19}$$

Now fix $t_0 \in \text{supp } \mu$ and set $A_0 := \Phi(t_0)$. Applying (19), with $F : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ given by

$$F(X) = \mathbb{1}_{\{X = A_0\}} := (\mathbb{1}_{\{X = A_0\}})_{1 \le i,j \le m},$$

we obtain that

$$\lim_{n \to \infty} \int_0^m \mathbb{1}_{\{\Phi_n(t) = \Phi(t_0)\}} d\mu_n(t) = (\mu(\{t_0\})). \tag{20}$$

Since $t_0 \in \text{supp } \mu$ we have $\text{Trace}(\Phi(t_0)) = t_0$. Thus the indicator function above is only non-zero in a subset of $\{t \in [0, m] : \text{Trace}(\Phi_n(t)) = t_0\}$. At the same time, for any $t \in \text{supp } \mu_n$, $\text{Trace}(\Phi_n(t)) = t$. These two observations, combined with (20), imply that for n sufficiently large, $\Phi_n(t_0) = \Phi(t_0)$ and $\mu_n(\{t_0\}) \to \mu(\{t_0\})$ as $n \to \infty$.

Similarly, for any $s_0 \notin \text{supp } \mu$, consider the function $G(X) = \mathbb{1}_{\{\text{trace } X = s_0\}} := (\mathbb{1}_{\{\text{trace } X = s_0\}})_{1 \le i,j \le m}$. Another application of (19) implies that

$$\lim_{n \to \infty} \int_0^m \mathbb{1}_{\{t = s_0\}} d\mu_n(t) = 0,$$

which leads to $\mu_n(\{s_0\}) \to 0$ as $n \to \infty$. Last, since $s_0 \notin \text{supp } \mu$, the value of $\Phi(s_0)$ will not affect the value of $\mathscr{C}(x, \Phi)$ (see Remark 1).

Looking back at (7), the facts that $\Phi_n(t) \to \Phi(t)$ for $t \in \text{supp } \mu$, and $\mu_n(\{s\}) \to \mu(\{s\})$ $\forall s$ imply $\mathscr{C}_{\beta, Q_n}(x_n, \Phi_n) \to \mathscr{C}_{\beta, Q}(x, \Phi)$ as $n \to \infty$ and thus continuity of $\mathscr{C}(Q)$ with respect to Q in $\mathcal{M}$. The second assertion in the lemma now follows from continuity of $\mathscr{C}(Q)$ and compactness of the space $\mathcal{M}$. $\quad\square$

Recall the definition of free energy with any constraint $Q \in \mathbb{S}_+^m$,

$$F_N^{\epsilon, Q}(\beta) = \frac{1}{N} \mathbb{E} \log \int_{\mathcal{Q}_N^\epsilon} \exp \beta \left(H_N(\vec{\sigma}) + \sum_{j=1}^m \vec{h}(j) \sum_{i=1}^N \vec{\sigma}_i(j) \right) d\lambda_N^m(\vec{\sigma}),$$

and denote the free energy with no constraint by

$$F_N(\beta) = \frac{1}{N} \mathbb{E} \log \int_{(S_N)^m} \exp \beta \left(H_N(\vec{\sigma}) + \sum_{j=1}^m \vec{h}(j) \sum_{i=1}^N \vec{\sigma}_i(j) \right) d\lambda_N^m(\vec{\sigma}).$$

The limiting free energy was obtained in [12]:

Theorem 13 (*[12], Theorem 1*). *For any $m \geq 1$, the limit of the free energy is given by*

$$\lim_{N \to \infty} F_N(\beta) = \sup_{Q \in \mathscr{M}} \inf_{x_Q, \Phi_Q, \Lambda_Q} \mathscr{P}(x_Q, \Phi_Q, \Lambda_Q, Q) = \sup_{Q \in \mathscr{M}} \inf_{x_Q, \Phi_Q} \mathscr{C}(x_Q, \Phi_Q, Q).$$

Let $\{Q_\beta\}_{\beta>0}$ be a sequence given by Lemma 12. Since $\{Q_\beta\}_{\beta>0}$ is bounded, there exists a subsequence $\{Q_{\beta_k}\}_{k \geq 0}$ and $Q_\infty \in \mathscr{M}$ such that $\{Q_{\beta_k}\}_{k \geq 0}$ converges to Q_∞ as $\beta_k \to \infty$. Without loss of generality, we will assume $\{Q_\beta\}$ converges to Q_∞. By Lemma 12, Theorems 1, and 13 we have

$$\lim_{N \to \infty} F_N(\beta) = \lim_{\epsilon \to 0} \lim_{N \to \infty} F_N^{\epsilon, Q_\beta}(\beta). \tag{21}$$

Moreover, since $Q_\beta \to Q_\infty$ as $\beta \to \infty$, then by dominated convergence theorem, we obtain

$$\lim_{\beta \to \infty} \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{\beta} F_N^{\epsilon, Q_\beta}(\beta) = \lim_{\beta \to \infty} \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{\beta} F_N^{\epsilon, Q_\infty}(\beta). \tag{22}$$

We now investigate the ground state energy

$$GSE := \lim_{N \to \infty} \max_{\vec{\sigma} \in (S_N)^m} \frac{H_N(\vec{\sigma})}{N}.$$

A standard computation (see [1, Section 5], for instance) implies that

$$GSE = \lim_{\beta \to \infty} \lim_{N \to \infty} \frac{1}{\beta} F_N(\beta) \text{ almost surely,}$$

and using (21) we obtain

$$GSE = \lim_{\beta \to \infty} \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{\beta} F_N^{\epsilon, Q_\beta}(\beta) \text{ almost surely.}$$

Combining with (22) we obtain

Proposition 14. *We have the following:*

$$GSE = \lim_{N \to \infty} \max_{R(\vec{\sigma}, \vec{\sigma}) \in Q_\infty} \frac{H_N(\vec{\sigma})}{N} = \lim_{\beta \to \infty} \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{\beta} F_N^{\epsilon, Q_\infty}(\beta).$$

5.2. Bounds on the minimizers and closed equations for points in the support of μ_β

For any matrix A and $\vec{x}_p$ satisfying the requirements in (1), we denote the corresponding Hamiltonian by

$$X_N^j(\vec{\sigma}) = \sum_{p \geq 2} \vec{x}_p(j) H_{N,p}(\vec{\sigma}(j)).$$

and the covariance by $\zeta(A) = \Sigma_{p \geq 2}(\vec{x}_p \otimes \vec{x}_p) \odot A^{\circ p}$.

Lemma 15. *There exists a constant C_ζ depending only on ζ such that for any $\beta > 0$,*

$$\beta x_{\beta,Q}(q) \leq \frac{C_\zeta}{Sum(\zeta(Q) - \zeta(\Phi_{\beta,Q}(q)))}, \forall q \in [0, m]. \tag{23}$$

Proof of Lemma 15. Note that for any $N \geq 1$, by Dudley's entropy integral,

$$\mathbb{E} \max_{\vec{\sigma} \in S_N^m} \frac{X_N(\vec{\sigma})}{N} \leq C_\zeta. \tag{24}$$

Here the constant $C_\zeta > 0$ depends only on ζ.

From Gaussian integration by parts, we obtain,

$$\beta(\zeta(Q) - \mathbb{E}\langle \zeta(R(\vec{\sigma}^1, \vec{\sigma}^2))\rangle_\beta) = \mathbb{E}\langle \frac{X_N(\vec{\sigma})}{N}\rangle_\beta \tag{25}$$

where $\langle \cdot \rangle_\beta$ is the Gibbs average with respect to the Gibbs measure $G_{N,\beta}(\sigma)$ defined by

$$G_{N,\beta}(\vec{\sigma}) = \frac{\exp \beta X_N(\vec{\sigma})}{Z_N(\beta)}.$$

From the differentiability of $\vec{\beta}_p$, we also have

$$\lim_{N \to \infty} \mathbb{E}\langle \zeta(R(\vec{\sigma}^1, \vec{\sigma}^2))\rangle_\beta = \int_0^m \zeta(\Phi_{\beta,Q}(s))x_{\beta,Q}(ds). \tag{26}$$

By (24)–(26), we then obtain,

$$\beta Sum(\zeta(Q) - \int_0^m \zeta(\Phi_{\beta,Q}(s))x_{\beta,Q}(ds)) = \mathbb{E}\langle \frac{X_N(\vec{\sigma})}{N}\rangle_\beta \leq \mathbb{E} \max_{\sigma \in S_N} \frac{X_N(\sigma)}{N} \leq C_\zeta.$$

Finally combining with the following two inequalities which can be derived from integration by parts,

$$\int_0^m \beta x_{\beta,Q}(s)\zeta(\Phi_{\beta,Q}(s))' \odot \Phi_{\beta,Q}'(s)ds = \beta(\zeta(Q) - \int_0^m \zeta(\Phi_{\beta,Q}(s))x_{\beta,Q}(ds))$$

and

$$\int_q^m \beta x_{\beta,Q}(s)\zeta(\Phi(s))' \odot \Phi_{\beta,Q}'(s)ds \geq \beta x_{\beta,Q}(q)(\zeta(Q) - \zeta(\Phi_{\beta,Q}(q))), \forall q \in [0, m]$$

we then obtain that,

$$\beta x_{\beta,Q}(q)Sum(\zeta(Q) - \zeta(\Phi_{\beta,Q}(q))) \leq C_\zeta, \forall q \in [0, m],$$

as desired. $\square$

Lemma 16. *There exist a constant $C_\xi' > 0$ and a positive semidefinite matrix A_ξ depending only on ξ such that*

$$\limsup_{\beta \to \infty} \beta(m - q_\beta) \leq C_\xi'$$

and

$$\limsup_{\beta \to \infty} \int_0^m \beta x_{\beta,Q}(s)\Phi_{i,j}'(s)ds \leq A_\xi.$$

Proof of Lemma 16. From Lemma 15, we have

$$\beta = \beta x_{\beta,Q}(q_\beta) \leq \frac{C_\zeta}{Sum(\zeta(Q) - \zeta(\Phi_{\beta,Q}(q_\beta)))}. \tag{27}$$

Thus the denominator on the right-side of (27) must go to 0 and we obtain

$$\lim_{\beta \to \infty} q_\beta = m. \tag{28}$$

On the other hand, as $Sum(\zeta)$ is non-decreasing, by the mean value theorem,

$$\beta Sum(\zeta(\Phi_{\beta,Q}(q_\beta))' \odot (Q - \Phi_{\beta,Q}(q_\beta))) \leq \beta Sum(\zeta(Q) - \zeta(\Phi_{\beta,Q}(q_\beta))) \leq C_\xi.$$

Consequently,

$$Sum(\zeta'(Q) \odot \limsup_{\beta \to \infty} \beta(Q - \Phi_{\beta,Q}(q_\beta))) \leq C_\zeta.$$

By the arbitrariness of ζ, the inequality above implies that $\limsup_{\beta \to \infty} \beta(Q - \Phi_{\beta,Q}(q_\beta))$ is a bounded matrix. Moreover, by integration by parts, we obtain,

$$C_\zeta \geq \int_0^m \beta x_{\beta,Q}(s)\zeta'(\Phi_{\beta,Q}(s)) \odot \Phi'_{\beta,Q}(s)ds \geq \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s)\zeta'(\Phi_{\beta,Q}(s)) \odot \Phi'_{\beta,Q}(s)ds$$

$$\geq \zeta'(\Phi_{\beta,Q}(\frac{m}{2})) \odot \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds.$$

Similarly, by the arbitrariness of ζ, the inequality above implies that $\limsup_{\beta \to \infty} \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s) \Phi'_{\beta,Q}(s)ds$ is a bounded matrix.

Finally,

$$\int_0^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds = \int_0^{\frac{m}{2}} \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds + \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds$$

$$\leq \int_0^{\frac{m}{2}} \frac{C_\xi}{Sum(\xi(Q) - \xi(\Phi_{\beta,Q}(q)))}dq + \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds$$

$$\leq \frac{m}{2} \frac{C_\xi}{Sum(\xi(Q) - \xi(\Phi_{\beta,Q}(\frac{m}{2})))} + \int_{\frac{m}{2}}^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds.$$

As $\lim_{\beta \to \infty} \Phi_{\beta,Q}(t) = \Phi_0(t)$ converges uniformly, we obtain that for β sufficiently large, $\int_0^m \beta x_{\beta,Q}(s)\Phi'_{\beta,Q}(s)ds$ is a uniformly bounded matrix. □

Combining the lemmas above, we can use the Helly's selection theorem combined with a diagonalization process to guarantee the vague convergence of $(\beta_n x_{\beta_n,Q})_{n \geq 1}$. As for $\Phi_0(t)$ satisfying (5), we can guarantee the existence of the following limit:

$$\lim_{k \to \infty} \beta_{n_k} x_{\beta_{n_k},Q} \in \mathcal{N} \text{ vaguely on } [0, m)$$

$$\lim_{k \to \infty} \beta_{n_k}(Q - \Phi_{\beta_{n_k},Q}(q_{\beta_{n_k}})),$$

$$\lim_{k \to \infty} \int_0^m \beta_{n_k} x_{\beta_{n_k},Q}(s) \Phi'_{\beta_{n_k},Q}(s)ds.$$

Without loss of generality, we can assume all these convergences hold for the sequence $(\beta_n)_{n \geq 1}$ and denote

$$\alpha_0 := \lim_{n \to \infty} \beta_n x_{\beta_n,Q} \in \mathcal{N} \text{ vaguely on } [0, m)$$

$$\Omega_0 := \lim_{n \to \infty} \beta_n (Q - \Phi_{\beta_n, Q}(q_{\beta_n})), \tag{29}$$

$$L_0 := \lim_{n \to \infty} \int_0^m \beta_n x_{\beta_n, Q}(s) \Phi'_{\beta_n, Q}(s) ds.$$

Lemma 17. *Let $(x_{\beta, Q}, \Lambda_{\beta, Q}, \Phi_{\beta, Q})$ be a minimizer of* (8) *and $\mu_\beta([0, t]) = x_{\beta, Q}(t)$. For any q in the support of μ_β, we have that*

$$\Phi_{\beta, Q}(q) = (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(0))^{-1} \vec{h} \vec{h}^T (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(0))^{-1}$$

$$+ \int_0^q (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(s))^{-1} (\xi''_\beta(\Phi_{\beta, Q}(s)) \odot \Phi'_{\beta, Q}(s)) (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(s))^{-1} ds \tag{30}$$

where $D_\beta^{x_{\beta, Q}}(q) = \int_q^m x(s) \xi''_\beta(\Phi(s)) \odot \Phi'_{\beta, Q}(s) ds$.

Proof of Lemma 17. Consider arbitrarily probability measure μ with distribution function y, and set $z(t) := (1 - \epsilon) x_{\beta, Q}(t) + \epsilon y(t)$. Then $y(t) - x_{\beta, Q}(t) = \int_0^t d(\mu - \mu_\beta)(s)$. By a straight-forward computation, we obtain that,

$$\partial_\epsilon \mathscr{P}_\beta(\Lambda_{\beta, Q}, z, \Phi_{\beta, Q})\Big|_{\epsilon=0} = \int_0^{\hat{T}} (y(t) - x_{\beta, Q}(t)) \langle \Gamma(t), \xi''(\Phi_{\beta, Q}(t)) \odot \Phi'_{\beta, Q}(t) \rangle dt \geq 0,$$

where

$$\Gamma(t) = (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(0))^{-1} \vec{h} \vec{h}^T (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(0))^{-1}$$

$$+ \int_0^t (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(s))^{-1} (\xi''_\beta(\Phi_{\beta, Q}(s)) \odot \Phi'_{\beta, Q}(s))$$

$$\times (\Lambda_{\beta, Q} - D_\beta^{x_{\beta, Q}}(s))^{-1} ds - \Phi_{\beta, Q}(t).$$

By Fubini's theorem,

$$\int_0^m \int_s^m \langle \Gamma(t), \xi''(\Phi_{\beta, Q}(t)) \odot \Phi'_{\beta, Q}(t) \rangle dt d\mu(s)$$

$$\geq \int_0^m \int_s^m \langle \Gamma(t), \xi''(\Phi_{\beta, Q}(t)) \odot \Phi'_{\beta, Q}(t) \rangle dt d\mu_\beta(s)$$

which implies that

$$\int_0^m \bar{\Gamma}(s) d\mu(s) \geq \int_0^m \bar{\Gamma}(s) d\mu_\beta(s)$$

where $\bar{\Gamma}(s) := \int_s^m \langle \Gamma(t), \xi''(\Phi_{\beta, Q}(t)) \odot \Phi'_{\beta, Q}(t) \rangle ds$. Since the inequality holds for all μ, this is equivalent to say that

$$\bar{\Gamma}(t) \geq \int_0^m \bar{\Gamma}(s) d\mu_\beta(s)$$

for all $t \in [0, m]$ and equality holds for every point in supp μ_β. Note that if $s \in$ supp $\mu_\beta \cap (0, m)$, we obtain

$$\frac{d}{dt} \bar{\Gamma}(t) = -\langle \Gamma(t), \xi''(\Phi_{\beta, Q}(t)) \odot \Phi'_{\beta, Q}(t) \rangle = 0.$$

If t is an isolated point in supp μ_β, then $\Phi'(t)$ can be any positive semidefinite matrix without changing the value of $\mathscr{C}$, which implies that $\Gamma(t) = 0$ and thus (30) holds. Moreover,

if t is not an isolated point, then as $\Phi_{\beta,Q}(t)$ is a continuously differentiable matrix path, we can derive the same conclusion that $\Gamma(t) = 0$ by continuity.   □

Lemma 18.   *For any q in the support of μ_β, we have that*

$$\Lambda_\beta - D_\beta^{x_\beta}(q) = \left(\int_q^m x_{\beta,Q}(s) \Phi_{\beta,Q}'(s) ds \right)^{-1}. \tag{31}$$

Proof of Lemma 18.   Recall the r step discretization of the Parisi functional given by (10). For $r \geq 1$, denote by $\mathscr{M}_r$ the space of all step functions $x \in \mathscr{N}$ with at most r jumps and by $\mathscr{M}_r'$ the space of all (x, Λ, Q) with $x \in \mathscr{M}_r$ and $\Lambda \in \mathbb{S}_+^m$ satisfying $\Lambda > \int_0^m \beta x(s) \xi_\beta''(\Phi(s)) \odot \Phi'(s) ds$.

Let (x_r, Λ^r, Q_r) be the minimizer of $\mathscr{P}_\beta$ restricted to $\mathscr{M}_r$. Based on the critical point of the Parisi formula, i.e., for the minimizer of the r step discretization, $\partial_{Q_p} \mathscr{P}_r = 0$ and $\partial_{Q_p} \mathscr{C}_r = 0$, we obtain the following equation:

$$\Lambda_1^{-1}(\vec{h}\vec{h}^T + \xi'(Q_1))\Lambda_1^{-1} + \Sigma_{1 \leq k \leq p-1} \frac{1}{x_k}(\Lambda_k^{-1} - \Lambda_{k+1}^{-1}) = Q_p$$

$$-\vec{h}\vec{h}^T + D_1^{-1}Q_1 D_1^{-1} + \sum_{1 \leq k \leq p-1} \frac{1}{x_k}(D_{k+1}^{-1} - D_k^{-1}) = \xi'(Q_p) \text{ for } 2 \leq p \leq r-1 \tag{32}$$

and for $p = 1$,

$$\Lambda_1^{-1}(\vec{h}\vec{h}^T + \xi'(Q_1))\Lambda_1^{-1} = Q_1, \quad \xi'(Q_1) = -\vec{h}\vec{h}^T + D_1^{-1}Q_1 D_1^{-1}.$$

Moreover, based on the extremality over Λ^k, we obtain

$$-\Lambda_1^{-1}(\vec{h}\vec{h}^T + \xi'(Q_1))\Lambda_1^{-1} + Q - (\Lambda^k)^{-1} + \Sigma_{1 \leq p \leq r-1} \frac{1}{x_p}(\Lambda_{p+1}^{-1} - \Lambda_p^{-1}) = 0.$$

Combining with 5.2 for $p = r - 1$ and set $D_{\beta,r}^{x,p} = \Sigma_{p \leq k \leq r-1} x_k(\xi'(Q_{k+1}) - \xi'(Q_k))$ to be the discrete form of $D_{\beta,Q}^x(q)$ in r-step discretization, then we get the following relation:

$$\Lambda_{r-1} = \Lambda - D_{\beta,r}^{x,r-1} = (Q - Q_{r-1})^{-1}.$$

Relation (32) also implies that $x_k(\xi'(Q_{k+1}) - \xi'(Q_k)) = D_{k+1}^{-1} - D_k^{-1}$ for $1 \leq k < r - 1$. Hence based on the equations above, we get

$$\Lambda_{p+1} - \Lambda_p = x_p(\xi_\beta'(Q_{p+1}) - \xi_\beta'(Q_p))$$
$$= D_{p+1}^{-1} - D_p^{-1}$$

Since $D_{r-1} = Q - Q_{r-1} = \Lambda_{r-1}^{-1}$, we then get $\Lambda_p = D_p^{-1}$ for $1 \leq p \leq r - 1$.

Then (x_r, Λ^r, Q_r) satisfies $\Lambda^n - \Lambda(q) = (\int_q^m x_r(s)\Phi'(s) ds)^{-1}$ for all q in the support of the probability measure μ_r induced by x_r. By the uniqueness of the minimizer (x_r, Λ^r), we may pass to a subsequence of (x_r, Λ^r) such that its limit equals $(x_\beta, \Lambda_\beta, Q_\beta)$. Thus we have the desired conclusion.   □

Recall the constants defined in (29).

Lemma 19.   *We have $0 < \Omega_0 < \infty$ and for any Φ satisfying (5), $L_0 > \int_0^m \alpha_0(s)\Phi_0'(s) ds$.*

Proof of Lemma 19. Assume by contradiction that $\Omega_0 = 0$. We first notice that Eq. (31) for $q = q_{\beta_n}$ reads as

$$\Lambda_{\beta_n} = \xi'(Q) - \xi'(\Phi_{\beta_n}(q_{\beta_n})) + (Q - \Phi_{\beta_n}(q_{\beta_n}))^{-1},$$

hence

$$\lim_{n \to \infty} \frac{\Lambda_{\beta_n}}{\beta_n} = \Omega_0^{-1} = \infty.$$

Combining with the fact that $D_{\beta_n}^{x_{\beta_n}}(q)$ is bounded for any $q \in [0, m]$, we obtain $[\beta_n^{-1}(\Lambda_{\beta_n} - D_{\beta_n}^{x_{\beta_n}}(q))]^{-1}$ converges to zero uniformly on $[0, m]$.

By (30), we can reach a contradiction as follows,

$$
\begin{aligned}
Q &= \lim_{n \to \infty} \Phi_{\beta_n}(q_{\beta_n}) \\
&= \lim_{n \to \infty} [\beta_n^{-1}(\Lambda_{\beta_n} - D_{\beta_n}^{x_{\beta_n}}(0))]^{-1} \vec{h}\vec{h}^T [\beta_n^{-1}(\Lambda_{\beta_n} - D_{\beta_n}^{x_{\beta_n}}(0))]^{-1} \\
&\quad + \int_0^{q_{\beta_n}} [\beta_n^{-1}(\Lambda_{\beta_n} - D_{\beta_n}^{x_{\beta_n}}(s))]^{-1} (\xi_{\beta_n}''(\Phi_{\beta_n,Q}(s)) \odot \Phi_{\beta_n,Q}'(s)) \\
&\quad \times [\beta_n^{-1}(\Lambda_{\beta_n} - D_{\beta_n}^{x_{\beta_n}}(s))]^{-1} ds.
\end{aligned}
$$

In conclusion, $0 < \Omega_0$. A similar argument shows that $\Omega_0 < \infty$.

Next, note that from (28),

$$\lim_{k \to \infty} q_{\beta_k} = m.$$

Therefore, for any fixed $q \in (0, m)$

$$
\begin{aligned}
\int_0^m \beta_n x_{\beta_n,Q}(s) \Phi_{\beta_n,Q}'(s) ds &= \int_0^{q_{\beta_n}} \beta_n x_{\beta_n,Q}(s) \Phi_{\beta_n,Q}'(s) ds + \int_{q_{\beta_n}}^m x_{\beta_n,Q}(s) \Phi_{\beta_n,Q}'(s) ds \\
&\geq \int_0^{q_{\beta_n}} \beta_n x_{\beta_n}(s) \Phi_0'(s) ds + \beta_n(Q - \Phi_{\beta_n}(q_{\beta_n}))
\end{aligned}
$$

Then by the dominated convergence theorem, we obtain

$$L_0 \geq \int_0^q \alpha_0(s) \Phi_0'(s) ds + \Omega_0.$$

Since this holds for all $q \in (0, m)$, by letting q tend to m, we obtain,

$$L_0 \geq \int_0^m \alpha_0(s) \Phi_0'(s) ds + \Omega_0.$$

As $\Omega_0 > 0$, we get the desired conclusion. $\quad\square$

5.3. Proof of Theorem 3

We are finally ready to prove Theorem 3.

We rewrite the Crisanti–Sommers functional (7) as follows:

$$
\begin{aligned}
\mathscr{C}_{\beta,Q}(x, \Phi) = \frac{1}{2}\Bigg[&\langle \xi_\beta'(\Phi(m)) + \vec{h}_\beta \vec{h}_\beta^T, \check{x}(m)\rangle - \int_0^m \langle \check{\Phi}(t), \xi_\beta''(\Phi(t)) \odot \Phi'(t)\rangle dt \\
&+ \int_0^{t_x} \langle (\check{\Phi}(m) - \check{\Phi}(t))^{-1}, \Phi'(t)\rangle dt + \log|\Phi(m) - \Phi(t_x)| \Bigg]
\end{aligned}
$$

where $\check{\Phi}(t) := \int_0^t x(s)\Phi'(s) ds$.

Proof of Theorem 3. We will consider that case $Q = Q_\infty$ as the general case is similar. We set (x_β, Φ_β) as the minimizer under the constraint $Q = Q_\infty$. Now we start with the lower bound.

For any fixed Φ satisfying (5), based on the definition of L_0 and α_0, we obtain that

$$
L_0 = \lim_{\beta \to \infty} \int_0^m \beta_n x_{\beta_n}(s) \Phi'_{\beta_n}(s) ds = \lim_{n \to \infty} \beta_n \check{\Phi}_{\beta_n}(m)
$$

$$
\lim_{n \to \infty} \beta_n \check{\Phi}_{\beta_n}(q) = \int_0^q \alpha_0(s) \Phi'_0(s) ds, q \in [0, m).
$$

In addition, from Lemma 19,

$$
\lim_{n \to \infty} \frac{\log |Q - \Phi_{\beta_n}(q_{\beta_n})|}{\beta_n} = \lim_{n \to \infty} \frac{\log |\beta_n(Q - \Phi_{\beta_n}(q_{\beta_n}))|}{\beta_n} - m \frac{\log \beta_n}{\beta_n} = 0.
$$

Hence by applying Fatou's lemma and the bounded convergence theorem, we obtain

$$
\begin{aligned}
GSE &= \lim_{n \to \infty} \frac{\mathscr{C}_{\beta_n}(x_{\beta_n}, \Phi_{\beta_n})}{\beta_n} \\
&\geq \frac{1}{2} \Bigg[\langle \xi'(Q) + \vec{h}\vec{h}^T, \lim_{n \to \infty} \beta_n \check{\Phi}_{\beta_n}(m) \rangle \\
&\quad - \int_0^m \langle \lim_{n \to \infty} \xi''(\Phi_{\beta_n}(t)) \odot \Phi'_{\beta_n}(t), \lim_{n \to \infty} \beta_n \check{\Phi}_{\beta_n}(t) \rangle dt \\
&\quad + \int_0^m \lim_{n \to \infty} \mathbb{1}_{[0,q_{\beta_n}]} \langle (\beta_n(\check{\Phi}_{\beta_n}(m) - \check{\Phi}_{\beta_n}(t)))^{-1}, \Phi'_{\beta_n}(t) \rangle dt \\
&\quad + \lim_{n \to \infty} \frac{\log |Q - \Phi_{\beta_n}(q_{\beta_n})|}{\beta_n} \Bigg] \\
&= \mathscr{C}(L_0, \alpha_0, \Phi_0)
\end{aligned}
$$

which implies that

$$
GSE \geq \inf_{(L,\alpha,\Phi) \in \mathscr{H}(Q)} \mathscr{C}(L, \alpha, \Phi).
$$

We now move the matching upper bound for GSE. Consider any $(L, \alpha, \Phi) \in \mathscr{H}(Q)$. As $\alpha \in \mathscr{N}_0$, there exists $c > 0$ such that $\alpha(t)$ is constant for all $t \in [m - c, m]$. Then for any $[m-c, m]$, we can set $\Phi(t) = \frac{t-m+c}{c} Q + \frac{m-t}{c} \Phi(m-c)$ without changing the value of $\mathscr{C}(L, \alpha, \Phi)$.

As $Q - \Phi(m - c)$ is positive definite, then $\Phi(m - \frac{c}{2}) = \frac{1}{2}(Q + \Phi(m-c))$ is positive definite. There exists ϵ sufficiently small such that $\frac{1}{2}(Q - \Phi(m - c)) - \epsilon \Omega_1$ is still positive definite, where $\Omega = L - \int_0^m \alpha(s) \Phi'(s) ds > 0$ and $\Omega_1 = \frac{\Omega}{\text{Trace}(\Omega)}$.

Take $v_\beta = m - \frac{1}{\beta} \text{Trace}(\Omega)$ and without loss of generality we can assume $v_\beta > m - \epsilon$ by choosing β sufficiently large. Consider $y_\beta \in \mathscr{M}$ defined by $y_\beta(s) = 1$ on $[v_\beta, m]$ and $y_\beta(s) = \min(\alpha(s)/\beta, 1)$ on $[0, v_\beta)$.

Next, for each v_β, we customize Φ by Φ_β as follows,

$$
\Phi_\beta(t) = \begin{cases}
\Phi(t) & \text{for } t \in [0, m - c], \\
Q - (m - v_\beta) \cdot \Omega_1 & \text{for } t = v_\beta, \\
\frac{v_\beta - t}{v_\beta - m + c} \Phi(m - c) + \frac{t - m + c}{v_\beta - m + c} \Phi(v_\beta) & \text{for } t \in [m - c, v_\beta] \\
Q - (m - t) \cdot \Omega_1 & \text{for } t \in [v_\beta, m]
\end{cases}
$$

Notice that $\lim_{\beta \to \infty} \Phi_\beta(t) = \Phi(t)$ and $\lim_{\beta \to \infty} \Phi'_\beta(t) = \Phi'(t)$ for $t \in [0, m]$.

Based on the definition y_β, we notice that (βy_β) converges vaguely to α on $[0, m)$ with

$$\int_0^m \beta y_\beta(s) \Phi'_\beta(s) ds = \beta(\Phi_\beta(m) - \Phi_\beta(v_\beta)) + \int_0^{v_\beta} \min(\alpha(s), \beta) \Phi'_\beta(s) ds \to \Omega$$

$$+ \int_0^m \alpha(s) \Phi'(s) ds = L.$$

We claim that

$$\lim_{\beta \to \infty} \frac{\mathscr{C}_{\beta, Q}(y_\beta, \Phi_\beta)}{\beta} = \mathscr{C}(L, \alpha, \Phi). \tag{33}$$

If (33) is valid, from the Crisanti–Sommers formula, we have

$$GSE = \lim_{\beta \to \infty} \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{F_N^{\epsilon, Q_\infty}(\beta)}{\beta} = \lim_{\beta \to \infty} \inf_{x, \Phi} \frac{\mathscr{C}_{\beta, Q_\infty}(x, \Phi)}{\beta}$$

$$\leq \lim_{\beta \to \infty} \frac{\mathscr{C}_{\beta, Q_\infty}(y_\beta, \Phi_\beta)}{\beta} = \mathscr{C}(L, \alpha, \Phi).$$

Since this is true for any α, Lemma 19 implies

$$GSE \leq \inf_{(L, \alpha, \Phi) \in \mathscr{K}(Q)} \mathscr{C}(L, \alpha, \Phi).$$

Now it suffices for us to prove the claim (33).

Firstly,

$$\frac{\log|\Phi_\beta(m) - \Phi_\beta(v_\beta)|}{\beta} = \frac{\log|\beta(\Phi_\beta(m) - \Phi_\beta(v_\beta))|}{\beta} - m \frac{\log \beta}{\beta} \to 0.$$

Secondly,

$$\frac{1}{\beta} \langle \xi'_\beta(Q) + \vec{h}_\beta \vec{h}_\beta^T, \Phi_\beta(m) \rangle = \langle \xi'(Q) + \vec{h} \vec{h}^T, \int_0^m \beta y_\beta(t) \Phi'_\beta(t) dt \rangle$$

$$\to \langle \xi'(Q) + \vec{h} \vec{h}^T, L \rangle.$$

Thirdly, since $\beta \check{\Phi}_\beta(q) \leq \int_0^m \beta y_\beta(s) \Phi'_\beta(s) ds$ for $q \in [0, m]$, by bounded convergence theorem and Lemma 19,

$$\frac{1}{\beta} \int_0^m \langle \xi''(\Phi_\beta(t)) \odot \Phi'_\beta(t), \check{\Phi}_\beta(t) \rangle dt = \int_0^m \langle \xi''(\Phi_\beta(t)) \odot \Phi'_\beta(t), \int_0^t \beta y_\beta(s) \Phi'_\beta(s) ds \rangle dt$$

$$\to \int_0^m \langle \xi''(\Phi(t)) \odot \Phi'(t),$$

$$\int_0^t \alpha(s) \Phi'(s) ds \rangle dt, \text{ as } \beta \to \infty.$$

Finally,

$$\left| \int_0^{v_\beta} \langle [\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t))]^{-1}, \Phi'_\beta(t) \rangle dt - \int_0^{v_\beta} \langle (L - \int_0^t \alpha(s) \Phi'(s) ds)^{-1}, \Phi'(t) \rangle dt \right|$$

$$\leq \left| \int_0^{v_\beta} \langle [\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t))]^{-1} - (L - \int_0^t \alpha(s) \Phi'(s) ds)^{-1}, \Phi'_\beta(t) \rangle dt \right|$$

$$- \left| \int_0^{v_\beta} \langle (L - \int_0^t \alpha(s) \Phi'(s) ds)^{-1}, \Phi'(t) - \Phi'_\beta(t) \rangle dt \right|.$$

As $L - \int_0^t \alpha(s)\Phi'(s)ds > \Omega$ and therefore $(L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1} > \Omega^{-1}$, we obtain that the second term in the inequality above tends to 0 as $\beta \to \infty$.

Now it remains to deal with the first term. Since for $t \in [0, m]$,

$$\lim_{\beta \to \infty} \beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t)) = L - \int_0^t \alpha(s)\Phi'(s)ds,$$

for $\epsilon > 0$ sufficiently small, there exists β_0 large enough, such that whenever $\beta > \beta_0$,

$$\|(\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t)))^{-1} - (L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}\|_\infty < \epsilon.$$

By the definition of Ω, we also get $L - \int_0^t \alpha(s)\Phi'(s)ds > \Omega$ and then $(L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1} > \Omega^{-1}$, for $t \in [0, m)$. Moreover, by the setting of Φ_β and Φ, they are uniformly bounded, *i.e.* there exists $M > 0$ such that for all $t \in [0, m]$, $\|\Phi(t)\|_\infty < M$.

Based on the relations above, we obtain

$$M_\beta := \left| \int_0^{v_\beta} \langle [\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t))]^{-1} - (L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}, \Phi'_\beta(t)\rangle dt \right|$$

$$\leq \left| \int_0^m \langle \mathbb{1}_{[0,v_\beta]}(t)[\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t))]^{-1}(L - \beta\check{\Phi}_\beta(m)) \right.$$

$$\left. \times (L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}, \Phi'_\beta(t)\rangle dt \right|$$

$$+ \left| \int_0^m \langle \mathbb{1}_{[0,v_\beta]}(t)[\beta(\check{\Phi}_\beta(m) - \check{\Phi}_\beta(t))]^{-1} \right.$$

$$\left. \times (\int_0^t \alpha(s)\Phi'(s)ds - \beta\check{\Phi}_\beta(t))(L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}, \Phi'_\beta(t)\rangle dt \right|$$

Then by dominated convergence theorem, we get that $M_\beta \to 0$ as $\beta \to \infty$. Theorem 3 immediately follows. $\square$

We end this section with the proof of (18).

Proof of (18). It suffices to show that each discrete path corresponds to a matrix path in $\mathscr{M}$. Consider a discrete monotone matrix path encoded by an increasing sequence of real numbers and a monotone sequence of $n \times n$ symmetric positive semidefinite matrices,

$$0 = x_0 \leq x_1 \leq \cdots \leq x_{r-2} \leq x_{r-1} \leq 1,$$
$$0 = Q_0 \leq Q_1 \leq \cdots \leq Q_{r-2} \leq Q_{r-1} \leq Q_r = Q,$$

where $r \geq 1$. As before, we denote $\underline{x} = (x_k)_{k=0}^{r-1}$ and $\underline{Q} = (Q_k)_{k=1}^r$.

Taking $t_k := \text{trace}\,(Q_k)$ we define a Lipschitz path Φ by taking $\Phi(t_k) = Q_k$ at each point t_k and interpolate by sine functions:

$$\Phi(t_k) = Q_k,$$
$$\Phi(t) = \frac{1}{2}[\Phi(t_k) + \Phi(t_{k+1})] + \frac{1}{2}[\Phi(t_{k+1}) - \Phi(t_k)]$$
$$\cdot \sin\left(\pi \frac{2t - t_k - t_{k+1}}{2(t_{k+1} - t_k)}\right) \text{ for } t_k \leq t \leq t_{k+1}.$$

Thus,

$$\Phi'(t) = \frac{\pi}{2} \cdot \frac{\Phi(t_{k+1}) - \Phi(t_k)}{t_{k+1} - t_k} \cdot \cos\left(\pi \frac{2t - t_k - t_{k+1}}{2(t_{k+1} - t_k)}\right)$$

and we set $x(t) = x_k$ for $t_k \leq t < t_{k+1}$.

It is not difficult to show from (9), that the Crisanti–Sommers functionals agree. From the construction above, we obtain that the desired Φ lies in $\mathcal{M}$, which means Φ is a continuously differentiable Lipschitz monotone matrix path with Lipschitz derivatives that satisfies (6) satisfying the assumption that for any $t \in \text{supp } \mu$, Trace $\Phi(t) = t$. □

6. Proofs of Propositions 5–11

In this section, without confusion of notations, we absorb the β into the model ξ in this section. Our starting point to prove all propositions is a critical point equation for points in the support of the Parisi measure.

Lemma 20. *Let (x, Φ) be a minimizer of* (7). *For $t \in \text{supp } \mu_\beta$, the following equation holds:*

$$\xi'(\Phi(t)) + \vec{h}\vec{h}^T = \int_0^t \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds.$$

Proof of Lemma 20. Let $a(t)$ be any non-zero continuous function on $[0, m]$ satisfying $0 \leq t + a(t) \leq m$ for all $t \in [0, m]$ and $|a(t) - a(t')| \leq |t - t'|$ for all $t, t' \in [0, m]$. Assume that $a(0) = 0$ and $a(t) = 0$, for all $t \in [\hat{T}, m]$.

Set $\Theta(\epsilon, t) = (1 - \epsilon)\Phi(t + a(t)) + \epsilon \Psi(t + a(t))$, where Ψ is an arbitrary matrix path in $\mathcal{M}$. Since (x, Φ) is a minimizer of the Crisanti–Sommers functional

$$0 \leq \partial_\epsilon \mathcal{C}(x, \Theta(\epsilon, t))\Big|_{\epsilon=0} \tag{34}$$

$$= \int_0^{\hat{T}} \left\langle \int_0^t \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds - \vec{h}\vec{h}^T - \xi'(\Phi(t)), \Psi(t) - \Phi(t) + a(t)\Phi'(t) \right\rangle \mu_P(dt).$$

Assume that the equality above holds (we provide the details of the computation of the derivative below). We now claim that (34) implies

$$\left\langle \int_0^t \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds - \vec{h}\vec{h}^T - \xi'(\Phi(t)), \Psi(t) - \Phi(t) \right\rangle \tag{35}$$

must vanish for all $t \neq 0$ and $t \in \text{supp } \mu_P$. Indeed, if (35) does not vanish, we can always modify $\Psi(t) - \Phi(t)$ and $a(t)$ so that (34) is negative.

Now, we turn to the proof of (34). By setting $\Theta(\epsilon, t) := (1 - \epsilon)\Phi(t + \epsilon a(t)) + \epsilon \Psi(t + \epsilon a(t))$, we start by calculating the differential of $\mathcal{C}$ at $\epsilon = 0$:

$$2\frac{d}{d\epsilon}\mathcal{C}(x, \Theta)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\Bigg[\int_0^n x(t)\langle \xi'(\Theta(t)) + \vec{h}\vec{h}^T, \Theta'(t)\rangle dt$$

$$+ \log|\Theta(n) - \Theta(\hat{T})| + \int_0^{\hat{T}} \langle \hat{\Theta}(t)^{-1}, \Theta'(t)\rangle dt \Bigg]\Bigg|_{\epsilon=0}$$

$$:= I + II + III. \tag{36}$$

Next, we notice that

$$\frac{d}{d\epsilon} \Theta(t) \Big|_{\epsilon=0} = \Psi(t) - \Phi(t) + a(t)\Phi'(t) \tag{37}$$

and

$$\frac{d}{d\epsilon} \Theta'(t) \Big|_{\epsilon=0} = (\Psi(t) - \Phi(t) + a(t)\Phi'(t))'. \tag{38}$$

Thus, combining (36), (37), and (38), we obtain

$$I = \int_0^{\hat{T}} x(t)\langle \xi''(\Phi(t)) \odot \Phi'(t), \Psi(t) - \Phi(t) + a(t)\Phi'(t)\rangle dt$$
$$+ \int_0^{\hat{T}} x(t)\langle \xi'(\Phi(t)) + \vec{h}\vec{h}^T, (\Psi(t) - \Phi(t) + a(t)\Phi'(t))'\rangle dt, \tag{39}$$

and

$$II = \left\langle (\Phi(m) - \Phi(\hat{T}))^{-1}, \Psi(m) - \Phi(m) + a(m)\Phi'(m)\right\rangle$$
$$- \left\langle (\Phi(m) - \Phi(\hat{T}))^{-1}, \Psi(\hat{T}) - \Phi(\hat{T}) + a(\hat{T})\Phi'(\hat{T})\right\rangle = 0, \tag{40}$$

and, by integration by parts,

$$III = -\int_0^{\hat{T}} \int_t^m x(s)\left\langle \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}, (\Psi(s) - \Phi(s) + a(s)\Phi'(s))'\right\rangle ds\, dt$$
$$+ \int_0^{\hat{T}} \left\langle \hat{\Phi}(t)^{-1}, (\Psi(t) - \Phi(t) + a(t)\Phi'(t))'\right\rangle dt$$
$$= -\int_0^{\hat{T}} x(t)\left\langle (\Psi(t) - \Phi(t) + a(t)\Phi'(t))', \int_0^t \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}ds\right\rangle dt$$
$$- \int_0^{\hat{T}} x(t)\left\langle \Psi(t) - \Phi(t) + a(t)\Phi'(t), \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}\right\rangle dt. \tag{41}$$

Here, we use the fact that $\Psi(t) = \Phi(t)$ for $t \in [\hat{T}, m]$ and $a(0) = 0$ and $a(t) = 0$, for all $t \in [\hat{T}, m]$.

By adding (39), (40), and (41), and using (36) we get

$$\frac{d}{d\epsilon} \mathscr{C}(x, \Theta) \Big|_{\epsilon=0} = \frac{1}{2} \Bigg[\int_0^{\hat{T}} x(t)\left\langle \Phi'(t) \odot \xi''(\Phi(t)) - \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}, \right.$$
$$\Psi(t) - \Phi(t) + a(t)\Phi(t)\Big\rangle dt$$
$$+ \int_0^{\hat{T}} x(t)\left\langle \vec{h}\vec{h}^T + \xi'(\Phi(t)) \right.$$
$$\left. - \int_0^t \hat{\Phi}(s)^{-1}\Phi'(s)\hat{\Phi}(s)^{-1}ds, (\Psi(t) - \Phi(t) + a(t)\Phi'(t))'\right\rangle dt \Bigg]$$
$$= \frac{1}{2} \Bigg[\int_0^{\hat{T}} x(t)\Big(\Big\langle \vec{h}\vec{h}^T + \xi'(\Phi(t)) - \int_0^t \hat{\Phi}(s)^{-1}\Phi'(s)\hat{\Phi}(s)^{-1}ds, \Psi(t)$$
$$\left. - \Phi(t) + a(t)\Phi(t)\right\rangle\Big)'dt \Bigg]$$

$$= \frac{1}{2} \left[\int_0^{\hat{T}} \left\langle \vec{h}\vec{h}^T + \xi'(\Phi(t)) - \int_0^t \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds, \ \Psi(t) \right. \right.$$

$$\left. \left. - \Phi(t) + a(t)\Phi'(t) \right\rangle \mu_P(dt) \right].$$

The critical point condition $\frac{d}{d\epsilon}\mathscr{C}(x, \Theta)|_{\epsilon=0} \geq 0$ combined with the above equation proves (34). $\square$

We now turn to the proofs of the propositions stated in Section 3.

Proof of Proposition 5. We will prove by contradiction. Recall that

$$s_x := x^{-1}(0) = \sup\{t \in [0, m] | x(t) = 0\}$$

and assume that $s_x \neq 0$. Then for all $t \in [0, s_x]$, $\hat{\Phi}(t) = \hat{\Phi}(s_x)$, and by Lemma 20 we have

$$\langle \xi''(\Phi(s_x)), \ \Phi'(s_x)^{\circ 2}\rangle \leq \langle \hat{\Phi}(s_x)^{-1} \Phi'(s_x)\hat{\Phi}(s_x)^{-1}, \ \Phi'(s_x)\rangle \tag{42}$$

and

$$\xi'(\Phi(s_x)) = \int_0^{s_x} \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds = \hat{\Phi}(s_x)^{-1} \Phi(s_x)\hat{\Phi}(s_x)^{-1}. \tag{43}$$

Moreover, since $s_x \neq 0$, we can modify $\Phi'(s_x)$ a little bit and set $\Phi'(s_x) = \Phi(s_x)$ without the change of value of $\mathscr{C}(x, \Phi)$. Then combining (42) and (43) yields

$$\left\langle \xi''(\Phi(s_x)), \ \Phi(s_x)^{\circ 2}\right\rangle \leq \langle \hat{\Phi}(s_x)^{-1} \Phi(s_x)\hat{\Phi}(s_x)^{-1}, \ \Phi(s_x)\rangle = \langle \xi'(\Phi(s_x)), \ \Phi(s_x)\rangle,$$

and thus

$$\text{Sum}(\xi''(\Phi(s_x)) \odot \Phi(s_x)^{\circ 2} - \xi'(\Phi(s_x)) \odot \Phi(s_x))$$
$$= \langle \xi''(\Phi(s_x)) \odot \Phi(s_x) - \xi'(\Phi(s_x)), \ \Phi(s_x)\rangle \leq 0. \tag{44}$$

However, straight-forward computation of the derivative implies

$$\xi''(\Phi(s_x)) \odot \Phi(s_x)^{\circ 2} - \xi'(\Phi(s_x)) \odot \Phi(s_x) = \sum_{p \geq 2} p(p-2)(\vec{\beta}_p \otimes \vec{\beta}_p) \odot \Phi(s_x)^{\circ p}$$

and

$$\text{Sum}(\vec{\beta}_p \otimes \vec{\beta}_p) \odot \Phi(s_x)^{\circ p} > 0 \text{ for even } p \text{ and } s_x = 0.$$

where the last inequality comes from the assumption that there exists $p \geq 4$ such that $\vec{\beta}_p \neq 0$, which leads to a contradiction. Thus $s_x = 0$, which is equivalent to say that $0 \in \text{supp } \mu_P$. $\square$

Proof of Proposition 6. Suppose $0 \in \text{supp } \mu_P$ and the existence of a sequence of $u_n \downarrow 0$ such that $u_n \in \text{supp } \mu_P$.

By considering the function $f : [0, m] \to \mathbb{R}$ and $g : [0, m] \to \mathbb{R}$ given respectively by

$$f(u) = \langle \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds, \ \Phi'(u)\rangle$$

and

$$g(u) = \langle \xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1} \Phi'(u)\hat{\Phi}(u)^{-1}, \ \Phi'(u)\rangle$$
$$+ \langle \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\Phi}(s)^{-1} \Phi'(s)\hat{\Phi}(s)^{-1} ds, \ \Phi''(u)\rangle.$$

Then $f(u_n) = 0$ for all $n \geq 1$. By mean value theorem, there exists a sequence $u'_n \downarrow 0$ such that $f'(u_n) = 0$. Notice that $f'(u_n) = g(u_n)$. By the continuity of g at 0, we obtain $g(0) = 0$, which implies that $2\langle \vec{\beta}_2 \otimes \vec{\beta}_2, \Phi'(0)^{\circ 2} \rangle = \langle \hat{\Phi}(0)^{-1} \Phi'(0), \hat{\Phi}(0)^{-1} \Phi'(0) \rangle$. $\square$

Proof of Proposition 7. As μ_P is non-decreasing, it contains at most countably many atoms. Now assume supp μ_P contains countably many isolated points, and we connect the points in supp μ_P by linear interpolation. Then for two consecutive points $s_1 < s_2$ in supp μ_P, define $a_0 := \mu_P([0, s_1])$.

Furthermore, we obtain that for $u \in (s_1, s_2)$, $\Phi''(u) = 0$ and therefore

$$g(u) = \langle \xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle.$$

We also have

$$f(s_1) = f(s_2) = 0 \quad \text{and} \quad \int_{s_1}^{s_2} f(q) dq = 0$$

where

$$f(u) = \langle \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\Phi}(s)^{-1} \Phi'(s) \hat{\Phi}(s)^{-1} ds, \Phi'(u). \rangle$$

Therefore $g(u) = 0$ has 2 solutions between s_1 and s_2.

The relation $g(u) = 0$ is equivalent to say that

$$\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2} \rangle = \langle \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle.$$

We then define $y(u) = \langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2} \rangle^{-\frac{1}{2}}$ and $z(u) = \langle \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle^{-\frac{1}{2}}$. Thus

$$z'(u) = -a_0 \langle \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^{-\frac{3}{2}} \cdot \langle \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle$$

and

$$\begin{aligned}
z''(u) = {} & 3a_0^2 \langle \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^{-\frac{5}{2}} \cdot \langle \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^2 \\
& - 3a_0^2 \langle \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^{-\frac{3}{2}} \\
& \cdot \langle \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t) \rangle \\
= {} & 3a_0^2 \langle \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^{-\frac{5}{2}} \cdot ((\langle \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle^2 \\
& - \langle \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \hat{\Phi}(t)^{-1} \Phi'(t) \rangle \\
& \cdot \langle \hat{\Phi}(t)^{-1} \Phi'(t), \hat{\Phi}(t)^{-1} \Phi'(t) \rangle) \leq 0,
\end{aligned}$$

where the last step uses the Cauchy inequality. Hence $z(u)$ is a concave function of u, this equation can have at most 2 roots in any interval where $\langle \xi''(\Phi(t)), \Phi'(t)^{\circ 2} \rangle^{-\frac{1}{2}}$ is convex.

Now to see that $\langle \xi''(\Phi(t)), \Phi'(t)^{\circ 2} \rangle^{-\frac{1}{2}}$ is convex for $\xi(A) = (\beta_p \otimes \beta_p) \odot A^{\circ p}$, we need to modify Φ without the change of $\mathscr{C}(x, \Phi)$.

For any $\epsilon > 0$, set

$$\Phi_\epsilon(t) = \begin{cases} \frac{t}{s_2} \cdot \Phi(s_2) & \text{for } t \in (s_1 + \epsilon, s_2) \\ \text{some smooth curve that connects } \Phi(s_1) \text{ and } \Phi(s_1 + \epsilon) & \text{for } t \in (s_1, s_1 + \epsilon). \end{cases}$$

Here we set Φ_ϵ mild enough so that $\hat{\Phi}_\epsilon$ is positive definite on $[s_1, s_1 + \epsilon]$. Note that (x, Φ_ϵ) is still a minimizer of $\mathscr{C}$.

Therefore for $u \in [s_1 + \epsilon, s_2]$,

$$y'(u) = -\frac{1}{2}\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{3}{2}} \cdot \langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3}\rangle$$

and

$$
\begin{aligned}
y''(u) &= -\frac{1}{2}\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{3}{2}} \cdot \langle \xi''''(\Phi(u)), \Phi'(u)^{\circ 4}\rangle \\
&\quad + \frac{3}{4}\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{5}{2}} \cdot \langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3}\rangle^2 \\
&= \frac{1}{4}\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{5}{2}} \cdot [3\langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3}\rangle^2 \\
&\quad - 2\langle \xi''''(\Phi(u)), \Phi'(u)^{\circ 4}\rangle \cdot \langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle] \\
&= \frac{1}{4}\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{5}{2}} \cdot [3\langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3}\rangle^2 \\
&\quad - 2\langle \xi''''(\Phi(u)), \Phi'(u)^{\circ 4}\rangle \cdot \langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle] \\
&= \frac{1}{4s_2^6}p^2(p-1)^2(p-2)\langle \xi''(\Phi(u)), \Phi'(u)^{\circ 2}\rangle^{-\frac{5}{2}} \\
&\quad \cdot [(3p-6)t^{2p-6}\langle \beta_p \otimes \beta_p, \Phi(s_2)^{\circ p}\rangle^2 - (2p-6)t^{2p-6}\langle \beta_p \otimes \beta_p, \Phi(s_2)^{\circ p}\rangle^2] \\
&= \frac{1}{4s_2^6}p^3(p-1)^2(p-2)t^{3p-8}\langle \beta_p \otimes \beta_p, \Phi(s_2)^{\circ p}\rangle^{-\frac{5}{2}} \cdot \langle \beta_p \otimes \beta_p, \Phi(s_2)^{\circ p}\rangle^2 \geq 0,
\end{aligned}
$$

which means that $y(u)$ is convex on $[s_1 + \epsilon, s_2]$. We then let ϵ tend to 0 and then get the desired conclusion. $\square$

Proof of Proposition 8. Since both $(u_l^+)_{l \geq 1}$ and $(u_l^-)_{l \geq 1}$ converges to u_0 and lies in supp μ_P, by mean value theorem and continuity of g, we obtain

$$
\begin{aligned}
0 &= \lim_{h \to 0^+} \frac{g(u_0 + h) - g(u_0)}{h} \\
&= \Big\langle \xi'''(\Phi(u_0)) \odot \Phi'(u_0)^{\circ 2} + \xi''(\Phi(u)) \odot \Phi''(u_0) \\
&\quad - 2\mu_P([0, u_0])\langle \hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}, \Phi'(u_0)\rangle \Big\rangle \\
&\quad + \langle \xi'(\Phi(u_0)) + \vec{h}\vec{h}^T - \int_0^{u_0} \hat{\Phi}(s)^{-1}\Phi'(s)\hat{\Phi}(s)^{-1}ds, \Phi'''(u_0)\rangle \\
&\quad + 2\langle \xi''(\Phi(u_0)) \odot \Phi'(u_0) - \hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}, \Phi''(u_0)\rangle.
\end{aligned}
$$

$$
\begin{aligned}
0 &= \lim_{h \to 0^-} \frac{g(u_0 + h) - g(u_0)}{h} \\
&= \Big\langle \xi'''(\Phi(u_0)) \odot \Phi'(u_0)^{\circ 2} + \xi''(\Phi(u)) \odot \Phi''(u_0) \\
&\quad - 2\mu_P([0, u_0))\langle \hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}, \Phi'(u_0)\rangle \Big\rangle \\
&\quad + \langle \xi'(\Phi(u_0)) + \vec{h}\vec{h}^T - \int_0^{u_0} \hat{\Phi}(s)^{-1}\Phi'(s)\hat{\Phi}(s)^{-1}ds, \Phi'''(u_0)\rangle \\
&\quad + 2\langle \xi''(\Phi(u_0)) \odot \Phi'(u_0) - \hat{\Phi}(u_0)^{-1}\Phi'(u_0)\hat{\Phi}(u_0)^{-1}, \Phi''(u_0)\rangle.
\end{aligned}
$$

By comparing the two equations, we obtain $\mu_P([0, u_0]) = \mu_P([0, u_0))$, which implies that μ_P is continuous at u_0. $\quad\square$

Proof of Proposition 9. If u_0 is an isolated point of supp μ_P, it must be a jump discontinuity of μ_P. Now assume that u_0 is not isolated and μ_P is continuous at the point u_0. Then by Lemma 20 and the mean value theorem we obtain

$$\langle \xi''(\Phi(u_0)), \Phi'(u_0)^{\circ 2} \rangle = \langle \hat{\Phi}(u_0)^{-1} \Phi'(u_0), \hat{\Phi}(u_0)^{-1} \Phi'(u_0) \rangle.$$

As $\hat{\Phi}(u_0) = \int_{u_0}^m x(t) \Phi'(t) dt$, which implies that $\hat{\Phi}(u_0)^{-1} \geq (Q - \Phi(u_0))^{-1} \geq Q^{-1}$, we have

$$
\begin{aligned}
\langle \xi''(\Phi(Q)), \Phi'(u_0)^{\circ 2} \rangle &\geq \langle \xi''(\Phi(u_0)), \Phi'(u_0)^{\circ 2} \rangle \\
&\geq \langle (Q - \Phi(u_0))^{-1} \Phi'(u_0), (Q - \Phi(u_0))^{-1} \Phi'(u_0) \rangle \\
&\geq \langle Q^{-1} \Phi'(u_0), Q^{-1} \Phi'(u_0) \rangle
\end{aligned}
$$

which leads to a contradiction. $\quad\square$

Proof of Proposition 10. First note that as $(a, b) \in supp \ \mu$, then by Lemma 20, for all $u \in (a, b)$,

$$F(u) := \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\Phi}(s)^{-1} \Phi'(s) \hat{\Phi}(s)^{-1} ds = 0,$$

and as Φ is differentiable for all $u \in (a, b)$, we can differentiate the equation above and obtain that, for all $u \in (a, b)$

$$G(u) := \xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1} = 0. \tag{45}$$

From (45) and the fact that Φ' is positive semi-definite, we obtain

$$\Phi'(u)^{\frac{1}{2}}(\xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}) \Phi'(u)^{\frac{1}{2}} = 0,$$

which implies that

$$\Phi'(u)^{\frac{1}{2}} [\xi''(\Phi(u)) \odot \Phi'(u)] \Phi'(u)^{\frac{1}{2}} = (\Phi'(u)^{\frac{1}{2}} \hat{\Phi}(u)^{-1} \Phi'(u)^{\frac{1}{2}})^2. \tag{46}$$

As both sides of the equality in (46) are positive semi-definite matrices, their positive semi-definite square root coincide. We thus get the relation

$$\Phi'(u)^{-\frac{1}{2}}(\Phi'(u)^{\frac{1}{2}}(\xi''(\Phi(u)) \odot \Phi'(u)) \Phi'(u)^{\frac{1}{2}})^{\frac{1}{2}} \Phi'(u)^{-\frac{1}{2}} = \hat{\Phi}(u)^{-1}.$$

As we know that $(a, b) \in$ supp μ_P and Φ is twice differentiable in (a, b), therefore $G''(u) = 0$. Consider the Frobenius inner product on $G''(u)$ and $\Phi'(u)$,

$$
\begin{aligned}
0 &= \langle G''(u), \Phi'(u) \rangle \\
&= \langle \xi'''(\Phi(u)) \odot \Phi'(u)^{\circ 2}, \Phi'(u) \rangle \\
&\quad - 2\mu_P([0, u]) \langle \hat{\Phi}(u)^{-1} \Phi'(u_0) \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle \\
&\quad + \langle \xi''(\Phi(u)) \odot \Phi''(u) - \hat{\Phi}(u)^{-1} \Phi''(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle \\
&= \langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3} \rangle - 2\mu_P([0, u]) \langle \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle \\
&\quad + \langle \xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi''(u) \rangle \\
&= \langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3} \rangle - 2\mu_P([0, u]) \langle \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1} \Phi'(u) \hat{\Phi}(u)^{-1}, \Phi'(u) \rangle \\
&= \langle \xi'''(\Phi(u)), \Phi'(u)^{\circ 3} \rangle - 2\mu_P([0, u]) \mathrm{Trace}(\Phi'(u)^{\frac{1}{2}}(\xi''(\Phi(u)) \odot \Phi'(u)) \Phi'(u)^{\frac{1}{2}})^{\frac{3}{2}}.
\end{aligned}
$$

Therefore we get the desired conclusion. $\quad\square$

Proof of Proposition 11. We will prove by contradiction and assume that the SK model is full replica symmetric breaking (FRSB). By Proposition 8, we know that any accumulation point u_0 in supp μ_P with $\Phi'(u_0) > 0$ must be of continuity at $u = u_0$.

Assume that $(a, b) \subset$ supp μ_P. Consider the function $F : [0, m] \to \mathbb{S}^m_+$ and $G : [0, n] \to \mathbb{S}^m_+$ given respectively by

$$F(u) = \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\Phi}(s)^{-1}\Phi'(s)\hat{\Phi}(s)^{-1}ds$$

and

$$G(u) = \xi''(\Phi(u)) \odot \Phi'(u) - \hat{\Phi}(u)^{-1}\Phi'(u)\hat{\Phi}(u)^{-1}.$$

First note by Lemma 20, for any $u \in (a, b)$,

$$F(u) = 0, G(u) = 0, f(u) = 0, g(u) = 0.$$

Therefore we obtain $g'(u) = 0$ for any $u \in (a, b)$.

We also note that on (a, b)

$$\begin{aligned}
g'(u) &= \langle \xi'''(\Phi(u)), \Phi'(u)^{\odot 3}\rangle - 2x(u)\langle \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}, \Phi'(t)\rangle \\
&= -2x(u)\langle \hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}\Phi'(t)\hat{\Phi}(t)^{-1}, \Phi'(t)\rangle \\
&< 0,
\end{aligned}$$

which leads to a contradiction. Here we use the assumption $\xi''' = 0$ derived from the setting of the SK model.

Now we obtain that if $\Phi' > 0$ on $[0, m]$, then the SK model is finitely atomic. Since by Proposition 7, the SK model contains at most 2 consecutive isolated points, we get the desired conclusion that the SK model is at most 1 replica symmetric breaking (1RSB). $\square$

7. Properties of the Crisanti–Sommers functional at zero temperature

In this section we provide some useful characterizations of minimizers of (12).

Theorem 21. *Let $(L, \alpha, \Phi) \in \mathcal{H}(Q)$. Define*

$$g(t) := \int_t^m \bar{g}(s)ds$$

and

$$\bar{g}(t) = \langle \Phi'(t), \bar{G}(t)\rangle$$

where

$$\bar{G}(t) = \xi'(\Phi(t)) - \int_0^t (L - \int_0^s \alpha(q)\Phi'(q)dq)^{-1}\Phi'(s)(L - \int_0^s \alpha(q)\Phi'(q)dq)^{-1}ds.$$

Then (L, α, Φ) is the minimizer of $\mathscr{C}$ if and only if the following equation holds,

$$\xi'(Q) + \vec{h}\vec{h}^T = \int_0^m (L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}\Phi'(t)(L - \int_0^t \alpha(s)\Phi'(s)ds)^{-1}dt$$

and the function g satisfies $\min_{u\in[0,m]} g(u) \geq 0$ and $\gamma_0(S) = \gamma_0([0, m))$, where $S := \{u \in [0, m)|g(u) = 0\}$.

Proof. The proof of this theorem is standard. Assume (L_0, α_0, Φ_0) is minimizer of $\mathscr{C}(L, \alpha, \Phi)$, γ_0 is the corresponding measure induced by $\alpha(s)$ and consider any $(L, \alpha, \Phi) \in \mathscr{K}$. For $0 \le \theta \le 1$, $(L_\theta, \alpha_\theta, \Phi_\theta)$ also lies in $\mathscr{K}$, where $L_\theta = (1 - \theta)L_0 + \theta L$, $\alpha_\theta = (1 - \theta)\alpha_0 + \theta\alpha$ and $\Phi_\theta = (1 - \theta)\Phi_0 + \theta\Phi$.

As (L_0, α_0, Φ_0) minimizes $\mathscr{C}$, we obtain

$$
\frac{\partial \mathscr{C}(L_\theta, \alpha_\theta, \Phi_\theta)}{\partial \theta}\Big|_{\theta=0} = \frac{1}{2}\Bigg[\langle \xi'(Q) + \vec{h}\vec{h}^T, L - L_0 \rangle
$$

$$
+ \int_0^m \langle (L - L_0 - \int_0^t (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds)^{-1}, \Phi_0'(t)\rangle dt
$$

$$
- \int_0^m \langle \xi''(\Phi_0(t)) \odot \Phi_0'(t), \int_0^t (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds\rangle dt
$$

$$
+ \int_0^m \langle (L - \int_0^t \alpha(s)\Phi_0'(s)ds)^{-1}, \Phi'(t) - \Phi_0'(t)\rangle dt
$$

$$
- \int_0^m \langle (L - \int_0^t \alpha(s)\Phi_0'(s)ds)^{-1}\Phi'(t)(L - \int_0^t \alpha_0(s)\Phi'(s)ds)^{-1},
$$

$$
\int_0^t \alpha(s)(\Phi'(s) - \Phi_0'(s))ds\rangle dt
$$

$$
- \int_0^m \langle \xi'''(\Phi_0(t)) \odot (\Phi(t) - \Phi_0(t)) \odot \Phi_0'(t) + \xi''(\Phi_0(t)) \odot (\Phi'(t) - \Phi_0'(t)),
$$

$$
\int_0^t \alpha(s)\Phi_0'(s)ds\rangle dt \Bigg]
$$

$$
\ge 0.
$$

Extra algebra leads to:

$$
\frac{\partial \mathscr{C}(L_\theta, \alpha_\theta, \Phi_\theta)}{\partial \theta}\Big|_{\theta=0}
$$

$$
= \frac{1}{2}\Bigg[\langle L - L_0, \xi'(Q) + \vec{h}\vec{h}^T - \int_0^m (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}\Phi_0'(t)
$$

$$
\times (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}dt\rangle
$$

$$
+ \int_0^m \langle \int_0^t (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds, (L - \int_0^t \alpha_0(s)\Phi'(s)ds)^{-1}\Phi_0'(t)
$$

$$
\times (L - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}\rangle dt
$$

$$
- \int_0^{m_0} \langle \int_0^{t_0} (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds, \xi''(\Phi_0(t) \odot \Phi_0'(t))\rangle dt
$$

$$
+ \int_0^m \langle (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}\Phi_0'(t)(L_0 - \int_0^t \alpha_0\Phi_0'(s)ds)^{-1},
$$

$$
\int_0^t \alpha_0(s)(\Phi'(s) - \Phi_0'(s))ds\rangle dt
$$

$$
- \int_0^m \langle \xi''(\Phi_0(t)) \odot \Phi_0'(t), \int_0^t \alpha_0(s)(\Phi'(s) - \Phi_0'(s))ds\rangle dt
$$

$$+ \int_0^m \langle (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}, \Phi'(t) - \Phi_0'(t)\rangle dt$$

$$- \int_0^m \langle \xi'''(\Phi_0(t)) \odot (\Phi(t) - \Phi_0(t)) \odot \Phi'(t) + \xi''(\Phi_0(t)) \odot (\Phi'(t) - \Phi_0'(t)),$$

$$\int_0^t \alpha_0(s)\Phi_0'(s)ds\rangle dt \Big].$$

Based on the arbitrariness of α, L and Φ, we obtain the relations:

$$\xi'(Q) + \vec{h}\vec{h}^T = \int_0^m (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1} \Phi_0'(t)(L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}dt,$$

$$\int_0^m \langle \int_0^t (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds, (L - \int_0^t \alpha_0(s)\Phi'(s)ds)^{-1}\Phi_0'(t)$$

$$\times (L - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}\rangle dt$$

$$- \int_0^m \langle \int_0^t (\alpha(s) - \alpha_0(s))\Phi_0'(s)ds, \xi''(\Phi_0(t)) \odot \Phi_0'(t)\rangle dt \geq 0,$$

and

$$\int_0^m \langle (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}, \Phi'(t) - \Phi_0'(t)\rangle dt$$

$$- \int_0^m \langle \xi'''(\Phi_0(t)) \odot (\Phi(t) - \Phi_0(t)) \odot \Phi'(t) + \xi''(\Phi_0(t)) \odot (\Phi'(t) - \Phi_0'(t)),$$

$$\int_0^t \alpha_0(s)\Phi_0'(s)ds\rangle dt$$

$$= \int_0^m \alpha_0(t)\langle \Phi(t) - \Phi_0(t), (L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}\Phi_0'(t)(L_0 - \int_0^t \alpha_0(s)\Phi_0'(s)ds)^{-1}dt\rangle$$

$$- \int_0^m \alpha_0(t)\langle \Phi(t) - \Phi_0(t), \xi''(\Phi_0(t)) \odot \Phi'(t)\rangle dt \geq 0$$

Hence, writing

$$\mathcal{Z}(s) = (L_0 - \int_0^s \alpha_0(q)\Phi_0'(q)dq)^{-1},$$

$$\int_0^m (\alpha(t) - \alpha_0(t))\langle \Phi'(t), \xi'(\Phi(t)) - \int_0^t \mathcal{Z}(s)\Phi_0'(s)\mathcal{Z}(s)ds\rangle dt \geq 0.$$

If t is an isolated point in supp γ_0, then $\Phi'(t)$ can be any symmetric matrix, which implies that $\bar{G}(t) = 0$. If t is not isolated, as Φ is continuously differentiable and Lipschitz, we can get the same conclusion based on approximation. Last, define $g(t) := \int_t^m \bar{g}(s)ds$, then $g(t)$ satisfies $\min_{u\in[0,m]} g(u) \geq 0$ and $\gamma_0(S) = \gamma_0([0, m))$ where $S := \{u \in [0, m)|g(u) = 0\}$. The converse direction can be proved by the uniqueness of the minimizer. $\quad\square$

Proposition 22. *The model is replica symmetric at zero temperature if and only if*

$$\xi'(Q) + \vec{h}\vec{h}^T \geq \xi''(Q) \odot Q$$

In this case, the minimizer (L_0, α_0, Φ_0) *is given by*

$$L_0 = Q^{\frac{1}{2}}(Q^{\frac{1}{2}}(\xi'(Q) + \vec{h}\vec{h}^T)Q^{\frac{1}{2}})^{-\frac{1}{2}}Q^{\frac{1}{2}}, \alpha_0 = 0 \text{ and } \Phi_0 = \frac{t}{m}Q.$$

Proof. Firstly, if the model is replica symmetric at zero temperature, then

$$\xi'(Q) + \vec{h}\vec{h}^T - L_0^{-1}QL_0^{-1} = 0.$$

Also as there is no point in supp γ_0, we can define Φ by $\Phi(t) = \frac{t}{m}Q$, and hence $\Phi'(t) = \frac{1}{m}Q$.

We prove this proposition by contradiction. Assume $\xi'(Q) + \vec{h}\vec{h}^T < \xi''(Q) \odot Q$. Then,

$$\bar{g}'(m) = \langle \xi''(Q) \odot \Phi'(m) - L_0^{-1}\Phi'(m)L_0^{-1}, \, \Phi'(m) \rangle$$

$$= \frac{1}{m^2} \langle \xi''(Q) \odot Q - \xi'(Q) - \vec{h}\vec{h}^T, \, Q \rangle > 0,$$

which implies that there exists $s_0 \in (0, m)$, such that $\bar{g}'(s) > 0$ for all $s \in [s_0, m]$. Therefore for all $s \in [s_0, m)$, we obtain $\bar{g}(s) < 0$ and $g(s) < 0$, which leads to a contradiction.

Conversely, if $\xi'(Q) + \vec{h}\vec{h}^T \geq \xi''(Q) \odot Q$, let

$$L_0 = Q^{\frac{1}{2}}(Q^{\frac{1}{2}}(\xi'(Q) + \vec{h}\vec{h}^T)Q^{\frac{1}{2}})^{-\frac{1}{2}}Q^{\frac{1}{2}}, \alpha_0 = 0 \text{ and } \Phi_0 = \frac{t}{m}Q.$$

Then $\int_0^m (L_0 - \int_0^s \alpha_0(q)\Phi'(q)dq)^{-1}\Phi'(s)(L_0 - \int_0^s \alpha_0(q)\Phi'(q)dq)^{-1}ds = L_0^{-1}QL_0^{-1} = \xi'(Q) + \vec{h}\vec{h}^T$. Furthermore,

$$\bar{g}(t) = \langle \xi'(\Phi(t)) - \int_0^t (L_0 - \int_0^s \alpha_0(q)\Phi'(q)dq)^{-1}\Phi'(s)$$

$$\times (L_0 - \int_0^s \alpha_0(q)\Phi'(q)dq)^{-1}ds, \, \Phi'(t)\rangle, \bar{g}(m)$$

$$= 0$$

and

$$\bar{g}'(t) = \frac{1}{m^2} \langle \xi''(\Phi(t)) \odot Q - L_0^{-1}QL_0^{-1}, \, Q \rangle \leq \frac{1}{m^2} \langle \xi''(Q) \odot Q - L_0^{-1}QL_0^{-1}, \, Q \rangle$$

$$= \frac{1}{m^2} \langle \xi''(Q) \odot Q - \xi'(Q) - \vec{h}\vec{h}^T, \, Q \rangle \leq 0,$$

which implies that $g(u) > 0$ for $u \in (0, m)$. Since $S = \emptyset$ and $\gamma_0(S) = 0 = \gamma([0, m))$, we conclude that (L_0, α_0, Φ_0) is minimizer which means that the model is replica symmetric at zero temperature. $\square$

Proof of the FRSB example (13). Recall the constraint

$$Q = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

and the matrix path as $\Phi(q) = \frac{q}{2}Q$ and $\phi(q) = \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle^{-\frac{1}{2}}$.

$$\phi''(q) = \frac{1}{4} \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle^{-\frac{5}{2}} [3\langle \xi'''(\Phi(q)), \Phi'(q)^{\circ 3} \rangle^2$$

$$- 2\langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle \langle \xi''''(\Phi(q)), \Phi'(q)^{\circ 4} \rangle].$$

Also, since $\phi(0) < \sqrt{2}$ and $\phi(2) > 0$, there exists a unique q_0 such that $\phi(q_0) = \frac{1}{\sqrt{2}}(2 - q_0)$. Define a distribution function as follows:

$$x(q) = \begin{cases} -\sqrt{2}\phi'(q) & 0 \leq q \leq q_0, \\ 1 & q \geq q_0. \end{cases}$$

Based on the definition of x and Φ, for $0 \leq q \leq q_0$ we obtain

$$
\begin{aligned}
\hat{\tilde{\Phi}}(q) &= Q - \Phi(q_0) - \sqrt{2}\phi(q_0)\frac{1}{2}Q + \sqrt{2}\phi(q)\frac{1}{2}Q \\
&= Q - \frac{q_0}{2}Q - \frac{1}{2}(2 - q_0)Q + \frac{\sqrt{2}}{2}\phi(q)Q \\
&= \sqrt{2}\phi(q)\Phi'(q)
\end{aligned}
$$

therefore

$$
\langle \hat{\tilde{\Phi}}(q)^{-1}\Phi'(q), \hat{\tilde{\Phi}}(q)^{-1}\Phi'(q) \rangle = \phi(q)^{-2} = \langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle.
$$

Recall the function

$$
f(u) = \langle \xi'(\Phi(u)) + \vec{h}\vec{h}^T - \int_0^u \hat{\tilde{\Phi}}(s)^{-1}\Phi'(s)\hat{\tilde{\Phi}}(s)^{-1}ds, \Phi'(u) \rangle.
$$

Since $f(0) = 0$, we then obtain that $f(q) = 0$ for $0 \leq q \leq q_0$. Moreover, since

$$
\phi(q) > \frac{1}{\sqrt{2}}(2 - q) = \langle (Q - \Phi(q))^{-1}\Phi'(q), (Q - \Phi(q))^{-1}\Phi'(q) \rangle,
$$

we obtain

$$
\langle \xi''(\Phi(q)), \Phi'(q)^{\circ 2} \rangle = \phi(q)^{-\frac{1}{2}} < \langle (Q - \Phi(q))^{-1}\Phi'(q), (Q - \Phi(q))^{-1}\Phi'(q) \rangle.
$$

Therefore if we define $h(s) = \int_0^s f(q)dq$, we get that $h(s) = 0$ for $s < q_0$ and $f(s) < 0$ for $s > q_0$.

Thus (x, Φ) minimizes the Crisanti–Sommers functional $\mathscr{C}$ in this case. $\quad\square$

Declaration of competing interest

Acknowledgements

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